Annals of Mathematics, 126 (1987), 35–108

The N-particle scattering problem: asymptotic completeness for short-range systems

By I. M. SIGAL* AND A. SOFFER†

Contents

0. Introduction
1. Hamiltonians
2. Asymptotic completeness
3. Results and strategy of proof
4. Geometry of N-body systems
5. Channel expansion
6. Propagation estimates I. The non-threshold case
7. Propagation estimates II. The threshold case
8. Phase-space partition of unity
9. Proof of asymptotic completeness
Appendix

Abstract

We prove the asymptotic completeness for short-range quantum-mechanical systems consisting of an arbitrary number of particles.

Introduction

Since the birth of quantum mechanics more than 60 years ago, most of the effort in understanding its mathematical structure was directed into two areas: stability of matter and scattering theory. It was already realized by the founders that the unique features of quantum mechanics, such as the uncertainty and Pauli principles, should account for the stability of quantum systems. The mathematical formulation and solution of the problem (the implosion part) was given in a few powerful strokes ([DL], [LT]; see also [Fe]; the explosion part of the problem was recently successfully attacked in [R], [Sig2, 4], [BL], [Lieb1, 2],

*Supported by NSF No. DMS-8507040.
†Weizmann Fellow of Mathematics and Physics.

[LSST]). On the other hand, the scattering theory went through a long and painful period of partial solutions. It took more than 25 years of involved and miraculous resolvent estimates (see [Sig3] for a review; here we mention only the pioneering works [P], [I], [KK] on the one-body problem and [F] on the three-body problem) before the dominant role of the classical behavior in one-body scattering was realized ([E1]). It took almost 10 more years to begin to unravel the unique feature of the many-body scattering, the interplay between the classical behavior of the centers-of-mass of clusters participating in scattering and the purely quantum (stable) motion within these clusters. The progress reported in the present work came about when the authors recognized that in some regions of the phase-space the quantum phenomena take over and for some arbitrary remote time intervals there exist probabilities, bounded away from 0, for the system to move into certain classically forbidden regions. As a result the uniform estimates suggested by the classical considerations break down and should be replaced by some sort of time average estimates. (Note that estimates of time averages are an integral part of the Enss approach [E1–6], [SKM], [K].)

We now describe the N-body scattering problem and the main ingredients of our approach. Consider an N-body system in the center-of-mass frame and let H be its self-adjoint Schrödinger operator (or Hamiltonian; see the next section for the definitions). The *scattering theory* studies large time behavior of the orbits $e^{-iHt}\psi$, for the states ($\equiv$ vectors) ψ orthogonal to the bound states ($\equiv$ eigenfunctions of H).

One expects that the probability that at time t the system is not broken into *independently moving*, *stable* subsystems vanishes as $t \to +\infty$. Cast in precise mathematical terms this problem is called *asymptotic completeness*.

In this paper we prove asymptotic completeness for short-range systems with an arbitrary number of particles (see announcement in [SigSof1]). Previously, asymptotic completeness was established for short-range systems with generic potentials ([F], [GM], [T], [How], [H], [HP], [Y] (3 particles), [H], [HP] (4 particles), [Sig1, 3, 5] (N particles). Note also pioneering works [He] on single-channel systems, [L1, 2] on repulsive potentials and [IO] on weak potentials which first utilized in the N-body situation important ideas of [Ka1, 2]). For explicitly defined classes of potentials it was shown for three-body short-range ([LS], [E2–6], see also [M2, 3], [MS], [SKM], [K]) and some long-range ([E6]) systems. There are also partial results for N-body systems due to [MS], [K]. A brief review of the papers up to 1983 can be found in [Sig3]. There are two basic ingredients to our approach: phase-space analysis and propagation estimates. The phase-space analysis was introduced into the scattering theory in [E1] (see [AH], [HS] for earlier ideas). Its main idea is to localize the Schrödinger states, $\psi_t \equiv e^{-iHt}\psi$, in the phase-space and to use different estimates in different phase-space regions. We realize this program by constructing a phase-space partition of unity $\{j_a(x, p)\}$ (here a runs through all break-ups and $p = -i\,\mathrm{grad}_x$), whose members are supported (in the sense made precise in Section 3) in phase-space regions which characterize different channels. Moreover, the boundaries of these regions lie in the classically forbidden (non-propagation) set. The next step is to introduce the modified wave operators,

$$W_a^{\pm} = s - \lim_{t \to \pm\infty} e^{iH_a t} j_a(x, p) e^{-iHt}.$$

Such operators for configuration space partitions of unity were introduced and applied to study of asymptotic completeness in [DS]. (This work, however, did not have the notion of a propagation set which is essential for the geometric decoupling of channels.) Then the existence of these operators is equivalent to the statement of asymptotic completeness. To prove the existence of $W_a^{\pm}$ we use, via the Cook-Kato argument, the second ingredient of our approach: the propagation estimates. These estimates pinpoint the phase-space behavior of the Schrödinger state ψ_t. They express the intuitive picture that after the collision a system in question disintegrates into a number of stable clusters whose centers-of-mass follow the classical trajectories $x = \nabla |k|^2 t$, where k is the classical momentum and $|k|^2$, the kinetic energy function (actually, we prove only $x \| \nabla |k|^2$ but the more precise statement $x = \nabla |k|^2 t$ follows from our estimates ([SigSof2, 3])). Again, we emphasize that we establish this picture for a certain time averaging and it probably breaks down if uniform (in time) estimates on decay of probabilities outside of the propagation region are required.

To prove the propagation estimates we study the evolution of certain operators (observables) characterizing the scattering process. This in turn is reduced to (local) positivity estimates of the commutators of H with these operators, restricted to energy intervals (energy shells). Our basic tool here is certain channel expansions of basic commutators. The positivity estimates of the commutators of H with $A = \frac{1}{2}(x \cdot p + p \cdot x)$ and with related operators were studied in [P], [Ka2], [L1–7] and developed into a powerful tool by Mourre ([M1, 2]; see also [PSS], [FH1, 2]). The culminating point of this development is the Mourre estimate and one of our main expansions can be considered as a further refinement of this estimate. Note that the positivity estimates for commutators of a pseudo-differential operator of interest with suitable pseudo-differential operators were used in PDE by [Hö] to prove the celebrated theorem on propagation of singularities. The latter problem is, in some sense, dual to the scattering problem: In scattering theory one studies the propagation of singularities in the Fourier (cotangent) space (i.e., the asymptotic behavior in the original, configuration space).

The distinguishing feature of many-body systems, compared to 2-body ones, is in the geometry of their potentials. The potential $V(x)$ of a many-body system does not vanish as $|x| \to \infty$ along certain planes X_a, where a labels different break-ups of the system in question into subsystems. Such a potential is not Δ-compact and, in general, changes the spectrum of $-\Delta$ qualitatively. In this language the statement of asymptotic completeness expresses the fact that as $t \to +\infty$, $e^{-iHt}\psi$ approaches a superposition of waves propagating freely (classically) along the planes X_a while committed to a bounded (quantum) motion in the perpendicular directions. The connection between the geometry of a many-body potential and spectral properties of the corresponding Schrödinger operator has been studied intensively in recent years (see e.g., [Z], [SS], [E2], [DHSV], [Sim], [Sig2]). The geometric methods developed in these papers play an important role in our work as well.

Finally, we mention some of the conventions we use. $F(A \in \Omega)$, where A is a self-adjoint operator and Ω, an interval in $\mathbf{R}$ of non-zero length, stands for $f(A)$, where f is a smoothed characteristic function of Ω. Construction of $f(A)$ will be specified in every case separately.

For an interval Δ and the Schrödinger operator H we use also the notation $F_\Delta(H)$ with $F_\Delta(s)$ a smooth function, $0 \le F_\Delta(s) \le 1$, $F_\Delta(s) = 1$ for $s \in \Delta$ and $= 0$ for $s \notin (\inf \Delta - \delta, \sup \Delta + \delta)$ with $\delta = \frac{1}{10}|\Delta|$.

$P_\Delta(A)$ stands for the spectral projection for a self-adjoint operator A.

$$D(A) = \text{domain of } A, \qquad R(A) = \text{range of } A.$$

Let A and B be two closed operators on $\mathscr{H}$. We say that B is *A-bounded* if $D(B) \supset D(A)$. We say that B is *A-compact* if it is A-bounded and, considered as an operator from $D(A)$, equipped with the graph norm, to $\mathscr{H}$, is compact.

$$\langle x \rangle = \left(1 + |x|^2\right)^{1/2},$$

$$B = O(|x|^{-\alpha}) \text{ means that the } \langle x \rangle^{\beta} B \langle x \rangle^{\alpha-\beta} \text{ are bounded}$$
$$\text{for any } \beta \in [0, \max(\alpha, 2)],$$

$$B_1 = O_1(|x|^{-\alpha}) \text{ means that } (H+i)^{-1}B_1 \text{ and } B_1(H+i)^{-1} \text{ are } O(|x|^{-\alpha}),$$

$$B_2 = O_2(|x|^{-\alpha}) \text{ means that } (H+i)^{-1}B_1(H+i)^{-1} \text{ is } O(|x|^{-\alpha}),$$

$$A \doteq B \text{ means that } A = B + O\left(|x|^{-1-\varepsilon}\right),$$

$$A \doteqdot B \text{ means that } A = B + O_1\left(|x|^{-1-\varepsilon}\right),$$

$$A \overset{\varepsilon}{=} B \text{ means that } \|A - B\| \le \varepsilon,$$

and similarly for the inequalities. For instance,

$A \overset{\varepsilon}{\geq} B$ means that there exists C, $\|C\| < \varepsilon$ such that $A \geq B + C$.

Acknowledgement. The authors are grateful to W. Hunziker and B. Simon for useful discussions. The research on this project began when both authors were at the Weizmann Institute.

1. Hamiltonians

Consider an N-body system in the physical space $\mathbf{R}^\nu$. The configuration space in the center-of-mass frame is ([SS])

$$X = \left\{ x \in R^{\nu N} \middle| \sum m_i x_i = 0 \right\} \tag{1.1}$$

with the inner product

$$\langle x, y \rangle = 2 \sum m_i x_i \cdot y_i. \tag{1.2}$$

Here $m_i > 0$ are masses of the particles in question. The *Schrödinger operator* of such a system is

$$H = -\Delta + V(x) \quad \text{on } L^2(X). \tag{1.3}$$

Here Δ is the Laplacian on X and

$$V(x) = \sum V_{ij}(x_i - x_j), \tag{1.4}$$

where (ij) runs through all the pairs satisfying $i < j$. We assume that the potentials V_{ij} are real and obey: $V_{ij}(y)$ are Δ_y-compact.

It is shown in [Com] (see also [RSII], [Sig3]) that under the conditions above the Kato theorem applies and H is self-adjoint on $D(H) = D(\Delta)$. Moreover, by a simple application of the Hölder and Young inequalities [RSII] and by a standard approximation argument (see [RSII], [Sig3]) one shows that if

$$V_{ij} \in L^r(\mathbf{R}^\nu) + \left(L^\infty(\mathbf{R}^\nu)\right)_\varepsilon, \quad \text{where } r > \frac{\nu}{2},$$

and the subscript ε indicates that the L^∞-component can be taken arbitrarily small, then V_{ij} is Laplacian compact.

Now we describe the decomposed systems. Denote by $a, b, \ldots,$ *partitions* of the set $\{1, \ldots, N\}$ into non-empty disjoint subsets, called clusters. The relation $b \subset a$ means that b is a refinement of a. Then $a_{\min}$ is the partition into N clusters $(1), \ldots, (N)$. Usually, we assume that partitions have at least two clusters. $\#(a)$ denotes the number of clusters in a. We also identify pairs $l = (ij)$ with partitions on $N - 1$ clusters: $(ij) \leftrightarrow \{(ij)(1) \ldots (\hat{i}) \ldots (\hat{j}) \ldots (N)\}$.

We define the *intercluster interaction* for a partition a as

(1.5) $I_a =$ sum of all potentials linking different clusters in a.

For each a we introduce the truncated Hamiltonian:

$$H_a = H - I_a. \tag{1.6}$$

These operators are clearly self-adjoint. They describe the motion of the original system broken into non-interacting clusters of particles.

Our method is based on the localization of operators in the *phase-space* $\mathscr{F} = X \times X'$. Here and henceforth, the prime stands for taking the dual of the space in question. The dual (momentum) space X' is identified with

$$X' = \left\{k \in R^{\nu N} \middle| \sum k_i = 0\right\}$$

which has the inner product

$$\langle k, u \rangle = \sum \frac{1}{2m_i} k_i \cdot u_i.$$

Thus $|k|^2$ is the symbol of $-\Delta$ and $-\Delta = |p|^2$ with $p = -i\,\mathrm{grad}_x$. We use extensively the natural bilinear form on $X \times X'$:

$$\langle x, k \rangle = \sum x_i \cdot k_i.$$

2. Asymptotic completeness

In this section we cast the intuitive notion of asymptotic completeness into precise mathematical terms and discuss different aspects of this notion. To this end we need the notion of channel. The channel is a scenario according to which the scattering process develops. Different scenarios are labeled by an asymptotic state of the system in question at $t = -\infty$ or $t = +\infty$. Thus a channel is described by a pair: $\alpha = (a, m)$, where a is a decomposition of the system into subsystems and m specifies a stable motion within each subsystem of this decomposition.

We proceed to the formal definitions. For each cluster decomposition a, define the configuration space of the relative motion of the clusters in a: $X_a = \{x \in X | x_i = x_j$ if i and j belong to same cluster of $a\}$ and the configuration space of the internal motion within those clusters:

$$X^a = \left\{x \in X \middle| \sum_{i \in C} m_i x_i = 0 \quad \text{for all } C \in a\right\}.$$

Clearly, X_a and X^a are orthogonal in inner product (1.2) and they span X:

$$X = X^a \oplus X_a.$$

Given generic vectors $x \in X$ and $k \in X'$, their projections on X^a, X_a and X'_a will be denoted by x^a, x_a and k_a, respectively. Accordingly, the momentum canonically conjugate to x_a and corresponding to k_a will be denoted by p_a.

If i and j belong to some cluster of a, then $x_i - x_j \in X^a$. This elementary fact yields the following decomposition [SS]:

$$H_a = H^a \otimes 1 + 1 \otimes T_a \quad \text{on } L^2(X) = L^2(X^a) \otimes L^2(X_a). \tag{2.1}$$

Here H^a is the Hamiltonian of the non-interacting a-clusters with their centers-of-mass fixed at the origin, acting on $L^2(X^a)$, and $T_a = |p_a|^2$, the kinetic energy of the center-of-mass motion of those clusters.

The eigenvalues of H^a, whenever they exist, will be denoted by ε_α, where $\alpha = (a, m)$ with m, the number of the eigenvalue in question counting the multiplicities. For $a = a_{\min}$, we set $\varepsilon_\alpha = 0$. The pairs α are in a one-to-one correspondence with the channels of H and will be used to label the latter. The set $\{\varepsilon_\alpha$, channels $\alpha\}$ is called the *threshold set* of H and ε_α's are called the *thresholds* of H. (For more details see [Sig3].)

For a channel α we define

(a) The channel Hilbert space

$$\mathscr{H}_\alpha = L^2(X_a);$$

(b) The channel Hamiltonian

$$H_\alpha = \varepsilon_\alpha + T_a \quad \text{acting on } \mathscr{H}_\alpha$$

(we denote by the same symbol also the natural extension of this operator to the entire $L^2(X)$);

(c) The channel identification operator

$$J_\alpha: \mathscr{H}_\alpha \to L^2(X): J_\alpha u = \psi^\alpha \otimes u$$

where ψ^α is the eigenfunction of H^a corresponding to ε_α.

We say that the many-body system in question is *short-range* if the pair potentials $V_{ij}(y)$ vanish at ∞ faster than $|y|^{-1}$ (the precise definition will be given in the next section). We say that the short-range many-body system is *asymptotically complete* if for any $\psi \in L^2(X)$ orthogonal to all eigenfunctions of H and for any $\varepsilon > 0$ there exist a finite subset, a_ε of channels and for each α from this subset, vectors $u^\pm_{\alpha,\varepsilon} \in \mathscr{H}_\alpha$ such that

$$\lim_{t \to \pm\infty} \left\| e^{-iHt}\psi - \sum_{\alpha \in a_\varepsilon} J_\alpha e^{-iM_\alpha t} u^\pm_{\alpha,\varepsilon} \right\| \le \varepsilon. \tag{2.2}$$

To establish a connection with the standard definition of asymptotic completeness we recall that the channel wave operators $\Omega^\pm_\alpha$ are defined as

$$\Omega^\pm_\alpha = s - \lim_{t \to \pm\infty} e^{iHt} J_\alpha e^{-iH_\alpha t}$$

on $\mathscr{H}_\alpha$, whenever these limits exist. Under the assumptions on potentials used in this paper, these strong limits do exist (see [Hu1, 2], [RSIII], [Sig3]). The existence of $\Omega_\alpha^\pm$ implies that they intertwine H and H_α: $e^{-iHt}\Omega_\alpha^\pm = \Omega_\alpha^\pm e^{-iH_\alpha t}$ and that their ranges are orthogonal:

$$\operatorname{Ran}\Omega_\alpha^\pm \perp \operatorname{Ran}\Omega_\beta^\pm \quad \text{for } \alpha \neq \beta.$$

The channel wave operators are said to be *complete* if

$$\bigoplus_\alpha \operatorname{Ran}\Omega_\alpha = \operatorname{Ran}(1 - P),$$

where P is the eigenprojection of H corresponding to point spectrum. Usually, the system is said to be asymptotically complete if its channel wave operators are complete (see e.g., [Hu1], [RSIII], [Sig3]). This definition is equivalent to ours as shown in:

PROPOSITION 2.1. *Assume the channel wave operators exist. Then the short-range many-body system is asymptotically complete if and only if its channel wave operators are complete.*

Proof. Assume the channel wave operators are complete. Then any ψ orthogonal to the eigenfunctions of H can be written as

$$\psi = \sum \psi_\alpha^\pm \quad \text{with } \psi_\alpha^\pm \in \operatorname{Ran}\Omega_\alpha^\pm.$$

Let $\psi_\alpha^\pm = \Omega_\alpha^\pm u_\alpha^\pm$. Then

$$\left\| e^{-iHt}\psi - \sum J_\alpha e^{-iH_\alpha t} u_\alpha^\pm \right\| = \left\| \psi - \sum e^{iHt} J_\alpha e^{-iH_\alpha t} u_\alpha^\pm \right\| \to 0,$$

which shows that the system is asymptotically complete.

Now assume that the system is asymptotically complete. Let $\psi \in \operatorname{Ran}(1 - P)$. Then for any $\varepsilon > 0$ there exist $u_{\alpha,\varepsilon}^\pm$ such that $\lim_{t\to\pm\infty}\|e^{-iHt}\psi - \sum J_\alpha e^{-iH_\alpha t}u_{\alpha,\varepsilon}^\pm\| < \varepsilon$. This implies

$$\left\| \psi - \sum \Omega_\alpha^\pm u_{\alpha,\varepsilon}^\pm \right\| \leq \varepsilon.$$

Since $\oplus \operatorname{Ran}\Omega_\alpha^\pm$ is closed, we have $\psi \in \oplus \operatorname{Ran}\Omega_\alpha^\pm$. □

Our next goal is to reduce the statement of asymptotic completeness to a certain "geometric" statement which does not involve the dynamical notion of bound state (for subsystems). We say that a short-range many-body system described by H is *asymptotically clustering* at an energy E, if there exists an interval Δ around E such that for any $\psi \in \operatorname{Ran} P_\Delta(H)$, orthogonal to the eigenfunctions of H there exist vectors $\psi_{a,E}^\pm \in L^2(X)$ such that

$$\left\| e^{-iHt}\psi - \sum e^{-iH_a t}\psi_{a,E}^\pm \right\| \to 0 \quad \text{as } t \to \pm\infty. \tag{2.3}$$

In other words the asymptotic clustering describes the fact that as $t \to \pm\infty$, the system breaks down into independently moving subsystems. However, it does not say anything about whether the resulting subsystems are stable or not. Nevertheless, we expect that these subsystems which are not stable split further into smaller fragments and so on till only stable clusters remain.

Henceforth, the channels corresponding to a cluster decomposition a will be called, sometimes, *a-channels*.

PROPOSITION 2.2. *Suppose the system in question and all of its subsystems are asymptotically clustering for all non-threshold energies. Then it is asymptotically complete.*

Proof. Let ψ be orthogonal to the eigenfunctions of H. Given ε_1 we can find a compact interval Δ avoiding all the thresholds of H and such that

$$\|\psi - P_\Delta\psi\| \le \varepsilon_1. \tag{2.4}$$

Let $\varphi = P_\Delta\psi$. For any $E \in \Delta$, the system is clustering. By the compactness argument there exists a finite number of intervals Δ_i with centers at E_i, covering Δ, and having the clustering property. Write

$$\varphi = \sum \varphi_i \quad \text{with } \varphi_i \in \operatorname{Ran} P_{\Delta_i},$$

orthogonal to the eigenfunctions of H. By the above there exist $\varphi_{a,i}^{\pm}$ with the property

$$\left\| e^{-iHt}\varphi_i - \sum e^{-iH_at}\varphi_{a,i}^{\pm} \right\| \to 0 \quad \text{as } t \to \pm\infty.$$

This yields

$$\left\| e^{-iHt}\varphi - \sum e^{-iH_at}\varphi_a^{\pm} \right\| \to 0 \tag{2.5}$$

with $\varphi_a^{\pm} = \sum_i \varphi_{a,i}^{\pm}$. Let now $P_a = \sum_{a\text{-chan}} P_\alpha$ be the eigenprojection of $H^a \otimes \mathbf{1}$ corresponding to its point spectrum. Let ψ^α be the corresponding bound states. Define $\chi_\alpha^{\pm} = \langle \psi^\alpha, \varphi_a^{\pm} \rangle$ and $\psi_a^{\pm} = (1 - P_a)\varphi_a^{\pm}$. With these notions and taking into account that

$$e^{-iH_at}P_\alpha = e^{-iH_\alpha t}P_\alpha \quad \text{and} \quad P_\alpha\varphi_a^{\pm} = J_\alpha\chi_\alpha^{\pm},$$

(2.4) and (2.5) imply

$$\lim_{t\to\pm\infty} \left\| e^{-iHt}\psi - \sum_{a\text{-chan}} J_\alpha e^{-iH_\alpha t}\chi_\alpha^{\pm} - \sum e^{-iH_at}\psi_a^{\pm} \right\| < \varepsilon.$$

Since $\psi_a^{\pm}$ as functions of x^a are orthogonal to the eigenfunctions of H^a and since the asymptotic clustering is assumed for all subsystems and therefore for all H^a, we can apply the above analysis to each $e^{-iH_at}\psi_a^{\pm} = e^{-iT_at}(e^{-iH^at}\psi_a^{\pm})$. Collecting into $u_\alpha^{\pm}$ all the $\chi_\alpha^{\pm}$'s coming from different a's we arrive at (2.2). □

Discussion 2.3. The statement of asymptotic clustering does not require any information about the point spectrum of the subsystems. In some sense it is a geometrical statement. We explain this point in more detail in Section 4. The asymptotic clustering is the statement we prove in what follows.

3. Results and strategy of proof

In this section we formulate our main result and outline its proof. We begin with the assumptions on potentials which we use in this work.

(A) $V_{ij}(y)$ are Δ_y-compact;

(B) $\langle y\rangle^{1+\theta}|\nabla V_{ij}(y)|$ are Δ_y-bounded for some $\theta > 0$;

(C) $|y|^2 \dfrac{\partial^2}{\partial y_k\, \partial y_l} V_{ij}(y)$ are Δ_y-bounded;

(D) $V_{ij}(y)\langle y\rangle^{\mu}$ are Δ_y-bounded.

By a short-range system we understand a system obeying (A)–(C) and (D) with $\mu > 1$.

The main result of this paper is:

THEOREM 3.1. *Assume an N-body system is described by potentials obeying conditions* (A)–(D) *with* $\mu > 1$. *Then asymptotic completeness holds for this system.*

The proof of Theorem 3.1 is given in Sections 4–9. Conditions (B) and (C) can be relaxed if we use more delicate methods of controlling various commutators of H. Condition (D) can be also relaxed if we exercise more care in our estimation. Since our priority in this work is presentation of the method, we use conditions more restrictive than necessary.

The feature which distinguishes an N-body Schrödinger operator from a 2-body one, creating a different level of complexity, is that in the many-body case, the potential $V(x)$ does not vanish as $|x| \to \infty$ along the planes X_a, where a runs through all the partitions. Note that operators of the form $-\Delta + V(x)$ with $V(x)$ vanishing at ∞ except for a certain system of planes, were first studied in [Ag]. Results of this paper can be generalized to such operators.

Strategy of the proof. We derive asymptotic completeness from two statements which summarize two basic ingredients of our approach: decoupling of channels and propagation estimates. The decoupling is realized by construction of an appropriate partition of unity on the phase-space (Section 5) and the propagation (or more precisely, non-propagation) estimates are proven by using positivity of commutators of H with appropriate observables. This positivity tells that the expectations of these observables along Schrödinger states increase (Sections 7–8).

A prominent role in our analysis is played by the operator

$$\gamma = \tfrac{1}{2}(\hat{x}\cdot p + p\cdot\hat{x}),$$

where $\hat{x}$ should be understood as

$$\hat{x} = x/\langle x\rangle \quad \text{with } \langle x\rangle = \sqrt{1+|x|^2}\,.$$

THEOREM 3.2. *The operator γ is self-adjoint on its natural domain and the sets $\mathcal{S}(X)$ and $D(|p|^n)$ (any $n = 1,2,\ldots$) are cores of γ.*

Proof. The vector field $v(x) \equiv x/\langle x\rangle$ is bounded and has bounded derivatives to any order. Hence it generates the global flow $\varphi_\theta(x)$ which is bounded and has derivatives in x to any order. Define the one-parameter family of unitary operators

$$U(\theta)\colon f \to \sqrt{J_\theta}\, f\circ\varphi_\theta,$$

where J_θ is the Jacobian of the transformation $x \to \varphi_\theta(x)$. Since φ_θ is a flow, J_θ is strictly positive. Clearly, for any θ, $U(\theta)$ maps $D(|p|^n)$ into $D(|p|^n)$ $(n = 1,2,\ldots)$ and $\mathcal{S}(X)$ into $\mathcal{S}(X)$. However, as a standard computation shows, γ is the generator of $U(\theta)$. Hence γ is self-adjoint and $D(|p|^n)$ and $\mathcal{S}(X)$ are cores of γ. □

Now we introduce a class of operators which appears naturally in our analysis. Those are operators of the form

$$J \equiv \sum j_i(x) f_i(p)\varphi_i(\gamma), \tag{3.1}$$

where the sum is finite, all functions are smooth and bounded and, in addition, the j_i obey the estimate $|\mathcal{D}^\alpha j_i(x)| \le C_\alpha\langle x\rangle^{-|\alpha|}$ for all α. Such operators will be called the *phase-space operators*. The function

$$j(x,k) = \sum j_i(x) f_i(k)\varphi_i(\hat{x}\cdot k) \tag{3.2}$$

will be called the *symbol* of the operator J. We say also that the phase-space operator, J, is *supported* in the phase-space region (which is denoted by f-supp J)supp j.

Fix energy $E \in \mathbf{R}$. A channel α is said to be *open* if $E - \varepsilon_\alpha \ge 0$. This condition is suggested by the law of conservation of energy applied to the channel α:

$$\underbrace{E}_{\text{total energy}} = \underbrace{\varepsilon_\alpha}_{\substack{\text{internal energy}\\ \text{of stable clusters}}} + \underbrace{|p_a|^2}_{\substack{\text{kinetic energy of}\\ \text{c-of-m motion of clusters}}} \tag{3.3}$$

Thus $E - \varepsilon_\alpha$ is the kinetic energy available to the clusters of α. For an open channel α, introduce $\kappa_\alpha = \sqrt{E - \varepsilon_\alpha}$. Furthermore, let $\Sigma_E = \{\pm\kappa_\alpha;\ \text{all open } \alpha\}$.

Below we use the symbol $u\|v$ to signify that the vectors u and v are parallel (multiples of each other). Recall also that channels corresponding to cluster decomposition a are called *a-channels*. Given E, define the set

$$PS_E = PS_E^+ \cup PS_E^- , \tag{3.4}$$

where

$$PS_E^\pm = \bigcup_{\text{open } \alpha} \{(x,k) \in \mathscr{F} | x^a = 0,\ x^c \neq 0\ \forall c \supsetneqq a,\ x_a \| \pm \nabla |k_a|^2,\ |k_a|^2 \in \Sigma_E\}. \tag{3.5}$$

The sets $PS_E^\pm$ describe different possibilities of outgoing/incoming free motion of non-interacting stable clusters of the total energy E. This free motion develops along classical trajectories: the coordinates and momenta of the cluster centers-of-mass are parallel or antiparallel depending on whether the clusters are outgoing or incoming. The restriction on the kinetic energy of this free classical motion stems from the energy conservation law and the fact that the stable internal motion of the clusters is described by bound state of their internal Hamiltonian and, consequently, the energy of the internal motion is given by the corresponding eigenvalue. This mixture of classical and quantum-mechanical pictures is the unique feature of the many-body scattering theory. The next theorem shows that asymptotically the quantum evolution $\psi_t = e^{-iHt}\psi$ with ψ having the energy near E becomes localized in the phase-space region $PS_E^\pm$.

Given E, we say that a subset of the phase space $\mathscr{F} = X \times X'$ *is a propagation set at the energy E* if for any phase-space operator J supported outside of the set in question there is a small interval Δ around E such that

$$\int_{-\infty}^{\infty} \left\| J \frac{1}{\sqrt{\langle x \rangle}} e^{-iHt}\psi \right\|^2 dt \leq C\|\psi\|^2 \tag{3.6}$$

for any $\psi \in \operatorname{Ran} P_\Delta(H)$ and with $C < \infty$ and independent of ψ.

Remark 3.3. This definition is equivalent to one in which the phase-space operators are replaced by more general pseudodifferential operators. However, since, as was mentioned above, the phase-space operators arise naturally in our approach we restrict our definitions to such operators.

Much more importantly, we can refine the definition of the propagation set by breaking it into two pieces each defined through one of the inequalities

$$\pm \int_0^{\pm\infty} \left\| J \frac{1}{\sqrt{\langle x \rangle}} e^{-iHt}\psi \right\|^2 dt \leq C\|\psi\|^2. \tag{3.7}$$

Respectively, PS_E^+ (resp. PS_E^-) is a candidate for the future (resp. past) propagation set. The further refinement of the notion of the propagation set would be to introduce the coefficient of proportionality in the relation $x_a \| \pm \nabla |k_a|^2$ – time. These notions and their ramifications will be discussed elsewhere ([SigSof3]).

In what follows, the energy E is assumed to be inside of the continuous spectrum of H; i.e., $E \geq \min\{\varepsilon_\alpha | \#(a(\alpha)) > 1\}$.

I. *First we show that under conditions* (A)–(C) (*note: condition* (D) *is not required*) *and for* E, *away from the thresholds and eigenvalues of* H, PS_E *is the propagation set at the energy* E.

This statement is proved in Sections 7–8. Here we explain some of the ingredients of this proof.

To prove inequalities of sort (3.7) we will estimate from below commutators of certain observables with H. To connect these estimates with the time-dependent propagation estimates we use an elementary lemma:

LEMMA 3.4. *Let for some fixed* Δ *and an* H^n*-bounded operator* F, *the estimate*

$$(3.8) \qquad P_\Delta i[H, F] P_\Delta \geq P_\Delta \frac{1}{\sqrt{\langle x \rangle}} F_0^2 \frac{1}{\sqrt{\langle x \rangle}} P_\Delta - \sum_{i=1}^{m} P_\Delta \frac{1}{\sqrt{\langle x \rangle}} F_i^2 \frac{1}{\sqrt{\langle x \rangle}} P_\Delta$$

hold for some self-adjoint, bounded operators F_i, *satisfying*

$$(3.9) \qquad \int_{-\infty}^{\infty} \left\| F_i \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C_1 \|\psi\|^2, \qquad i \geq 1,$$

for all $\psi \in \operatorname{Ran} P_\Delta$ *and with* C_1 *finite and independent of* ψ. *Then*

$$(3.10) \qquad \int_{-\infty}^{\infty} \left\| F_0 \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C_2 \|\psi\|^2$$

for all $\psi \in \operatorname{Ran} P_\Delta$ *and with* C_2 *finite and independent of* ψ.

Proof. Inequality (3.8) implies

$$\int_{-T}^{T} \left\| F_0 \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq \int_{-T}^{T} \langle \psi_t, i[H, F] \psi_t \rangle \, dt + \sum_{i=1}^{m} \int_{-T}^{T} \left\| F_i \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt.$$

For the first integral on the right-hand side, we have

$$\begin{aligned} \int_{-T}^{T} \langle \psi_t, i[H, F] \psi_t \rangle \, dt &= \int_{-T}^{T} \frac{d}{dt} \langle \psi_t, F\psi_t \rangle \, dt \\ &= \langle \psi_T, F\psi_T \rangle - \langle \psi_{-T}, F\psi_{-T} \rangle. \end{aligned}$$

Let φ be a C_0^∞ function, equal to 1 on Δ. Then $\psi_{\pm T} = \varphi(H)\psi_{\pm T}$. Using that $F\varphi(H)$ is bounded, we obtain

$$|\langle \psi_{\pm T}, F\psi_{\pm T}\rangle| \leq C\|\psi\|^2.$$

This implies

$$\int_{-T}^{T} \left\| F_0 \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C_3\|\psi\|^2.$$

Since the left-hand side is monotonically increasing as $T \to \infty$ and uniformly bounded from above, it converges as $T \to \infty$ and is bounded from above by the same bound. □

Remark 3.5. An example of an F_i, $i \geq 1$, is $O(\langle x \rangle^{-\varepsilon})$ for any $\varepsilon > 0$. A result of Mourre [M1] (three particles) and Perry-Sigal-Simon [PSS] (the general case (see also [SigSof2] for a different proof)) implies that $\langle x \rangle^{-1/2-\varepsilon}$, $\varepsilon > 0$, is *locally* H-smooth for any compact interval Δ when it avoids the thresholds and eigenvalues of H; i.e.,

$$\int \left\| \langle x \rangle^{-1/2-\varepsilon} e^{-iHt}\psi \right\|^2 dt \leq C\|\psi\|^2$$

with $C < \infty$ independent of ψ and for all $\psi \in \operatorname{Ran} P_\Delta(H)$.

The lemma reduces the proof of propagation estimates to estimating from below the commutators of H with appropriate bounded operators which we call the *propagation observables*. We distinguish two principal cases:

(a) $\gamma \notin \Sigma_E$. This case we call the *non-threshold case*.

(b) $\gamma \in \Sigma_E$. This case we call the *threshold case*.

The propagation observables used in these cases are entirely different.

Now we proceed to the second ingredient of our approach. We use the following standard notation:

$$|x|_a = \min\{|x_i - x_j| \,|\, i \text{ and } j \text{ belong to different clusters of } a\}.$$

This is the *intercluster distance* in the decomposition a. We mention separately that $F(|x|_a > \delta|x|)$ stands for a smoothed characteristic function of the region in which $|x|_a > \delta|x|$. It can be defined, for instance, as $\prod_{l \subseteq a} F(|x^l|/|x| > \delta)$. Here a pair l is identified with the partition containing $N - 1$ clusters, one of which is l itself and $x^l = x_i - x_j$ for $l = (ij)$.

II. *On the next step we show that for any non-threshold energy E, there are phase-space operators $j_{a,E}(x, p)$, forming a partition of unity:*

$$\sum_a j_{a,E}(x, p) = 1 + O(|x|^{-1}), \tag{3.11}$$

(which we call a *phase-space partition of unity*) with the following properties:

(i) $j_{a,E}$ depend on p^a only through γ, (3.12)

(ii) $j_{a,E}$ are supported in $\{(x,k) \in X \times X' \| x|_a > \delta |x|\}$ for some $\delta > 0$, (3.13)

(iii) $\nabla j_{a,E} = \langle x \rangle^{-1} j_a + O_1(|x|^{-2})$ (3.14)

where $\nabla j(x,k) = \nabla_{(x,s)} I(x,k,s)|_{s=\hat{x}\cdot k}$ with I defined using the right-hand side of (3.2) and the j_a are phase-space operators supported away from PS_E.

Relation (3.13) shows that $j_{a,E}$ are supported in the region where the clusters move away from each other and relation (3.14) shows that the "boundary", $f-\operatorname{supp}(\nabla j_{a,E})$, of $j_{a,E}$ lives in the region where *no* propagation takes place in x.

III. *Derivation of asymptotic completeness* (*Theorem* 3.1). First we introduce the *Deift-Simon wave operators* for given E and sufficiently small $\Delta \supset E$:

$$W_{a,E}^{\pm} = s-\lim e^{iH_a t} j_{a,E}(x,p) e^{-iHt}, \tag{3.15}$$

if the limits exist on $\operatorname{Ran} P_\Delta(H)$. (Deift and Simon [DS] introduced such operators for a partition of unity depending only on x.)

LEMMA 3.6 (DEIFT-SIMON ARGUMENT). *If the limits* (3.15) *exist on* Ran P_Δ, *then the system is asymptotically clustering at the energy E.*

Proof. Let ψ be orthogonal to the bound state subspace of H. Write

$$e^{-iHt}\psi = \sum j_{a,E} e^{-iHt}\psi = \sum e^{-iH_a t} e^{iH_a t} j_{a,E} e^{-iHt}\psi,$$

where we have omitted the term $O(|x|^{-1})e^{-iHt}\psi$ which tends to 0 by [PSS]. Introducing $\varphi_a^{\pm} = W_a^{\pm}\psi$ we rewrite this as

$$e^{-iHt}\psi = \sum e^{-iH_a t}\varphi_a^{\pm} + R^{\pm}(t), \tag{3.16}$$

where

$$R^{\pm}(t) = \sum e^{-iH_a t}\left[e^{iH_a t} j_{a,E} e^{-iHt} - W_a^{\pm}\right]\psi.$$

Since the $W_{a,E}^{\pm}$ exist, we have

$$\|R^{\pm}(t)\| \to 0 \quad \text{as } t \to \pm\infty.$$

Hence the system is asymptotically clustering. □

Before proceeding to the next step we mention the following property of the symbol $O(|x|^{-\alpha})$:

For any function f with C_0^∞ derivatives,

$$O(|x|^{-\alpha})f(B) = O(|x|^{-\alpha}) = f(B)O(|x|^{-\alpha}). \tag{3.17}$$

Here B stands for one of the operators H_a, p_a, γ_a and x (with arbitrary a). These relations follow from Lemma A.4(ii).

In our proof of asymptotic completeness we do not use the entire result of statement I but only the following part of it:

(E) Let E be a fixed energy away from the thresholds and eigenvalues of H. Then there is a small interval Δ around E such that the following estimate holds:

$$\int_{-\infty}^{\infty}\left\| j_a \frac{1}{\sqrt{\langle x\rangle}} e^{-iHt}\psi\right\|^2 dt \le C\|\psi\|^2$$

with $C < \infty$ and independent of ψ. Here the j_a are the phase-space operators appearing on the right-hand side of (3.14) (which are supported away from PS_E).

PROPOSITION 3.7. *Let* (A)–(E), *with* $\mu > 1$, *hold. Let the energy* E *be away from the thresholds and eigenvalues of* H. *Then the Deift-Simon wave operators* $W_{a,E}^{\pm}$ *exist on* Ran P_Δ, *provided* Δ *is a sufficiently small interval around* E *and disjoint from the thresholds and eigenvalues of* H.

Proof. Omit the subindex E from the partition of unity. Let

$$W_a(t) = e^{iH_at} j_a(x,p) e^{-iHt}.$$

By the Fundamental Theorem of Calculus

$$W_a(t) = j_a + i\int_0^t e^{iH_as}(H_a j_a - j_a H)e^{-iHs}\,ds. \tag{3.18}$$

We have

$$H_a j_a - j_a H = [H_a, j_a] - j_a I_a. \tag{3.19}$$

Due to property (3.13) of j_a, condition (D) on the potentials and (3.19),

$$j_a I_a (H+i)^{-1} = O(|x|^{-\mu}). \tag{3.20}$$

(This is the only place where condition (D) is used!)

Now we analyze $[H_a, j_a]$. Pick n such that $H + n \ge 1$. We use that

$$[H_a, j_a] = -(H_a + n)\left[\frac{1}{H_a+n}, j_a\right](H_a+n).$$

We compute

$$-\left[\frac{1}{H_a+n}, j(x)\right] = \frac{1}{H_a+n}[p^2, j(x)]\frac{1}{H_a+n}$$
$$= \frac{1}{H_a+n}(2p\cdot\nabla j(x) - \Delta j(x))\frac{1}{H_a+n}.$$

If j obeys $|\mathscr{D}^\alpha j(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}$ for all α, then

$$|\nabla j| \leq c_1 \langle x \rangle^{-1} \quad \text{and} \quad |\Delta j| \leq c_2 \langle x \rangle^{-2}.$$

Finally, equation (A.39) with $A = \gamma$ and $B = 1/(H_a + n)$ and estimate (A.46) yield

$$\left[\frac{1}{H_a + n}, f(\gamma)\right] = f'(\gamma)\left[\frac{1}{H_a + n}, \gamma\right] + O(|x|^{-2}).$$

Using (for details see (A.42a), (A.48)) the fact that

$$i\left[\frac{1}{H_a + n}, \gamma\right] = \frac{1}{\sqrt{\langle x \rangle}} B_4 \frac{1}{\sqrt{\langle x \rangle}} + O(|x|^{-2}),$$

where

$$B_4 = i\left[\frac{1}{H_a + n}, A\right] + \frac{1}{H_a + n}\gamma^2 \frac{1}{H_a + n}, \tag{3.21}$$

we see that simple estimates of commutators involving $\langle x \rangle^{-\alpha}$ (e.g., $[\langle x \rangle^{-\alpha}, f'(\gamma)]$; see Lemma A.4ii) and relation (3.17) yield

$$i\left[\frac{1}{H_a + n}, f(\gamma)\right] = \frac{1}{\sqrt{\langle x \rangle}} B_4 f'(\gamma) \frac{1}{\sqrt{\langle x \rangle}} + O(|x|^{-2}). \tag{3.21a}$$

Collecting these estimates, using that, due to (i), H_a commutes with the p-factors in the expression for $j_{a,E}$ and writing the operators j_a of (3.14) (do not confuse them with $j_a = j_{a,E}$) as $j_a = (j_{a,i}^* j_{a,i},\ i = 1, \ldots, 4) + O(|x|^{-1})$, we obtain

$$i[H_a, j_a] = (H_a + n)\left\{\frac{1}{\sqrt{\langle x \rangle}} \sum j_{a,i}^* B_i j_{a,i} \frac{1}{\sqrt{\langle x \rangle}} + O(|x|^{-2})\right\}(H_a + n), \tag{3.22}$$

where B_i are bounded operators, and $j_{a,i}$ are phase-space operators supported away from PS_E. Let $g \in \operatorname{Ran} P_\Delta(H)$, $f \in \operatorname{Ran} P_{\Delta'}(H_a)$ with $\Delta' \supset \Delta$. Consider $\langle [H_a, j_a] g, f \rangle$. Using equation (3.22), and denoting $g_H = (H + n)g$ and $f_{H_a} = (H_a + n)f$, we obtain

$$|\langle [H_a, j_a] g, f \rangle| \leq C \sum \left(\left\| j_{a,i} \frac{1}{\sqrt{\langle x \rangle}} g_H \right\| \left\| j_{a,i} \frac{1}{\sqrt{\langle x \rangle}} f_{H_a} \right\| + \left\| \frac{1}{\langle x \rangle} g_H \right\| \left\| \frac{1}{\langle x \rangle} f_{H_a} \right\| \right). \tag{3.23}$$

Collecting (3.20) and (3.22) and using (3.19) we obtain

$$|\langle (H_a j_a - j_a H)g, f\rangle| \le C\sum\left(\left\|j_{a,i}\frac{1}{\sqrt{\langle x\rangle}}g_H\right\|\left\|j_{a,i}\frac{1}{\sqrt{\langle x\rangle}}f_{H_a}\right\| + \left\|\langle x\rangle^{-\min(\mu/2,1)}g_H\right\|\left\|\langle x\rangle^{-\min(\mu/2,1)}f_{H_a}\right\|\right).$$

Let Δ' be slightly larger than Δ so that $\Delta'\setminus\bar{\Delta}$ is open. Let $\psi \in \operatorname{Ran} P_\Delta(H)$ and $u \in \operatorname{Ran} P_{\Delta'}(H_a)$. We have

$$(3.24)\quad \int_0^{|t|}|\langle (H_a j_a - j_a H)e^{-iHs}\psi, e^{-iH_a s}u\rangle|\,ds$$
$$\le C\left[\sum\int_0^{|t|}\left\|j_{a,i}\frac{1}{\sqrt{\langle x\rangle}}e^{-iHs}\psi_H\right\|^2 ds\right]^{1/2}$$
$$\times\left[\sum\int_0^{|t|}\left\|j_{a,i}\frac{1}{\sqrt{\langle x\rangle}}e^{-iH_a s}u_{H_a}\right\|^2 ds\right]^{1/2}$$
$$+C\left[\int_0^{|t|}\left\|\langle x\rangle^{-\alpha}e^{-iHs}\psi_H\right\|^2 ds\right]^{1/2}\times\left[\int_0^{|t|}\left\|\langle x\rangle^{-\alpha}e^{-iH_a s}u_{H_a}\right\|^2 ds\right]^{1/2},$$

where $\alpha = \min(1, \mu/2)$. Now due to condition (E) the first product on the right side of (3.24) is bounded by $\mathrm{const}\|\psi\|\,\|u\|$, provided Δ and Δ' are sufficiently small. Furthermore, the Perry-Sigal-Simon result ([PSS]; see also [SifSof2]) implies that $\langle x\rangle^{-\alpha}$ with $\alpha > \frac{1}{2}$ is locally H- (and therefore also H_a-) smooth (Δ' avoids the thresholds and eigenvalues of H and, consequently, also of H_a). (This is the only place where the condition $\mu > 1$ is used!) Hence the second product on the right side of (3.24) is bounded by the same $\mathrm{const}\|\psi\|\,\|u\|$. This proves the estimate:

$$(3.25)\qquad \int_0^{|t|}|\langle (H_a j_a - j_a H)e^{-iHs}\psi, e^{-iH_a s}u\rangle|\,ds \le \mathrm{const}\|\psi\|\,\|u\|$$

for all $\psi \in \operatorname{Ran} P_\Delta(H)$ and $u \in \operatorname{Ran} P_{\Delta'}(H_a)$. Hence the integral on the right side of (3.18), multiplied from the left by $P_{\Delta'}(H_a)$, converges strongly on $\operatorname{Ran} P_\Delta(H)$, which proves that $s-\lim_{t\to\pm\infty}P_{\Delta'}(H_a)W_a(t)$ exists on $\operatorname{Ran} P_\Delta(H)$.

Finally, we show that $\chi(H_a)W_a(t) \xrightarrow{s} 0$ as $|t|\to\infty$ for any smooth $\chi(s)$ supported away from $\bar{\Delta}$. Using that $\chi(H)P_\Delta(H) = 0$ we obtain $\chi(H_a)W_a(t)P_\Delta(H) = e^{iH_a t}Ke^{-iHt}P_\Delta(H)$, where

$$K = \{j_a(x,p)(\chi(H_a)-\chi(H)) + [\chi(H_a), j_a(x,p)]\}\varphi(H)$$

(with $\varphi(s) \in C_0^\infty$ and equal to 1 on Δ'). Now note that, Lemma A.4(ii) and

A.5(i) and the fact that $j_a(x, p)$ is a phase-space operator yield

$$[\chi(H_a), j_a(x, p)] = O(|x|^{-1}).$$

This implies that $[\chi(H_a), j_a]\varphi(H)$ is compact. Furthermore, equation (3.13) and the standard result

$$(\chi(H_a) - \chi(H))F(|x|_a > \delta|x|) \text{ is compact}$$

(see e.g., [PSS] and Section 4) imply that $(\chi(H_a) - \chi(H))j_a(x, p)$ is compact. Hence the operator K is compact. Since, by [PSS], H is absolutely continuous and since Δ avoids the eigenvalues of H, $Ke^{-iHt}P_\Delta(H) \xrightarrow{s} 0$ as $|t| \to \infty$ and therefore the desired result follows. This proves the existence of $s - \lim_{t \to \pm\infty} W_a(t)$ on Ran $P_\Delta(H)$. □

Lemma 3.6 and Proposition 3.7 imply that the system is asymptotically clustering for any energy E, away from the thresholds and eigenvalues of H, provided (A)–(E) and statement II hold. Since this result applies to any number of particles, it implies, due to Proposition 2.2, that the system in question is asymptotically complete with the proviso above. □

4. Geometry of N-body systems

The geometry of the N-body potential $V(x) = \sum V_{ij}(x_i - x_j)$ is determined by the system of planes X_a, where a runs through all cluster decompositions. These are exactly the planes along which $V(x)$ does not vanish. The main property of these planes is the relation

$$X_a \cap X_b = X_{a \cup b}. \tag{4.1}$$

This relation follows directly from the definition of X_a. Note that $X_{a_{\max}} = \{0\}$.

Our first exercise in the N-body geometry is to prove the existence of a partition of unity $\{\psi_a(x)\}$ on the configuration space X (here a runs through all non-trivial cluster decompositions) which describes geometrically different break-ups of the system.

THEOREM 4.1. *There exists a partition of unity $\{\psi_a(x)\}$ with the following properties*:

(i) *The ψ_a are smooth, $0 \le \psi_a \le 1$, homogeneous, degree 0 for $|x| \ge 1$ and are normalized as $\sum_a \psi_a(x)^2 = 1$,*

(ii) supp $\psi_a \subset \{|x|_a \ge \delta|x|\}$ *for some $\delta > 0$.*

The second property shows that each ψ_a lives in the region where the clusters in a move away from each other with the mutual distances proportional to the distance to the origin.

There are many ways to prove this theorem. A simple analysis of the unit sphere S in X and its intersection with the interaction planes X_l provides one such proof. Roughly speaking we will define ψ_a so that

$$\operatorname{supp} \psi_a \subset \operatorname{nbhd}(X_a) \Big\backslash \Big[\bigcup_{d \supsetneq a} \operatorname{nbhd}(X_d) \Big].$$

Due to (4.1) the domains on the right side of these relations cover X so that a partition of unity with this property exists.

Proof. Let ε_a be small, positive numbers obeying

$$\varepsilon_a \le \frac{1}{20\alpha} \varepsilon_f \quad \text{for } \#(f) > \#(a), \tag{4.2}$$

where the number $\alpha \ge 1$, depending only on the masses (and the number of particles), is chosen so that

$$|x^f| \le \alpha(|x^a| + |x^b|) \quad \text{with } f = a \cup b. \tag{4.3}$$

We also set up

$$\varepsilon_0 = \tfrac{1}{10} \varepsilon_{a_{\min}}^5.$$

Define the open homogeneous sets

$$B_\varepsilon(X_a) = \{ x \in X \,|\, |x^a| < \varepsilon |x| \}.$$

Then equations (4.2) and (4.3) imply

$$B_{\varepsilon_a}(X_a) \cap B_{\varepsilon_b}(X_b) \subset B_{(1/8)\varepsilon_f}(X_f) \quad \text{with } f = a \cup b, \tag{4.4}$$

provided $f \ne a, b$. Here the right-hand side is empty if $\#(f) = 1$. This for instance is always the case, if $\#(a) = \#(b) = 2$ and $a \ne b$. Equation (4.4) will play an important role in our construction.

Equation (4.1) implies

$$X = \bigcup_{a \ne a_{\max}} \Big[X_a \Big\backslash \bigcup_{b \supsetneq a} X_b \Big]. \tag{4.5}$$

Thus the open (conical) sets

$$\Gamma_{a,\varepsilon} = B_{(10/9)\varepsilon_a}(X_a) \Big\backslash \bigcup_{b \supsetneq a} \overline{B_{(1/2)\varepsilon_b}(X_b)}, \tag{4.6}$$

where ε stands for the multi-index with components ε_a, form an open covering of X.

LEMMA 4.2. *There is a number $\delta > 0$ such that*

$$\Gamma_{a,\varepsilon} \subset \{ x \in X \,|\, |x|_a > \delta |x| \}. \tag{4.7}$$

Proof. Let a pair $l \not\subseteq a$. Let $b = a \cup l$. Then $b \supsetneqq a$. Due to equation (4.4),

$$B_{\varepsilon_l}(X_l) \cap B_{\varepsilon_a}(X_a) \subset B_{(1/8)\varepsilon_b}(X_b).$$

Since $b \supsetneqq a$, we have that

$$\Gamma_{a,\varepsilon} \cap B_{\varepsilon_l}(X_l) \subset \left[B_{\varepsilon_l}(X_l) \cap B_{\varepsilon_a}(X_a)\right] \setminus B_{(1/2)\varepsilon_b}(X_b) = \varnothing.$$

In other words

$$\Gamma_{a,\varepsilon} \subset \left\{x \in X \middle| |x^l| > \varepsilon_l |x| \right\}.$$

Since this is true for all $l \not\subseteq a$, (4.7) follows with $\delta = \min_{l \not\subseteq a} \varepsilon_l$. □

Now, let $\{\psi_a\}$ be a smooth partition of unity on X with respect to the covering (4.6); i.e.,

$$0 \le \psi_a(x) \le 1, \qquad \sum \psi_a(x)^2 = 1$$

and

$$\text{(4.8)} \qquad \operatorname{supp} \psi_a \subset \Gamma_{a,\varepsilon}.$$

Due to Lemma 4.2, (ii) holds. Since $\Gamma_{a,\varepsilon}$ are *homogeneous* sets, ψ_a can be chosen to be homogeneous degree 0 for $|x| \ge 1$. Hence they obey also (i). □

It is convenient to regroup the partition of unity $\{\psi_a\}$ as follows:

$$\text{(4.9)} \qquad \varphi_a = \left[\sum_{b \subseteq a} \psi_b^2\right]^{1/2} \quad \text{for each two-cluster } a.$$

The new cut-off functions φ_a are smooth, non-negative and satisfy

$$\text{(4.10)} \qquad \sum_{\text{2-cluster } a} \varphi_a(x)^2 = 1$$

and

$$\text{(4.11)} \qquad \operatorname{supp} \varphi_a \subset \{x \in X \| x|_a > \delta |x|\} \quad \text{for some } \delta > 0.$$

Now we list the basic properties of the introduced partitions of unity (the proof can be found in [PSS]),

$$\text{(4.12)} \qquad I_a \varphi_a = O_1(|x|^{-\mu})$$

and for any smooth function g with a compact support

$$\text{(4.13)} \qquad (g(H) - g(H_a))\varphi_a = O(|x|^{-\mu})$$

and is compact, where the power μ is the same as in Condition (D).

Finally, we mention that a similar partition of unity $\{\varphi_b^a(x^a),\ b \subset a\}$ exists for each configuration space X^a:

$$\text{(4.14)} \qquad \sum_{b \subsetneqq a} \varphi_b^a(x^a)^2 = 1.$$

With a suitable adjustment, it has the same properties as the partition $\{\varphi_a\}$. For instance,

$$\operatorname{supp} \varphi_b^a \subset \{x^a \in X^a \| x^a|_b \geq \delta |x^a|\},$$
$$I_b \varphi_b^a = O_1(|x^a|^{-\mu}),$$

etc.

5. Phase-space partition of unity

To decouple (geometrically) different channels we use a phase-space partition of unity whose members are supported on different channels with the x-boundary located in the non-propagation set. In this section we construct such a partition.

Henceforth, E is a fixed energy away from the thresholds and eigenvalues of H and the positive numbers ε_a are the same as those defined in Section 4 (see equations (4.2) and (4.3)). We also use, without special mentioning, other definitions from Section 4. We will also use the notation

$$\kappa_\varepsilon = \sqrt{1 - \varepsilon^2}\,\kappa \quad \text{for } \kappa \in \Sigma_E.$$

We use the following cut-off functions:

$$F_\varepsilon(s \in \Omega) = \begin{cases} 1 & \operatorname{dist}(s, \Omega^c) \geq 2\varepsilon \\ 0 & \operatorname{dist}(s, \Omega) \geq 2\varepsilon \end{cases} \tag{5.1a}$$

if $|\Omega| > 0$ and $|\Omega^c| > 0$, and

$$F_\varepsilon(s \in \Omega) = \begin{cases} 1 & \operatorname{dist}(s, \Omega) \leq \varepsilon \\ 0 & \operatorname{dist}(s, \Omega) \geq 2\varepsilon \end{cases} \tag{5.1b}$$

if $|\Omega| = 0$, and

$$F_\varepsilon(s \in \Omega) = 1 - F_\varepsilon(s \in \Omega^c) \tag{5.1c}$$

if $|\Omega^c| = 0$. Here $\Omega \subset \mathbf{R}$ (usually, an interval), $\Omega^c = \mathbf{R} \setminus \Omega$ and the relation $s \notin \Omega$ is treated as $s \in \Omega^c$. Besides, the F_ε are assumed smooth, $0 \leq F_\varepsilon \leq 1$ and for Ω a semi-finite interval,

$$F_\varepsilon'(s \in \Omega) = \text{const} \quad \text{for } \operatorname{dist}(s, \partial\Omega) \leq \varepsilon.$$

In particular, the F_ε can be adjusted in such a way as to satisfy for Ω, a semi-finite interval,

$$F_\varepsilon'(s \in \Omega) = \text{const} \cdot F_\varepsilon(s \in \partial\Omega). \tag{5.2}$$

Henceforth we will use the notation

$$r^a = \frac{\langle x^a \rangle}{\langle x \rangle} \quad \text{and} \quad r_a = \frac{\langle x_a \rangle}{\langle x \rangle}.$$

Next we define our basic cut-off functions on the phase-space:

(5.3a) $$\tilde{j}_{a,\kappa}(x,k) = F_{\varepsilon_0}(r^a < \varepsilon_a)\left[1 - F_{\varepsilon_0}\left(r^a > \tfrac{3}{4}\varepsilon_a\right)F_{\sqrt{\varepsilon_0}}\left(|k_a| = \kappa_{\varepsilon_a}\right)\right]$$

if $\#(a) < N$ and

(5.3b) $$\tilde{j}_{a,\kappa}(x,k) = \prod_{\text{all } l} F_{\varepsilon_0}(r^l > \varepsilon_a)$$

if $\#(a) = N$. Here, recall, l denotes a pair of indices and $x^l = x_i - x_j$ for $l = (ij)$. (Remember, also, that l is identified with an $(N-1)$-cluster decomposition.) We define

(5.4) $$j_{a,\kappa}(x,k) = \tilde{j}_{a,\kappa}(x,k) \prod_{n=2}^{\#(a)-1}\left(1 - \sum_{\#(b)=n} \tilde{j}_{b,\kappa}(x,k)\right),$$

where $\prod_{n=2}^{\#(a)-1}(\ldots) = 1$ for $\#(a) = 2$.

Henceforth we willl use an abbreviated notation for subsets of the phase-space $\mathscr{F} = X \times X'$, displaying only inequalities defining these subsets. Denote

$$\Omega_{a,\kappa} = \bigcup_{\substack{b \supseteq a \\ \alpha = \frac{3}{4}, 1}} \left\{|r^b - \alpha\varepsilon_b| < \varepsilon_0, \left||k_b| - \kappa_{\alpha\varepsilon_b}\right| > \sqrt{\varepsilon_0}\right\} \quad \text{for } \#(a) < N$$

and

$$\Omega_{a,\kappa} = \bigcup_l \Omega_{l,\kappa} \quad \text{for } \#(a) = N.$$

We will show later (though it is not used directly in our analysis) that the intersection of this set with $\{|\hat{x}\cdot k - \kappa| < \varepsilon_0\}$ lies outside of PS_E. We have:

THEOREM 5.1. (i) *$j_{a,\kappa}(x,k)$ is homogeneous of degree 0 in x for $|x| \geq 1$ and independent of k^a,*

(ii) $\sum j_{a,\kappa}(x,k) = 1$,

(iii) supp $j_{a,\kappa} \subset \{|x|_a > \delta|x|\}$ *for some* $\delta > 0$,

(iv) $\operatorname{supp}(\nabla_x j_{a,\kappa}) \subset \Omega_{a,\kappa}$.

Proof. (i) The homogeneity in x follows readily from the definitions (5.4), (5.3). The fact that $j_{a,\kappa}(x,k)$ is independent of k^a follows from:

LEMMA 5.2. *The terms with $b \not\supset a$ can be dropped from the right-hand side of* (5.4).

Proof. Our geometric analysis is based on the relations

(5.5) $$B_{\varepsilon_a}(X_a) \cap B_{\varepsilon_b}(X_b) \subset B_{(1/8)\varepsilon_f}(X_f)$$

with $f = a \cup b$, and provided $f \neq a, b$,

(5.6) $$B_{\varepsilon_f}(X_f) \subset \bigcup_{d \supseteq f} \Gamma_{d,\varepsilon};$$

i.e., the sets on the right-hand side form an open (homogeneous) covering of the set on the left-hand side, and

$$B_{\varepsilon_a}(X_a) \cap \Gamma_{b,(1/2)\varepsilon} = \varnothing \quad \text{if } a \cup b \neq a, b. \tag{5.7}$$

Equation (5.5) follows from equations (4.2) and (4.3). It was already used in Section 4 (see equation (4.4)). Equation (5.6) follows from the definition (4.6) of $\Gamma_{a,\varepsilon}$. Equation (5.7) follows from equations (5.5) and (4.6) (e.g. the latter implies that

$$\Gamma_{b,(1/2)\varepsilon} \cap B_{(1/8)\varepsilon_f}(X_f) = \varnothing \quad \text{for } f \supsetneqq b). \tag{5.8}$$

Note also that equations (5.5) and (5.6) are homogeneous in the ε_a's in the sense that they hold if all ε_a are replaced by $\alpha\varepsilon_a$ with $\alpha > 0$. Equations (5.5) and (5.6) (the latter with $\varepsilon_f \to \frac{1}{2}\varepsilon_f$ and $\varepsilon \to \frac{1}{2}\varepsilon$) imply that

$$B_{\varepsilon_a}(X_a) \cap B_{\varepsilon_b}(X_b) \subset \bigcup_{d \supseteq a \cup b} \Gamma_{d,(2/3)\varepsilon}, \tag{5.9}$$

provided $a \cup b \neq a, b$. The latter is the case if $\#(b) < \#(a)$ and $b \not\supset a$. Due to Definition (5.3),

$$\tilde{j}_{d,\kappa}(x,k) = 1 \quad \text{on } \Gamma_{d,(2/3)\varepsilon} \text{ for } \#(d) \leq N-1. \tag{5.10}$$

Moreover, due to equation (5.7),

$$\tilde{j}_{c,\kappa}(x,k) = 0 \quad \text{on } \Gamma_{d,(2/3)\varepsilon} \text{ if } c \neq d,\ \#(c) = \#(d). \tag{5.11}$$

Therefore

$$\left(1 - \sum_{\#(c)=\#(d)} \tilde{j}_{c,\kappa}(x,k)\right) = 0 \quad \text{on } \Gamma_{d,(2/3)\varepsilon}. \tag{5.12}$$

This relation together with equation (5.9) and the fact that $\Gamma_{d,\varepsilon} = \{0\}$ for $\#(d) = 1$ yields

$$\tilde{j}_{a,\kappa}(x,k)\tilde{j}_{b,\kappa}(x,k) \prod_{n=2}^{\#(b)-1} \left(1 - \sum_{\#(c)=n} \tilde{j}_{c,\kappa}(x,k)\right) = 0 \tag{5.13}$$

if $a \cup b \neq a, b$, which proves the lemma. □

(ii) We begin with:

LEMMA 5.3.

$$\prod_{n=2}^{N} \left(1 - \sum_{\#(b)=n} \tilde{j}_{b,\kappa}(x,k)\right) = 0. \tag{5.14}$$

Proof. Due to Definition (5.3),

$$\operatorname{supp}\left(1 - \tilde{j}_{a_{\min},\kappa}\right) \subset \bigcup_{\text{all } l} B_{\varepsilon_{a_{\min}}}(X_l).$$

Hence equation (5.7) and the inequalities on the ε_a's imply

$$\operatorname{supp}\left(1 - \tilde{j}_{a_{\min},\kappa}\right) \subset \bigcup_{\#(d) \le N-1} \Gamma_{d,(1/2)\varepsilon}.$$

This together with equation (5.12) implies

$$\prod_{n=2}^{N-1}\left(1 - \sum_{\#(c)=n} \tilde{j}_{c,\kappa}(x,k)\right) = 0 \quad \text{on } \operatorname{supp}\left(1 - \tilde{j}_{a_{\min},\kappa}\right). \qquad \square$$

Now we derive (ii) from Lemma 5.3. Expanding the product, we find

$$\begin{aligned} 0 &= \prod_{n=2}^{N}\left(1 - \sum \tilde{j}_{a,\kappa}\right) \\ &= \prod_{n=2}^{N-1}\left(1 - \sum \tilde{j}_{a,\kappa}\right) - \tilde{j}_{a_{\min}} \prod_{n=2}^{N-1}\left(1 - \sum \tilde{j}_{a,\kappa}\right) \\ &= \prod_{n=2}^{N-2}\left(1 - \sum \tilde{j}_{a,\kappa}\right) - \left(\sum_{\#(b)=N-1} \tilde{j}_{b,\kappa}\right)\prod_{n=2}^{N-2}\left(1 - \sum \tilde{j}_{a,\kappa}\right) - j_{a_{\min},\kappa} \\ &= \cdots = 1 - \sum j_{a,\kappa}. \end{aligned}$$

This implies (ii).

(iii) Equation (5.12) yields

$$\prod_{n=2}^{s}\left(1 - \sum_{\#(c)=n} \tilde{j}_{c,\kappa}(x,k)\right) = 0 \quad \text{on } \Gamma_{b,(2/3)\kappa} \text{ if } \#(b) \le s. \tag{5.15}$$

Next we use a more detailed version of (5.6):

$$B_{\varepsilon_a}(X_a) = \Gamma_{a,\varepsilon} \bigcup_{b \supsetneq a} \Gamma_{b,(2/3)\varepsilon}. \tag{5.16}$$

The last two equations imply

$$j_{a,\kappa}(x,k) = F(\Gamma_{a,\varepsilon}) \prod_{n=2}^{\#(a)-1}\left(1 - \sum_{\#(c)=n} \tilde{j}_{a,\kappa}(x,k)\right), \tag{5.17}$$

where $F(\Gamma_{a,\varepsilon})$ is an appropriate smooth function living in $\Gamma_{a,\varepsilon}$. The latter equation yields: $\operatorname{supp} j_{a,\kappa} \subset \Gamma_{a,\varepsilon}$ which, due to (4.7), implies (iii).

(iv) As in Lemma 5.3 we show that

$$\nabla_x \cdot \tilde{j}_{a,\kappa}(x) \times \prod_{n=2}^{N-1}\left(1 - \sum_{\#(b)=n} \tilde{j}_{b,\kappa}(x,k)\right) = 0$$

for $\#(a) = N$. Hence it suffices to consider $\nabla_x \tilde{j}_{a,\kappa}(x,k)$ with $\#(a) < N$. We differentiate

$$\nabla_x \tilde{j}_{a,\kappa}(x,k) = \Big\{F'_{\varepsilon_0}(r^a < \varepsilon_a)\Big[1 - F_{\varepsilon_0}\big(r^a > \tfrac{1}{2}\varepsilon_a\big)F_{\sqrt{\varepsilon_0}}\big(|k_a| = \kappa_{\varepsilon_a}\big)\Big] - F_{\varepsilon_0}\big(r^a < \varepsilon_a\big)F'_{\varepsilon_a}\big(r^a > \tfrac{1}{2}\varepsilon_a\big)F_{\sqrt{\varepsilon_0}}\big(|k_a| = \kappa_{\varepsilon_a}\big)\Big\}\nabla_x(r^a).$$

Using equations (5.1c) and (5.2) and using that

$$F_{\varepsilon_0}\big(r^a \geq \tfrac{1}{2}\varepsilon_a\big) = 1 \quad \text{on supp } F'_{\varepsilon_0}\big(r^a \leq \varepsilon_a\big)$$
$$F_{\varepsilon_0}\big(r^a \leq \varepsilon_a\big) = 1 \quad \text{on supp } F'_{\varepsilon_0}\big(r^a \geq \tfrac{1}{2}\varepsilon_a\big),$$

we obtain

$$\nabla_x \tilde{j}_{a,\kappa}(x,k) = \text{const}\Big[F_{\varepsilon_0}\big(r^a = \varepsilon_a\big)F_{\sqrt{\varepsilon_0}}\big(|k_a| \neq \kappa_{\varepsilon_a}\big) + F_{\varepsilon_0}\big(r^a = \tfrac{1}{2}\varepsilon_a\big)F_{\sqrt{\varepsilon_0}}\big(|k_a| = \kappa_{\varepsilon_a}\big)\Big]\nabla_x(r^a). \tag{5.18}$$

This relation together with Definition (5.3) yields (iv). □

Now we quantize the partition of unity $j_{a,\kappa}(x,k)$ defined above. Using the standard procedure of pseudodifferential calculus we define $J_{a,\kappa} \equiv j_{a,\kappa}(x,p)$ by the formula

$$j_{a,\kappa}(x,p)\psi = \frac{1}{(2\pi)^{n/2}}\int j_{a,\kappa}(x,k)\hat{\psi}(k)e^{ik\cdot x}\,dx, \tag{5.19}$$

where $\hat{\psi}$ is the Fourier transform of ψ and $n = \nu(N-1)$, the dimension of X.

Theorem 5.4. *The operators $j_{a,\kappa}(x,p)$ form a partition of unity on $L^2(X)$:*

$$\sum_a j_{a,\kappa}(x,p) = 1 + O\big(|x|^{-1}\big). \tag{5.20}$$

Moreover, they have the following properties:

(i) *$j_{a,\kappa}(x,p)$ are phase-space operators independent of γ and p^a,*

(ii) *$j_{a,\kappa}(x,p)$ are supported in $\{|x|_a > \delta|x|\}$ for some $\delta > 0$,*

(iii) *$\nabla_x j_{a,\kappa}(x,p)$ are supported in $\Omega_{a,\kappa}$.*

Proof. (i)–(iii) follow readily from the definition of $j_{a,\kappa}(x,p)$ (equations (5.3), (5.4) and (5.19)) and Theorem 5.1. (Recall that the support, or f-support, of $j_{a,\kappa}(x,p)$ is defined as the phase-space support of its symbol $j_{a,\kappa}(x,k)$.) Now we prove equation (5.20). Denote by $j'_{a,\kappa}(x,p)$ the pseudodifferential operators obtained from the symbols $j_{a,\kappa}(x,k)$ by the direct substitution of $F_{\sqrt{\varepsilon_0}}(|p_a| = \kappa_{\varepsilon_a})$ into (5.3) instead of $F_{\sqrt{\varepsilon_0}}(|k_a| = \kappa_{\varepsilon_a})$ and then using the resulting operators in equation (5.4). Going through the proof of Theorem 5.1(ii) but with the operators instead of the symbols we find that

$$\sum j'_{a,\kappa}(x,p) = 1 + O\big(|x|^{-1}\big),$$

where the term $O(|x|^{-1})$ results from commuting the functions of r^a through functions of $|p_b|$, $b \supseteq a$. Since $j_{a,\kappa}(x,p)$ is obtained from $j'_{a,\kappa}(x,p)$ by a rearrangement of the r^a- and $|p_a|$-factors, we have

$$j_{a,\kappa}(x,p) = j'_{a,\kappa}(x,p) + O(|x|^{-1}).$$

The last two equations yield (5.20). □

Now we construct a phase-space partition of unity $\{j_{a,E}\}$ which decouples the channels. Let $\{\psi_a\}$ be a partition of unity on the configuration space as constructed in Section 4 but with the usual normalization: $\Sigma\psi_a(x) = 1$. Recall that ψ_a are homogeneous functions, degree 0, for $|x| \geq 1$ with

$$\operatorname{supp}\psi_a \subset \{|x|_a > \delta|x|\} \quad \text{for some } \delta > 0.$$

Let $\tilde{\Sigma}_E$ be a finite subset of Σ_E such that the intervals $\{|s-\kappa| < \varepsilon_0\}$ with $\kappa \in \tilde{\Sigma}_E$ cover Σ_E while the boundaries of these intervals lie outside of Σ_E. (If necessary, ε_0 can be adjusted a little.) Such a set exists since there is only a finite number of generations of thresholds (which can be labeled by $\#(a) = k$). Let $\{F(\gamma \notin \Sigma_E),\ F(\gamma = \kappa),\ \kappa \in \tilde{\Sigma}_E\}$ be a partition of unity with the properties

$$(5.21) \qquad F(s \notin \Sigma_E) \text{ and } F'(s = \kappa) \quad \text{are supported away from } \Sigma_E$$

and

$$F(s = \kappa) \text{ are supported in } \{|s-\kappa| < \varepsilon_0\}.$$

We define

$$(5.22) \qquad j_{a,E}(x,p) = \psi_a(x)F(\gamma \notin \Sigma_E) + \sum_{\kappa \in \tilde{\Sigma}_E} j_{a,\kappa}(x,p)F(\gamma = \kappa),$$

where $\{j_{a,\kappa}(x,p)\}$ is the partition of unity defined above. Clearly $\{j_{a,E}\}$ is a partition of unity on the phase-space:

$$(5.23) \qquad \sum j_{a,E}(x,p) = 1 + O(|x|^{-1}).$$

For symbols $j(x,k) = \phi(x,k,\hat{x}\cdot k)$ (with the obvious meaning for ϕ) we use the notation

$$(5.24) \qquad \nabla j(x,k) = \nabla_{(x,s)}\phi(x,k,s)|_{s=\hat{x}\cdot k}.$$

THEOREM 5.5. *A phase-space partition of unity constructed above (equation (5.22)) has the following properties*

(i) $j_{a,E}$ *depend on* p^a *only through* γ,
(ii) $j_{a,E}$ *are supported in* $\{|x|_a > \delta|x|\}$,
(iii) $\nabla j_{a,E} = \langle x\rangle^{-1} j_a + O_1(|x|^{-2})$,

where j_a are phase-space operators supported in

$$\{\text{dist}(\hat{x}\cdot k, \Sigma_E) \geq \delta_0\} \bigcup_{k\in\tilde{\Sigma}_E} (\Omega_{a,\kappa} \cap \{|\hat{x}\cdot k - \kappa| < \varepsilon_0\}) \tag{5.25}$$

where δ_0 is a fixed number.

Proof. (i) and (ii) follow readily from the definition of $j_{a,E}(x,p)$ and Theorem 5.4. (iii) follows from Theorem 5.5 and equation (5.21). In fact, δ_0 is the distance between the sets Σ_E and $\text{supp}\, F(s \notin \Sigma_E) \bigcup_{s\in\tilde{\Sigma}_E} \text{supp}\, F'(s=\kappa)$. □

In fact, the proof of Theorem 5.1 allows us to make a more precise statement. Let $\nabla j_{a,E}(x,p)$ be the phase-space operator with the symbol $\nabla j_{a,E}(x,k)$ (see equation (5.24)).

THEOREM 5.6.

$$\nabla j_{a,E}(x,p) = \sum T_b j_{b,\kappa,\alpha}(x,p)\langle x\rangle^{-1} + T_0 F_{\delta_0}(\gamma \notin \Sigma_E), \tag{5.26}$$

where the sum extends over all b, $b \supseteq a$, $\#(b) < N$ and $\alpha = \frac{3}{4}, 1$; T_b, T_0 are γ-independent phase-space operators and

$$j_{b,\kappa,\alpha}(x,p) = F(\Gamma_{b,\varepsilon}) F_{\sqrt{\varepsilon_0}}(r^b = \alpha\varepsilon_b) F_{\sqrt{\varepsilon_0}}(|p_b| \neq \kappa_{\alpha\varepsilon_b}) \times F(\gamma = \kappa) \tag{5.27}$$

with $F(\Gamma_{b,\varepsilon})$ a smooth function supported in $\Gamma_{b,\varepsilon}$.

Proof. The statement follows from the definition (5.22), (5.7) of $j_{a,E}(x,p)$, equation (5.18) and an equation similar to (5.17). The latter allows us to insert $F(\Gamma_{a,\varepsilon})$. □

For the sake of interpretation we show that $\nabla j_{a,E}$ lives away from the propagation set. Define

$$\Omega_\kappa = \{\hat{x}\cdot k = \kappa,\ r^a = \bar{\varepsilon},\ |k_a| \neq \kappa_\varepsilon\} \cap \Gamma_{a,\varepsilon}, \qquad \bar{\varepsilon} \leq \varepsilon_a.$$

LEMMA 5.7. *If E is away from the thresholds of H, then $\Omega_\kappa \cap PS_E = \varnothing$.*

Proof. Denote

$$PS_E^b = \{x^b = 0,\ x_b \| \nabla |k_b|^2,\ |k_b| = \kappa\}.$$

We consider four cases:

i) $\Omega_\kappa \cap PS_E^b = \varnothing$ for all $b \supseteq a$.

Indeed, $r^a = \varepsilon$ implies that $x^b \neq 0$ for any $b \supseteq a$.

ii) $\Omega_\kappa \cap PS_E^b = \varnothing$ for all β: $\kappa_\beta \neq \kappa$.

Indeed, on PS_E^b, $\hat{x}\cdot k = \hat{x}_b \cdot k_b$ since $x^b = 0$. Furthermore, $\hat{x}_b \cdot k_b = |k_b|$ since $x_b \| \nabla |k_b|^2$. Since $|k_b| = \kappa_\beta$, we conclude that $\hat{x}\cdot k = \kappa_\beta$ on PS_E^b.

iii) $\Omega_\kappa \cap PS_E^b = \varnothing$ for all $b \subset a$, $\kappa_\beta = \kappa$. Indeed, we have on PS_E^b for $b \subset a$,

$$r^a = \frac{|x_b^a|}{|x_b|} = \frac{|k_b^a|}{|k_b|},$$

which implies

$$|k_a| = \sqrt{|k_b|^2 - |k_b^a|^2} = |k_b|\sqrt{1-(r^a)^2}.$$

Thus $|k_a| = \kappa_\beta\sqrt{1-(r^a)^2}$ on PS_E^b for $b \subset a$. On the other hand $|k_a| \neq \kappa\sqrt{1-(r^a)^2}$ on Ω_κ.

iv) $\Omega_\kappa \cap PS_E^b = \varnothing$ for all $b \not\supset a$ and $b \not\subset a$.

Indeed, let $f = a \cup b$. On one hand, due to equation (4.4), $\Omega_\kappa \cap PS_E^b \subset B_{(1/8)\varepsilon_f}(X_f)$. The latter set is non-empty only if $\#(f) > 1$. In the latter case, $B_{(1/4)\varepsilon_f} \cap \Gamma_{a,\varepsilon} = \varnothing$ and therefore $B_{(1/4)\varepsilon_f} \cap \Omega_\kappa = \varnothing$, so the result follows. □

6. Expansion in channels

In this section we describe the structure of the commutator $i[H, A]$ localized near a fixed energy E. This will reveal the entire channel composition of the N-body system in question. The result of this section is one of the basic ingredients in the proof of the propagation estimates.

We begin by introducing necessary notation and definitions. We fix a labelling of eigenvalues $\varepsilon_{a,j}$ of H^a. Let $P^a = \sum P_j^a$ be the projections on the pure point spectrum subspaces for H^a. Here P_j^a is the projection on a finite dimensional eigenspace of H^a corresponding to the eigenvalue $\varepsilon_{a,j}$. In particular, for $a = a_{\max}$, the 1-cluster partition, $P = \sum P_j$ (we remove the label a for $\#(a) = 1$). We use the following notation:

$$P^N = \sum_{i \le N} P_j, \qquad \bar{P}^N = 1 - P^N \quad \text{and} \quad P^{\text{cont}} = 1 - \sum_{\text{all } j} P_j$$

and

$$F_\Delta^N = F_\Delta \bar{P}^N.$$

Recall, H^a is the Hamiltonian of the decomposed system (i.e., non-interacting clusters) with the center-of-mass of each cluster in a fixed at the origin. All the quantities pertaining to H^a will be *distinguished by the superindex* a. Sometimes we consider these quantities on the entire space $L^2(X)$ without a change of notation. Introduce

$$\bar{P}_a^N = 1 - P_a^N \quad \text{with } P_a^N = P^{a,N} \otimes 1,$$

the extension of the corresponding operators from $L^2(X^a)$ to the entire $L^2(X)$. We review some notation concerning the channels α.

$a(\alpha)$ = the cluster decomposition corresponding to α;
$\#(\alpha)$ = the number of clusters, $\#(a)$, in $a(\alpha)$;
$m(\alpha)$ = the order number of the eigenvalue $+\varepsilon_\alpha$ of $H^{a(\alpha)}$ associated with this channel;
P_α = the projection on the channel bound state ($P_\alpha = 1$ for the free channel);
p_α = the channel momentum: $p_\alpha = p_a$ for $a = a(\alpha)$;
T_α = the channel kinetic energy: $T_\alpha = T_a = |p_\alpha|^2$ for $a = a(\alpha)$.
γ_α = the channel γ-operator: $\gamma_\alpha = \gamma_a$ for $a = a(\alpha)$ where $\gamma_a = \frac{1}{2}(\hat{x}_a \cdot p_a + p_a \cdot \hat{x}_a)$.

The *strings* are tuples of the form

$$S = (a_2, a_3, \dots, a_k, \alpha),$$

where $\#(a_i) = i$, $a_i \supset a_{i+1}$ and $a(\alpha) = a_k$. We let $\alpha(S) = \alpha$.

After this lengthy review of notations we can proceed to the main result of this section: channel expansion of the commutator $i[H, A]$.

The result below, though somewhat different in spirit, is closely related to the Mourre estimate ([M1], [PSS], [FH2]).

THEOREM 6.1. *For any $\varepsilon > 0$, $\delta > 0$ and a compact region I, there are $\delta_0 > 0, N_1, N_2, \dots$ and functions F_α with $0 \le F_\alpha \le 1$, $\operatorname{supp} F_\alpha \subset (-\delta/2, \delta/2)$ such that for any $\Delta \subset I$ with $|\Delta| \le \delta_0$ and any $N \ge N_1$, the following estimate holds*:

$$F_\Delta^N i[H, A] F_\Delta^N \overset{\varepsilon}{\gtrsim} F_\Delta^N \Bigg(\sum_{\substack{m(\alpha) \le N_{\#(\alpha)} \\ E - \varepsilon_\alpha \ge 0}} 2\kappa_\alpha^2 \phi_{\alpha, E} \Bigg) F_\Delta^N \tag{6.1}$$

and

$$\left(F_\Delta^N\right)^2 \overset{\varepsilon}{=} F_\Delta^N \Bigg(\sum_{\substack{m(\alpha) \le N_{\#(\alpha)} \\ E - \varepsilon_\alpha \ge 0}} \phi_{\alpha, E} \Bigg) F_\Delta^N \tag{6.2}$$

where $E = \inf \Delta + \delta$, κ_α is defined for this E, the sums extend over all open channels with $m(\alpha) \le N_{\#(\alpha)}$ and

$$\phi_{\alpha, E} = \sum_{\alpha(S) = \alpha} j_S P_\alpha F(|\gamma_\alpha| < \kappa_\alpha + \delta) F_\alpha(|p_\alpha| - \kappa_\alpha) F(|\gamma_\alpha| < \kappa_\alpha + \delta) P_\alpha j_S^* \tag{6.3}$$

with $F(s \le \kappa_\alpha + \delta) = 0$ for $s > \kappa_\alpha + \frac{5}{4}\delta$ and $= 1$ for $s < \kappa_\alpha + \frac{3}{4}\delta$ and j_S are bounded operators defined by

$$j_S = \varphi_{a_2} \bar{P}_{a_2}^{N_2} \varphi_{a_3}^{a_2} \bar{P}_{a_3}^{N_3} \cdots \varphi_{a_k}^{a_{k-1}} P_\alpha \quad \text{for } S = (a_2, \dots, a_k, \alpha), \tag{6.4}$$

where $\{\varphi_{a_{k+1}}^{a_k}\}$ *are configuration space partitions of unity on* X^a *defined in Section* 4.

Remark 6.2. The bounded operators j_S form a partition of unity on $L^2(X)$:

$$\sum j_S j_S^* = 1. \tag{6.5}$$

To prove this relation we note first the following recursive relation, which we state, for the notation's sake, only for the last step $k = 2$:

$$j_S = \begin{cases} \varphi_{a_2} P_\alpha & \text{if } S = (a_2, \alpha) \\ \varphi_{a_2} \bar{P}_{a_2}^{N_2} & \text{if } S = (a_2, S_2) \end{cases} \tag{6.6}$$

where $S_l = (a_l, \dots, a_k, \alpha)$ and $j_{S_l}^{a_l}$ are the "l-th tail" of (6.4).

Now we prove (6.5). We use the induction in the string length k (we do only the last step $k = 2$). Using (6.6), we obtain

$$\sum j_S j_S^* = \sum_{a_2} \varphi_{a_2} \Bigg(\sum_{\substack{a(\alpha) = a_2 \\ m(\alpha) \le N_{\#(\alpha)}}} P_\alpha \Bigg) \varphi_{a_2} + \sum_{a_2} \varphi_{a_2} \bar{P}_{a_2}^{N_2} \Bigg(\sum_{S_2} j_{S_2}^{a_2} \big(j_{S_2}^{a_2} \big)^* \Bigg) \bar{P}_{a_2}^{N_2} \varphi_{a_2}.$$

Using that

$$\sum_{\substack{a(\alpha) = a_2 \\ m(\alpha) \le N_2}} P_\alpha = P_{a_2}^{N_2} \quad \text{and} \quad \sum_{S_2} j_{S_2}^{a_2} \big(j_{S_2}^{a_2} \big)^* = 1$$

(the latter is the induction assumption), we obtain

$$\sum j_S j_S^* = \sum_{a_2} \varphi_{a_2} P_{a_2}^{N_2} \varphi_{a_2} + \sum_{a_2} \varphi_{a_2} \bar{P}_{a_2}^{N_2} \varphi_{a_2} = \sum_{a_2} (\varphi_{a_2})^2 = 1.$$

Remark 6.3. If E is such that all the channels are closed then a result stronger than the one stated in the theorem holds:

$$F_\Delta^N i[H, A] F_\Delta^N = 0 \quad \text{for all } N \text{ sufficiently large.}$$

Indeed, in this case $F_\Delta^N(H) = 0$, since $\bar{\Delta}$ is away from the continuum of H and all the discrete eigenvalues of H, except for those in a small neighborhood of $\inf \sigma_{ess.}(H)$ (and therefore away from $\bar{\Delta}$), are subtracted.

Proof. We conduct the proof by induction in decompositions a. We first prove the theorem with $F(|\gamma_\alpha| < \kappa_\alpha + \delta)$ replaced by 1 in the definition of the $\phi_{\alpha, E}$. In the next step we prove the localization lemma that allows one to insert $F(|\gamma_\alpha| < \kappa_\alpha + \delta)$ on both sides of $F_\alpha(|p_\alpha| - \kappa_\alpha)$, to complete the proof of the theorem. For $a = a_{\min}$ the statement is trivial. Because of the notation, we demonstrate only the last step of the induction process. Assume the statement of

Theorem 6.1 holds for any H^a with $\#(a) = 2$ and show it for H. The induction hypothesis:
Given ε, δ and a compact interval I there exist δ_2, and $N_2, N_3, \ldots$ and functions F_α with $0 \le F_\alpha \le 1$, $\operatorname{supp} F_\alpha \subset (-\delta/2, \delta/2)$ such that for any $N \ge N_2$ and for any $\Delta_1 \subset I$ with $|\Delta_1| < \delta_2$,

$$(6.7) \qquad F_{\Delta_1}^{a,N} i[H^a, A^a] F_{\Delta_1}^{a,N} \overset{\varepsilon}{\ge} F_{\Delta_1}^{a,N}\Big(\sum_{H^a\text{-chan.}} 2\kappa_\alpha^2 \sum_{H^a\text{-chan.}} j_S^a P_\alpha F_\alpha(|p_\alpha^a| - \kappa_\alpha)(j_S^a)^*\Big) F_{\Delta_1}^{a,N}$$

and

$$(6.8) \qquad \big(F_{\Delta_1}^{a,N}\big)^2 \overset{\varepsilon}{=} F_{\Delta_1}^{a,N}\Big(\sum j_S^a P_\alpha F_\alpha(|p_\alpha^a| - \kappa_a)(j_S^a)^*\Big) F_{\Delta_1}^{a,N}$$

with the quantities pertaining to H^a distinguished by the super index a and the sums over H^a-channels (i.e. the channels of the Hamiltonian H^a) restricted by $m(\alpha) \le N_{\#(\alpha)}$ and κ_α defined for the energy E from the middle of the interval Δ_1. Pick an arbitrary compact subset $I \subset \mathbf{R}$ and fix it. Let $\Delta \subset I$, arbitrary otherwise.
Denote

$$(6.9) \qquad B_{\Delta,N} \equiv F_\Delta^N i[H, A] F_\Delta^N$$

The IMS localization formula ([Sig1], [CFKS]) yields

$$(6.10) \qquad B_{\Delta,N} = \sum_{2\text{-cluster}} F_\Delta^N \varphi_a i[H, A] \varphi_a F_\Delta^N + O(|x|^{-2}),$$

where $\{\varphi_a\}$ is a configuration space partition of unity:

$$(6.11) \qquad \sum_{2\text{-cluster}} (\varphi_a(x))^2 = 1,$$

defined at the end of Section 4. Next, property (4.12) of the cut-off functions φ_a and the equation

$$(6.12) \qquad [A, I_a] = x \cdot \nabla I_a$$

yield

$$(6.13) \qquad \varphi_a[H, A]\varphi_a = \varphi_a[H_a, A]\varphi_a + O_1(|x|^{-1-\theta}).$$

Applying this relation to equation (6.10) we obtain

$$(6.14) \qquad B_{\Delta,N} = \sum_{2\text{-clusters}} F_\Delta^N \varphi_a i[H_a, A]\varphi_a F_\Delta^N + O(|x|^{-1-\theta}),$$

where we have used that $F_\Delta^N = F_\Delta^N F_\Omega$ for some fixed compact interval $\Omega \supset \Delta$ and that $F_\Omega O_1(|x|^{-1-\theta}) = O(|x|^{-1-\theta})$.

Now, since $F_\Delta^N \xrightarrow{s} 0$ as $|\Delta| \to 0$ and $N \to \infty$ and $O(|x|^{-1-\theta})$ is H-compact, we have: $F_\Delta^N O = F_\Delta^N F_\Omega O \to 0$. Here $\Omega \supset \Delta$ is fixed and compact and O stands for the operator $O(|x|^{-1-\theta})$. Since I is compact, we can find δ_0' and N_1' so that $\|F_{\Delta'}^{N'} O F_{\Delta'}^{N'}\| < \varepsilon$ for any $\Delta' \subset I$ (or a small neighborhood of I, if necessary) with $|\Delta'| < \delta'$ and $N' > N_1'$. Multiplying (6.14) from both sides by $F_{\Delta'}^{N'}$, using the above inequality, we obtain, after dropping the primes in Δ' and N' the equation:

$$(6.15) \qquad B_{\Delta, N} \overset{\varepsilon}{=} \sum_{\text{2-cluster}} F_\Delta^N \varphi_a i[H_a, A]\varphi_a F_\Delta^N$$

$$\text{for any } \Delta \subset I,\ |\Delta| < \delta_0' \text{ and } N > N_1'.$$

LEMMA 6.4. *For any ε, δ_2, and a compact set $I \subset \mathbf{R}$ there exist δ_0'' and N_1'' (independent of δ_2) such that*

$$(6.16) \qquad \|F_\Delta^N \varphi_a - F_\Delta^N \varphi_a F_{\Delta_1}(H_a)\| \le \varepsilon$$

for any $N \ge N_1''$ and any $\Delta \subset I$ with $|\Delta| < \delta_0''$ and some $\Delta_1 \supset \Delta$ with $|\Delta_1| < \delta_1$.

Proof. Henceforth, all intervals are open and $\Delta \subset \Delta'$ means $\bar{\Delta} \subset \Delta'$. Recall that for an interval Δ and number $\alpha < \frac{1}{10}|\Delta|$, $\Delta^\alpha = (\inf \Delta - \alpha, \sup \Delta + \alpha)$.

Fix $E \in I$. Pick an open interval Δ_E around E with $|\Delta_E| < \delta_2$. Let $\alpha = \frac{1}{10}|\Delta|$ and $\Delta^\alpha \subset \Delta_E$ so that

$$F_\Delta(H) = F_\Delta(H) F_{\Delta_E}(H).$$

Then for each Δ_E

$$\begin{aligned} A_{\Delta, N, \Delta_E} &\equiv F_\Delta^N \varphi_a\big(1 - F_{\Delta_E}(H_a)\big) \\ &= F_\Delta^N\big[F_{\Delta_E}(H)\varphi_a - \varphi_a F_{\Delta_E}(H_a)\big] \to 0 \quad \text{as } |\Delta| \to 0 \text{ and } N \to \infty \end{aligned}$$

since $F_\Delta^N \xrightarrow{s} 0$ as $|\Delta| \to 0$ and $N \to \infty$ and the operator in square parentheses is compact. Hence for any $\varepsilon > 0$ and small α ($\alpha < \frac{1}{10}\delta_2$) there are N_1'' and δ_E such that

$$(6.17) \quad \|A_{\Delta, N, \Delta_E}\| \le \varepsilon \quad \text{for any } N \ge N_1'' \text{ and } \Delta^\alpha \subset \Delta_E \text{ with } |\Delta| < \delta_E.$$

Since $\{\Delta_E,\ E \in I\}$ covers I and since I is compact there is a finite subcovering $\{\Delta_i\}$ of I. Rearrange $\{\Delta_i\}$, if necessary, so that $\Delta_i \cap \Delta_{i+1} \neq 0$. Due to (6.17)

$$(6.18) \quad \|A_{\Delta, N, \Delta_i}\| \le \varepsilon \quad \text{for any } N \ge N_1'' \text{ and } \Delta^\alpha \subset \Delta_i \text{ with } |\Delta^\alpha| < \delta_i$$

where $\delta_i = \delta_E$ for i: $\Delta_i = \Delta_E$. Take $\delta_0'' = \frac{1}{10}\min(\delta_i, \sup \Delta_i - \inf \Delta_{i+1})$. Take now any $\Delta^\alpha \subset I$ with $|\Delta^\alpha| < \delta_0''$. Due to the choice of δ_0'', Δ^α is entirely in one of the Δ_i's, say Δ_j. Let $\Delta_1 = \Delta_j$. Thus (6.18) yields $\|A_{\Delta, N, \Delta_1}\| \le \varepsilon$ for any $N \ge N_1''$, which is the desired result. □

Applying Lemma 6.4 to equation (6.15) we obtain:
For any $\varepsilon, \delta_2 > 0$ and compact interval I there are δ_0 $(= \min(\delta_0', \delta_0''))$ and N_1 $(= \max(N_1', N_1''))$ such that

$$B_{\Delta, N} \stackrel{\varepsilon}{=} \sum_{2\text{-cluster}} F_\Delta^N \varphi_a F_{\Delta_1}(H_a) i[H_a, A] F_{\Delta_1}(H_a) \varphi_a F_\Delta^N$$

for any $\Delta \subset I$ with $|\Delta| < \delta_0$ and any $N > N_1$ and some $\Delta_1 \supset \Delta$ with $|\Delta_1| < \delta_2$. Take this δ_2 the same as in the induction statement. Now, using the equation

$$H_a = H^a \otimes 1 + 1 \otimes T_a \tag{6.19}$$

and its consequence

$$i[H_a, A] = i[H^a, A^a] \otimes 1 + 1 \otimes 2T_a$$

(here we have used that $i[T_a, A] = 2T_a$) and introducing the notation $C^a = i[H^a, A^a]$, we obtain

$$B_{\Delta, N} \stackrel{\varepsilon}{=} F_\Delta^N \Big(\sum \varphi_a F^a_{\Delta_1 - T_a} (C^a + 2T_a) F^a_{\Delta_1 - T_a} \varphi_a \Big) F_\Delta^N \tag{6.20}$$

where $F^a_{\Delta - T_a}$ stands for $F_\Delta(H_a)$ (cf. equation (6.19)).

LEMMA 6.5. *Given ε, N_2 and compact I, there is δ_3 so that for any $\Delta_1 \subset I$ with $|\Delta_1| < \delta_3$,*

$$F^a_{\Delta_1} C^a F^a_{\Delta_1} \stackrel{\varepsilon}{=} F^{a, N_2}_{\Delta_1} C^a F^{a, N_2}_{\Delta_1}. \tag{6.21}$$

Proof. In the proof we drop the superindex a. Write

$$\begin{aligned} F_{\Delta_1} C F_{\Delta_1} = {} & F_{\Delta_1}^{N_2} C F_{\Delta_1}^{N_2} + P^{N_2} F_{\Delta_1} C F_{\Delta_1} \bar{P}^{N_2} \\ & + \bar{P}^{N_2} F_{\Delta_1} C F_{\Delta_1} P^{N_2} + P^{N_2} F_{\Delta_1} C F_{\Delta_1} P^{N_2} \end{aligned} \tag{6.22}$$

and estimate the last three terms.

Let $P^N_{\text{tail}} = \sum_{j \geq N+1} P_j$. Since $P^{N_2} C$ is compact and since $P^N_{\text{tail}} \xrightarrow{s} 0$ as $N \to \infty$, for any ε we can choose N such that

$$\| P^{N_2} C P^N_{\text{tail}} \| \leq \varepsilon.$$

Fix this N. Next, we choose

$$\delta_3 \leq \tfrac{1}{3} \min_{\substack{i, j \leq N+1 \\ \varepsilon_i \neq \varepsilon_j}} |\varepsilon_i - \varepsilon_j|.$$

Then for any Δ_1 with $|\Delta_1| < \delta_3$ and $|\text{supp}\, F_{\Delta_1}| \leq \frac{11}{10} |\Delta_1|$,

$$F_{\Delta_1}(\varepsilon_i) F_{\Delta_1}(\varepsilon_j) = 0 \quad \text{for } \varepsilon_i \neq \varepsilon_j;\ i, j \leq N + 1 \tag{6.23}$$

and

$$P^{N_2, N} F_{\Delta_1}(H) F_{\Delta_1}(\varepsilon_j) = 0 \quad \text{for any } j \leq N_2,$$

where

$$P^{N_2, N} = \sum_{N \geq j \geq N_2 + 1} P_j.$$

This gives

(6.24) $$F_{\Delta_1} P^{N_2} C P^{N_2}_{\text{tail}} F_{\Delta_1} = \sum_{j \leq N_2} P_j C P^{N_2}_{\text{tail}} F_{\Delta_1} F_{\Delta_1}(+\varepsilon_j) = 0.$$

Hence

$$F_{\Delta_1} P^{N_2} C P^{N_2}_{\text{tail}} F_{\Delta_1} = F_{\Delta_1} P^{N_2} C P^{N}_{\text{tail}} F_{\Delta_1} \stackrel{\varepsilon}{=} 0.$$

Furthermore

$$F_{\Delta_1} P^{N_2} C P^{N_2} F_{\Delta_1} = \sum_{i,\, j \leq N_2} F_{\Delta_1} P_i C P_j F_{\Delta_1}.$$

Now

$P_i C P_j = 0 \quad$ for $\varepsilon_i = \varepsilon_j$ by the Virial theorem (see e.g., [PSS], [CFKS])

and

$$F_{\Delta_1} P_i C P_j F_{\Delta_1} = P_i C P_j F_{\Delta_1}(\varepsilon_j) F_{\Delta_1}(\varepsilon_i) = 0$$

for $i, j \leq N$ and $\varepsilon_i \neq \varepsilon_j$ by (6.23). Hence

(6.25) $$F_{\Delta_1} P^N C P^N F_{\Delta_1} = 0.$$

Finally, since for sufficiently large but bounded $\Omega \subset \mathbf{R}$, $P^N = P^N F_\Omega$ and since $F_\Omega C$ is bounded, $P^N C$ is compact. Since, in addition, $P^{\text{cont}} F_{\Delta_1} \xrightarrow{s} 0$ as $|\Delta_1| \to 0$, we have that for any ε and any $E \in I$ there is an interval Δ_E around E so that

$$\| P^{N_2} C P^{\text{cont}} F_{\Delta_E} \| \leq \varepsilon.$$

By a similar but simpler compactness argument than in the one in the proof of Lemma 3.4, this implies that there is δ_3 so that

$$\| P^{N_2} C P^{\text{cont}} F_{\Delta'} \| \leq \varepsilon \quad \text{for any } \Delta' \subset I \text{ with } |\Delta'| \leq 2\delta_3.$$

Equations (6.24)–(6.25) imply

$$F_{\Delta'} C F_{\Delta'} \stackrel{\varepsilon}{=} F^{N_2}_{\Delta'} C F^{N_2}_{\Delta'}$$

for any $\Delta' \subset I$ with $|\Delta'| \leq 2\delta_3$. This yields (6.21). □

Lemma 6.5 implies that for any ε and N_2 there is δ_3 such that

(6.26) $$F^a_{\Delta_1 - T_a} C^a F^a_{\Delta_1 - T_a} \stackrel{\varepsilon}{=} F^{a,\, N_2}_{\Delta_1 - T_a} C^a F^{a,\, N_2}_{\Delta_1 - T_a}$$

for any $\Delta_1 \subset I$ and $|\Delta_1| < \delta_3$. To demonstrate this we write the left side as a fiber integral with respect to T_a,

$$\int^{\oplus} F^a_{\Delta_1 - |k_a|^2} C^a F^a_{\Delta_1 - |k_a|^2},$$

observe that, since H_a is bounded from below, the integral ranges over a compact region and then apply Lemma 6.5.

Applying (6.26) to (6.20) we obtain that for any ε, δ_0 and N_2 there are δ_2, δ_3 such that

$$B_{\Delta, N} \overset{\varepsilon}{=} \sum_{\#(a)=2} F^N_\Delta \varphi_a \Big[F^{a, N_2}_{\Delta_1 - T_a} C^a F^{a, N_2}_{\Delta_1 - T_a} + 2T_a \big(F^a_{\Delta - T_a} \big)^2 \Big] \varphi_a F^N_\Delta \tag{6.27}$$

for any $\Delta \subset I$ with $|\Delta| < \delta_0$ and some Δ_1, $\Delta \subset \Delta_1 \subset I$, with $|\Delta_1| < \min(\delta_2, \delta_3)$. Take now

$$\delta_1 = \min\big(\delta_2, \delta_3, \delta^2/100\big)$$

and assume $|\Delta_1| < \delta_1$.

Consider the last term in equation (6.27):

$$\Big(F^a_{\Delta_1 - T_a}\Big)^2 = \Big(F^a_{\Delta_1 - T_a}\Big)^2 P^{a, N_2} + \Big(F^{a, N_2}_{\Delta_1 - T_a}\Big)^2, \tag{6.28}$$

We expand

$$\begin{aligned}\Big(F^a_{\Delta_1 - T_a}\Big)^2 P^{a, N_2} &= \sum_{j \le N_2} F_{\Delta_1}\big(\varepsilon_{a, j} + T_a\big) P^a_j \\ &= \sum_{\substack{a(\alpha) = a \\ m(\alpha) \le N_2}} F_{\Delta_1}\big(\varepsilon_\alpha + |p_\alpha|^2\big) P_\alpha\end{aligned}$$

(recall that $T_\alpha = |p_\alpha|^2$). Note that

$$F_{\Delta_1}\big(\varepsilon_\alpha + |p_\alpha|^2\big) = 0 \quad \text{if } \sup \Delta_1 - \varepsilon_\alpha < 0.$$

Furthermore,

$$T_\alpha F_{\Delta_1}(\varepsilon_\alpha + T_\alpha) \ge (\inf \Delta_1 - \varepsilon_\alpha) F_{\Delta_1}(\varepsilon_\alpha + T_\alpha)$$

Since $E = \inf \Delta + \delta$, $|\Delta_1| < \delta_1 < \delta$ and $\Delta \subset \Delta_1$, we have that

$$E > \sup \Delta_1 \quad \text{and} \quad E - \delta_1 < \inf \Delta_1,$$

Hence

$$F_{\Delta_1}\big(\varepsilon_2 + |p_\alpha|^2\big) = 0 \quad \text{if } E - \varepsilon_\alpha \le 0$$

and

$$\begin{aligned}T_\alpha F_{\Delta_1}\big(\varepsilon_\alpha + |p_\alpha|^2\big) &\ge \big(E - \varepsilon_\alpha - \delta_1\big) F_{\Delta_1}\big(\varepsilon_\alpha + |p_\alpha|^2\big) \\ &= \big(\kappa^2_\alpha - \delta_1\big) F_{\Delta_1}\big(\varepsilon_\alpha + |p_\alpha|^2\big).\end{aligned}$$

Define

$$F_\alpha(s) = F_{\Delta_1}\left(\varepsilon_\alpha + |p_\alpha|^2\right)$$

for α with $\#(\alpha) = 2$ and $E - \varepsilon_\alpha \geq 0$. Then

$$F_\alpha(|k_\alpha| - \kappa_a) = F_{\Delta_1}\left(\varepsilon_\alpha + |k_\alpha|^2\right)$$

and

$$(*) \qquad \operatorname{supp} F_\alpha \subset \left(-\frac{\delta}{2}, \frac{\delta}{2}\right).$$

The first relation is obvious. We show the latter relation. Since $|\Delta_1| < \delta_1$ and since $E \in \Delta_1$, we have that

$$\Delta_1 \subset (E - \delta_1, E + \delta_1).$$

Since $\kappa_\alpha^2 = E - \varepsilon_\alpha$ (provided $E \geq \varepsilon_\alpha$) we have

$$(**) \qquad \operatorname{supp} F_\alpha \subset \left\{ s \,|\, (s + \kappa_\alpha)^2 - \kappa_\alpha^2 \in (-\delta_1, \delta_1) \right\}.$$

Considering separately the cases $|\kappa_\alpha| < \sqrt{\delta_1/2}$ and $|\kappa_\alpha| \geq \sqrt{\delta_1/2}$, one finds easily that the right side of $(**)$ is contained in the interval $(-4\sqrt{\delta_1}, 4\sqrt{\delta_1})$. Since $\delta < \delta^2/100$, $(*)$ follows.

As a result we have

$$(6.29) \qquad \left(F^a_{\Delta_1 - T_a}\right)^2 P^{a, N_2} = \sum_{\substack{a(\alpha) = a \\ m(\alpha) \leq N_2}} F_\alpha(|p_\alpha| - \kappa_\alpha) P_\alpha.$$

Combining (6.28) and (6.29) and substituting the result into (6.27) we obtain

$$(6.30) \qquad B_{\Delta, N} \overset{\varepsilon}{=} \sum_{\substack{\#(\alpha) = 2 \\ m(\alpha) \leq N_2}} F_\Delta^N 2\kappa_\alpha \varphi_{a(\alpha)} P_\alpha F_\alpha(|p_\alpha| - \kappa_\alpha) \varphi_{a(\alpha)} F_\Delta^N + K_{\Delta, N}$$

where

$$(6.31) \qquad K_{\Delta, N} = \sum_{\#(a) = 2} F_\Delta^N \varphi_a \left[F^{a, N_2}_{\Delta_1 - T_a} C^a F^{a, N_2}_{\Delta_1 - T_a} + 2T_a \left(F^{a, N_2}_{\Delta_1 - T_a} \right)^2 \right] \varphi_a F_\Delta^N.$$

Now we use the induction hypothesis (6.7) and (6.8) in order to estimate (6.31). First, we represent $F^{a, N_2}_{\Delta_1 - T_a} C^a F^{a, N_2}_{\Delta_1 - T_a}$ and $T_a(F^{a, N_2}_{\Delta_1 - T_a})$ as fiber integrals

$$\int^\oplus F^{a, N_2}_{\Delta_1 - |k_a|^2} C^a F^{a, N_2}_{\Delta_1 - |k_a|^2}\, dk_a \quad \text{and} \quad \int^\oplus |k_a|^2 \left(F^{a, N_2}_{\Delta_1 - |k_a|^2} \right)^2 dk_a$$

and then apply (6.7) and (6.8), respectively, to the integrands. This amounts, in fact, to replacing E in (6.7) and (6.8) by $E - |k_a|^2$. Since $2(E - |k_a|^2 - \varepsilon_\alpha)_+ \geq$

$\kappa_\alpha^2 - 2|k_a|^2$, the result is

$$(6.32)\qquad F^{a,N_2}_{\Delta_1-T_a}(C^a + 2T_a)F^{a,N_2}_{\Delta_1-T_a} \geq F^{a,N_2}_{\Delta_1-T_a} \sum_{\substack{H^a\text{-chan.}\\ E-\varepsilon_\alpha\geq 0}} 2\kappa_\alpha^2\tilde{\phi}^a_{\alpha,E-T_a}F^{a,N_2}_{\Delta_1-T_a}$$

where

$$(6.33)\qquad \tilde{\phi}^a_{\alpha,E-T_a} = \sum_{\alpha(S)=\alpha} j^a_S P_\alpha F_\alpha(|p_\alpha| - \kappa_\alpha)j^{a*}_S.$$

This inequality and the relations $|p^a_\alpha|^2 + T_a = |p_\alpha|^2$ and

$$\{\alpha|\alpha \text{ open for } E - |k_a|^2\} \subset \{\alpha|\alpha \text{ open for } E\}$$

yield

$$(6.34)\qquad K_{\Delta,N} \overset{\varepsilon}{\geq} F^N_\Delta \sum_{\#(a)=2}\sum_{H^a\text{-chan.}} 2\kappa_\alpha^2\varphi_a F^a_{\Delta_1-T_a}\bar{P}^{N_2}_a\tilde{\phi}^a_{\alpha,E-T_a}F^a_{\Delta_1-T_a}\varphi_a F^N_\Delta.$$

Now using Lemma 6.4 to remove $F^a_{\Delta_1-T_a} = F_1(H_a)$ back to F^N_Δ, we obtain

$$(6.35)\qquad K_{\Delta,N} \overset{\varepsilon}{\geq} F^N_\Delta\Big(\sum_{\#(\alpha)\geq 3} 2\kappa_\alpha^2\tilde{\phi}_{\alpha,E}\Big)F^N_\Delta,$$

where

$$\tilde{\phi}_{\alpha,E} = \sum_{\substack{\#(a)=2\\ a\supsetneqq a(\alpha)}} \varphi_a\bar{P}^{N_2}_a\big(\tilde{\phi}^a_{\alpha,E-T_a}\big)\bar{P}^{N_2}_a\varphi_a.$$

Due to equations (6.6) and (6.33), the latter expression can be rewritten as

$$(6.36)\qquad \tilde{\phi}_{\alpha,E} = \sum_{\alpha(S)=\alpha} j_S P_\alpha F_\alpha(|p_\alpha| - \kappa_\alpha)|j^*_S.$$

Together with (6.30) and (6.6) this yields

$$(6.37)\qquad B_{\Delta,N} \overset{\varepsilon}{\geq} F^N_\Delta\Big(\sum 2\kappa_\alpha^2\tilde{\phi}_{a,E}\Big)F^N_\Delta.$$

Next we need the *Localization Lemma*:

LEMMA 6.6. *Let F_i, $i = 1, 2$, be smooth functions such that* supp $F_1 \subset (-\delta/2, \delta/2)$ *and* supp $F_2 \subset [\frac{3}{4}\delta, \infty)$. *Then*

$$(6.38)\qquad F_1(|p_\alpha| - \kappa)F_2(\pm\gamma_\alpha - \kappa) = O(|x|^{-1}).$$

Proof. In the proof we omit the subindex α. Pick F_3 so that supp $F_3 \subset (-(2/3)\delta, (2/3)\delta)$ and $F_3 = 1$ on supp F_1. Denote $g = F_3(|p| - \kappa)$, $F_1(p) = F_1(|p| - \kappa)$ and $F_2(\gamma) = F_2(\pm\gamma - \kappa)$. The operator $\gamma_g \equiv g\gamma g$ is symmetric and bounded:

$$|\langle\hat{x}\cdot pgu, gu\rangle| \leq \|pgu\|\,\|\hat{x}gu\| \leq \big(\kappa + \tfrac{2}{3}\delta\big)\|gu\|^2,$$

where we have used that

$$\|\hat{x}f\| \le \|f\|.$$

This shows that

$$\pm g\gamma g \le \kappa + \tfrac{2}{3}\delta.$$

Now observe that

$$F_1(p)F_2(\gamma) = F_1(p)\big(F_2(\gamma) - F_2(\gamma_g)\big).$$

Using the Fourier representation

$$F(\gamma) = \int \hat{F}_2(s)e^{i\gamma s}\,ds$$

and using a continuity argument in order to extend the following result from $\mathscr{L}(\mathbf{R})$-functions to smooth bounded functions F_2 with C_0^∞ derivative, we obtain

$$\begin{aligned} F_2(\gamma) - F_2(\gamma_g) &= \int_{-\infty}^{\infty} \hat{F}_2(s)(e^{i\gamma s} - e^{i\gamma_g s})\,ds \\ &= -i\int_{-\infty}^{+\infty} ds\,\hat{F}_2(s)e^{i\gamma s}\int_0^s du\,e^{-i\gamma u}(\gamma - g\gamma g)e^{i\gamma_g u}. \end{aligned}$$

This implies

$$F_1(p)F_2(\gamma) = -i\int_{-\infty}^{\infty} ds\,\hat{F}_2(s)\int_0^s du\,F_1(p)e^{i\gamma(s-u)}(\gamma - g\gamma g)e^{i\gamma_g u}.$$

We commute $F_1(p)$ on the right side of this expression through $e^{i\gamma(s-u)}$. The result is

$$F_1(p)F_2(\gamma) = B_1 + B_2, \tag{6.39}$$

where

$$B_1 = -i\int_{-\infty}^{\infty} ds\,\hat{F}_2(s)\int_0^s du\big[F_1(p), e^{i\gamma(s-u)}\big](\gamma - g\gamma g)e^{i\gamma_g u}$$

and

$$B_2 = i\int_{-\infty}^{\infty} ds\,\hat{F}_2(s)\int_0^s du\,e^{i\gamma(s-u)}F_1(p)\gamma(1-g)e^{i\gamma_g u}.$$

Next we show that

$$\big[F_1(p), \gamma\big] = O(|x|^{-1}). \tag{6.40}$$

Using that

$$\big[F_1(p), \gamma\big] = \sum\left[p_i F_1(p), \frac{x_i}{\langle x\rangle}\right] + O(|x|^{-1}) \tag{6.41}$$

and using Lemma A.4(ii), we obtain (6.40).

Now the relation

$$[F_1(p), e^{i\gamma t}] = -i\int_0^t e^{i\gamma(t-s)}[F_1(p), \gamma]e^{i\gamma s}\,ds, \tag{6.42}$$

equations (6.40) and (A.11) of the appendix imply that

$$[F_1(p), e^{i\gamma t}] = O(|x|^{-1}t^2). \tag{6.43}$$

Equations (6.40) and (6.43) together with the relations

$$F_1(p)(1-g) = 0$$

and

$$\int_{-\infty}^{\infty} |\hat{F}_2(s)|\,|s|^n\,ds < \infty \quad \text{for } n = 1,2,3,$$

imply

$$B_i = O(|x|^{-1}), \qquad i = 1,2. \qquad \square$$

Since $\operatorname{supp} F_\alpha \subset (-\delta/2, \delta/2)$ and $F(s \le \kappa_\alpha + \delta) = 0$ for $s > \kappa_\alpha + 5/4\delta$ by the definition, Lemma 6.6 implies

$$F_\alpha(|p_\alpha| - \kappa_\alpha) = F(|\gamma_\alpha| \le \kappa_\alpha + \delta)F_\alpha(|p_\alpha| - \kappa_\alpha)F(|\gamma_\alpha| \le \kappa_\alpha + \delta) + O(|x|^{-1}).$$

This gives (see equations (6.3) and (6.36) for the definition)

$$\tilde{\phi}_{\alpha, E} = \phi_{\alpha, E} + K, \tag{6.44}$$

where K is a compact operator.

Take $\bar{\delta}_0 < \delta_0$ and $\bar{N}_1 \le N_1$ so that $\|F_{\bar{\Delta}}^{\bar{N}} K F_{\bar{\Delta}}^{\bar{N}}\| \le \bar{\varepsilon}$ for any $\bar{N} > \bar{N}_1$ and $|\bar{\Delta}| \le \bar{\delta}_0$. Multiplying (6.37) from both sides by $F_{\bar{\Delta}}^{\bar{N}}$ and absorbing $\bar{\varepsilon}$ into the ε, we obtain, after dropping the bars over δ_0, Δ, N_1 and N, the desired result (6.1).

Now we prove (6.2). The proof is similar to the proof of (6.1) but simpler; it uses only Lemma 6.4:

$$\begin{aligned}
(F_\Delta^N)^2 &= \sum_{\#(a)=2} F_\Delta^N \varphi_a^2 F_\Delta^N \\
&\overset{\varepsilon}{=} \sum_{\#(a)=2} F_\Delta^N \varphi_a F_\Delta^N (H_a)^2 \varphi_a F_\Delta^N \\
&= \sum_{\substack{\#(a)=2\\ m(\alpha)=2}} F_\Delta^N \varphi_a P_\alpha F_\alpha(|p_\alpha| - \kappa_\alpha) P_\alpha \varphi_a F_\Delta^N \\
&= \sum_{\#(a)=2} F_\Delta^N \varphi_a \bar{P}^{a, N_2} F_{\Delta_1}(H_a)^2 \bar{P}^{a, N_2} \varphi_a F_\Delta^N.
\end{aligned}$$

Applying the induction hypothesis (6.8) to the last term and using expression

(6.6) for j_S we obtain

$$(F_\Delta^N)^2 \geq F_\Delta^N\left(\sum \tilde{\phi}_{\alpha,E}\right)F_\Delta^N. \tag{6.45}$$

Using (6.44) and the analysis following it again, we arrive at (6.2). □

COROLLARY 6.7. *Under the condition of Theorem 6.1 and with the same notation,*

$$F_\Delta^N i[H, A] F_\Delta^N \geq F_\Delta^N \sum_{\substack{m(\alpha) \leq N_{\#(\alpha)} \\ \alpha\text{-open}}} \left(2\kappa_\alpha^2 - \varepsilon\right)\phi_{\alpha,E}F_\Delta^N \tag{6.46}$$

and

$$(F_\Delta^N)^2 \gtrless (1 \pm \varepsilon)F_\Delta^N \sum_{\substack{m(\alpha) \leq N_{\#(\alpha)} \\ \alpha\text{-open}}} \phi_{\alpha,E}F_\Delta^N \tag{6.47}$$

Proof. Take $\Delta' \subset \Delta$ and $N' > N$ so that

$$F_{\Delta'}^{N'}F_\Delta^N = F_{\Delta'}^{N'}.$$

Multiply (6.1) by $F_{\Delta'}^{N'}$, and use the channel expansion formula 6.2 for

$$\varepsilon\left(F_{\Delta'}^{N'}\right)^2.$$

The result, after dropping the $' - s$ over Δ and N is equation (6.46). Similarly, we obtain (6.47). □

7. Propagation estimates I. Non-threshold case

In this section we prove the propagation estimates for the region of the phase space which is, essentially, $\{(x, k) | \hat{x} \cdot k \notin \Sigma_E\}$.

THEOREM 7.1. *Let E be a fixed energy which is not a threshold or an eigenvalue of H. Then for any smooth bounded function $F(s)$ supported away from Σ_E there is a small interval Δ around E such that*

$$\int_{-\infty}^{\infty} \left\| F(\gamma)\frac{1}{\sqrt{\langle x \rangle}}\psi_t \right\|^2 dt \leq c\|\psi\|^2 \tag{7.1}$$

for any $\psi \in \operatorname{Ran} P_\Delta$ and $\psi_t = e^{-iHt}\psi$ with c $(< \infty)$ independent of ψ.

According to our general convention $F(s \geq s_0)$ is a smooth non-negative function, ≤ 1 and satisfies

$$F(s \geq s_0) = \begin{matrix} 1 & \text{for } s \geq s_0 + \delta_1 \\ 0 & \text{for } s \leq s_0 - \delta_1. \end{matrix}$$

with δ_1 specified in an appropriate place. According to our general strategy we connect, using Lemma 3.4, propagation estimates with estimates from below of commutators of H with certain observables. In particular, the proof of Theorem 7.1 will be based on the following:

PROPOSITION 7.2. *Let E be away from the thresholds and eigenvalues of H and let $\gamma_0 \notin \Sigma_E$, $|\gamma_0| < \max \kappa_\alpha$. Then there is a small interval around E such that the estimate*

$$F_\Delta i[H, F(\gamma \geq \gamma_0)] F_\Delta \dot{\geq} \theta F_\Delta \frac{1}{\sqrt{\langle x \rangle}} F'(\gamma \geq \gamma_0) \frac{1}{\sqrt{\langle x \rangle}} F_\Delta, \tag{7.2}$$

holds for some $\theta > 0$ and for any $F(s \geq \gamma_0)$ with derivative $F'(s \geq \gamma_0) \geq 0$, supported in $[\gamma_0 - \delta_1, \gamma_0 + \delta_1]$ with

$$\delta_1 < \frac{1}{10} \frac{1}{10 + |\gamma_0|} \text{dist.}(\gamma_0, \Sigma_E).$$

Proof. Take Δ' slightly larger than Δ so that

$$F_{\Delta'} F_\Delta = F_\Delta$$

and fix it. It is proven in the appendix (Corollary A.8) that for such Δ and Δ' and for any bounded smooth function f with a C_0^∞ derivative,

$$F_\Delta i[H, f(\gamma)] F_\Delta = F_\Delta f'(\gamma)^{1/2} F_{\Delta'} i[H, \gamma] F_{\Delta'} f'(\gamma)^{1/2} F_\Delta + O(|x|^{-2}). \tag{7.3}$$

Note also that a similar relation holds if H and γ are replaced by H_a and γ_a. Next we use

$$i[H, \gamma] = \frac{1}{\sqrt{\langle x \rangle}} (i[H, A] - \gamma^2) \frac{1}{\sqrt{\langle x \rangle}} + O(|x|^{-2}). \tag{7.4}$$

Note that the last two estimates show that the commutator of any two factors in the first term on the right side of (7.3) is $O(|x|^{-2})$ so that the order in which they stand is, in fact, inessential. We will estimate the first term on the right-hand side of (7.4). This estimate is based on the channel expansion result proved in Section 6 (Theorem 6.1). Roughly speaking, we show that $f'(\gamma) i[H, \gamma]$ is positive on each channel separately.

Equations (7.3) and (7.4) and relation (3.17) imply

$$\begin{aligned} &F_\Delta i[H, F(\gamma \geq \gamma_0)] F_\Delta \\ &\doteq F_\Delta F'(\gamma \geq \gamma_0)^{1/2} F_{\Delta'} \frac{1}{\sqrt{\langle x \rangle}} (i[H, A] - \gamma^2) \frac{1}{\sqrt{\langle x \rangle}} F_{\Delta'} F'(\gamma \geq \gamma_0)^{1/2} F_\Delta. \end{aligned} \tag{7.5}$$

Next, using the relations

$$\left[F_{\Delta'}, \frac{1}{\sqrt{\langle x\rangle}}\right] = O(|x|^{-3/2}) \tag{7.6}$$

and

$$\left[F'(\gamma \ge \gamma_0)^{1/2}, \frac{1}{\sqrt{\langle x\rangle}}\right] = O(|x|^{-3/2}), \tag{7.7}$$

which follow from Lemma A.3(ii) we move $1/\sqrt{\langle x\rangle}$ outside of $F'(\gamma \ge \gamma_0)^{1/2}$. After that, we estimate the γ^2-term. First, we observe that, due to Lemma A.4(i),

$$\left[F'(\gamma \ge \gamma_0)^{1/2}, F_{\Delta'}\right] = O(|x|^{-1}). \tag{7.8}$$

Next we use the localization of $F'(\gamma \ge \gamma_0)^{1/2}$ and the spectral theorem to deduce

$$\gamma^2 F'(\gamma \ge \gamma_0)^{1/2} \le (\gamma_0^2 + \varepsilon) F'(\gamma \ge \gamma_0)^{1/2} \tag{7.9}$$

with $\varepsilon = 4|\gamma_0|\delta_1$. The last two relations yield

$$\begin{aligned} &- F'(\gamma \ge \gamma_0)^{1/2} F_{\Delta'} \gamma^2 F_{\Delta'} F'(\gamma \ge \gamma_0)^{1/2} \\ &\quad \ge - F'(\gamma \ge \gamma_0)^{1/2} F_{\Delta'} (\gamma_0^2 + \varepsilon) F_{\Delta'} F'(\gamma \ge \gamma_0)^{1/2}. \end{aligned}$$

Taking this into account we arrive at

$$F_\Delta i[H, F] F_\Delta \dot{\ge} F_\Delta \frac{1}{\sqrt{\langle x\rangle}} (F')^{1/2} F_{\Delta'} (i[H, A] - 2\gamma_0^2 - 2\varepsilon) F_{\Delta'} (F')^{1/2} \frac{1}{\sqrt{\langle x\rangle}} F_\Delta,$$

where we have used the obvious abbreviations. Applying to this inequality expansions (6.1) and (6.2), with ε as given above, we arrive at

$$\begin{aligned} F_\Delta i[H, F] F_\Delta \dot{\ge} 2F_\Delta \frac{1}{\sqrt{\langle x\rangle}} (F')^{1/2} F_{\Delta'} & \\ \sum_{\alpha\text{-open}} (\kappa_\alpha^2 - \gamma_0^2 - 2\varepsilon) \phi_\alpha F_{\Delta'} (F')^{1/2} \frac{1}{\sqrt{\langle x\rangle}} F_\Delta & \end{aligned} \tag{7.10}$$

$$\phi_\alpha = \sum_{\alpha(S)=\alpha} j_S P_\alpha F(|\gamma_\alpha| \le \kappa_\alpha + \delta) F_\alpha(|p_\alpha| = \kappa_\alpha) F(|\gamma_\alpha| \le \kappa_\alpha + \delta) j_S^*. \tag{7.11}$$

Here δ is a given small number (see Theorem 6.1) and

$$F(s < \kappa_\alpha + \delta) = 0 \quad \text{for } s > \kappa_\alpha + 3/2\delta. \tag{7.12}$$

We pick $\delta = 10\delta_1$.

LEMMA 7.3. *Assume f is smooth and*

$$\operatorname{supp} f \subset \left\{ s \| s - \gamma_0 | < \tfrac{1}{4}\delta \right\} \tag{7.13}$$

with δ as given in Theorem 6.1 (see equation (7.12)). Then for any ε_1,

$$f(\gamma)\phi_\alpha \overset{\varepsilon_1}{=} O(|x|^{-1}) \quad \text{for } \alpha \text{ such that } |\gamma_0| > \kappa_\alpha + 2\delta. \tag{7.14}$$

Here $O(|x|^{-1})$ depends on ε_1.

Proof. We prove first the following relation: For f as above and any ε_1,

$$f(\gamma) j_S \overset{\varepsilon_1}{=} j_S f(\gamma_\alpha) + O(|x|^{-1}), \tag{7.15}$$

where $\alpha(S) = \alpha$ and $O(|x|^{-1})$ depends on ε_1.

To prove (7.15) we analyse the commutators of $f(\gamma)$ with the operators $\varphi_b^a(x^a)$ and P_a^N out of which j_S is built. Lemma A.3(i) yields

$$[f(\gamma), \varphi_b^a(x^a)] = O(|x|^{-1}). \tag{7.16}$$

Next, using the same argument as in the proof of Lemma A.9(ii) we obtain that for any ε_1

$$P_a^N O(|x|^{-1}) \overset{\varepsilon_1}{=} O(|x|^{-1}). \tag{7.17}$$

Now, Lemma A.9(iii) yields that for any ε_1,

$$[P_a^N, f(\gamma)] \overset{\varepsilon_1}{=} O(|x|^{-1}). \tag{7.18}$$

Define $\tilde{j}_S$ by $j_S = \tilde{j}_S P_\alpha$. Due to equation (6.4), $\tilde{j}_S$ is a finite sum of terms each of which is product of some φ_b^a and P_b. Then equations (7.16)–(7.18) show that for any ε_1,

$$[f(\gamma), \tilde{j}_S] \overset{\varepsilon_1}{=} O(|x|^{-1}). \tag{7.19}$$

Finally, Lemma A.6(ii) implies that for any ε_1,

$$f(\gamma) P_\alpha \overset{\varepsilon}{=} P_\alpha f(\gamma_\alpha) + O(|x|^{-1}). \tag{7.20}$$

The last two equations imply the desired result.

Now to deduce (7.14) we recall the definition of ϕ_α; note that $j_S P_\alpha = j_S$ for $\alpha(S) = \alpha$. Using (7.15), pull $f(\gamma)$ through j_S inside of ϕ_α next to $F(|\gamma_\alpha| \le \kappa_\alpha + \delta)$. The latter cut-off function satisfies

$$F(s \le \kappa_\alpha + \delta) = 0 \quad \text{for } s \ge \kappa_\alpha + 3/2\delta$$

and therefore, due to (7.13),

$$f(\gamma_\alpha)F(|\gamma_\alpha| \le \kappa_\alpha + \delta) = 0$$

and we arrive at (7.14). □

Let $\kappa(\varepsilon)$ be the number of terms on the right-hand side of (7.10). Take $\varepsilon_1 < \varepsilon[10\kappa(\varepsilon)\max \kappa_\alpha^2]^{-1}$. Pull $F_1(\gamma = \gamma_0)$ satisfying (7.13) out of $F'(\gamma \ge \gamma_0)^{1/2}$ (i.e. $F_1 = 1$ on supp F') and use equation (7.14) to obtain

$$(7.21) \quad (F')^{1/2}F_{\Delta'} \sum_{\alpha \text{ open}} \left(\kappa_\alpha^2 - \gamma_0^2 - 2\varepsilon\right)\phi_\alpha F_{\Delta'}(F')^{1/2}$$
$$\ge (F')^{1/2}F_{\Delta'} \sum_{\substack{\alpha \text{ open} \\ |\gamma_0| < \kappa_\alpha + 2\delta}} \left(\kappa_\alpha^2 - \gamma_0^2 - 2\varepsilon\right)\phi_\alpha F_{\Delta'}(F')^{1/2}$$
$$- \frac{\varepsilon}{10}(F')^{1/2}F_{\Delta'}^2(F')^{1/2} + O\left(|x|^{-1}\right).$$

Now we use that $\gamma_0 \notin \Sigma_E$. Since $\delta < \frac{1}{10}\text{dist}(\gamma_0, \Sigma_E)$, we have

$$(7.22) \qquad |\gamma_0| \le \kappa_\alpha + 2\delta \Leftrightarrow |\gamma_0| < \kappa_\alpha.$$

Since $\varepsilon < \frac{1}{10}\theta$, where

$$(7.23) \qquad \theta = \min_{\kappa_\alpha > |\gamma_0|} \left(\kappa_\alpha^2 - \gamma_0^2\right),$$

we have that the first term on the right-hand side of (7.21) is greater than or equal to

$$(7.24) \qquad \tfrac{2}{3}\theta(F')^{1/2}F_{\Delta'} \sum_{\alpha:\, |\gamma_0| < \kappa_\alpha} \phi_\alpha F_{\Delta'}(F')^{1/2}.$$

Now using (7.14) again we restore the summation on the right side of (7.24) to over all open α:

$$\text{r.h.s. of (7.24)} \ge -\frac{\varepsilon}{10}(F')^{1/2}F_{\Delta'}^2(F')^{1/2}$$
$$+ \frac{2}{3}\theta(F')^{1/2}F_{\Delta'} \sum_{\alpha \text{ open}} \phi_\alpha F_{\Delta'}(F')^{1/2} + O\left(|x|^{-1}\right).$$

Folding back the sum on the right by using (6.47):

$$F_{\Delta'} \sum_{\alpha \text{ open}} \phi_\alpha F_{\Delta'} \ge (1 - \varepsilon)F_{\Delta'}^2$$

and remembering that $\varepsilon < \theta/10$, we obtain finally

$$(F')^{1/2}F_{\Delta'}\sum_{\alpha}\left(\kappa_\alpha^2 - \gamma_0^2 - 2\varepsilon\right)\phi_\alpha F_{\Delta'}(F')^{1/2} \geq \frac{\theta}{2}(F')^{1/2}(F_{\Delta'})^2(F')^{1/2} + O(|x|^{-1}) \tag{7.25}$$

Combining this inequality with (7.10) and pulling $(F_{\Delta'})^2$ through $(F')^{1/2}$ and absorbing it into one of the F_Δ, we arrive at (7.2). □

Proof of Theorem 7.1. We observe that $F_0(\gamma = \gamma_0) \equiv [F'(\gamma \geq \gamma_0)]^{1/2}$ is, in fact, an arbitrary positive function supported in a small but fixed neighborhood of γ_0. Hence, inequality (7.2) implies, by Lemma 3.4, the statement of Theorem 7.1 with an additional restriction that $f(\gamma)$ have compact support.

It remains to prove (7.1) for a bounded smooth function supported in a neighborhood of infinity. Let $M > \max_\alpha \kappa_\alpha$ and let $F(s \geq M)$ be supported away from $[\max \kappa_\alpha, \infty)$. Consider the observable

$$\phi = -\gamma F(\gamma \geq M). \tag{7.26}$$

This observable is not bounded, but H-bounded.

PROPOSITION 7.5. *Let E be away from the thresholds and eigenvalues of H. Then there is a small interval Δ around E such that*

$$F_\Delta i[H, \phi] F_\Delta \gtrsim \delta F_\Delta \frac{1}{\sqrt{\langle x\rangle}} F(\gamma \geq M) \frac{1}{\sqrt{\langle x\rangle}} F_\Delta. \tag{7.27}$$

Proof. We follow the proof of Proposition 7.2 with the following simplifications:

Instead of subtle channel analysis of $i[H, A]$ (before equations (7.21)–(7.25)) we use its following simple consequence:

$$F_{\Delta'} i[H, A] F_{\Delta'} \leq \left(2\max\kappa_\alpha^2 + \varepsilon_2\right)F_{\Delta'}^2,$$

where ε_2 depends on $|\Delta'|$.

Equation (7.14) and the channel expansion for $F_{\Delta'}$ imply

$$F'(\gamma \geq M)F_{\Delta'} \overset{\varepsilon_1}{=} O(|x|^{-1})$$

Taking these into account we obtain the desired inequality. □

This proposition together with Lemma 3.4 and Remark 3.5 imply that

$$\int_{-\infty}^{\infty}\left\|F(\gamma \geq M)\frac{1}{\sqrt{\langle x\rangle}}\psi_t\right\|^2 dt \leq c\|\psi\|^2$$

for all $\psi \in \operatorname{Ran} P_\Delta$, which completes the proof of Theorem 7.1. □

8. Propagation estimates II. Threshold case

In this section we prove the propagation estimates for, essentially, the phase-space region

$$\{(x, k)|\hat{x}\cdot k \in \Sigma_E\} \cap (\mathscr{F}\setminus PS_E), \tag{8.1}$$

where E is away from the thresholds and eigenvalues of H. We begin with more special estimates which, however, suffice for our proof of asymptotic completeness. Below we use definitions and notation of Sections 4 and 5.

We use the cut-off functions defined in Section 5: equations (5.1), (5.2). Besides, we use the notation $F(s = \kappa)$, $\kappa \in \Sigma_E$, and $F(s \notin \Sigma_E)$ for generic C^∞, bounded cut-off functions satisfying (5.21). Thus, in particular,

$$F'(s = \lambda) = F(s \notin \Sigma_E). \tag{8.2}$$

We will also use the following notation:

$$\gamma^a = \tfrac{1}{2}(\hat{x}^a\cdot p^a + p^a\cdot\hat{x}^a) \quad \text{with } \hat{x}^a = \frac{x^a}{\langle x^a\rangle},$$

$$\gamma_a = \tfrac{1}{2}(\hat{x}_a\cdot p_a + p_a\cdot\hat{x}_a) \quad \text{with } \hat{x}_a = \frac{x_a}{\langle x_a\rangle}.$$

All these γ-operators are related as

$$\gamma = \gamma^a r^a + \gamma_a r_a - \frac{i(n-1)}{2\langle x\rangle}. \tag{8.3}$$

Recall some definitions from Section 4:

$$\Gamma_a = \left\{r^a < \tfrac{10}{9}\varepsilon_a,\ r^b > \tfrac{1}{2}\varepsilon_b \text{ for all } b \supsetneq a\right\} \tag{8.4}$$

with ε_a satisfying (4.2):

$$\varepsilon_a < \frac{1}{20\varepsilon_{b,a}}\varepsilon_b \quad \text{for } \#(a) > \#(b). \tag{8.5}$$

(We drop here the subindex $\varepsilon = (\varepsilon_a)$ in Γ_a.) Note here two special cases

$$\Gamma_a = \left\{r^a < \tfrac{10}{9}\varepsilon_a\right\} \quad \text{for } \#(a) = 2, \tag{8.6}$$

$$\Gamma_a = \left\{r^l > \tfrac{1}{2}\varepsilon_l \text{ all pairs } l\right\} \quad \text{for } \#(a) = N. \tag{8.7}$$

Recall that $F(\Gamma_a)$ denotes a smooth function supported in Γ_a (see Section 5). Recall also that

$$\varepsilon_0 = \tfrac{1}{10}\left[\varepsilon_{a_{\min}}\right]^5. \tag{8.8}$$

Introduce the operators

$$J_{a,E,\varepsilon} = F(\Gamma_a)F_{\varepsilon_0}(r^a = \varepsilon)F_{\sqrt{\varepsilon_0}}(|p_a| \neq \kappa_\varepsilon)F(\gamma = \kappa) \quad \text{for } a \neq a_{\min}. \tag{8.9}$$

These operators do not cover the entire region (8.1) in the sense that

$$\bigcup_{k \in \Sigma_E,\, a,\, \varepsilon > 0} f - \operatorname{supp} J_{a,E,\varepsilon} \subsetneq (8.1). \tag{8.10}$$

However, they suffice for the proof of asymptotic completeness. We will consider more general operators which cover (8.1) at the end of this section. The main result of this section is:

THEOREM 8.1. *Let E be away from the thresholds and eigenvalues of H. Let $J_{a,E,\varepsilon}$ be an operator defined in* (8.9). *Then, there exists a small interval Δ around E such that*

$$\int_{-\infty}^{\infty} \left\| J_{a,E,\varepsilon} \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C\|\psi\|^2 \tag{8.11}$$

for all $\psi \in \operatorname{Ran} P_\Delta$ and with C independent of ψ.

Recall that Lemma 3.4 reduces the proof of propagation estimates of the form (8.11) to estimates from below of i times commutators of H with suitable propagation observables, restricted to an energy shell. In this section we use propagation observables of the form

$$\phi = j(x)f(p_a)g(\gamma_a)\phi(\gamma), \tag{8.12}$$

where all the functions are real, smooth and bounded. Besides, j is homogeneous

degree 0 for $|x| \geq 1$ and

$\operatorname{supp} j \subset \{x \in X \,|\, |x|_a > \delta|x| \text{ and } |x_a| > \delta|x|\}$ for some $\delta > 0$,

f and ϕ have compact supports,

g' has a compact support.

The extra factor—$g(\gamma_a)$—these operators contain, is needed in order to control $i[H, j]$. The factors on the right side of (8.12) commute modulo $O(|x|^{-1})$. This fact follows from Lemma A.4 and the relation

$$\langle x \rangle \langle x_a \rangle^{-1} F\left(\frac{|x_a|}{|x|} > \delta\right) \text{ is bounded} \tag{8.13}$$

(this relation is needed only for the commutators of $g(\gamma_a)$ with $f(p_a)$). This implies, for instance, that

$$\phi = \operatorname{Re}\phi + O(|x|^{-1}),$$

where $\operatorname{Re}\phi = \frac{1}{2}(\phi + \phi^*)$ is a self-adjoint operator. Here we use also relation (3.17):

$$f(B)O(|x|^{-1}) = O(|x|^{-1}) = O(|x|^{-1})f(B),$$

where B stands for one of the operators H_a, $|p_a|$, γ_a (with arbitrary a).

Moreover, since different factors on the right side of (8.12) are positive operators, whose square roots commute modulo $O(|x|^{-1})$, we have

(8.14) $$\phi = \text{positive oper.} + O(|x|^{-1}).$$

Indeed, the difference between a symmetrized expression,

(8.15) $$\phi^{\text{sym}} = \phi_1^*\phi_1, \quad \text{where } \phi_1 = j(x)^{1/2}f(p_a)^{1/2}g(\gamma_a)^{1/2}\phi(\gamma)^{1/2}$$

and ϕ is $O(|x|^{-1})$.

The philosophy of the analysis presented below hinges on the fact that terms of higher orders in $|x|^{-1}$ can be dropped. They lead to contributions into the time integrals which are convergent by a result of [PSS] (see Remark 3.5). This allows us to treat factors on the right side of (8.12) as commuting, reducing our analysis, essentially, to classical phase-space estimates. We give justification of such estimates in an example considered below. The argument presented below is typical and can be easily adapted to other cases treated later.

Consider, for instance, the operator

$$\phi = F_{\varepsilon_0}(\gamma_a \geq \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)$$

with some fixed numbers $\lambda > \mu > 0$. Our task is to estimate the commutator $i[p_a^2, \phi]$. Lemma 8.4 below yields

(*)

$$i[p_a^2, \phi] = 2\frac{1}{\sqrt{\langle x\rangle}}F'_{\varepsilon_0}(\gamma_a \geq \mu)(p_a^2 - \gamma_a^2)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)\frac{1}{\sqrt{\langle x\rangle}} + O_2(|x|^{-2}).$$

Observe

(**) $$p_a^2 - \gamma_a^2 \geq \theta \quad \text{on } f-\operatorname{supp}\left(F'_{\varepsilon_0}(\gamma_a \geq \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)\right),$$

where $\theta = (\lambda - 2\sqrt{\varepsilon_0})^2 - (\mu + 2\varepsilon_0)^2$.

We claim that

(***)

$$i[p_a^2, \phi] \geq 2\theta F'_{\varepsilon_0}(\gamma_a \geq \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) + O_2(|x|^{-2}) + O(|x|^{-1}|x_a|^{-1})$$

Here, as in all such estimates below, $\phi = F_{\varepsilon_0} F_{\sqrt{\varepsilon_0}}$ and $F'_{\varepsilon_0} F_{\sqrt{\varepsilon_0}}$ are identified with their real (self-adjoint) parts. To justify $(**)$ we proceed as follows. Due to Lemma A.5(ii) and equation (3.17), we get

$$F'_{\varepsilon_0}(\gamma_a \geq \mu) p_a^2 F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)$$
$$= \left(F'_{\varepsilon_0}\right)^{1/2} p_a^2 F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)\left(F'_{\varepsilon_0}\right)^{1/2} + O\left(|x_a|^{-1}\right).$$

Using the Fourier transform (i.e. the spectral theorem for $|p_a|$), we obtain

$$p_a^2 F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) \geq \left(\lambda - 2\sqrt{\varepsilon_0}\right)^2 F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda).$$

Furthermore, Lemma A.5(ii) and equation (3.17) yield

$$\left(F'_{\varepsilon_0}\right)^{1/2} F_{\sqrt{\varepsilon_0}}\left(F'_{\varepsilon_0}\right)^{1/2} = F'_{\varepsilon_0} F_{\sqrt{\varepsilon_0}} + O\left(|x_a|^{-1}\right).$$

The last three estimates imply

$$F'_{\varepsilon_0} p_a^2 F_{\sqrt{\varepsilon_0}} > \left(\lambda - 2\sqrt{\varepsilon_0}\right)^2 F'_{\varepsilon_0} F_{\sqrt{\varepsilon_0}} + O\left(|x_a|^{-1}\right),$$

where, remember, both sides should be understood in terms of their self-adjoint realizations. Similarly

$$\begin{aligned} F'_{\varepsilon_0}(\gamma_a \geq \mu)\gamma_a^2 F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) &= \left(F_{\sqrt{\varepsilon_0}}\right)^{1/2} F'_{\varepsilon_0}\gamma_a^2\left(F_{\sqrt{\varepsilon_0}}\right)^{1/2} + O\left(|x_a|^{-1}\right) \\ &\leq (\mu + 2\varepsilon_0)^2\left(F_{\sqrt{\varepsilon_0}}\right)^{1/2} F'_{\varepsilon_0}\left(F_{\sqrt{\varepsilon_0}}\right)^{1/2} + O\left(|x_a|^{-1}\right) \\ &= (\mu + 2\varepsilon_0) F'_{\varepsilon_0} F_{\sqrt{\varepsilon_0}} + O\left(|x_a|^{-1}\right). \end{aligned}$$

The last two equations together with $(*)$ yield $(***)$.

In what follows we proceed from estimates of type $(**)$ to estimates of type $(***)$ (i.e. replacing simple polynomials of r^a, r_a, γ_a, γ, p_a^2 standing next to operators of type (8.12) by their minima on the phase-space supports of the latter operators) without giving a justification. The latter is always a variant of the argument presented above.

Remark 8.2. If the potentials are twice continuously differentiable, then the commutators of H with the first three factors in (8.12) can be computed classically, using the Poisson bracket of appropriately defined symbols. However, since we assume that the derivatives of the potentials are H-bounded, we stick to operator (i.e., quantum) estimates (see, however, [SigSof1]).

Finally in our computations we encounter terms of the form

$$(H + i)\frac{1}{\sqrt{\langle x \rangle}} F(\gamma \notin \Sigma_E)(\text{bndd opr}) F(\gamma \notin \Sigma_E)\frac{1}{\sqrt{\langle x \rangle}}(H + i).$$

These terms are treated by a combination of Theorem 7.1 and Lemma 3.4. Hence we introduce the following equivalence relation:

$$(8.16)\quad A \triangleq B \quad \text{if } A - B = O_2(|x|^{-1-\varepsilon}) + (H+i)\frac{1}{\sqrt{\langle x\rangle}} \times F(\gamma \notin \Sigma)(H\text{-bndd opr})F(\gamma \notin \Sigma)\frac{1}{\sqrt{\langle x\rangle}}(H+i)$$

and similarly for inequalities.

Proof of Theorem 8.1. In what follows E is a fixed number away from the thresholds and eigenvalues of H and we fix $\kappa \in \Sigma_E$. To fix ideas we assume $\kappa > 0$. We *suppress the subindex* E from our notation. Our task is to estimate $i[H, \phi]$ for certain operators ϕ of the form (8.12). We begin with some technical statements.

Using the equation

$$[H, F] = (H+i)\left[F, (H+i)^{-1}\right](H+i)$$

and equation (3.21a) (with n replaced by i), we obtain

$$(8.17)\quad [H, F(\gamma = \kappa)] = (H+i)\frac{1}{\sqrt{\langle x\rangle}}F'(\gamma=\kappa)^{1/2}BF'(\gamma=\kappa)^{1/2} \times \frac{1}{\sqrt{\langle x\rangle}}(H+i) + O_2(|x|^{-2}),$$

where B is a bounded operator. By equation (8.2),

$$(8.18)\quad F'(\gamma=\kappa)^{1/2} = F(\gamma \notin \Sigma_E).$$

Hence

$$(8.19)\quad [H, F(\gamma=\kappa)] \triangleq 0.$$

We will use the following estimate proved in the appendix (Lemma A.6): Assume $V_{ij}(y) = O_1(|y|^{-\varepsilon})$ and $\nabla V_{ij}(y) = O_1(|y|^{-1-\varepsilon})$; then

$$(8.20)\quad \begin{aligned} \left[f_1(p_b^2), I_a\right]F(|x|_a > \delta|x|) &= O(|x|^{-1-\varepsilon}),\\ \left[f_2(\gamma_b), I_a\right]F(|x|_a > \delta|x|) &= O_2(|x|^{-1-\varepsilon}) \end{aligned}$$

for any C_0^∞ function f_1 and any bounded function f_2 having a C_0^∞ derivative.

Thus to compute the commutators of H with the first three factors on the right side of (8.12) it suffices to compute the commutators of H_a, and therefore just of p^2, with these factors.

LEMMA 8.3.

$$i[p^2, f(r^a)] \doteq \frac{1}{\langle x \rangle} f'(r^a)(\gamma^a - \gamma r^a) \tag{8.21}$$

for any smooth bounded function with bounded derivatives.

Proof. Compute

$$i[p^2, f(r^a)] = 2p \cdot \nabla f(r^a) - \Delta f(r^a). \tag{8.22}$$

Now

$$\Delta f(r^a) = O(|x|^{-2}) \tag{8.23}$$

and

$$\nabla f(r^a) = \left(\frac{x^a}{\langle x^a \rangle} - r_a \frac{x}{\langle x \rangle} \right) \frac{1}{\langle x \rangle} f'(r^a). \tag{8.24}$$

The last three relations imply (8.21). □

LEMMA 8.4.

$$i[p_a^2, f(\gamma_a)] = 2\frac{1}{\langle x \rangle} f'(\gamma_a)(p_a^2 - \gamma_a^2) + O_2(|x|^{-2}) \tag{8.25}$$

with the same restrictions on f as in Lemma 8.3.

Proof. This expression follows from the equation

$$i[p_a^2, f(\gamma_a)] = i[p_a^2, \gamma_a] f'(\gamma_a) \frac{1}{\langle x \rangle} + O_2(|x|^{-2}),$$

which is a special case of equations (A.45), (A.46) and the equation

$$i[p_a^2, \gamma_a] \doteq 2(p_a^2 - \gamma_a^2) \frac{1}{\langle x \rangle}.$$ □

To demonstrate our main ideas we consider first the case of two-cluster decompositions a. The only difference between this case and the general one is that for a 2-cluster a,

$$\frac{\langle x^a \rangle}{\langle x \rangle} < \varepsilon \quad \text{for suff. small } \varepsilon > 0 \Rightarrow |x|_a > \delta |x| \text{ for some } \delta > 0, \tag{8.26}$$

i.e. small intracluster distance implies that the clusters are well-separated. This is, in general, not true for k-cluster a's with $k \geq 3$. As a result we will have to insert in the latter case some extra geometric cut-off functions.

Now we proceed directly to estimates of basic commutators. We will use the induction in cluster decompositions: k-cluster decompositions $\to$ $(k-1)$-cluster decompositions. We consider separately the regions $\{\gamma_a > \sqrt{1-\varepsilon^2}\kappa\}$, $\{\gamma_a < \sqrt{1-\varepsilon^2}\kappa\}$ and $\{\gamma_a = \sqrt{1-\varepsilon^2}\kappa\}$. Below we handle subindices liberally, dropping sometimes those which are not needed.

1) We introduce the observables

$$\phi_{\varepsilon,\lambda}^{(1,2)} = \pm F_{\varepsilon_0}(r^a \le \varepsilon) F_{\varepsilon_0}(\pm\gamma_a \ge \mu_\pm) F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) F(\gamma = \kappa),$$

where $\mu_- < \sqrt{1-\varepsilon^2}\kappa < \mu_+$ and $\mu_\pm \neq \lambda$. Define the cut-off operator

$$\tilde{\phi}_{\varepsilon,\lambda}^{(1,2)} = F_{\varepsilon_0}(r^a = \varepsilon) F_{\varepsilon_0}(\pm\gamma_a \ge \mu_\pm) F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) F(\gamma = \kappa). \tag{8.27}$$

LEMMA 8.5.

$$i[H, \phi_{\varepsilon,\lambda}^{(i)}] \overset{\Delta}{\ge} \frac{1}{\sqrt{\langle x\rangle}} \theta \tilde{\phi}_{\varepsilon,\lambda}^{(i)} \frac{1}{\sqrt{\langle x\rangle}} \tag{8.28}$$

for some $\theta > 0$ and $i = 1, 2$.

Proof. To fix the ideas we consider only the case $i = 1$ (the plus sign). The other case is treated in exactly the same way. We *drop all the indices* at the F, ϕ, $\tilde{\phi}$ and μ.

By equations (8.20) and (8.26), we have

$$[I_a, \phi] = O_2(|x|^{-1-\varepsilon}). \tag{8.29}$$

Hence

$$[H, \phi] \doteq [H_a, \phi]. \tag{8.30}$$

To compute the commutators of H_a with the first two factors in ϕ we use the relations:

$$[H_a, f(x)] = [p^2, f(x)] \tag{8.31}$$

and

$$[H_a, f(\gamma_a)] = [p_a^2, f(\gamma_a)], \tag{8.32}$$

and Lemmas 8.3 and 8.4. To compute the commutator with the last factor in ϕ we use (8.19). The result is (observe that $F'_{\varepsilon_0}(s \le \varepsilon) \le 0$)

$$i[H, \phi] \overset{\Delta}{\ge} \frac{1}{\sqrt{\langle x\rangle}} \left[(-F')(r^a\gamma - \gamma^a)FF + FF'(p_a^2 - \gamma_a^2)FFF\right] \frac{1}{\sqrt{\langle x\rangle}}. \tag{8.33}$$

First we analyze the first term on the right. We use the equation

$$\gamma = r^a\gamma^a + r_a\gamma_a + \frac{i(n-1)}{2\langle x\rangle}. \tag{8.34}$$

Using that $r_a = \sqrt{1-(r^a)^2}$, we obtain

$$r^a\gamma - \gamma^a = \frac{\sqrt{1-(r^a)^2}}{r^a}\left(\gamma_a - \sqrt{1-(r^a)^2}\,\gamma\right) + \frac{i(n-1)}{2\langle x^a\rangle}. \tag{8.35}$$

On the phase-space support of the operator

$$F_{\varepsilon_0}'(r^a \le \varepsilon)F_{\varepsilon_0}(\gamma_a > \mu)F(\gamma = \kappa)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda), \tag{8.36}$$

we compute

$$r^a\gamma - \gamma^a \ge \frac{\sqrt{1-\varepsilon^2}}{2\varepsilon}(\mu - \kappa_\varepsilon) = \theta \tag{8.37}$$

since $\varepsilon_0 \ll \varepsilon$. Thus, by the argument given at the beginning of this section (before Remark 8.2),

$$(-F')(r^a\gamma - \gamma^a)FF \gtrsim \theta(-F')FF$$

(remember also our agreement about the symmetrization).

Next if $\lambda < \mu$, then by Lemma 6.6,

$$F_{\varepsilon_0}'(\gamma_a \ge \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) = O(|x|^{-1}), \tag{8.38}$$

provided $\sqrt{\varepsilon_0} < \frac{1}{10}(\mu - \lambda)$. If $\lambda > \mu$, then we have

$$F_{\varepsilon_0}'(\gamma_a \ge \mu)(p_a^2 - \gamma_a^2)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) \ge \tfrac{1}{2}(\lambda^2 - \mu^2)(F'F) + O(|x|^{-1}), \tag{8.39}$$

provided $\varepsilon_0 \le (\lambda - \mu^2)/80$. (Here we have used again the argument presented before Remark 8.2 and identified the operators on both sides with their symmetized expressions.) The last relation implies:

(8.40) The second term on the right side of (8.33)

$$\gtrsim \frac{1}{2}(\lambda^2 - \mu^2)\frac{1}{\sqrt{\langle x\rangle}}(FF'FF)\frac{1}{\sqrt{\langle x\rangle}}.$$

These estimates together give

$$i[H, \phi] \overset{\Delta}{\ge} -\theta\frac{1}{\sqrt{\langle x\rangle}}F'FFF\frac{1}{\sqrt{\langle x\rangle}}. \tag{8.41}$$

This inequality together with the inequality

$$C(-F'FFF) \ge \tilde{\phi} + O(|x|^{-1}) \tag{8.42}$$

yield (8.28). □

2) We introduce the observable

(8.43)

$$\phi^{(3)}_{\varepsilon_1, \lambda} = F_{\varepsilon_0}(r^a \leq \varepsilon_1)F_{\varepsilon_0}(\gamma_a \geq \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)F(\gamma = \kappa) \quad \text{with } \kappa_{\varepsilon_1} < \mu < \lambda,$$

and the cut-off operator

$$\tilde{\phi}^{(3)}_{\varepsilon_1, \lambda} = F_{\varepsilon_0}(r^a \leq \varepsilon_1)F_{\varepsilon_0}(\gamma_a = \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)F(\gamma = \kappa); \tag{8.44}$$

then

$$C(-FF'FF) \dot{\geq} C\tilde{\phi}^{(3)}. \tag{8.45}$$

LEMMA 8.6.

$$i[H, \phi^{(3)}] \overset{\Delta}{\geq} \frac{1}{\sqrt{\langle x \rangle}}\left[\theta\tilde{\phi}^{(3)} - C\tilde{\phi}^{(1)}\right]\frac{1}{\sqrt{\langle x \rangle}} \tag{8.46}$$

for some $\theta > 0$.

Proof. Equations (8.29)–(8.32) hold in this case as well. Thus as in the derivation of equation (8.33) we obtain

$$i[H, \phi^{(3)}] \overset{\Delta}{\geq} \frac{1}{\sqrt{\langle x \rangle}}\left[F'(\gamma^a - \varepsilon\gamma)FFF + FF'(p_a^2 - \gamma_a^2)FF\right]\frac{1}{\sqrt{\langle x \rangle}}. \tag{8.47}$$

By (8.45), the second term on the right-hand side $\dot{\geq} \theta(1/\sqrt{\langle x \rangle})\tilde{\phi}^{(3)}(1/\sqrt{\langle x \rangle})$, while the first term $\dot{\geq}$

$$\begin{aligned} &-\frac{1}{\sqrt{\langle x \rangle}}F_{\varepsilon_0}(r^a = \varepsilon_1)F_{\varepsilon_0}(\gamma_a \geq \mu)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)F(\gamma = \kappa)\frac{1}{\sqrt{\langle x \rangle}} \\ &\geq \frac{1}{\sqrt{\langle x \rangle}}\tilde{\phi}^{(1)}_{\varepsilon_1, \lambda}\frac{1}{\sqrt{\langle x \rangle}}, \end{aligned} \tag{8.48}$$

where we have used that the second cut-off function enforces

$$\gamma_a \geq \mu - \varepsilon_0 > \sqrt{1 - \varepsilon_1^2}\,\kappa - \varepsilon_0. \tag{8.49}$$

Note also that on $f-\operatorname{supp} F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)$

$$|p_a| > (1 + 2\varepsilon_0)\sqrt{1 - \varepsilon_1^2}\,\kappa. \tag{8.50}$$

The mentioned inequalities imply (8.46). □

Now we finish the proof of Theorem 8.1. First by the property discussed at the beginning of this section (see equations (8.14) and (8.15)) we rewrite our phase-space operators as

$$\tilde{\phi}^{(k)}_{\varepsilon, \lambda} = \Lambda^{(k)*}_{\varepsilon, \lambda}\Lambda^{(k)}_{\varepsilon, \lambda} + O(|x|^{-1}) \tag{8.51}$$

and

$$(8.52) \qquad J_{a,E,\varepsilon} = G^*_{a,E,\varepsilon} G_{a,E,\varepsilon} + O(|x|^{-1}),$$

where

$$(8.53) \quad \Lambda^{(1,2)}_{\varepsilon,\lambda} = F_{\varepsilon_0}(r^a = \varepsilon)^{1/2} F_{\varepsilon_0}(\pm\gamma_a \geq \mu)^{1/2} F(\gamma = \kappa)^{1/2} F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda),$$

etc. Then Lemmas 8.5 and 3.4 (see also Remark 3.5) imply (we drop temporarily some of the subindices)

$$(8.54) \qquad \int_0^\infty \left\| \Lambda^{(i)} \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C\|\psi\|^2.$$

Lemmas 8.6 and 3.4 imply

$$(8.55) \qquad \int_0^\infty \left\| \Lambda^{(3)} \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C\|\psi\|^2 + c\int_0^\infty \left\| \tilde{\phi}^{(1)} \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt$$

for all $\psi \in \operatorname{Ran} P_\Delta$, where Δ is chosen so that the contribution of the last two terms in (8.16) is finite. Substituting (8.54) with $i = 1$ into the second term on the r.h.s. of (8.55) yields

$$(8.56) \qquad \int_0^\infty \left\| \Lambda^{(i)} \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C\|\psi\|^2 \qquad \text{for } i = 1, 2, 3.$$

Now we connect $J_{a,E,\varepsilon}$ with $\tilde{\phi}^{(k)}_{\lambda,\varepsilon}$. Let $\{\chi_k(\gamma_a)\}$ be a partition of unity ($\Sigma\chi_k(\gamma_a) = 1$) with the members

$$\chi_{1,2}(\gamma_a) = F_{(1/\mu)\varepsilon_0}(\pm\gamma_a \geq \pm\kappa_\varepsilon + \tfrac{3}{2}\varepsilon_0) \quad \text{and} \quad \chi_3(\gamma_a) = F_{\varepsilon_0}(\gamma_a = \kappa_\varepsilon)$$

(the participating cut-off functions must be, of course, appropriately adjusted). Next, let $M \geq 2\max(\sup_{E\in\Delta}|\kappa_\alpha|) + 1$. We choose a finite number of

$$\lambda_i \in \left\{\lambda \in [0, M+1] \,\middle|\, |\lambda - \kappa_\varepsilon| > \tfrac{3}{2}\sqrt{\varepsilon_0}\right\}$$

so that with appropriately adjusted cut-off functions

$$F_{\sqrt{\varepsilon_0}}(|p_a| \neq \kappa_\varepsilon) = \sum_i F_{(1/\mu)\sqrt{\varepsilon_0}}(|p_a| = \lambda_i) + F(|p_a| \geq M).$$

Then

$$J_{a,E,\varepsilon} = F(\Gamma_a) \sum_{i,k} F_{\varepsilon_0}(r^a = \varepsilon) \chi_k(\gamma_a) F_{(1/\mu)\sqrt{\varepsilon_0}}(|p_a| = \lambda_i) F(\gamma = \kappa) Y_a.$$

By Lemma 6.6,

$$\chi_3(\gamma_a) F_{(1/\mu)\varepsilon_0}(|p_a| = \lambda_i) = O(|x|^{-1}) \quad \text{if } \lambda_i < \kappa_\varepsilon + \sqrt{\varepsilon_0}\,.$$

Hence for $k = 3$ the sum in i can be restricted, modulo $O(|x|^{-1})$, to the terms with $\lambda_i > \kappa_\varepsilon + \frac{3}{2}\sqrt{\varepsilon_0}$. Hence

$$J_{a,E,\varepsilon} = \sum_{i,k} F(\Gamma_a)\tilde{\phi}^{(k)}_{\lambda_i,\varepsilon} + J_{a,E,\varepsilon} F(|p_a| > \mu) + O(|x|^{-1}) \tag{8.57}$$

with $\mu_\pm = \kappa_\varepsilon \pm \frac{3}{2}\varepsilon_0$ for $k = 1, 2$ and $\mu = \kappa_\varepsilon$ for $k = 3$ and with trivial renormalization of the subindices in the definition of $\phi^{(k)}_{\lambda,\varepsilon}$ (e.g. $F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)$ should be replaced by $F_{(1/\mu)\sqrt{\varepsilon_0}}(|p_a| = \lambda)$, etc). Next, since $M \geq 2\max_\alpha(\sup_{E\in\Delta}|\kappa_\alpha|) + 1$, we have that

$$F(\Gamma_a)F(|p_a| \geq M)F_\Delta(H) = O(|x|^{-\mu}) \tag{8.58}$$

by equation (4.7) and the relation

$$F(|x|_a > \delta|x|)(g(H) - g(H_a)) = O(|x|^{-\mu}),$$

valid for any C_0^∞ function g. Here μ is the same as in condition (D). This together with a result of [PSS] gives

$$\int_0^\infty \left\| F(\Gamma_a)F(|p_a| \geq M)\frac{1}{\sqrt{\langle x\rangle}}\psi_t \right\|^2 dt \leq C\|\psi\|^2 \tag{8.59}$$

for any $\psi \in \operatorname{Ran} F_\Delta(H)$. Estimates (8.56)–(8.59) and a result of [PSS] yield

$$\int_0^\infty \left\| G_a \frac{1}{\sqrt{\langle x\rangle}}\psi_t \right\|^2 dt \leq C\|\psi\|^2. \tag{8.60}$$

Now we consider $J_{a,E,\varepsilon}$ with cluster decompositions a having more than 2 clusters. In this case we introduce operators needed to assure a proper localization of our observables in the configuration space (inside of Γ_a). They are constructed so as to facilitate our inductive proof. Define for $\#(a) \geq 3$

(8.61)
$$Y_a(x,p) = \prod_{b\supset a} F_{\varepsilon_0}\left(r^b \geq \tfrac{1}{10}\varepsilon_b\right)\left[1 - F_{\varepsilon_0}\left(r_b < \tfrac{1}{5}\varepsilon_b\right)F_{2\sqrt{\varepsilon_0}}\left(|p_b| = \kappa_{(1/10)\varepsilon_b}\right)\right],$$

and

$$X_{a,\lambda}(x,p) = Y_a(x,p)F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda)F(\gamma = \kappa).$$

Since

$$F_{\varepsilon_0}\left(r^b > \tfrac{1}{10}\varepsilon_b\right) = 1 \quad \text{on } \Gamma_a,\ a \subsetneq b, \tag{8.62}$$

we obtain that

$$Y_a(x,p) = 1 \quad \text{on } \Gamma_a,\ a \subsetneq b$$

and therefore

$$F(\Gamma_a)F_{\varepsilon_0}(r^a \le \varepsilon_a)Y_a(x,p) = F(\Gamma_a) + O(|x|^{-n}) \quad \text{for all } n > 0. \tag{8.63}$$

An easy consequence of this relation is:

LEMMA 8.7. *For any $\varepsilon_1 < \varepsilon_a$ and sufficiently small,*

$$F_{\varepsilon_0}(r^a < \varepsilon_1)Y_a(x,p)\big(1 - F(|x|_a > \delta|x|)\big) = O(|x|^{-1}) \tag{8.64}$$

for some fixed $\delta > 0$.

Proof. The relation follows from equations (8.62) and (4.7) and Lemma A.5 used for estimating commutators of functions of x, p and γ. □

LEMMA 8.8.

$$i[H_a, X_{a,\lambda}] \overset{\Delta}{\ge} -\frac{C}{\sqrt{\langle x\rangle}}\left[\sum_{b\supset a,\ \alpha=\frac15,\frac1{10}} J_{b,E,\alpha\varepsilon_b}\right]\frac{1}{\sqrt{\langle x\rangle}}. \tag{8.65}$$

Proof. Since $V_a = V - I_a$ commutes with Y_a we have

$$i[H_a, Y_a] = i[p^2, Y_a]. \tag{8.66}$$

By equations (8.22) and (8.23) we have

$$i[p^2, Y_a] \doteq 2p\cdot\nabla_x Y_a. \tag{8.67}$$

Compute the derivative of each factor in (8.61). In order to simplify the notation we abuse it first by setting (contrary to the letter of our convention (5.1c) but not to its spirit)

$$F_{\sqrt{\varepsilon_0}}(|p_b| = \kappa_{(1/10)\varepsilon_b}) = F_{\sqrt{\varepsilon_0}}(|p_b| \ne \kappa_{(1/5)\varepsilon_b}).$$

Observing that

$$F_{\varepsilon_0}\left(r^b \le \tfrac15\varepsilon_b\right) = 1 \quad \text{on supp } F_{\varepsilon_0}\left(r^b \ge \tfrac1{10}\varepsilon_b\right)',$$

and using (5.1) and (5.2), we obtain

$$\begin{aligned} p\cdot\nabla_x\Big[F_{\varepsilon_0}\left(r^b > \tfrac1{10}\varepsilon_b\right)\Big(1 - F_{\varepsilon_0}\left(r^b < \tfrac15\varepsilon_b\right)F_{\sqrt{\varepsilon_0}}(|p_b| = \kappa_{(1/10)\varepsilon_b})\Big)\Big] \\ \doteq (\gamma^b - \gamma r^b)\sum_{\alpha=\frac15,\frac1{10}} F_{\varepsilon_0}(r^b = \alpha\varepsilon_b)F_{\sqrt{\varepsilon_0}}(|p_b| \ne \kappa_{\alpha\varepsilon_b})\langle x\rangle^{-1}\text{const}. \end{aligned} \tag{8.68}$$

The latter relation implies

(8.69)

$$p\cdot\nabla_x Y_a \doteq \sum_{\substack{b\supset a\\ \alpha=\frac15,\frac1{10}}} (\gamma^b - \gamma r^b)Z_b F_{\varepsilon_0}(r^b = \alpha\varepsilon_b)F_{\sqrt{\varepsilon_0}}(|p_b| \ne \kappa_{\alpha\varepsilon_b})Y_b\langle x\rangle^{-1},$$

where Z_b are phase-space operators. Next we use that

$$\gamma^b = \left(\gamma - \gamma_b r_b - \frac{1(n-1)}{2\langle x\rangle}\right)\frac{1}{r^b},$$

that $(r^b)^{-1}$ is bounded on supp $F_{\varepsilon_0}(r^b = \varepsilon)$ and that $\gamma_b = \hat{x}_b \cdot p_b + O(|x|^{-1})$. Commuting p_b toward $F(|p_a| = \lambda)$ and γ toward $F(\gamma = \kappa)$ and bounding them with the help of the spectral theorem, and using equation (8.63), we obtain

$$(8.70)\quad F(\gamma = \kappa)p \cdot \nabla_x Y_a F_{\sqrt{\varepsilon_0}}(|p_a| = \lambda) \dot{\geq} -C\frac{1}{\sqrt{\langle x\rangle}} \sum_{\substack{b \supseteq a \\ \alpha = \frac{1}{5}, \frac{1}{10}}} J_{b, E, \alpha\varepsilon_b}\frac{1}{\sqrt{\langle x\rangle}}$$

(remember our agreement about the symmetrization).

Finally using equations (8.67)–(8.70), using that H_a commutes with $|p_a|$ and taking into account equation (8.19) we arrive at (8.65). □

For ease of notation we write, in what follows, the sum on the right-hand side of (8.66) as $\Sigma_{b \supsetneqq a} J_b^2$, ignoring the summation over α and omitting the subindices E and ε.

Now, in the case $\#(a) > 3$ we use the observables

$$\psi^{(k)}_{\varepsilon,\lambda} = \phi^{(k)}_{\varepsilon,\lambda} Y_a \quad \text{and} \quad \tilde{\psi}^{(k)}_{\varepsilon,\lambda} = \tilde{\phi}^{(k)}_{\varepsilon,\lambda} Y_a.$$

Taking into account Lemma 8.8 we obtain exactly as in Lemma 8.5 and 8.6 (remember that $\#(a) \geq 3$)

$$i[H, \psi^{(k)}_{\varepsilon,\lambda}] \overset{\Delta}{\geq} \frac{1}{\sqrt{\langle x\rangle}}\left(\theta\tilde{\psi}^{(k)}_{\varepsilon,\lambda} - \delta_{k,3}C\tilde{\psi}^{(1)}_{\varepsilon,\lambda} - C\sum_{b \supsetneqq a} J_b\right)\frac{1}{\sqrt{\langle x\rangle}},$$

for $k = 1, 2, 3$ and some $\theta > 0$. Proceeding exactly as above (see equations (8.54)–(8.60)), we obtain for $\Lambda_i = \Lambda^{(i)} Y_a^{1/2}$,

$$(8.71)\qquad \int_0^\infty \left\|\Lambda_i \frac{1}{\sqrt{\langle x\rangle}}\psi_t\right\|^2 dt + \int_0^\infty \left\|G_a \frac{1}{\sqrt{\langle x\rangle}}\psi_t\right\|^2 dt$$

$$\leq C\|\psi\|^2 + \sum_{b \supsetneqq a}\int_0^\infty \left\|G_b \frac{1}{\sqrt{\langle x\rangle}}\psi_t\right\|^2 dt.$$

This inequality applied successively and combined with (8.60) gives

$$(8.72)\qquad \int_0^\infty \left\|G_a \frac{1}{\sqrt{\langle x\rangle}}\psi_t\right\|^2 dt \leq C\|\psi\|^2$$

for all a and all $\psi \in \operatorname{Ran} P_\Delta$. Since the operators $G_{a,E,\varepsilon}$ are exactly of the same form as $J_{a,E,\varepsilon}$ this completes the proof of Theorem 8.1. □

COROLLARY 8.9. *If E is away from the thresholds and eigenvalues of H, then*

$$\int_0^\infty \left\| \Lambda_i \frac{1}{\sqrt{\langle x \rangle}} \psi_t \right\|^2 dt \leq C \|\psi\|^2 \tag{8.73}$$

for all a, i and $\psi \in \operatorname{Ran} P_\Delta$.

Proof. This estimate follows from (8.71) and (8.72). □

Discussion 8.10. This corollary together with the estimate of Theorem 8.1 for $a = a_{\min}$ implies the (non)-propagation estimates for entire $\mathscr{F} \setminus PS_E$ as was stated in Section 3. We omit here the geometric analysis leading to this conclusion, but just mention that the missing region of $\mathscr{F} \setminus PS_E$ is covered by the following operators:

$$F(\Gamma_a) F(|p_a| \geq M) \quad \text{with } M > \max|\kappa_a| \tag{8.74}$$

and

$$F(\Gamma_a) F_{\varepsilon_0}(\gamma_a > \mu) F_{\sqrt{\varepsilon_0}}(|p_a| \leq \lambda) \quad \text{for } \mu > \lambda \tag{8.75}$$

for $a \neq a_{\min}$. We have the propagation estimates for these operators due to (8.58) which is valid for Δ sufficiently small interval around E (so that $M > \max_\alpha(\sup_{E \in \Delta} |\kappa_\alpha|)$) (see (8.59)), and due to the fact that

$$(8.75) = O\big(|x|^{-1}\big).$$

The latter relation holds by the localization Lemma 6.6. Similarly, the case $r^a \sim \varepsilon = 0$ is treated as case (3) for $\lambda > \kappa$ and by the above geometric analysis for $\lambda < \kappa$.

9. Proof of asymptotic completeness

Consider an N-body system with pair potentials satisfying conditions (A)–(D) with $\mu > 1$ (see Section 3). First turn to the phase-space geometry of such a system. Theorems 5.5 and 5.6 provide a partition of unity with required properties (3.11)–(3.13). (Property (3.14)-(iii) also holds as follows from Lemma 5.7. However, we do not use this fact.) No conditions on the potentials are required here. Moreover, Theorems 5.5 and 5.6 and Theorems 7.1 and 8.1 imply that condition (E), formulated in the paragraph preceding Proposition 3.7 holds, provided conditions (A)–(C) are satisfied (note condition (D) is not required). (Condition (E) shows that the constructed partition does decouple the channels.) Hence the statement of Proposition 3.7 is valid under conditions (A)–(D) with $\mu > 1$ only. This together with Lemma 3.6 shows that the system in question is asymptotically clustering at any non-threshold energy E from the continuous spectrum of H. Then by Proposition 2.2 this N-body system is asymptotically complete. □

Appendix

In this appendix we prove various estimates related to commutators of functions of self-adjoint operators. We use the following class of functions:

f is a bounded and smooth function on $\mathbf{R}$ whose (distributional) Fourier transform $\hat{f}$ obeys

$$\int_{-\infty}^{\infty} |\hat{f}(s)||s|^n \, ds < \infty \quad \text{with } n = 1,2,3. \tag{A.1}$$

In the main text the estimates derived in this appendix are used only for bounded smooth functions f with C_0^∞ derivatives.

To estimate the commutators of $f(A)$, where A is a self-adjoint operator, with different operators we use the representation

$$f(A) = \int_{-\infty}^{\infty} \hat{f}(s) e^{iAs} \, ds. \tag{A.2}$$

Clearly, this representation is valid for functions f with L^1 Fourier transforms $\hat{f}$. To apply this representation we assume first that f belongs to the Schwartz class $\mathscr{S}(\mathbf{R})$ and then extend the obtained results to the class defined in (A.1) (clearly, $\mathscr{S}(\mathbf{R})$ is dense in $\bigcap_{n=1}^{3} \mathbf{L}^1(\mathbf{R}, |s|^n \, ds)$).

Lemma A.1. *Let A and B be self-adjoint operators and let B be, in addition, bounded. Let f obey* (A.1). *Then*

$$[B, f(A)] = i \int_{-\infty}^{\infty} ds \, \hat{f}(s) \int_0^s du \, e^{i(s-u)A} [A, B] e^{iuA}. \tag{A.3}$$

Equation (A.3) *can be first defined in the sense of a quadratic form on* $D(A) \times D(A)$.

Proof. Using the representation (A.2) we obtain

$$\begin{aligned}[B, f(A)] &= \int_{-\infty}^{\infty} ds \, \hat{f}(s) [B, e^{isA}] \\ &= \int_{-\infty}^{\infty} ds \, \hat{f}(s) e^{isA} (e^{-isA} B e^{isA} - B).\end{aligned}$$

Finally using that

$$e^{-isA} B e^{isA} - B = i \int_0^s e^{-iuA} [A, B] e^{+iuA} \, du \tag{A.4}$$

we arrive at (A.3). □

Lemma A.2. *Let f obey* (A.1) *but only for* $n = 1, 2$. *Then*

$$\left[f(\gamma), (n + H)^{-1} \right] = O(|x|^{-1}). \tag{A.5}$$

Proof. We use the representation (A.3) with $A = \gamma$ and $B = (n + H)^{-1}$. Clearly

$$[\gamma, (n + H)^{-1}] = O(|x|^{-1}). \tag{A.6}$$

Hence it remains to show that

$$\| |x| e^{i\gamma t} \langle x \rangle^{-1} \| \le c|t|. \tag{A.7}$$

To demonstrate this we use the equations

$$[e^{iCt}, \langle x \rangle^{-1}] = \langle x \rangle^{-1} [\langle x \rangle, e^{iCt}] \langle x \rangle^{-1} \tag{A.8}$$

and

$$[\langle x \rangle, e^{iCt}] = i \int_0^t e^{iC(t-s)} [\langle x \rangle, C] e^{iCs} \, ds, \tag{A.9}$$

valid for any self-adjoint operator C. Since

$$[\langle x \rangle, \gamma] \text{ is bounded,}$$

the last two relations with $C = \gamma$ give

$$\| \langle x \rangle [e^{i\gamma t}, \langle x \rangle^{-1}] \langle x \rangle \| \le c|t|. \tag{A.10}$$

The latter relation implies (A.7).

Now we are ready to complete the proof. Indeed, (A.7) shows that

$$\int_0^s e^{i(s-u)\gamma} O(|x|^{-1}) e^{iu\gamma} \, du = s^2 O(|x|^{-1}) \tag{A.11}$$

which implies (A.5). □

Remark A.3. In fact, the proof of this lemma implies a more general result. Let f obey (A.1) with $n = 1, 2$, and let a bounded operator B satisfy

$$[B, \gamma] = O(|x|^{-1}).$$

Then

$$[B, f(\gamma)] = O(|x|^{-1}).$$

Indeed, this follows directly from the proof of Lemma (A.2) with equation (A.6) replaced by

$$[B, \gamma] = O(|x|^{-1}).$$

Equation (A.11) will often be used in our estimates. We will also need its following generalization:

$$\int_0^s e^{i(s-u)\gamma} O(|x|^{-\alpha}) e^{iu\gamma} \, du = s^{2([\alpha]+1)} O(|x|^{-\alpha}) \tag{A.12}$$

for $\alpha > 0$. To prove (A.12) we use the equations obtained from (A.8) and (A.9) by replacing $\langle x \rangle$ by $\langle x \rangle^\sigma$ and using that

(A.13) $$[\langle x \rangle^\sigma, \gamma] \text{ is bounded for } \sigma \le 1.$$

This gives immediately

(A.14) $$\left\| \langle x \rangle^\sigma \left[e^{i\gamma t}, \langle x \rangle^{-\sigma} \right] \langle x \rangle^\sigma \right\| \le c|t|$$

for $\sigma \le 1$. If $\alpha \le 1$, then this estimate with $\sigma = \alpha$ yields (A.12). If $\alpha > 1$, then we pull $\langle x \rangle^{-\alpha}$ through $e^{i\gamma t}$ in pieces. This proves (A.12).

LEMMA A.4. *Let f obey* (A.1). *Consider a class of bounded differentiable functions j obeying*

(A.15) $$|\nabla j(y)| \le c \langle y \rangle^{-\alpha}.$$

(i) *If $j(x^a)$ obeys* (A.15), *then*

(A.16) $$[f(\gamma), j(x^a)] = O(|x|^{-\alpha}).$$

(ii) *If $j(x)$ obeys* (A.15), *then*

(A.17) $$[f(A), j(x)] = O(|x|^{-\alpha})$$

for $A = \gamma_a, p_a^2, H_a$.

(iii) *If $\varphi \in C_0^\infty$ and the derivatives of $j(x)$ up to the order k are $O(|x|^{-\alpha})$, then*

(A.18) $$\gamma^n [\varphi(\gamma), j(x)] \gamma^m = O(|x|^{-\alpha})$$

for any n and m satisfying $n + m \le k$.

Proof. (i) First we observe that

(A.19) $$[\gamma, j(x^a)] = O(|x|^{-\alpha}).$$

This follows from the computations

$$[\gamma, j(x^a)] = \frac{x^a}{\langle x \rangle} \cdot [p^a, j] + [p^a, j] \cdot \frac{x^a}{\langle x \rangle}$$

and

$$[p^a, j] = -i \nabla j$$

and estimate (A.15). Now representing $[f(\gamma), j(x^a)]$ according to (A.3) (with $A = \gamma$ and $B = j(x^a)$) and using (A.19) and (A.11) we arrive at (A.16).

(ii) Consider first $A = \gamma_a$. We use equation (A.3) with $B = j(x)$ and $A = \gamma_a$. Observing that

$(*)$ $$[j(x), \gamma_a] = O(|x|^{-\alpha}),$$

and using (A.11) (with γ replaced by γ_a) we conclude that

$$[f(\gamma_a), j(x)] = O(|x|^{-\alpha}). \tag{A.20}$$

The case $A = p_a^2, H_a$ is slightly more complicated. Let n be sufficiently large so that $A + n \geq 1$ (remember that A is bounded from below). We introduce the function

$$\varphi(t) = \begin{cases} f\left(\frac{1}{t} - n\right) & \text{if } t \in \left[0, (\mu + n)^{-1}\right] \\ 0 & \text{if } t \notin \left(-\varepsilon, (\mu + n)^{-1} + \varepsilon\right), \end{cases} \tag{A.21}$$

where $\mu = \inf \sigma(A)$. Besides we assume that φ is smooth. Clearly such a function is well-defined and it satisfies

$$\varphi\left(\frac{1}{s+n}\right) = f(s) \quad \text{for } \mu \leq s < \infty.$$

Hence

$$f(A) = \varphi(B), \quad \text{where } B = (A + n)^{-1}. \tag{A.22}$$

Now we use again (A.3) but in the form

$$[f(A), j(x)] = i\int_{-\infty}^{\infty} ds\, \hat{\varphi}(s) \int_0^s e^{i(s-u)B}[j(x), B] e^{iuB}\, du. \tag{A.23}$$

Since $j(x)$ is homogeneous degree 0 for $|x| \geq 1$, we have that

$$[j(x), B] = O(|x|^{-\alpha}). \tag{A.24}$$

Next we claim that

$$\| |x| e^{iBt} \langle x \rangle^{-1} \| \leq c|t|. \tag{A.25}$$

Indeed, since $[\langle x \rangle, B]$ is bounded for $B = (p_a^2 + n)^{-1}, (H_a + n)^{-1}$, equations (A.8) and (A.9) (with $C = B$) yield (A.25).

Now equations (A.23)–(A.25) imply (A.17).

(iii) This relation follows from (ii) with $A = \gamma$ and the estimate

$$ad_\gamma^n(j(x)) = (\hat{x} \cdot \nabla)^n j(x) = O(|x|^{-\alpha}),$$

where $ad_\gamma(j) = i[\gamma, j]$. This estimate holds provided that the n-th order derivatives of $j(x)$ are $O(|x|^{-\alpha})$. □

Before proceeding we derive a general formula we will use later. Let f_1 and f_2 be functions obeying (A.1) and let A_1 and A_2 be self-adjoint operators such that $[A_1, A_2]$ is densely defined and there is a dense set Ω such that $e^{iA_1t_1}e^{iA_2t_2}$ maps Ω into the quadratic form domain of $[A_1, A_2]$. Then the following equation holds

$$\text{(A.26)}\quad [f_1(A_1), f_2(A_2)] = -\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dr \hat{f}_1(s)\hat{f}_2(r) \times \int_0^s dp \int_0^r dq\, e^{i(s-p)A_1} e^{i(r-q)A_2} i[A_1, A_2] e^{iqA_2} e^{ipA_1}$$

in the form sense on Ω.

Indeed, using representation (A.2) we derive

$$[f_1(A_1), f_2(A_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}_1(s)\hat{f}_2(r)[e^{isA_1}, e^{irA_2}]\, ds\, dr.$$

Next, observing that

$$[e^{isA_1}, e^{irA_2}] = -i\int_0^s dp \int_0^r dq\, e^{i(s-p)A_1} e^{i(r-q)A_2} \times [A_1, A_2] e^{iqA_2} e^{ipA_1},$$

where the right-hand side is understood as a quadratic form on Ω, we obtain (A.26).

LEMMA A.5. *Let f and g obey* (A.1) *and* $A = p_a^2, H_a$. *Then*

(i) $$[g(\gamma), f(A)] = O(|x|^{-1}),$$

(ii) $$[g(\gamma_a), f(A)] = O(|x_a|^{-1}),$$

(iii) $$[g(\gamma_a), f(\gamma)] = O(|x|^{-1}).$$

Proof. (i) Let $B \equiv (A+n)^{-1}$, where n is such that $A + n \geq 1$. Let $\varphi(B) = f(A)$, where φ is defined in (A.21). First we observe that

$$\text{(A.27)}\quad [B, f(\gamma)] = O(|x|^{-1})$$

by Lemma A.2. This estimate, representation (A.3) (with $A = B$, $B = f(\gamma)$ and $f = \varphi$) and estimates (A.11) imply (i).

(ii) The proof of this case is exactly the same as that of (i).

(iii) In this case we take a different route. Using equation (A.3) we derive

$$\text{(A.28)}\quad [\varphi(\gamma), g(\gamma_a)] = \int_{-\infty}^{\infty} dt\, \hat{\varphi}(t) \int_0^t ds\, e^{i\gamma(t-s)} i[\gamma, g(\gamma_a)] e^{i\gamma s}.$$

Now we estimate the commutator inside of the integral. Using that

$$\gamma = r_a\gamma_a + r^a\gamma^a + O(|x|^{-1})$$

and

$$[r_a, \varphi(\gamma_a)]\gamma_a = [r_a, \varphi(\gamma_a)\gamma_a] + O(|x|^{-1}) = O(|x|^{-1}),$$

we derive

$$\text{(A.29)}\quad [\gamma, g(\gamma_a)] = [r^a, g(\gamma_a)]\gamma^a + O(|x|^{-1}).$$

Representing now

$$\gamma^a = \frac{1}{r^a}\gamma - \frac{r_a}{r^a}\gamma_a + O(|x|^{-1})$$

and using that

$$[r^a, g(\gamma_a)]\frac{1}{r^a} = [\langle x\rangle^{-1}, g(\gamma_a)]\langle x\rangle = O(|x|^{-1}),$$

by Lemma A.4(ii), we obtain

$$[r^a, g(\gamma_a)]\gamma^a = O(|x|^{-1})\gamma + O(|x|^{-1}).$$

Together with (A.29) this gives

$$[\gamma, g(\gamma_a)] = O(|x|^{-1})\gamma + O(|x|^{-1}). \tag{A.30}$$

Plugging this into equation (A.28) we find

$$[\varphi(\gamma), g(\gamma_a)] = I_1 + I_2,$$

where

$$I_1 = \int_{-\infty}^{\infty} dt\, \hat{\varphi}(t) \int_0^t ds\, e^{i\gamma(t-s)} O(|x|^{-1}) e^{i\gamma s}$$

and

$$I_2 = \int_{-\infty}^{\infty} dt\, \hat{\varphi}(t) \int_0^t ds\, e^{i\gamma(t-s)} \gamma O(|x|^{-1}) e^{i\gamma s}.$$

By equation (A.11) we have

$$I_1 = O(|x|^{-1}).$$

Next, observing that

$$I_2 = -i\int_{-\infty}^{\infty} dt\, \hat{\varphi}(t) \frac{d}{dt} e^{i\gamma t} \int_0^t ds\, e^{-i\gamma s} O(|x|^{-1}) e^{i\gamma s}$$

and integrating by parts we obtain

$$I_2 = i\int_{-\infty}^{\infty} dt\, \hat{\varphi}'(t) \int_0^t ds\, e^{i\gamma(t-s)} O(|x|^{-1}) e^{i\gamma s} + i\int_{-\infty}^{\infty} dt\, \hat{\varphi}(t) O(|x|^{-1}) e^{i\gamma t}.$$

Again, by equation (A.11),

$$I_2 = O(|x|^{-1}).$$

Hence, (iii) holds. □

LEMMA A.6. *Assume* $V_{ij}(y) = O_1(|y|^{-\varepsilon})$ *and* $\nabla V_{ij}(y) = O_1(|y|^{-1-\varepsilon})$. *Then*

$$[f_1(\gamma_b), I_a] F(|x|_a > \delta|x|) = O_2(|x|^{-1-\varepsilon}), \tag{A.31}$$

for any f_1 *obeying* (A.1) *and*

$$[f_2(p_b^2), I_a]F(|x|_a \geq \delta|x|) = O(|x|^{-1-\varepsilon}) \tag{A.32}$$

for every C_0^∞ *function* f_2.

Proof. Let $F_a = F(|x|_a > \delta|x|)(1+p^2)^{-1}$. For $A = \gamma_b$ we compute

$$[f(A), I_a]F_a = [f(A), I_aF_a] - I_a[f(A), F_a]. \tag{A.33}$$

Consider first the second term on the right-hand side. Pick δ' so small that

$$F(|x|_a > \delta'|x|)F(|x|_a > \delta|x|) = F(|x|_a > \delta|x|).$$

By Lemma A.4, it is easy to show that

$$f(A)F(|x|_a > \delta|x|) = F(|x|_a > \delta'|x|)f(A)F(|x|_a > \delta|x|) + O(|x|^{-2}).$$

The last two relations imply

$$I_a[f(A), F_a] = I_aF(|x|_a > \delta'|x|)[f(A), F_a] + (p^2+1)O(|x|^{-2}).$$

Since $V_{ij}(y) = O_1(|y|^{-\varepsilon})$, we have that

$$I_aF(|x|_a > \delta'|x|) = O_1(|x|^{-\varepsilon}) \tag{A.34}$$

by the definition of $|x|_a$. Next, using Lemma A.2 and Lemma A.4(ii), one shows readily that $[f(A), F_a] = O(|x|^{-1})$. This together with (A.34) yields

$$I_a[f(A), F_a] = (p^2+1)O(|x|^{-1-\varepsilon}). \tag{A.35}$$

Consider now the first term on the right side of (A.32). Using that $\nabla V_{ij}(y) = O_1(|y|^{-1-\varepsilon})$ by the assumptions, and a simple computation, we obtain

$$[A, I_aF_a] = O(|x|^{-1-\varepsilon}). \tag{A.36}$$

Representing the commutator $[f(A), I_aF_a]$ according to (A.3) (with $B \to I_aF_a$) and using (A.36) and (A.11), we arrive at

$$[f(A), I_aF_a] = O(|x|^{-1-\varepsilon}). \tag{A.37}$$

This estimate together with (A.34) and (A.32) implies

$$[f(A), I_a]F_a = (p^2+1)O(|x|^{-1-\varepsilon}), \tag{A.38}$$

which yields (A.31).

To prove (A.32) we take $f_2(p_b^2) = \varphi(1/p_b^2+1)$ with φ defined in (A.21). Proceeding as above but with F_a replaced by $F = F(|x|_a > \delta|x|)$ and observing that

$$\left[(p_b^2+1)^{-1}, I_aF\right] = O(|x|^{-1-\varepsilon})$$

and

$$I_a\Big[\varphi\big((p_b^2+1)^{-1}\big), F\Big] = O(|x|^{-1-\varepsilon}),$$

we arrive at (A.32). □

LEMMA A.7. *Let A and B be two self-adjoint operators and let B, in addition, be bounded. Let f obey* (A.1). *Then*

$$[B, f(A)] = f'(A)[B, A] + R, \tag{A.39}$$

where

$$R = -i\int_{-\infty}^{\infty} ds\, \hat{f}(s)\int_0^s du \int_0^u dp\, e^{i(s-p)A}[A,[A,B]]e^{ipA}. \tag{A.40}$$

These equations are understood in the form sense with the form domains $D(A^2)$.

Proof. Using equation (A.3), we obtain

$$\begin{aligned}[B, f(A)] &= i\int_{-\infty}^{\infty} ds\, \hat{f}(s)e^{isA}s[A,B]\\ &\quad + \int_{-\infty}^{\infty} ds\,\hat{f}(s)e^{isA} i\int_0^s du\big(e^{-iuA}[B,A]e^{iuA} - [B,A]\big).\end{aligned}$$

Representing the expression in parentheses as an integral of its derivative we arrive at (A.37). □

LEMMA A.8. *Let f obey* (A.1). *Then*

$$[H, f(\gamma)] = f'(\gamma)[H,\gamma] + O_2(|x|^{-2}). \tag{A.41}$$

Proof. Choose n so that $H + n \geq 1$ and compute

$$[H, f(\gamma)] = -(H+n)\big[(H+n)^{-1}, f(\gamma)\big](H+n). \tag{A.42}$$

Applying equation (A.37) with $A = \gamma$ and $B = (H+n)^{-1}$ to the commutator on the right side of this expression and observing that

$$-\big[(H+n)^{-1}, \gamma\big] = (H+n)^{-1}[H,\gamma](H+n)^{-1} \tag{A.42a}$$

and that

$$(H+n)f'(\gamma)(H+n)^{-1} = f'(\gamma) + (H+n)O(|x|^{-1}), \tag{A.43}$$

by Lemma A.2, we obtain

(A.44)

$$[H, f(\gamma)] = f'(\gamma)[H,\gamma] + (H+n)R(H+n) + (H+n)O(|x|^{-1})[H,\gamma]$$

where the remainder R is given by (A.40) with $A = \gamma$ and $B = (H+n)^{-1}$.

Since, by (7.4),

$$[H, \gamma] = O_1(|x|^{-1}),$$

we have that

(A.45) $$(H+n)O(|x|^{-1})[H, \gamma] = O_2(|x|^{-2}).$$

Thus it suffices to show that

(A.46) $$R = O(|x|^{-2}).$$

To this end it suffices to demonstrate (see (A.12)) that

(A.47) $$[\gamma, [\gamma, H]](n+H)^{-1} = O(|x|^{-2}).$$

This is just a straightforward computation. For instance

(A.48) $$-i[\gamma, H] = \langle x\rangle^{-1}(2p^2 - x\cdot\nabla V(x)) + \langle x\rangle^{-1}\sum_{i=0}^{2} f_i(x)\gamma^i$$

with some functions f_i obeying $|D^\alpha f_i(x)| \leq c_\alpha \langle x\rangle^{-2+i-|\alpha|}$, and so forth. (This expression is so involved partly due to the fact that we use $\langle x\rangle$ and not $|x|$.) Thus (A.41) results. □

COROLLARY A.9. *Let f and $(f')^{1/2}$ satisfy* (A.1) *and $f' \geq 0$. Then*

(A.49) $$F_\Delta[H, f(\gamma)]F_\Delta = F_\Delta f'(\gamma)^{1/2}F_{\Delta'}[H, \gamma]F_{\Delta'}f'(\gamma)^{1/2}F_\Delta + O(|x|^{-2}),$$

where Δ' is larger than Δ so that $F_\Delta F_{\Delta'} = F_\Delta$.

Proof. We begin with an auxiliary estimate. We claim that

(A.50) $$\left[\varphi(\gamma), \left[\frac{1}{H+n}, \gamma\right]\right] = O(|x|^{-2}),$$

provided φ obeys (A.1). This follows from equation (3.21) and Remark (A.3).

Now we are ready for the derivation of (A.47). Write in equation (A.39), $f'(\gamma) = [f'(\gamma)^{1/2}]^2$ and commute $f'(\gamma)^{1/2}$ through $[H, \gamma]$ to the other side of the latter. By equation (A.48) with $\varphi = (f')^{1/2}$ this yields

(A.51)

$$[H, f(\gamma)] = -(H+n)f'(\gamma)^{1/2}\left[\frac{1}{H+n}, \gamma\right]f'(\gamma)^{1/2}(H+n) + O_2(|x|^{-2}).$$

Take now Δ' slightly larger than Δ so that

$$F_\Delta F_{\Delta'} = F_\Delta.$$

Sandwich (A.51) by F_Δ on both sides and pull one $F_{\Delta'}$ out of each F_Δ on the right-hand side and commute it through $f'(\gamma)^{1/2}$ using that, by Lemma A.5(i),

$$[F_{\Delta'}, f'(\gamma)^{1/2}] = O(|x|^{-1}). \tag{A.52}$$

Since this commutator is multiplied by

$$\left[\frac{1}{H+n}, \gamma\right] = O(|x|^{-1}), \tag{A.53}$$

it contributes $O(|x|^{-2})$ to the resulting expression. It remains to observe that $[(1/H+n), \gamma] = -(1/H+n)[H,\gamma](1/H+n)$ and to pull the $(H+n)^{-1}$'s through $F'(\gamma)^{1/2}$'s to cancel them against the $(H+n)$'s. The result is (A.49). □

Finally, we proceed to:

LEMMA A.10. *Let f obey* (A.1).

(i) *If the bound state $\psi^\alpha, \nabla\psi^\alpha \in \langle x^\alpha\rangle^{-\eta}L^2(X^\alpha)$, then*

$$f(\gamma)P_\alpha - P_\alpha f(\gamma_\alpha) = O(|x|^{-\eta}). \tag{A.54}$$

(ii) *If no fall off at ∞ for ψ^α is assumed, then*

$$f(\gamma)P_\alpha - P_\alpha f(\gamma_\alpha) \overset{\varepsilon}{=} O(|x|^{-1}), \tag{A.55}$$

where ε is arbitrary and $O(|x|^{-1})$ depends on ε.

(iii) *Statements* (i) *and* (ii) *still hold after γ_α in* (A.52) *and* (A.53) *is replaced by γ.*

Proof. (i) We have

$$\gamma = r^\alpha\gamma^\alpha + r_\alpha\gamma_\alpha + O(|x|^{-1}). \tag{A.56}$$

Using that γ_α commutes with P_α and that

$$r^\alpha\gamma^\alpha P_\alpha = O(|x|^{-\eta}), \tag{A.57}$$

we obtain

$$(\gamma - \gamma_\alpha)P_\alpha = O(|x|^{-\eta}) + Q_\alpha\gamma_\alpha, \tag{A.58}$$

where

$$Q_\alpha \equiv (-1 + r_\alpha)P_\alpha. \tag{A.59}$$

Next we use

$$f(\gamma) - f(\gamma_\alpha) = \int_{-\infty}^{\infty} ds\,\hat{f}(s)\int_0^s dp\,e^{i\gamma p}(\gamma - \gamma_\alpha)e^{i\gamma_\alpha(s-p)},$$

which together with (A.56) and the estimate

$$\left\| |x|^\eta e^{iBt}\langle x\rangle^{-\eta}\right\| \le c|t| \quad \text{with } B = \gamma, \gamma_\alpha, \tag{A.60}$$

which is an obvious variant of (A.7) (see also (A.12)–(A.14)), implies that it suffices to estimate

$$\int_{-\infty}^{\infty} \hat{f}(s) \int_0^s dp\, e^{i\gamma p} Q_\alpha \gamma_\alpha e^{i\gamma_\alpha(s-p)}.$$

Observing that $\gamma_\alpha e^{i\gamma_\alpha(s-u)} = -i\,d/ds\, e^{i\gamma_\alpha(s-u)}$ and integrating by parts we convert this term into

$$f(\gamma)Q_\alpha + \int_{-\infty}^{\infty} ds\, \hat{f}'(s) \int_0^s dp\, e^{i\gamma p} Q_\alpha e^{i\gamma_\alpha(s-p)}$$

which is $O(|x|^{-\eta})$ due to (A.58) and

$$Q_\alpha = -\frac{r^\alpha}{1 + r_\alpha} P_\alpha = O(|x|^{-\eta}).$$

(ii) Let χ_R be a smooth cut-off function of x^α obeying: $\chi_R = 1$ for $\langle x^\alpha \rangle \le R$ and $\chi_R = 0$ for $\langle x^\alpha \rangle \ge 2R$. Let $\bar{\chi}_R = 1 - \chi_R$ and retain the same symbols for the operators of multiplication by χ_R and $\bar{\chi}_R$. We derive as in (i):

$$f(\gamma) P_\alpha \chi_R - P_\alpha \chi_R f(\gamma_\alpha) = O_R(|x|^{-1}),$$

where the dependence of the error term on R is exhibited explicitly (in fact, it behaves as $R/|x|$). This yields

$$f(\gamma) P_\alpha = P_\alpha f(\gamma_\alpha) + A_R + O_R(|x|^{-1}),$$

where $A_R = -P_\alpha \bar{\chi}_R f(\gamma_\alpha) + f(\gamma) P_\alpha \bar{\chi}_R$. Clearly

$$\|A_R\| \le c \|P_\alpha \bar{\chi}_R\| \equiv \varepsilon(R),$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.

(iii) We use expression (A.3) with $A = \gamma$ and $B = P_\alpha$ and equation (A.54) to compute the commutator on the right side of (A.3):

$$[\gamma, P_\alpha] = [r^\alpha \gamma^\alpha, P_\alpha] + [r_\alpha - 1, P_\alpha]\gamma_\alpha + O(|x|^{-1}).$$

After this we proceed as in (i) and (ii). □

Remark A.11. The idea of cutting bound states at infinity in order to avoid the question of fall-off properties of the threshold bound states comes from [E5, 6] (we are grateful to V. Enss for pointing out this idea to us. The idea of approximating bound states by fast decaying functions was also used in a similar context, the many-body scattering theory in [Sig2, 3], and dates back to earlier works of the first author (I.M.S.)).

University of California, Irvine
California Institute of Technology, Pasadena

References

[AG] S. Agmon, *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations*, Math. Notes, Princeton Univ. Press, 1982.

[AH] H. Araki and R. Haag, Collision cross section in terms of local observables, CMP **4** (1967), 77–91.

[BL] R. Benguria and E. Lieb, Proof of the stability of highly negative ions in the absence of the Pauli principle, Phys. Rev. Lett. **50** (1983), 1771.

[Com] J. M. Combes, Relatively compact interactions in many particle systems, Comm. Math. Phys. **12** (1969), 283.

[CFKS] H. Cycon, R. Froese, W. Kirsch and B. Simon, Lectures on the Schrödinger Equation (to appear).

[DL] F. Dyson and A. Lenard, Stability of matter I, II, J. Math. Phys. **8** (1967), 423–434; **9** (1968), 698–711.

[DMSV] P. Deift, W. Hunziker, B. Simon and E. Vock, Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems, Comm. Math. Phys. **64** (1978), 1–34.

[DS] P. Deift and B. Simon, A time-dependent approach to the completeness of multiparticle quantum systems, Comm. Pure Appl. Math. **30** (1977), 573–583.

[E1] V. Enss, A note on Hunziker's theorem, Comm. Math. Phys. **52** (1977), 233–238.

[E2] ______, Asymptotic completeness for quantum mechanical scattering I. Short range potentials, Comm. Math. Phys. **61** (1978), 285–291.

[E3] ______, Propagation properties of quantum scattering states, J. Funct. Anal. **52** (1983), 219–268.

[E4] ______, in: *Differential Equations*. I. W. Knowles and R. T. Lewis, ed., North Holland Math. Studies **92** (1984), 173–204.

[E5] ______, Topics in scattering theory for multiparticle quantum mechanics, a progress report, Physica **124** A (1984), 269–292.

[E6] ______, in Proc. Como summer school on the Schrödinger equation, Como, 1984 (to be published).

[F] L. D. Faddeev, Mathematical aspects of the three-body problem in the quantum theory of scattering, Israel program for scientific translations, Jerusalem (1965).

[Fe] P. Federbush, A new approach to the stability of matter problem, I, II, J. Math. Phys. **16** (1975) 347–351; 706–709.

[FH1] R. Froese and I. Herbst, Exponential bounds and absence of positive eigenvalues of N-body Schrödinger operators, Comm. Math. Phys. **87** (1982), 429–447.

[FH2] ______, A new proof of the Mourre estimate, Duke Math. J. 49 (1982), 1975.

[G] J. Ginibre, in *Schrödinger Equation*, W. Thirring and P. Urban, Eds., Springer-Verlag, 1977.

[GM] J. Ginibre and M. Moulin (Combescure), Hilbert space approach to the quantum mechanical three body problem, Ann. Inst. H. Poincaré, **21** (1974), 97–145.

[H] G. Hagedorn, Asymptotic completeness for classes of two, three and four particle Schrödinger operators, Trans. AMS **258** (1980), 1–75.

[HP] G. Hagedorn and P. Perry, Asymptotic completeness for certain three body Schrödinger operators, Comm. Pure Appl. Math. **36** (1983), 213–233.

[HS] R. Haag and J. A. Swieca, When does a quantum field theory describe particles?, Comm. Math. Phys. 1 (1965), 308.

[He] K. Hepp, On the quantum mechanical N-body problem, Helv. Phys. Acta **42** (1969), 425–458.

[Hö] L. Hörmander, On the existence and the regularity of solutions of linear pseudo-differential equations, L'Enseign Math. **17** (1971), 99–163.

[How] J. Howland, Abstract stationary theory of multichannel scattering, J. Funct. Anal. **22** (1976), 250–282.
[Hu1] W. Hunziker, in *Advances in Theoretical Physics*, Vol X, A. Barut, W. Britten (ed.) (1968).
[Hu2] ______, Time dependent scattering theory for singular potentials, Helv. Phys. Acta **40** (1967), 1052–1062.
[I] T. Ikebe, Eigenfunction expansions associated with the Schrödinger operators and their application to scattering theory,' Arch. Rat. Mech. Anal. **5** (1960), 1–34.
[IO] R. J. Iorio, Jr. and M. O'Carroll, Asymptotic completeness of multiparticle Schrödinger Hamiltonians with weak potentials, Comm. Math. Phys. **27** (1972), 137–145.
[Ka1] T. Kato, Wave operators and similarity for some non-self-adjoint operators, Math. Anal. **162** (1966), 258–279.
[Ka2] ______, Smooth operators and commutators, Studia Math. XXXI (1968), 535–546.
[K] M. Krishna, Large time behavior of some N-body systems, Comm. Math. Phys. **101** (1985), 129–152.
[KK] T. Kato and S. T. Kuroda, The abstract theory of scattering, Rocky Mount. J. Math. **1** (1971), 127–171.
[L1] R. Lavine, Absolute continuity of Hamiltonian operators with repulsive potentials, Proc. AMS **22** (1969), 55–60.
[L2] ______, Scattering theory for long range potentials, J. Funct. Anal. **5** (1970), 368–382.
[L3] ______, Commutators and scattering theory I: Repulsive interactions, Comm. Math. Phys. **20** (1971), 302–323.
[L4] ______, Commutators and scattering II: A class of one body problems, Indiana Univ. Math. J. **21** (1972), 643–656.
[L5] ______, Absolute continuity of positive spectrum for Schrödinger operators with long range potentials, J. Funct. Anal. **12** (1973), 30–54.
[L6] ______, Completeness of the wave operators in the repulsive N-body problem, J. Math. Phys. 14 (1973), 376–379.
[L7] ______, in *Scattering Theory in Mathematical Physics*, J. A. Lavita and J.-P. Marchand, eds., D. Reidel Publ. Co., 1973.
[Lieb1] E. H. Lieb, The stability of matter, Rev. Mod. Phys. **48** (1976), 553–569.
[Lieb2] ______, Bound on the maximum negative ionization of atoms and molecules, Phys. Rev. A. **29** (1984), 3018.
[LS] M. Loss and I. M. Sigal, The three-body problem with threshold singularities, ETH preprint, 1980.
[LSST] E. H. Lieb, I. M. Sigal, B. Simon and W. Thirring, Asymptotic neutrality of large Z-ions, Phys. Rev. Lett. **52** (1984), 994–996.
[LT] E. M. Lieb and W. Thirring, Bound for the kinetic energy of fermions which proves the stability of matter, Phys. Rev. Lett. **35** (1975), 687–689.
[M1] E. Mourre, Absence of singular spectrum of certain self-adjoint operators, Comm. Math. Phys. **78** (1981), 391–408.
[M2] ______, Opérateurs conjugués et propriétés de propagation, Comm. Math. Phys. **91** (1983), 279–300.
[M3] ______, Opérateurs conjugué et propriété de propagation II, remarques sur la completude asymptotique des systemes à trois corps, Marseille 1983, preprint.
[MS] E. Mourre and I. M. Sigal, Phase space analysis and scattering theory for N-particle systems (microlocal analysis of propagation of particles), 1983, preprint.
[P] A. J. Povsner, On the expansion of arbitrary functions in terms of the eigenfunctions of the operator $-\Delta u + cu$, Math. Sb. **32** (1953), 109–156; AMS Transl. **60** (1967), 1–49.

[PSS] P. Perry, I. M. Sigal and B. Simon, Spectral analysis of N-body Schrödinger operators, Ann. of Math. **114** (1981), 519–567.

[Pu] C. R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer-Verlag, 1967.

[R] M. B. Ruskai, Absence of highly negative ions I, II, Comm. Math. Phys. **82** (1982), 457 and **85** (1982), 325.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* II, III, Academic Press, (1975), (1979).

[Sig1] I. M. Sigal, Mathematical foundations of quantum scattering theory for multiparticle systems, a Memoir of AMS N209 (1978).

[Sig2] ———, Geometric methods in the quantum many body problem. Non-existence of very negative ions, Comm. Math. Phys. **85** (1982), 309–324.

[Sig3] ———, *Scattering Theory for Many-Body Quantum Mechanical Systems*, Springer Lecture Notes in Math. N1011, 1983.

[Sig4] ———, How many electrons can a nuclear bind?, Anal. Phys. **157** (1984), 307–320.

[Sig5] ———, Asymptotic completeness of many-body short range systems, Lett. in Math. Phys. 8 (1984), 181–188.

[SigSof1] I. M. Sigal and A. Soffer, Asymptotic completeness for short range many-body systems, Bulletin AMS **14**, No. 1 (1986), 107–110.

[SigSof2] ———, Local decay estimates for N-body systems, Caltech preprint.

[SigSof3] ———, Sharp propagation estimates for N-body systems, Caltech preprint.

[Sim] B. Simon, Geometric methods in multiparticle quantum systems, Comm. Math. Phys. **55** (1977), 259–274.

[SKM] K. B. Sinha, M. Krishna, Pl. Muthuramalingam, On the completeness in three-body scattering, Ann. Inst. M. Poincaré **41** (1984), 79–101.

[SS] A. G. Sigalov and I. M. Sigal, Description of the spectrum of the energy operator of quantum mechanical systems that is invariant with respect to permutations of identical particles. Theor. Math. Phys. **5** (1970), 990–1005.

[T] L. E. Thomas, Asymptotic completeness in two and three particle quantum mechanical scattering, Ann. Phys. **90** (1975), 127–165.

[Y] K. Yajima, An abstract stationary approach to three-body scattering, J. Fac. Sci. Univ. Tokyo, Sic 1A, **25** (1978), 109–132.

[Z] G. M. Zhislin, Discussion of the spectrum of the Schrödinger operator for systems of many particles, Tr. Mosk. Mat. Obs. **9** (1960), 81–128 (in Russian).

(Received July 1, 1985)
(Revised December 23, 1986)