# Renormalization Group and Non-relativistic Quantum Electrodynamics Les Houches, August, 2010 

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#### Abstract

We review some basic notions, results and techniques in non-relativistic quantum electrodynamics (QED). The review is based on joint work with Volker Bach and Jürg Fröhlich and with Walid Abou Salem, Thomas Chen, Faupin, and Marcel Griesemer. References to these and other contributions will be given at the end of these notes.

The sections marked with $*$ did not appear in the original presentation.


## 1 Overview

I will describe mathematical theory of the non-relativistic QED. This theory was developed in the last 10 or so years.

It deals with quantum-mechanical particle systems coupled to quantized electromagnetic field at the energies $\ll m c^{2}$ (the rest energy of electron).

Sample of results it addresses are

- Stability;
- Radiation;
- Renormalization of mass;
- One-particle states.

We translate some of the physical notions above into mathematical terms:

- Stability $\Longleftrightarrow$ Existence of the ground state;
- Radiation $\Longleftrightarrow$ Turning of the excited states of particle systems into resonances, Scattering theory.

One of the key notions here is that of the resonance. It gives a clear-cut mathematical description of processes of emission and absorption of the electro-magnetic radiation.

The key and unifying technique I will concentrate on is the spectral renormalization group. It is easily combined with other techniques, e.g. complex deformations (for resonances), the Mourre estimate (for dynamics) and fiber integral decompositions and Ward identities (for translationally invariant systems).

The techniques used here extend to analysis of existence and stability of thermal states.

## 2 Non-relativistic QED

### 2.1 Schrödinger equation

The starting point of the non-relativistic QED is the time-dependent Schrödinger equation with

$$
i \partial_{t} \psi=H \psi
$$

where $\psi$ is a differentiable path in the Hilbert space $\mathcal{H}=\mathcal{H}_{p} \otimes \mathcal{H}_{f}$, which is the tensor product of the state spaces of the particles $\left(\mathcal{H}_{p}\right.$, say, $\left.\mathcal{H}_{p}=L^{2}\left(R^{3 n}\right)\right)$ and of the quantized electromagnetic field ( $\mathcal{H}_{f}=$ Bosonic Fock space), and $H$ is the standard quantum Hamiltonian on $\mathcal{H}=\mathcal{H}_{p} \otimes \mathcal{H}_{f}$, given by

$$
H=\sum_{j=1}^{n} \frac{1}{2 m_{j}}\left(i \nabla_{x_{j}}-g A\left(x_{j}\right)\right)^{2}+V(x)+H_{f} .
$$

Here $m_{j}$ and $x_{j}, j=1, \ldots, n$, are the particle masses and positions, $x=\left(x_{1}, \ldots, x_{n}\right), V(x)$ is the total potential affecting particles $g>0$ is a coupling constant related to the particle charge. (For more details see Appendix B.)

For simplicity we omitted the interaction of the spin with magnetic field -

$$
\sum_{j=1}^{n} \frac{g}{2 m_{j}} \sigma_{j} \cdot \operatorname{curl} A\left(x_{j}\right)
$$

If we fix the particle potential $V(x)$, then the Hamiltonian $H_{g}$ depends on two free parameters:

- The coupling constant $g$ (related to the electron charges);
- The ultraviolet cut-off $\kappa$ (related to the electron renormalized mass).


### 2.2 Stability and Radiation*

For a large class of potentials $V(x)$, including Coulomb potentials,the operator $H$ is self-adjoint and bounded below.

- The stability of the system under consideration is equivalent to the statement of existence of the ground state of $H$, i.e. an eigenfunction with the smallest possible energy.
- The physical phenomenon of radiation is expressed mathematically as emergence of resonances out of excited states of a particle system due to coupling of this system to the quantum electromagnetic field.

Our goal is to develop the spectral theory of the Hamiltonian $H$ and relate to the properties of the relevant evolution. Namely, for a quantum-mechanical system of particles coupled to quantized electromagnetic field we would like to show that

1) The ground state of the particle system is stable when the coupling is turned on, while
2) The excited states, generically, are not. They turn into resonances.

### 2.3 Particle system

The matter system considered consists of $n$ charged particles interacting between themselves and with external fields. Its Hamiltonian operator is given by

$$
\begin{equation*}
H_{p}:=-\sum_{j=1}^{n} \frac{1}{2 m_{j}} \Delta_{x_{j}}+V(x), \tag{1}
\end{equation*}
$$

where $\Delta_{x_{j}}$ is the Laplacian in the variable $x_{j}$ and, recall, $V(x)$ is the total potential of the particle system. $H_{p}$ acts on a Hilbert space of the particle system, $\mathcal{H}_{p}$, which is either $L^{2}\left(\mathbb{R}^{3 n}\right)$ or a subspace of this space determined by a symmetry group of the particle system.

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### 2.4 Quantized Electromagnetic Field

The quantized electromagnetic field is described by the quantized vector potential, which, in the Coulomb gauge $(\operatorname{div} A(y)=0)$, is given by

$$
\begin{equation*}
A(y)=\int\left(e^{i k y} a(k)+e^{-i k y} a^{*}(k)\right) \chi(k) \frac{d^{3} k}{\sqrt{|k|}}, \tag{2}
\end{equation*}
$$

where $\chi$ is an ultraviolet cut-off: $\chi(k)=1$ in a neighborhood of $k=0$ and is decaying sufficiently fast at infinity.

The dynamics of the quantized electromagnetic field is given through the quantum Hamiltonian

$$
\begin{equation*}
H_{f}=\int d^{3} k \omega(k) a^{*}(k) \cdot a(k) \tag{3}
\end{equation*}
$$

where $\omega(k)=|k|$ is the dispersion law connecting the energy of the field quantum with its wave vector $k$.

Here $a(k)$ and $a^{*}(k)$ are annihilation and creation operators acting on the Fock space $\mathcal{H}_{f} \equiv \mathcal{F}$ (see Appendix A for the definitions).

### 2.5 Ultra-violet Cut-off

Assuming the ultra-violet cut-off $\chi(k)$ decays on the scale $\kappa$, in order to correctly describe the phenomena of interest, such as emission and absorption of electromagnetic radiation, i.e. for optical and rf modes, we have to assume that the cut-off energy,
$\hbar c \kappa \gg \alpha^{2} m c^{2}$, ionization energy, characteristic energy of the particle motion,
or $\alpha^{2} \ll \kappa$ in our units. On the other hand, we assume

$$
\hbar c \kappa \ll m c^{2} \text {, the rest energy of the the electron, }
$$

where the relativistic effects, such as electron-positron pair creation, vacuum polarization and relativistic recoil, take place. Combining the last two conditions we arrive at

$$
\alpha^{2} \ll \kappa \ll 1 \quad\left(\alpha^{2} m c / \hbar \ll \kappa \ll m c / \hbar\right) .
$$

After the rescaling $x \rightarrow \alpha^{-1} x$ and $k \rightarrow \alpha^{2} k$ the new cut-off momentum scale, $\kappa^{\prime}=\alpha^{-2} \kappa$, satisfies

$$
1 \ll \kappa^{\prime} \ll \alpha^{-2},
$$

which is easily accommodated by our estimates (e.g. we can have $\kappa=O\left(\alpha^{-1 / 3}\right)$ ).

### 2.6 Units*

We use the units in which the Planck constant divided by $2 \pi$, the speed of light and the electron mass are equal to $1(\hbar=1, c=1$ and $m=1)$. In these units the electron charge is equal to $-\sqrt{\alpha}$, where $\alpha=\frac{e^{2}}{4 \pi \hbar c} \approx \frac{1}{137}$ (the fine-structure constant) and the distance, time and energy are measured in units of $\hbar / m c=3.86 \cdot 10^{-11} \mathrm{~cm}, \hbar / m c^{2}=1.29 \cdot 10^{-21} \mathrm{sec}$ and $m c^{2}=0.511 \mathrm{MeV}$, respectively (natural units).

## 3 Resonances

As was mentioned above, the mathematical language which describes the physical phenomenon of radiation is that of quantum resonances. We expect that the latter emerge out of excited states of a particle system due to coupling of this system to the quantum electro-magnetic field.

Quantum resonances manifest themselves in three different ways:

1) Eigenvalues of complexly deformed Hamiltonian;
2) Poles of the meromorphic continuation of the resolvent across the continuous spectrum;
3) Metastable states.

### 3.1 Complex Deformation

To define the resonances for the Hamiltonian $H$ we pass to the one-parameter (deformation) family

$$
\begin{equation*}
H_{\theta}:=U_{\theta} H U_{\theta}^{-1} \tag{4}
\end{equation*}
$$

where $\theta$ is a real parameter and $U_{\theta}$, on the total Hilbert space $\mathcal{H}:=\mathcal{H}_{p} \otimes \mathcal{F}$, is the one-parameter group of unitary operators, whose action is rescaling particle positions and of photon momenta:

$$
x_{j} \rightarrow e^{\theta} x_{j} \text { and } k \rightarrow e^{-\theta} k .
$$

One can show show that:

1) Under a certain analyticity condition on coupling functions, the family $H_{\theta}$ has an analytic continuation in $\theta$ to the disc $D\left(0, \theta_{0}\right)$, as a type A family in the sense of Kato;
2) The real eigenvalues of $H_{\theta}, \operatorname{Im} \theta>0$, coincide with eigenvalues of $H$ and that complex eigenvalues of $H_{\theta}, \operatorname{Im} \theta>0$, lie in the complex half-plane $\mathbb{C}^{-}$;
3) The complex eigenvalues of $H_{\theta}, \operatorname{Im} \theta>0$, are locally independent of $\theta$.

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We call complex eigenvalues of $H_{\theta}, \operatorname{Im} \theta>0$ the resonances of $H$.
Exercise.* Find the complex deformation of the hydrogen atom and photon Hamiltonians $H:=$ $-\Delta-\frac{\alpha}{|x|}$ and $H_{f}$. Find the spectra of the deformations $H_{\theta}$ and $H_{f \theta}$. (Answer:

$$
\begin{gathered}
H_{\theta}=e^{-2 \theta}(-\Delta)-e^{-\theta} \frac{\alpha}{|x|}, H_{f \theta}=e^{-\theta} H_{f} \\
\sigma\left(H_{\theta}\right)=\left\{e_{j}^{h y d r}\right\} \cup e^{-2 \operatorname{Im} \theta}[0, \infty) \\
\sigma\left(H_{f \theta}\right)=\{0\} \cup e^{-\operatorname{Im} \theta}[0, \infty)
\end{gathered}
$$

where $e_{j}^{h y d r}$ are the eigenvalues of the hydrogen atom.)

### 3.2 Resonances as Poles

Let $\Psi_{\theta}=U_{\theta} \Psi$, etc., for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^{+}$. Use the unitarity of $U_{\theta}$ for real $\theta$, to obtain (the Combes argument)

$$
\begin{equation*}
\left\langle\Psi,(H-z)^{-1} \Phi\right\rangle=\left\langle\Psi_{\bar{\theta}},\left(H_{\theta}-z\right)^{-1} \Phi_{\theta}\right\rangle . \tag{5}
\end{equation*}
$$

Assume now that $\Psi_{\theta}$ and $\Phi_{\theta}$ have analytic continuations into a complex neighbourhood of $\theta=0$ and continue the r.h.s analytically first in $\theta$ into the upper half-plane and then in $z$ across the continuous spectrum.

- The real eigenvalues of $H_{\theta}$ give real poles of the r.h.s. of (5) and therefore they are the eigenvalues of $H$.
- The complex eigenvalues of $H_{\theta}$ are poles of the meromorphic continuation of the l.h.s. of (5) across the spectrum of $H$ onto the second Riemann sheet.

The poles manifest themselves physically as bumps in the scattering cross-section or poles in the scattering matrix.

Exercise.* Use complex deformation in order to continue the integral $\int_{0}^{\infty} \frac{f(\omega)}{\omega-z} d \omega$ across the semi-axis $(0, \infty)$ from the upper semi-plane $\mathbb{C}^{+}$to the second Riemann sheet. Formulate conditions on $f(\omega)$ so that such a continuation is possible. (Answer to the first part: $\int_{0}^{\infty} \frac{f\left(e^{-\theta} \omega\right)}{\omega-e^{\theta} z} d \omega, \operatorname{Im} \theta>$ $0, \operatorname{Im} z<0$.

Remark. The r.h.s. of (5) has an analytic continuation into a complex neighbourhood of $\theta=0$, if $\Psi, \Phi \in \mathcal{D}$, where

$$
\begin{equation*}
\mathcal{D}:=\bigcup_{n>0, a>0} \operatorname{Ran}\left(\chi_{N \leq n} \chi_{|T| \leq a}\right) \tag{6}
\end{equation*}
$$

Here $N=\int d^{3} k a^{*}(k) a(k)$ be the photon number operator and $T$ be the self-adjoint generator of the one-parameter group $U_{\theta}, \theta \in \mathbb{R}$. It is easy to show that the set $\mathcal{D}$ is dense.

### 3.3 Resonance States as Metastable States

Let $z_{*}, \operatorname{Im} z_{*} \leq 0$, be the ground state or resonance eigenvalue. One expects that for an initial condition, $\psi_{0}$, localized in a small energy interval around the ground state or resonance energy, $\operatorname{Re} z_{*}$, the solution, $\psi=e^{-i H t} \psi_{0}$, of the time-dependent Schrödinger equation, $i \partial_{t} \psi=H \psi$, is of the form

$$
\begin{equation*}
\psi=e^{-i z_{*} t} \phi_{*}+O_{\mathrm{loc}}\left(t^{-\alpha}\right)+O_{\mathrm{res}}\left(g^{\beta}\right), \tag{7}
\end{equation*}
$$

for some $\alpha, \beta>0$ (depending on $\psi_{0}$ ), where

- $\phi_{*}$ is either the ground state or an excited state of the unperturbed system, depending on whether $z_{*}$ is the ground state energy or a resonance eigenvalue;
- The error term $O_{\mathrm{loc}}\left(t^{-\alpha}\right)$ satisfies $\left\|(\mathbf{1}+|T|)^{-\nu} O_{\mathrm{loc}}\left(t^{-\alpha}\right)\right\| \leq C t^{-\alpha}$, where $T$ is the generator of the group $U_{\theta}$, with an appropriate $\nu>0$;
- The error term $O_{\text {res }}\left(g^{\beta}\right)$ is absent in the ground state case.
(7) implies that, for the resonance, $-\operatorname{Im} z_{*}$ gives the decay probability per unit time, and $\left(-\operatorname{Im} z_{*}\right)^{-1}$, as the life-time, of the resonance.


### 3.4 Relation between Poles and Asymptotics*

- To determine the asymptotic behaviour of solutions $e^{-i H t} \psi_{0}$ we use the formula connecting the propagator and the resolvent:

$$
e^{-i H t} f(H)=\frac{1}{\pi} \int_{-\infty}^{\infty} d \lambda f(\lambda) e^{-i \lambda t} \operatorname{Im}(H-\lambda-i 0)^{-1}
$$

- For the ground state the absolute continuity of the spectrum outside the ground state energy, or a stronger property of the limiting absorption principle, suffices to establish the result above.
- For the resonances, one performs a suitable deformation of the contour of integration to the second Riemann sheet to pick up the contribution of poles there. This works when the resonances are isolated. In the present case, they are not. This is a consequence of the infrared problem.

Hence, determining the long-time behaviour of $e^{-i H t} \psi_{0}$ is a subtle problem.

### 3.5 Comparison with Quantum Mechanics*

This situation is quite different from the one in Quantum Mechanics (e.g. Stark effect or tunneling decay) where the resonances are isolated eigenvalues of complexly deformed Hamiltonians. This makes the proof of their existence and establishing their properties, e.g. independence of $\theta$ (and, in fact, of the transformation group $U_{\theta}$ ), relatively easy. In the non-relativistic QED (and other massless theories), giving meaning of the resonance poles and proving independence of their location of $\theta$ is a rather involved matter.

### 3.6 Infrared Problem

The resonances arise from the eigenvalues of the non-interacting Hamiltonian $H_{g=0}$. The low energy spectrum of the operator $H_{0}$ consists of branches $\left[\epsilon_{i}^{(p)}, \infty\right)$ of absolutely continuous spectrum and of the eigenvalues $\epsilon_{i}^{(p)}$ 's, sitting at the continuous spectrum 'thresholds' $\epsilon_{i}^{(p)}$ 's.

The eigenvalues $\epsilon_{i}^{(p)}$ 's correspond to the eigenfunctions $\phi_{i}^{(p)} \otimes \Omega$, where $\phi_{i}^{(p)}$ are the eigenfunctions of the particle system, while $\Omega$ is the photon vacuum. The branches $\left[\epsilon_{i}^{(p)}, \infty\right)$ of absolutely continuous spectrum are associated with generalized eigenfunctions of the form $\phi_{i}^{(p)} \otimes g_{\lambda}$, where $g_{\lambda}$ are the generalized eigenfunctions of $H_{f}: H_{f} g_{\lambda}=\lambda g_{\lambda}, 0<\lambda<\infty$.

The absence of gaps between the eigenvalues and thresholds is a consequence of the fact that the photons are massless.

To address this problem we use the spectral RG. The problem here is that the leading part of the perturbation in $H$ is marginal.

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## 4 Existence of the ground and resonance states

### 4.1 Bifurcation of Eigenvalues and Resonances

Let $\epsilon_{0}^{(p)}<\epsilon_{1}^{(p)}<\ldots$ be the isolated eigenvalues of the particle Hamiltonian $H_{p}$. Stated informally what we show is

- The ground state of $\left.H\right|_{g=0} \Rightarrow$ the ground state $H\left(\epsilon_{0}=\epsilon_{0}^{(p)}+O\left(g^{2}\right)\right.$ and $\left.\epsilon_{0}<\epsilon_{0}^{(p)}\right)$;
- The excited states of $\left.H\right|_{g=0} \Rightarrow$ (generically) the resonances of $H\left(\epsilon_{j, k}=\epsilon_{j}^{(p)}+O\left(g^{2}\right)\right)$;
- There is $\Sigma>\inf \sigma(H)$ (the ionization threshold) s.t. for energies $<\Sigma$ that particles are exponentially localized around the common center of mass.

For energies $>\Sigma$ the system either sheds off the excess of energy and descends into a localized state or some breaks apart with some of the particles flying off to infinity.

To formulate this result more precisely, denote

$$
\epsilon_{g a p}^{(p)}(\nu):=\min \left\{\left|\epsilon_{i}^{(p)}-\epsilon_{j}^{(p)}\right| \mid i \neq j, \epsilon_{i}^{(p)}, \epsilon_{j}^{(p)} \leq \nu\right\}
$$

Theorem (Fate of particle bound states) Fix $e_{0}^{(p)}<\nu<\inf \sigma_{e s s}\left(H_{p}\right)$ and let $g \ll \epsilon_{g a p}^{(p)}(\nu)$. Then for $g \neq 0$,

- $H$ has a ground state, originating from a ground state of $\left.H\right|_{g=0}\left(\epsilon_{0}=\epsilon_{0}^{(p)}+O\left(g^{2}\right), \epsilon_{0}<\epsilon_{0}^{(p)}\right)$;
- Generically, $H$ has no other bound state (besides the ground state);
- Eigenvalues, $\epsilon_{j}^{(p)}<\nu, j \neq 0$, of $\left.H\right|_{g=0} \Longrightarrow$ resonance eigenvalues, $\epsilon_{j, k}$, of $H$;
- $\epsilon_{j, k}=\epsilon_{j}^{(p)}+O\left(g^{2}\right)$ and the total multiplicity of $\epsilon_{j, k}$ equals the multiplicity of $\epsilon_{j}^{(p)}$;
- There is $\Sigma>\inf \sigma(H)$ (the ionization threshold) s.t. for any energy interval in $\Delta \subset(\inf \sigma(H), \Sigma)$,

$$
\left\|e^{\delta|x|} \psi\right\|<\infty, \forall \psi \in \operatorname{Ran} E_{\Delta}(H), \delta<\Sigma-\sup \Delta
$$

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Remark. The relation $\epsilon_{0}<\epsilon_{0}^{(p)}$ is due to the fact that the electron surrounded by clouds of photons become heavier.

### 4.2 Meromorphic Continuation Across Spectrum

Theorem. (Meromorphic continuation of the matrix elements of the resolvent) Let $\epsilon_{0}:=\inf \sigma\left(H_{g}\right)$ be the ground state energy of $H_{g}$. Assume $g \ll \epsilon_{g a p}^{(p)}(\nu)$. Then

- For a dense set (defined in (10) below) of vectors $\Psi$ and $\Phi$, the matrix elements

$$
F(z, \Psi, \Phi):=\left\langle\Psi,(H-z)^{-1} \Phi\right\rangle
$$

have meromorphic continuations from $\mathbb{C}^{+}$across the interval $\left(\epsilon_{0}, \nu\right) \subset \sigma_{\text {ess }}(H)$ into

$$
\left\{z \in \mathbb{C}^{-} \mid \epsilon_{0}<\operatorname{Re} z<\nu\right\} / \bigcup_{0 \leq j \leq j(\nu)} S_{j, k}
$$

where $S_{j, k}$ are the wedges starting at the resonances

$$
\begin{equation*}
S_{j, k}:=\left\{z \in \mathbb{C}\left|\frac{1}{2} \operatorname{Re}\left(e^{\theta}\left(z-\epsilon_{j, k}\right)\right) \geq\left|\operatorname{Im}\left(e^{\theta}\left(z-\epsilon_{j, k}\right)\right)\right|\right\}\right. \tag{8}
\end{equation*}
$$

- This continuation has poles at $\epsilon_{j, k}: \lim _{z \rightarrow \epsilon_{j, k}}\left(\epsilon_{j, k}-z\right) F(z, \Psi, \Phi)$ is finite and $\neq 0$.


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### 4.3 Resonance Poles*

Can we make sense of the resonance poles in the present context? Let

$$
Q:=\left\{z \in \mathbb{C}^{-} \mid \epsilon_{0}<\operatorname{Re} z<\nu\right\} / \bigcup_{j \leq j(\nu), k} S_{j, k}
$$

Theorem. For each $\Psi$ and $\Phi$ from a dense set of vectors, the meromorphic continuation, $F(z, \Psi, \Phi)$, of the matrix element $\left\langle\Psi,(H-z)^{-1} \Phi\right\rangle$ is of the following form near the resonance $\epsilon_{j}$ of $H$ :

$$
\begin{equation*}
F(z, \Psi, \Phi)=\left(\epsilon_{j, k}-z\right)^{-1} p(\Psi, \Phi)+r(z, \Psi, \Phi) \tag{9}
\end{equation*}
$$

Here $p$ and $r(z)$ are sesquilinear forms in $\Psi$ and $\Phi$, s.t.

- $r(z)$ is analytic in $Q$ and bounded on the intersection of a neighbourhood of $\epsilon_{j, k}$ with $Q$ as

$$
|r(z, \Psi, \Phi)| \leq C_{\Psi, \Phi}\left|\epsilon_{j, k}-z\right|^{-\gamma} \text { for some } \gamma<1
$$

- $p \neq 0$ at least for one pair of vectors $\Psi$ and $\Phi$ and $p=0$ for a dense set of vectors $\Psi$ and $\Phi$ in a finite co-dimension subspace.


### 4.4 Local Decay?

### 4.5 Analyticity?

### 4.6 Discussion*

- Generically, excited states turn into the resonances, not bound states. is generically satisfied.
- The second theorem implies the absolute continuity of the spectrum and its proof gives also the limiting absorption.
- The meromorphic continuation in question is constructed in terms of matrix elements of the resolvent of a complex deformation, $H_{\theta}, \operatorname{Im} \theta>0$, of the Hamiltonian $H$.
- The proof of first theorem gives fast convergent expressions in the coupling constant $g$ for the ground state energy and resonances.
- The dense set mentioned in the second theorem is

$$
\begin{equation*}
\mathcal{D}:=\bigcup_{n>0, a>0} \operatorname{Ran}\left(\chi_{N \leq n} \chi_{|T| \leq a}\right), \tag{10}
\end{equation*}
$$

where $N=\int d^{3} k a^{*}(k) a(k)$ be the photon number operator and $T$ be the self-adjoint generator of the one-parameter group $U_{\theta}, \theta \in \mathbb{R}$. (It is dense, since $N$ and $T$ commute.)

## 5 Analysis of QED Hamiltonian

We want to understand the spectral structure of the quantum Hamiltonian

$$
H=\sum_{j=1}^{n} \frac{1}{2 m_{j}}\left(i \nabla_{x_{j}}-g A\left(x_{j}\right)\right)^{2}+V(x)+H_{f}
$$

The main steps in our analysis are:

- Perform a new canonical transformation (a generalized Pauli-Fierz transform)

$$
H \rightarrow e^{-i g F} H e^{i g F}
$$

- Apply the spectral RG on new - momentum anisotropic - Banach spaces.


### 5.1 Generalized Pauli-Fierz transformation

for simplicity, consider one particle of mass 1. We define the generalized Pauli-Fierz transformation as:

$$
\begin{equation*}
H^{P F}:=e^{-i g F} H e^{i g F}, \tag{11}
\end{equation*}
$$

where $F(x)$ is the self-adjoint operator given by

$$
\begin{equation*}
F(x)=\sum_{\lambda} \int\left(\bar{f}_{x, \lambda}(k) a_{\lambda}(k)+f_{x, \lambda}(k) a_{\lambda}^{*}(k)\right) \frac{\chi(k) d^{3} k}{\sqrt{|k|}} \tag{12}
\end{equation*}
$$

with the coupling function $f_{x, \lambda}(k)$ chosen as

$$
\begin{equation*}
f_{x, \lambda}(k):=e^{-i k x} \frac{\varphi\left(|k|^{\frac{1}{2}} e_{\lambda}(k) \cdot x\right)}{\sqrt{|k|}} \tag{13}
\end{equation*}
$$

with $\varphi \in C^{2}$ and $\varphi^{\prime}(0)=1$.
The standard Pauli-Fierz transformation: $\varphi(s)=s$.

### 5.2 Generalized Pauli-Fierz Hamiltonian

The Hamiltonian $H^{P F}$ is of the same form as $H$. Indeed, compute:

$$
\begin{equation*}
H^{P F}=\frac{1}{2}\left(p+g A_{1}(x)\right)^{2}+V_{g}(x)+H_{f}+g G(x) \tag{14}
\end{equation*}
$$

where

$$
A_{1}(x)=\sum_{\lambda} \int\left(\bar{\chi}_{x, \lambda}(k) a_{\lambda}(k)+\chi_{x, \lambda}(k) a_{\lambda}^{*}(k)\right) \frac{d^{3} k}{\sqrt{|k|}},
$$

with the new coupling function $\chi_{\lambda, x}(k):=\left(e_{\lambda}(k) e^{-i k x}-\nabla_{x} f_{x, \lambda}(k)\right) \chi(k)$ and

$$
\begin{gathered}
V_{g}(x):=V(x)+2 g^{2} \sum_{\lambda} \int|k|\left|f_{x, \lambda}(k)\right|^{2} d^{3} k, \\
G(x):=-i \sum_{\lambda} \int|k|\left(\bar{f}_{x, \lambda}(k) a_{\lambda}(k)-f_{x, \lambda}(k) a_{\lambda}^{*}(k)\right) \frac{d^{3} k}{\sqrt{|k|}} .
\end{gathered}
$$

The potential $V_{g}(x)$ is a small perturbation of $V(x)$ and the operator $G(x)$ is easy to control. The new coupling function has better infrared behaviour for bounded $|x|$ :

$$
\begin{equation*}
\left|\chi_{\lambda, x}(k)\right| \leq \text { const } \min (1, \sqrt{|k|}\langle x\rangle) \tag{15}
\end{equation*}
$$

Remark.* The formula (14) can be obtained by using the commutator expansion

$$
e^{-i g F(x)} H_{f} e^{i g F(x)}=-i g\left[F, H_{f}\right]-g^{2}\left[F,\left[F, H_{f}\right]\right] .
$$

## 6 Renormalization Group

To find the spectral structure of $H_{\theta}$ we use the spectral renormalization group (RG):

- Pass from a single operator $H_{\theta}$ to a Banach space $\mathcal{B}$ of Hamiltonian-type operators;
- Construct a map, $\mathcal{R}_{\rho}$, (RG transformation) on $\mathcal{B}$, with the following properties:
(a) $\mathcal{R}_{\rho}$ is 'isospectral';
(b) $\mathcal{R}_{\rho}$ removes the photon degrees of freedom related to energies $\geq \rho$.
- Relate the dynamics of semi-flow, $\mathcal{R}_{\rho}^{n}, n \geq 1$, (called renormalization group) to spectral properties of individual operators in $\mathcal{B}$.


### 6.1 RG Map

The renormalization map is defined on Hamiltonians acting on $\mathcal{H}_{f}$ which as follows

$$
\begin{equation*}
\mathcal{R}_{\rho}=\rho^{-1} S_{\rho} \circ F_{\rho} \tag{16}
\end{equation*}
$$

where $\rho>0, S_{\rho}: \mathcal{B}[\mathcal{H}] \rightarrow \mathcal{B}[\mathcal{H}]$ is the scaling transformation:

$$
\begin{equation*}
S_{\rho}(\mathbf{1}):=\mathbf{1}, \quad S_{\rho}\left(a^{\#}(k)\right):=\rho^{-d / 2} a^{\#}\left(\rho^{-1} k\right) \tag{17}
\end{equation*}
$$

and $F_{\rho}$ is the (smooth) Feshbach-Schur map, or decimation, map,

$$
\begin{equation*}
F_{\rho}(H):=\chi_{\rho}\left(H-H \bar{\chi}_{\rho}\left(\bar{\chi}_{\rho} H \bar{\chi}_{\rho}\right)^{-1} \bar{\chi}_{\rho} H\right) \chi_{\rho} \tag{18}
\end{equation*}
$$

where $\chi_{\rho}$ and $\bar{\chi}_{\rho}$ is a pair of orthogonal projections, defined as

$$
\chi_{\rho}=\chi_{H_{p}=e_{j}} \otimes \chi_{H_{f} \leq \rho} \quad \text { and } \quad \bar{\chi}_{\rho}:=\mathbf{1}-\chi_{\rho}
$$

Remark. The construction of the map $F_{\rho}$ can be generalized to 'smooth' projections which form a partition of unity $\chi_{\rho}^{2}+\bar{\chi}_{\rho}^{2}=\mathbf{1}$.

### 6.2 Isospectrality of $F_{\rho}$

The map $F_{\rho}$ is isospectral in the sense of the following theorem:
Theorem 6.1. (i) $\lambda \in \rho(H) \Leftrightarrow 0 \in \rho\left(F_{\rho}(H-\lambda)\right)$;
(ii) $H \psi=\lambda \psi \Longleftrightarrow F_{\rho}(H-\lambda) \varphi=0$;
(iii) $\operatorname{dim} \operatorname{Ker}(H-\lambda)=\operatorname{dim} \operatorname{Ker} F_{\rho}(H-\lambda)$;
(iv) $(H-\lambda)^{-1}$ exists $\Leftrightarrow F_{\rho}(H-\lambda)^{-1}$ exists.

For the proof of this theorem as well as for the relation between $\psi$ and $\varphi$ in (ii) and between $(H-\lambda)^{-1}$ and $F_{\rho}(H-\lambda)^{-1}$ in (iv) see Appendix C.

## 7 A Banach Space of Hamiltonians

Operators on the subspace Ran $\chi_{1}$ of the Fock space $\mathcal{F}$ are said to be in the generalized normal form if they can be written as:

$$
\begin{equation*}
H=\sum_{m+n \geq 0} \int_{B_{1}^{m+n}} \prod_{i=1}^{m+n} d^{3} k_{i} \prod_{i=1}^{m} a^{*}\left(k_{i}\right) w_{m, n}\left(H_{f} ; k_{(m+n)}\right) \prod_{i=m+1}^{m+n} a\left(k_{i}\right) \tag{19}
\end{equation*}
$$

where $B_{1}^{r}$ denotes the cartesian product of $r$ unit balls in $\mathbb{R}^{3}$ and $k_{(m)}:=\left(k_{1}, \ldots, k_{m}\right)$.
We assume that the functions $w_{m, n}\left(r, k_{(m+n)}\right) \in C^{2}$ are symmetric w. r. t. the variables $\left(k_{1}, \ldots, k_{m}\right)$ and $\left(k_{m+1}, \ldots, k_{m+n}\right)$ and obey

$$
\begin{equation*}
\left\|w_{m, n}\right\|_{\mu}:=\left.\max _{j} \sup _{r \in I, k_{(m+n)} \in B_{1}^{m+n}}| | k_{j}\right|^{-\mu} \prod_{i=1}^{m+n}\left|k_{i}\right|^{1 / 2} w_{m, n}\left(r ; k_{(m+n)}\right) \mid \tag{20}
\end{equation*}
$$

where $\mu \geq-1 / 2$ and $I:=[0,1]$. (We write $H=\sum_{m+n \geq 0} H_{m, n}$.) For $\mu \geq 0$ and $0<\xi<1$ we define the Banach space

$$
\begin{equation*}
\mathcal{B}^{\mu \xi}:=\left\{H:\|H\|_{\mu, \xi}:=\sum_{m+n \geq 0} \xi^{-(m+n)}\left\|w_{m, n}\right\|_{\mu}<\infty\right\} \tag{21}
\end{equation*}
$$

### 7.1 Basic Bound

The following bound shows that our Banach space norm control the operator norm:
Theorem 7.1. Let $\chi_{\rho} \equiv \chi_{H_{f} \leq \rho}$. Then for all $\rho>0$ and $m+n \geq 1$

$$
\begin{equation*}
\left\|\chi_{\rho} H_{m, n} \chi_{\rho}\right\| \leq \frac{\rho^{m+n+\mu}}{\sqrt{m!n!}}\left\|w_{m, n}\right\|_{\mu} \tag{22}
\end{equation*}
$$

This shows that the terms with higher numbers of creation and annihilation operators are easier to control.

### 7.2 Sketch of Proof of Basic Bound*

For simplicity we prove this inequality for $m=n=1$. We have

$$
\left\langle\chi_{\rho} \phi, H_{1,1} \chi_{\rho} \phi\right\rangle=\int_{B_{1}^{2}} \prod_{i=1}^{2} d^{3} k_{i}\left\langle\Phi_{k_{1}}, w_{1,1}\left(H_{f} ; k_{1}, k_{2}\right) \Phi_{k_{2}}\right\rangle
$$

where $\Phi_{k}=a(k) \chi_{\rho} \phi$. Now we write $\chi_{\rho}=\chi_{\rho} \chi_{2 \rho}$ and pull $\chi_{2 \rho}$ toward $w_{1,1}\left(H_{f} ; k_{1}, k_{2}\right)$ using

$$
a(k) F\left(H_{f}\right)=F\left(H_{f}+|k|\right) a(k), F\left(H_{f}\right) a^{*}(k)=a^{*}(k) F\left(H_{f}+|k|\right)
$$

(the pull-trough formulae). This gives

$$
\begin{aligned}
& \left|\left\langle\phi, \chi_{\rho} H_{1,1} \chi_{\rho} \phi\right\rangle\right| \\
= & \left|\int_{\left|k_{i}\right| \leq 2 \rho, i=1,2} \prod_{i=1}^{2} d^{3} k_{i}\left\langle\Phi_{k_{1}}, \chi_{2 \rho-\left|k_{1}\right|} w_{1,1}\left(H_{f} ; k_{1}, k_{2}\right) \chi_{2 \rho-\left|k_{2}\right|} \Phi_{k_{2}}\right\rangle\right| \\
\leq & \int_{\left|k_{i}\right| \leq 2 \rho, i=1,2} \prod_{i=1}^{2} d^{3} k_{i}\left\|\Phi_{k_{1}}\right\|\left\|w_{1,1}\left(H_{f} ; k_{1}, k_{2}\right)\right\|\left\|\Phi_{k_{2}}\right\| \\
\leq & \left(\int_{\left|k_{i}\right| \leq 2 \rho, i=1,2} \prod_{i=1} d^{3} k_{i} \frac{\left\|w_{1,1}\left(H_{f} ; k_{1}, k_{2}\right)\right\|^{2}}{\left|k_{1}\right|\left|k_{2}\right|}\right)^{1 / 2} \int d^{3} k\left\|\sqrt{|k|} \Phi_{k}\right\|^{2}
\end{aligned}
$$

Now, using $\left\|w_{1,1}\left(H_{f} ; k_{1}, k_{2}\right)\right\| \leq\left\|w_{1,1}\right\|_{\mu} \frac{\left|k_{1}\right|^{\mu}+\left|k_{2}\right|^{\mu}}{\left|k_{1}\right|^{\frac{1}{2}}\left|k_{2}\right|^{\frac{1}{2}}}$ and $\int d^{3} k\left\|\sqrt{|k|} \Phi_{k}\right\|^{2}=\left\|\sqrt{H_{f}} \chi_{\rho} \phi\right\|^{2}$, we find

$$
\left|\left\langle\phi, \chi_{\rho} H_{1,1} \chi_{\rho} \phi\right\rangle\right| \leq \rho^{2+\mu}\left\|w_{1,1}\right\|_{\mu}
$$

### 7.3 Unstable, Neutral and Stable Components

Decompose $H \in \mathcal{B}^{\mu \xi}$ as

$$
\begin{equation*}
H=E \mathbf{1}+T+W \tag{23}
\end{equation*}
$$

where

$$
E:=\langle\Omega, H \Omega\rangle, T:=H_{0,0}-\langle\Omega, H \Omega\rangle \sim H_{f}, W:=\sum_{m+n \geq 1} H_{m, n}
$$

These terms of the Hamiltonians scale as follows

- $\rho^{-1} S_{\rho}\left(H_{f}\right)=H_{f}\left(H_{f}\right.$ is a fixed point of $\left.\rho^{-1} S_{\rho}\right)$;
- $\rho^{-1} S_{\rho}(E \cdot \mathbf{1})=\rho^{-1} E \cdot \mathbf{1}\left(E \cdot \mathbf{1}\right.$ expand under $\rho^{-1} S_{\rho}$ at a rate $\left.\rho^{-1}\right)$;
- $\left\|S_{\rho}\left(W_{m, n}\right)\right\|_{\mu} \leq \rho^{\alpha}\left\|w_{m, n}\right\|_{\mu}, \alpha:=m+n-1+\mu \delta_{m+n=1}\left(W_{m n}\right.$ contract under $\rho^{-1} S_{\rho}$, if $\mu>0)$.
Thus for $\mu>0, E, T, W$ behave as relevant, marginal, and irrelevant operators, respectively. For $\mu=0$, the operators $W_{m n}, m+n=1$, become marginal.


## 8 Action of Renormalization Map

To control these components we introduce, for $\alpha, \beta, \gamma>0$, the following polydisc:

$$
\begin{aligned}
\mathcal{D}^{\mu}(\alpha, \beta, \gamma) \quad & :=\left\{H=E+T+W \in \mathcal{W}^{\mu}| | E \mid \leq \alpha\right. \\
& \left.\sup _{r \in[0, \infty)}\left|T^{\prime}(r)-1\right| \leq \beta,\|W\|_{\mu, \xi} \leq \gamma\right\}
\end{aligned}
$$

Theorem 8.1. Let $0<\rho<1 / 2, \alpha, \beta, \gamma \leq \rho / 8$ and $\mu_{0}=1 / 2$. Then there is $c>0$, s.t.

- $\mathcal{R}_{\rho}\left(H_{\theta}\right) \in \mathcal{D}^{\mu_{0}}\left(\alpha_{0}, \beta_{0}, \gamma_{0}\right), \alpha_{0}=O\left(g^{2} \rho^{\mu_{0}-2}\right), \beta_{0}=O\left(g^{2} \rho^{\mu_{0}-1}\right), \gamma_{0}=O\left(g \rho_{0}^{\mu}\right)$, provided $g \ll 1$;
- $\mathcal{D}^{\mu}(\alpha, \beta, \gamma) \subset D\left(\mathcal{R}_{\rho}\right)$, provided $\mu>0$;
- $\mathcal{R}_{\rho}: \mathcal{D}^{\mu}(\alpha, \beta, \gamma) \rightarrow \mathcal{D}^{\mu}\left(\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}\right)$,
continuously, with
$\alpha^{\prime}=\rho^{-1} \alpha+c\left(\gamma^{2} / 2 \rho\right), \beta^{\prime}=\beta+c\left(\gamma^{2} / 2 \rho\right), \gamma^{\prime}=c \rho^{\mu} \gamma$.


### 8.1 Idea of Proof of $\mathcal{D}^{\mu}(\rho / 8,1 / 8, \rho / 8) \subset D\left(\mathcal{R}_{\rho}\right)^{*}$

Since $W:=H-E-T$ defines a bounded operator on $\mathcal{F}$, we only need to check the invertibility of $H_{\tau \chi_{\rho}}$ on $\operatorname{Ran} \bar{\chi}_{\rho}$. The operator $E+T$ is invertible on $\operatorname{Ran} \bar{\chi}_{\rho}$ : for all $r \in[3 \rho / 4, \infty)$

$$
\begin{aligned}
\operatorname{Re} T(r)+\operatorname{Re} E & \geq r-|T(r)-r|-|E| \\
& \geq r\left(1-\sup _{r}\left|T^{\prime}(r)-1\right|\right)-|E| \\
& \geq \frac{3 \rho}{4}(1-1 / 8)-\frac{\rho}{8} \geq \frac{\rho}{2} \\
& \Rightarrow E+T \text { is invertible and }\left\|(E+T)^{-1}\right\| \leq 2 / \rho
\end{aligned}
$$

Now, by the basic estimate, $\|W\| \leq \rho / 8$ and therefore,

$$
\begin{aligned}
& \left\|\bar{\chi}_{\rho} W \bar{\chi}_{\rho}(E+T)^{-1}\right\| \leq 1 / 4 \\
\Rightarrow & E+T+\bar{\chi}_{\rho} W \bar{\chi}_{\rho} \text { is invertible on } \operatorname{Ran} \bar{\chi}_{\rho} \\
\Rightarrow & \mathcal{D}^{\mu}(\rho / 8,1 / 8, \rho / 8) \subset D\left(F_{\rho}\right)=D\left(\mathcal{R}_{\rho}\right) .
\end{aligned}
$$

### 8.2 Sketch of Proof of $\mathcal{R}_{\rho}: \mathcal{D}^{\mu, s}(\alpha, \beta, \gamma) \rightarrow \mathcal{D}^{\mu, s}\left(\alpha^{\prime}, \beta^{\prime}, \gamma^{\prime}\right)^{*}$

Normal form of $\mathcal{R}_{\rho}(H)$. Recall that $\chi_{\rho} \equiv \chi_{H_{f} \leq \rho}$ and $\bar{\chi}_{\rho}:=1-\chi_{\rho}$. Let $H_{0}:=E+T$, so that $H=H_{0}+W$. We have shown above

$$
\left\|H_{0}^{-1} \bar{\chi}_{\rho}\right\| \leq \frac{2}{\rho} \quad \text { and } \quad\|W\| \leq \frac{\rho}{8}
$$

In the Feshbach-Schur map, $F_{\rho}$,

$$
F_{\rho}(H)=\chi_{\rho}\left(H_{0}+W-W \bar{\chi}_{\rho}\left(\bar{\chi}_{\rho}\left(H_{0}+W\right) \bar{\chi}_{\rho}\right)^{-1} \bar{\chi}_{\rho} W\right) \chi_{\rho}
$$

we expand the resolvent $\left(\bar{\chi}_{\rho}\left(H_{0}+W\right) \bar{\chi}_{\rho}\right)^{-1}$ in the norm convergent Neumann series

$$
F_{\rho}(H)=\chi_{\rho}\left(H_{0}+\sum_{s=0}^{\infty}(-1)^{s} W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}\right) \chi_{\rho}
$$

Next, we transform the right side to the generalized normal form using generalized Wick's theorem.
Generalized Wick's Theorem. To write the product $W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}$ in the generalized normal form we pull the annihilation operators, $a$, to the right and the creation operators, $a^{*}$, to the left, apart from those which enter $H_{f}$. We use the rules:

$$
a(k) a^{*}\left(k^{\prime}\right)=a^{*}\left(k^{\prime}\right) a(k)+\delta\left(k-k^{\prime}\right)
$$

$$
a(k) F\left(H_{f}\right)=F\left(H_{f}+|k|\right) a(k), F\left(H_{f}\right) a^{*}(k)=a^{*}(k) F\left(H_{f}+|k|\right) .
$$

Some of the creation and annihilation operators reach the extreme left and right positions, while the remaining ones contract. The terms with $m$ creation operators on the left and $n$ annihilation operators on the right contribute to the $(m, n)$ - formfactor, $w_{m, n}^{(s)}$, of the operator $W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}$. As the result we obtain the generalized normal form of $F_{\rho}(H)$ :

$$
F_{\rho}(H)=\sum_{m+n \geq 0} W_{m, n}^{\prime} .
$$

The term $W_{0,0}^{\prime}=\left\langle W_{0,0}^{(s)}\right\rangle_{\Omega}+\left(W_{0,0}^{(s)}-\left\langle W_{0,0}^{(s)}\right\rangle_{\Omega}\right)$ contributes the corrections to $E+T$.
This is the standard way for proving the Wick theorem, taking into account the presence of $H_{f}-$ dependent factors. (See [11] for a different, more formal proof.)

Estimating Formfactors. The problem here is that the number of terms generated by various contractions is $O(s!)$. Therefore a simple majoration of the series for the $(m, n)-$ formfactor, $w_{m, n}^{(s)}$, of the operator $W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}$ will diverge badly.

To overcome this we re-sum the series by, roughly, representing the sum over all contractions, for a given $m$ and $n$, as

$$
w_{m, n}^{(s)} \sim\left\langle\Omega,\left[W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}\right]_{m, n} \Omega\right\rangle
$$

where $\left[W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}\right]_{m, n}$ is $W\left(H_{0}^{-1} \bar{\chi}_{\rho}^{2} W\right)^{s}$, with $m$ escaping creation operators and $n$ escaping annihilation operators deleted. Now the estimate of $w_{m, n}^{(s)}$ is straightforward:

$$
\left\|w_{m, n}^{(s)}\right\| \lesssim\left\|\chi_{\rho^{\prime}} W^{\prime} \chi_{\rho^{\prime}}\right\|^{s+1}
$$

(symbolically), for the operator norm, and similarly for $\mathcal{B}^{\mu, \xi}$ - norm.

## PICTURE

## 9 Renormalization Group

### 9.1 Iteration of $\mathcal{R}_{\rho}$

To iterate $\mathcal{R}_{\rho}$ we have to control the expanding direction: $\mathcal{R}_{\rho}(\zeta \mathbf{1})=\rho^{-1} \zeta \mathbf{1}$. To control this direction, we adjust, inductively, at each step the constant component $\langle H\rangle_{\Omega}:=\langle\Omega, H \Omega\rangle$ of the initial Hamiltonian, $H$ :

$$
\left|\langle H\rangle_{\Omega}-e_{n-1}\right| \leq \frac{1}{12} \rho^{n+1}
$$

$e_{n-1}$ is the unique zero of the function $\left\langle\mathcal{R}_{\rho}^{n-1}\left(H-\langle H\rangle_{\Omega}+\lambda\right)\right\rangle_{\Omega}$,
so that

$$
H \in \quad \text { the domain of } \quad \mathcal{R}_{\rho}^{n} .
$$

### 9.2 RG Dynamics

The results outlined above can be reformulated as $\mathcal{R}_{\rho}^{n}$ has

- the fixed-point manifold $\mathcal{M}_{f p}:=\mathbb{C} H_{f}$,
- an unstable manifold $\mathcal{M}_{u}:=\mathbb{C} 1$,
- a (complex) co-dimension 1 stable manifold $\mathcal{M}_{s}$ for $\mathcal{M}_{f p}$ foliated by (complex) co-dimension 2 stable manifolds for each fixed point.


## PICTU RE

Stable and unstable manifolds.

### 9.3 RG and Spectral Properties

Adjust the parameter $\langle\Omega, H \Omega\rangle$ iteratively, so that $H \in D\left(\mathcal{R}_{\rho}^{n}\right) \Longrightarrow H^{(n)}:=\mathcal{R}_{\rho}^{n}(H)$ :

$$
H \Longrightarrow \text { RG BOX } \Longrightarrow H^{(n)}
$$

Use that for $n$ sufficiently large, $H^{(n)}(\lambda) \approx \zeta H_{f}$, for some $\zeta \in \mathbb{C}$, $\operatorname{Re} \zeta>0$, to find spectral information about $H^{(n)}$
$\Longrightarrow$ Spectral information about $H^{(n-1)}$ (by 'isospectrality' of $\mathcal{R}_{\rho}$ )
$\Longrightarrow$ Spectral information about $H$ :

$$
\text { Spec info }(H) \Longleftarrow \mathbf{R G} \mathbf{B O X} \Longleftarrow \text { Spec } \operatorname{info}\left(H^{(n)}\right)
$$

$\Longrightarrow$ Spectral information about $H_{\theta}$.

## 10 Open Problems

Connection between the ground state and resonance eigenvalues to poles of the scattering matrix;

Minimal and maximal velocity of photons;

Asymptotic completeness;

Bohr photon frequency laws.

## 11 Comments on Literature*

These lectures follow the papers $[74,31,1]$, which in turn are based on $[11,12,5]$. The papers $[74,31,1]$, use the smooth Feshbah-Schur map ( $[5,36]$ ), which is much more powerful (see Appendix D), while in these lecture we use, for simplicity, the the original, Feshbach-Schur map (see [11, 12]), which is simpler to formulate.

Theorems 1 and 2 were proven in $[11,12,13]$ for 'confined particles' (the exact conditions are somewhat technical) and in the present form in [74].

The binding results are given in [13, 35].
The results of [13] on existence (and uniqueness) of the ground state were considerably improved in $[62,48,49,50,55,4]$ (by compactness techniques) and [7] (by multiscale techniques), with the sharpest result given in $[38]^{1}$.

[^0]Related results:
The asymptotic stability of the ground state (local decay): $[15,30,32]$.
The survival probabilities of excited states: $[13,43,1]$.
Atoms with dynamic nuclei: [24, 2, 65].
Analyticity of the ground state eigenvalues in parameters and asymptotic expansions: [13, 7, 9], [37, 44, 45, 46].
Enhanced binding: $[56,64,39,40,41]$.
Electron mass renormalization: $[40,57,63,64,6,18]$.
One particle states: $[26,19,20,33]$.
Scattering amplitudes: [8].
Scattering theory: [27, 28, 29].
Positive temperatures: [14, 66, 67].

## A ${ }^{*}$ Creation and Annihilation Operators

The Bosonic Fock space, $\mathcal{F}$, over $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)\left(\right.$ or $\left.L^{2}\left(\mathbb{R}^{3}, \mathbb{C}^{2}, d^{3} k\right)\right)$ is defined by

$$
\begin{equation*}
\mathcal{F}:=\bigoplus_{n=0}^{\infty} \mathcal{S}_{n} L^{2}\left(\mathbb{R}^{3}, \mathbb{C}^{2}, d^{3} k\right)^{\otimes n} \tag{24}
\end{equation*}
$$

and $a^{*}(k)$ and $a(k)$ denote the creation and annihilation operators on $\mathcal{F}$. The families $a^{*}(k)$ and $a(k)$ are operator-valued generalized, transverse vector fields: (below $a_{\lambda}^{\#}=a_{\lambda}$ or $a_{\lambda}^{*}$.)

$$
a^{\#}(k):=\sum_{\lambda \in\{-1,1\}} e_{\lambda}(k) a_{\lambda}^{\#}(k)
$$

where $e_{\lambda}(k)$ are polarization vectors, i.e. orthonormal vectors in $\mathbb{R}^{3}$ satisfying $k \cdot e_{\lambda}(k)=0$, and $a_{\lambda}^{\#}(k)$ are scalar creation and annihilation operators, satisfying the canonical commutation relations:

$$
\begin{equation*}
\left[a_{\lambda}^{\#}(k), a_{\lambda^{\prime}}^{\#}\left(k^{\prime}\right)\right]=0, \quad\left[a_{\lambda}(k), a_{\lambda^{\prime}}^{*}\left(k^{\prime}\right)\right]=\delta_{\lambda \lambda^{\prime}} \delta^{3}\left(k-k^{\prime}\right) \tag{25}
\end{equation*}
$$

where $\mathcal{S}_{n}$ is the orthogonal projection onto the subspace of totally symmetric $n$-particle wave functions contained in the $n$-fold tensor product $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)^{\otimes n}$ of $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)$; and $\mathcal{S}_{0} L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)^{\otimes 0}:=$ $\mathbb{C}$. The vector $\Omega:=1 \bigoplus_{n=1}^{\infty} 0$ is called the vacuum vector in $\mathcal{F}$. Vectors $\Psi \in \mathcal{F}$ can be identified with sequences $\left(\psi_{n}\right)_{n=0}^{\infty}$ of $n$-particle wave functions, $\psi_{n}\left(k_{1}, \ldots, k_{n}\right)$, which are totally symmetric in their $n$ arguments, and $\psi_{0} \in \mathbb{C}$.

The scalar product of two vectors $\Psi$ and $\Phi$ is given by

$$
\begin{equation*}
\langle\Psi, \Phi\rangle:=\sum_{n=0}^{\infty} \int \prod_{j=1}^{n} d^{3} k_{j} \overline{\psi_{n}\left(k_{1}, \ldots, k_{n}\right)} \varphi_{n}\left(k_{1}, \ldots, k_{n}\right) . \tag{26}
\end{equation*}
$$

Given a one particle dispersion relation $\omega(k)$, the energy of a configuration of $n$ non-interacting field particles with wave vectors $k_{1}, \ldots, k_{n}$ is given by $\sum_{j=1}^{n} \omega\left(k_{j}\right)$. We define the free-field Hamiltonian, $H_{f}$, giving the field dynamics, by

$$
\begin{equation*}
\left(H_{f} \Psi\right)_{n}\left(k_{1}, \ldots, k_{n}\right)=\left(\sum_{j=1}^{n} \omega\left(k_{j}\right)\right) \psi_{n}\left(k_{1}, \ldots, k_{n}\right) \tag{27}
\end{equation*}
$$

for $n \geq 1$ and $\left(H_{f} \Psi\right)_{n}=0$ for $n=0$. Here $\Psi=\left(\psi_{n}\right)_{n=0}^{\infty}$ (to be sure that the r.h.s. makes sense we can assume that $\psi_{n}=0$, except for finitely many $n$, for which $\psi_{n}\left(k_{1}, \ldots, k_{n}\right)$ decrease rapidly at
infinity). Clearly that the operator $H_{f}$ has the single eigenvalue 0 with the eigenvector $\Omega$ and the rest of the spectrum absolutely continuous.

With each function $\varphi \in L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)$ one associates an annihilation operator $a(\varphi)$ defined as follows. For $\Psi=\left(\psi_{n}\right)_{n=0}^{\infty} \in \mathcal{F}$ with the property that $\psi_{n}=0$, for all but finitely many $n$, the vector $a(\varphi) \Psi$ is defined by

$$
\begin{equation*}
(a(\varphi) \Psi)_{n}\left(k_{1}, \ldots, k_{n}\right):=\sqrt{n+1} \int d^{3} k \overline{\varphi(k)} \psi_{n+1}\left(k, k_{1}, \ldots, k_{n}\right) \tag{28}
\end{equation*}
$$

These equations define a closable operator $a(\varphi)$ whose closure is also denoted by $a(\varphi)$. Eqn (28) implies the relation

$$
\begin{equation*}
a(\varphi) \Omega=0 \tag{29}
\end{equation*}
$$

The creation operator $a^{*}(\varphi)$ is defined to be the adjoint of $a(\varphi)$ with respect to the scalar product defined in Eq. (26). Since $a(\varphi)$ is anti-linear, and $a^{*}(\varphi)$ is linear in $\varphi$, we write formally

$$
\begin{equation*}
a(\varphi)=\int d^{3} k \overline{\varphi(k)} a(k), \quad a^{*}(\varphi)=\int d^{3} k \varphi(k) a^{*}(k) \tag{30}
\end{equation*}
$$

where $a(k)$ and $a^{*}(k)$ are unbounded, operator-valued distributions. The latter are well-known to obey the canonical commutation relations (CCR):

$$
\begin{equation*}
\left[a^{\#}(k), a^{\#}\left(k^{\prime}\right)\right]=0, \quad\left[a(k), a^{*}\left(k^{\prime}\right)\right]=\delta^{3}\left(k-k^{\prime}\right) \tag{31}
\end{equation*}
$$

where $a^{\#}=a$ or $a^{*}$.
Now, using this one can rewrite the quantum Hamiltonian $H_{f}$ in terms of the creation and annihilation operators, $a$ and $a^{*}$, as

$$
\begin{equation*}
H_{f}=\int d^{3} k a^{*}(k) \omega(k) a(k) \tag{32}
\end{equation*}
$$

acting on the Fock space $\mathcal{F}$.
More generally, for any operator, $t$, on the one-particle space $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)$ we define the operator $T$ on the Fock space $\mathcal{F}$ by the following formal expression $T:=\int a^{*}(k) t a(k) d k$, where the operator $t$ acts on the $k$-variable ( $T$ is the second quantization of $t$ ). The precise meaning of the latter expression can obtained by using a basis $\left\{\phi_{j}\right\}$ in the space $L^{2}\left(\mathbb{R}^{3}, \mathbb{C}, d^{3} k\right)$ to rewrite it as $T:=$ $\sum_{j} \int a^{*}\left(\phi_{j}\right) a\left(t^{*} \phi_{j}\right) d k$.

## B *Rescaling and Conditions on Potentials

First, we consider the Hamiltonian $H_{g}$ for an atom or molecule interacting with radiation field,

$$
H(\alpha)=\sum_{j=1}^{n} \frac{1}{2 m_{j}}\left(i \nabla_{x_{j}}-\sqrt{\alpha} A_{\chi^{\prime}}\left(x_{j}\right)\right)^{2}+\alpha V(x)+H_{f}
$$

where $\alpha V(x)$ is the total Coulomb potential of the particle system, and $A_{\chi^{\prime}}(y)$ is the original vector potential with the ultraviolet cut-off $\chi^{\prime}$. Rescaling $x \rightarrow \alpha^{-1} x$ and $k \rightarrow \alpha^{2} k$ we arrive at the Hamiltonian

$$
H_{g}=\sum_{j=1}^{n} \frac{1}{2 m_{j}}\left(i \nabla_{x_{j}}-g A\left(x_{j}\right)\right)^{2}+V(x)+H_{f}
$$

where $g:=\alpha^{3 / 2}$ and $A(y)=A_{\chi}(\alpha y)$, with $\chi(k):=\chi^{\prime}\left(\alpha^{2} k\right) .{ }^{2}$ After that we relax the restriction on $V(x)$ by allowing it to be a standard generalized $n$-body potential (see below). Note that though

[^1]this is not displayed, $A(x)$ does depend on $g$. This however does not effect the analysis of the Hamiltonian $H_{g}$. (If anything, this makes certain parts of it simpler, as derivatives $A(x)$ bring down g.)

Generalized $n$-body potentials :
(V) $V(x)=\sum_{i} W_{i}\left(\pi_{i} x\right)$, where $\pi_{i}$ are a linear maps from $\mathbb{R}^{3 n}$ to $\mathbb{R}^{m_{i}}, m_{i} \leq 3 n$ and $W_{i}$ are KatoRellich potentials (i.e. $W_{i}\left(\pi_{i} x\right) \in L^{p_{i}}\left(\mathbb{R}^{m_{i}}\right)+\left(L^{\infty}\left(\mathbb{R}^{3 n}\right)\right)_{\varepsilon}$ with $p_{i}=2$ for $m_{i} \leq 3, p_{i}>2$ for $m_{i}=4$ and $p_{i} \geq m_{i} / 2$ for $\left.m_{i}>4\right)$.

Under the assumption (V), the operator $H_{g}^{S M}$ is self-adjoint. In order to tackle the resonances we choose the ultraviolet cut-off, $\chi(k)$, so that

The function $\theta \rightarrow \chi\left(e^{-\theta} k\right)$ has an analytic continuation from the real axis, $\mathbb{R}$, to the strip $\left\{\theta \in \mathbb{C}||\operatorname{Im} \theta|<\pi / 4\}\right.$ as a $L^{2} \bigcap L^{\infty}\left(\mathbb{R}^{3}\right)$ function,
e.g. $\chi(k)=e^{-|k|^{2} / \kappa^{2}}$. Furthermore, we assume that the potential, $V(x)$, satisfies the condition:
(DA) The the particle potential $V(x)$ is dilation analytic in the sense that the operator-function $\theta \rightarrow V\left(e^{\theta} x\right)(-\Delta+1)^{-1}$ has an analytic continuation from the real axis, $\mathbb{R}$, to the strip $\left\{\theta \in \mathbb{C}\left||\operatorname{Im} \theta|<\theta_{0}\right\}\right.$ for some $\theta_{0}>0$.
$* * * *$
In order not to deal with the problem of center-of-mass motion which is not essential in the present context, we assume that either some of the particles (nuclei) are infinitely heavy or the system is placed in a binding, external potential field. This means that the operator $H_{p}$ has isolated eigenvalues below its essential spectrum. However, we expect that the techniques we discuss here can be extended to translationally invariant particle systems.

## C *Proof of Theorem 6.1

In this appendix we omit the subindex $\rho$ at $\chi_{\rho}$ and $\bar{\chi}_{\rho}$, and replace the subindex $\rho$ in other operators by the subindex $\chi$. Moreover, we replace $H-\lambda$ by $H$. Though $\chi$ and $\bar{\chi}$ we deal with are projections, we often keep the powers $\chi^{2}$ and $\bar{\chi}^{2}$, which occur often below, having in mind showing possible generalization to $\chi$ and $\bar{\chi}$ which are 'almost (or smooth) projections' satisfying $\chi^{2}+\bar{\chi}^{2}=1$ (see Appendix D).

First we note that the relation between $\psi$ and $\varphi$ in Theorem 6.1 (ii) is $\varphi=\chi \psi, \psi=Q_{\chi}(H) \varphi$, and between $H^{-1}$ and $F_{\chi}(H)^{-1}$ in (iv) is

$$
\begin{equation*}
H^{-1}=Q_{\chi}(H) F_{\chi} H^{-1} Q_{\chi}(H)^{\#}+\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} \tag{33}
\end{equation*}
$$

where $H_{\bar{\chi}}:=\bar{\chi}_{\rho} H \bar{\chi}_{\chi}$ and $Q_{\chi}(H)$ and $Q_{\chi}(H)^{\#}$ are the operators, given by

$$
Q_{\chi}(H):=\chi-\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} H \chi
$$

$$
Q_{\chi}^{\#}(H):=\chi-\chi H \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi}
$$

Proof of Theorem 6.1. Throughout the proof we use the notation $F:=F_{\chi}(H), Q:=Q_{\chi}(H)$, and $Q^{\#}:=Q_{\chi}^{\#}(H)$. Note that (i) $\left(0 \in \rho(H) \Leftrightarrow 0 \in \rho\left(F_{\chi}(H)\right)\right.$ ) follows from (iv) ( $H^{-1}$ exists $\Leftrightarrow$ $F_{\chi}(H)^{-1}$ exists) and (iv) follows from (33), so we start with the latter.
Proof of (33). We observe the relations

$$
\begin{equation*}
H \chi=\chi H_{\chi}+\bar{\chi}^{2} H \chi, \quad \text { and } \quad H \bar{\chi}=\bar{\chi} H_{\bar{\chi}}+\chi^{2} H \bar{\chi} \tag{34}
\end{equation*}
$$

which follow from $\chi^{2}+\bar{\chi}^{2}=\mathbf{1}$. The next two identities,

$$
\begin{equation*}
H Q=\chi F \quad \text { and } \quad Q^{\#} H=F \chi \tag{35}
\end{equation*}
$$

are of key importance in the proof. They both derive from a simple computation, which we give only for the first equality in (35),

$$
\begin{align*}
H Q & =H \chi-H \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} H \chi \\
& =\chi H_{\chi}+\bar{\chi}^{2} H \chi-\left(\bar{\chi} H_{\bar{\chi}}+\chi^{2} H \bar{\chi}\right) H_{\bar{\chi}}^{-1} \bar{\chi} H \chi \\
& =\chi F . \tag{36}
\end{align*}
$$

Suppose first that the operator $F$ has bounded invertible and define

$$
\begin{equation*}
R:=Q F^{-1} Q^{\#}+\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} \tag{37}
\end{equation*}
$$

Using (35) and (34), we obtain

$$
\begin{align*}
H R & =H Q F^{-1} Q^{\#}+\left(\bar{\chi} H_{\bar{\chi}}+\chi^{2} H \bar{\chi}\right) H_{\bar{\chi}}^{-1} \bar{\chi}  \tag{38}\\
& =\chi Q^{\#}+\bar{\chi}^{2}+\chi^{2} H \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} \\
& =\chi^{2}+\bar{\chi}^{2}=\mathbf{1}
\end{align*}
$$

and, similarly, $R H=1$. Thus $R=H^{-1}$, and (33) holds true.
Conversely, suppose that $H$ is bounded invertible. Then, using the definition of $F$ and the relation $\chi^{2}+\bar{\chi}^{2}=1$, we obtain

$$
\begin{align*}
F \chi H^{-1} \chi & =\chi H \chi^{2} H^{-1} \chi-\chi H \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} H \chi^{2} H^{-1} \chi  \tag{39}\\
& =\chi H \chi^{2} H^{-1} \chi-\chi H \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} H H^{-1} \chi+\chi H \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} H \bar{\chi}^{2} H^{-1} \chi \\
& =\chi H \chi^{2} H^{-1} \chi+\chi H \bar{\chi}^{2} H^{-1} \chi \\
& =\chi^{2}
\end{align*}
$$

Similarly, one checks that $\chi H^{-1} \chi F=\mathbf{1}$. Thus $F$ is invertible on $\operatorname{Ran} \chi$ with inverse $F^{-1}=\chi H^{-1} \chi$. Proof of (ii) $\left(H \psi=\lambda \psi \Longleftrightarrow F_{\rho}(H-\lambda) \varphi=0\right)$. If $\psi \in \mathcal{H} \backslash\{0\}$ solves $H \psi=0$ then (35) implies that

$$
\begin{equation*}
F \chi \psi=Q^{\#} H \psi=0 \tag{40}
\end{equation*}
$$

Furthermore, by (34), $0=\bar{\chi} H \psi=H_{\bar{\chi}} \bar{\chi} \psi+\bar{\chi} H \chi^{2} \psi$, and hence

$$
\begin{equation*}
Q \chi \psi=\chi^{2} \psi-\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} H \chi^{2} \psi=\chi^{2} \psi+\bar{\chi}^{2} \psi=\psi \tag{41}
\end{equation*}
$$

Therefore, $\psi \neq 0$ implies $\chi \psi \neq 0$.
If $\varphi \in \operatorname{Ran} \chi \backslash\{0\}$ solves $F \varphi=0$ then the definition of $Q$ implies that

$$
\begin{equation*}
\chi Q \varphi=\chi \varphi=\varphi \tag{42}
\end{equation*}
$$

which implies that $Q \varphi \neq 0$ provided $\varphi \neq 0$.
Proof of (iii) ( $\left.\operatorname{dim} \operatorname{Ker}(H-\lambda)=\operatorname{dim} \operatorname{Ker} F_{\rho}(H-\lambda)\right)$. By (i), $\operatorname{dim} \operatorname{Ker} H=0$ is equivalent to $\operatorname{dim} \operatorname{Ker} F=0$, assuming that $H \in D(F)$. We may therefore assume that $\operatorname{Ker} H \neq 0$ and $\operatorname{Ker} F \neq 0$ are both nontrivial. Eq. (41) shows that $\chi: \operatorname{Ker} H \rightarrow \operatorname{Ker} F$ is injective, hence $\operatorname{dim} \operatorname{Ker} H \leq \operatorname{dim} \operatorname{Ker} F$, and Eq. (42) shows that $Q: \operatorname{Ker} F \rightarrow \operatorname{Ker} H$ is injective, hence $\operatorname{dim} \operatorname{Ker} H \geq \operatorname{dim} \operatorname{Ker} F$. This establishes (iv) and moreover that $\chi: \operatorname{Ker} H \rightarrow \operatorname{Ker} F$ and $Q: \operatorname{Ker} F \rightarrow \operatorname{Ker} H$ are actually bijections.

## D Smooth Feshbach-Schur Map

We define the smooth Feshbach-Schur map and formulate its important isospectral property Let $\chi$, $\bar{\chi}$ be a partition of unity on a separable Hilbert space $\mathcal{H}$, i.e. $\chi$ and $\bar{\chi}$ are positive operators on $\mathcal{H}$ whose norms are bounded by one, $0 \leq \chi, \bar{\chi} \leq 1$, and $\chi^{2}+\bar{\chi}^{2}=1$. We assume that $\chi$ and $\bar{\chi}$ are nonzero. Let $\tau$ be a (linear) projection acting on closed operators on $\mathcal{H}$ with the property that operators in its image commute with $\chi$ and $\bar{\chi}$. We also assume that $\tau(\mathbf{1})=\mathbf{1}$. Let $\bar{\tau}:=\mathbf{1}-\tau$ and define

$$
\begin{equation*}
H_{\tau, \chi^{\#}}:=\tau(H)+\chi^{\#} \bar{\tau}(H) \chi^{\#} . \tag{43}
\end{equation*}
$$

where $\chi^{\#}$ stands for either $\chi$ or $\bar{\chi}$.
Given $\chi$ and $\tau$ as above, we denote by $D_{\tau, \chi}$ the space of closed operators, $H$, on $\mathcal{H}$ which belong to the domain of $\tau$ and satisfy the following three conditions:
(i) $\tau$ and $\chi$ (and therefore also $\bar{\tau}$ and $\bar{\chi}$ ) leave the domain $D(H)$ of $H$ invariant:

$$
\begin{equation*}
D(\tau(H))=D(H) \text { and } \chi D(H) \subset D(H) \tag{44}
\end{equation*}
$$

(ii)

$$
\begin{equation*}
H_{\tau, \bar{\chi}} \text { is (bounded) invertible on } \operatorname{Ran} \bar{\chi} \tag{45}
\end{equation*}
$$

and
(iii)

$$
\begin{equation*}
\bar{\tau}(H) \chi \text { and } \chi \bar{\tau}(H) \text { extend to bounded operators on } \mathcal{H} . \tag{46}
\end{equation*}
$$

(For more general conditions see [5, 36].)
The smooth Feshbach-Schur map (SFM) maps operators on $\mathcal{H}$ belonging to $D_{\tau, \chi}$ to operators on $\mathcal{H}$ by $H \mapsto F_{\tau, \chi}(H)$, where

$$
\begin{equation*}
F_{\tau, \chi}(H):=H_{0}+\chi H_{1} \chi-\chi W \bar{\chi} H_{\tau, \bar{\chi}}^{-1} \bar{\chi} W \chi \tag{47}
\end{equation*}
$$

Here $H_{0}:=\tau(H)$ and $W:=\bar{\tau}(H)$. Note that $H_{0}$ and $W$ are closed operators on $\mathcal{H}$ with coinciding domains, $D\left(H_{0}\right)=D(W)=D(H)$, and $H=H_{0}+W$. We remark that the domains of $\chi W \chi, \bar{\chi} W \bar{\chi}$, $H_{\tau, \chi}$, and $H_{\tau, \bar{\chi}}$ all contain $D(H)$.

The following result ([5]) generalizes Theorem 6.1 above; its proof is similar to the one of that theorem:

Theorem D. 1 (Isospectrality of SFM). Let $0 \leq \chi \leq 1$ and $H \in D_{\tau, \chi}$ be an operator on a separable Hilbert space $\mathcal{H}$. Then we have the following results:
(i) $H$ is bounded invertible on $\mathcal{H}$ if and only if $F_{\tau, \chi}(H)$ is bounded invertible on Ran $\chi$. In this case

$$
\begin{align*}
H^{-1} & =Q_{\tau, \chi}(H) F_{\tau, \chi}(H)^{-1} Q_{\tau, \chi}(H)^{\#}+\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi}  \tag{48}\\
F_{\tau, \chi}(H)^{-1} & =\chi H^{-1} \chi+\bar{\chi} \tau(H)^{-1} \bar{\chi} \tag{49}
\end{align*}
$$

(ii) If $\psi \in \mathcal{H} \backslash\{0\}$ solves $H \psi=0$ then $\varphi:=\chi \psi \in \operatorname{Ran} \chi \backslash\{0\}$ solves $F_{\tau, \chi}(H) \varphi=0$.
(iii) If $\varphi \in \operatorname{Ran} \chi \backslash\{0\}$ solves $F_{\tau, \chi}(H) \varphi=0$ then $\psi:=Q_{\tau, \chi}(H) \varphi \in \mathcal{H} \backslash\{0\}$ solves $H \psi=0$.
(iv) The multiplicity of the spectral value $\{0\}$ is conserved in the sense that $\operatorname{dim} \operatorname{Ker} H=\operatorname{dim} \operatorname{Ker} F_{\tau, \chi}(H)$.
(v) Assume, in addition, that $H=H^{*}$ and $\tau(H)=\tau(H)^{*}$ are self-adjoint, and introduce the bounded operators

$$
\begin{align*}
M & :=H_{\tau, \chi}^{-1} \bar{\chi}(H-\tau(H)) \chi \quad \text { and }  \tag{50}\\
N & :=\left(\mathbf{1}+M^{*} M\right)^{-1 / 2} \tag{51}
\end{align*}
$$

Then, for any $\psi \in \mathcal{H}$,

$$
\begin{align*}
& \lim _{\varepsilon \searrow 0} \operatorname{Im}\left\langle\psi,(H-i \varepsilon)^{-1} \psi\right\rangle=  \tag{52}\\
& \quad \lim _{\varepsilon \searrow 0} \operatorname{Im}\left\langle N Q_{\tau, \chi}(H)^{*} \psi,\left(N F_{\tau, \chi}(H) N-i \varepsilon\right)^{-1} N Q_{\tau, \chi}(H)^{*} \psi\right\rangle
\end{align*}
$$

and

$$
\begin{align*}
\lim _{\varepsilon \searrow 0} & \operatorname{Im}\left\langle\psi,\left(N F_{\tau, \chi}(H) N-i \varepsilon\right)^{-1} \psi\right\rangle=  \tag{53}\\
& \lim _{\varepsilon \searrow 0} \operatorname{Im}\left\langle\chi N^{-1} \psi,(H-i \varepsilon)^{-1} \chi N^{-1} \psi\right\rangle .
\end{align*}
$$

## E Transfer of Local Decay

We formulate an important property of the smooth Feshbach-Schur map which allows us to reduce the proof of the local decay for the original operator, $H-\lambda$, to the proof of this property for a much simpler one, $\mathcal{R}_{\rho}^{n}(H-\lambda)$.

One can transfer through the renormalization group also other properties of hamiltonians, e.g. the limiting absorption principle and local decay ([32], cf. Statement (v) of Theorem D.1):

Theorem E.1. Under certain conditions on a self-adjoint operator $B$ and a $C^{1}$ family $H(\lambda), \lambda \in \Delta$, of self-adjoint operators, we have that

- $F_{\tau, \chi}(H(\lambda))$ is a self-adjoint operator;
- $\langle B\rangle^{-\theta}\left[F_{\tau, \chi}(H(\lambda)-i \varepsilon)\right]^{-1}\langle B\rangle^{-\theta}$ converges in norm, as $\varepsilon \rightarrow 0+$;
- $\langle B\rangle^{-\theta}\left(F_{\tau, \chi}(H(\lambda))-i 0\right)^{-1}\langle B\rangle^{-\theta} \in C^{\nu}(\Delta)$
$\Rightarrow\langle B\rangle^{-\theta}(H(\lambda)-i 0)^{-1}\langle B\rangle^{-\theta} \in C^{\nu}(\Delta)$,
where $0 \leq \nu \leq 1$ and $0<\theta \leq 1$.


## F Analyticity of all Parts of $H$

In this appendix we state a useful result, due to [37], about families Hamiltonians $H_{\lambda} \equiv H\left(\underline{w^{\lambda}}\right)$ of the form (19). The result says that if $H_{\lambda}$ is analytic, then so are every component, $E_{\lambda}, T_{\lambda}, W_{\lambda}$, of it. Here, recall, that $E_{\lambda}:=w_{0,0}^{\lambda}(0), T_{\lambda}:=w_{0,0}^{\lambda}\left(H_{f}\right)-w_{0,0}^{\lambda}(0), W_{\lambda}:=H_{\lambda}-E_{\lambda}-T_{\lambda}$ (see (23)).

Proposition F. 1 ([37]). Suppose that $\lambda \mapsto H\left(\underline{w}^{\lambda}\right)$ is analytic in $\lambda \in S \subset \mathbb{C}$ and that $H\left(\underline{w}^{\lambda}\right)$ belongs to some polydisc $\mathcal{D}(\alpha, \beta, \gamma)$ for all $\lambda \in S$. Then $\lambda \mapsto E_{\lambda}, T_{\lambda}$, $W_{\lambda}$ are analytic in $\lambda \in S$.

## References

[1] W. Abou Salem, J. Faupin, J. Fröhlich, I.M.Sigal. On theory of resonances in non-relativisitc QED. Adv. Appl. Math., 2009, to appear; arXiv, 2007.
[2] L. Amour, B. Grébert, J.-C. Guillot. The dressed mobile atoms and ions. J. Math. Pures Appl. (9) 86, no. 3, 177-200, 2006.
[3] A. Arai. Mathematical analysis of a model in relativistic quantum electrodynamics. Applications of renormalization group methods in mathematical sciences (in Japanese) Kyoto, 1999.
[4] A. Arai, Masao Hirokawa. Ground states of a general class of quantum field Hamiltonians. Rev. Math. Phys. 12 , no. 8, 1085-1135, 2000.
[5] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal. Smooth Feshbach map and operator-theoretic renormalization group methods. Journal of Functional Analysis, 203, 44-92, 2003.
[6] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal. The renormalized electron mass in non-relativistic quantum electrodynamics. Journal of Functional Analysis, 243, 426-535, 2007.
[7] V. Bach, J. Fröhlich, and A. Pizzo. Infrared-Finite Algorithms in QED: The Groundstate of an atom interacting with the quantized radiation field. Communications in Mathematical Physics 264, Issue: 1, 145-165, 2006.
[8] V. Bach, J. Fröhlich, and A. Pizzo. An infrared-finite algorithm for Rayleigh scattering amplitudes, and Bohr's frequency condition. Comm. Math. Phys. 274, no. 2, 457-486, 2007.
[9] V. Bach, J. Fröhlich, and A. Pizzo. Infrared-finite algorithms in QED II. The expansion of the groundstate of an atom interacting with the quantized radiation field, Adv. in Math., 220 (4), 1023-1074, 2009. mp_arc.
[10] V. Bach, J. Fröhlich, and I. M. Sigal. Mathematical theory of non-relativistic matter and radiation. Letters in Math. Physics, 34:183-201, 1995.
[11] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. Adv. in Math. , 137:299-395, 1998.
[12] V. Bach, J. Fröhlich, and I. M. Sigal. Renormalization group analysis of spectral problems in quantum field theory. Adv. in Math. , 137:205-298, 1998.
[13] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. Commun. Math. Phys., 207(2):249-290, 1999.
[14] V. Bach, J. Fröhlich, and I. M. Sigal. Return to equilibrium. J. Math. Phys., 41(6):3985-4060, 2000.
[15] V. Bach, J. Fröhlich, I. M. Sigal, and A. Soffer. Positive commutators and spectrum of PauliFierz Hamiltonian of atoms and molecules. Commun. Math. Phys., 207(3):557-587, 1999.
[16] I. Catto, P. Exner, Ch. Hainzl. Enhanced binding revisited for a spinless particle in nonrelativistic QED. J. Math. Phys. 45, no. 11, 4174-4185, 2004.
[17] I. Catto, Ch. Hainzl. Self-energy of one electron in non-relativistic QED. J. Funct. Anal. 207, no. 1, 68-110, 2004.
[18] T. Chen. Infrared renormalization in non-relativistic QED and scaling criticality. J. Funct. Anal. 254, no. 10, 2555-2647, 2008.
[19] T. Chen, J. Fröhlich, A. Pizzo. Infraparticle Scattering states in non-relativistic QED: I. The Bloch-Nordsieck paradigm. e-print, arXiv:0709.2493
[20] T. Chen, J. Fröhlich, A. Pizzo. Infraparticle scattering states in non-relativistic QED: II. Mass Shell Properties. e-print, arXiv:0709.2812
[21] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. Photons and Atoms - Introduction to Quantum Electrodynamics. John Wiley, New York, 1991.
[22] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. Atom-Photon Interactions - Basic Processes and Applications. John Wiley, New York, 1992.
[23] J. Dereziński and V. Jakšić. Spectral theory of Pauli-Fierz operators. J. Funct. Anal., 180(2):243-327, 2001.
[24] J. Faupin. Resonances of the confined hydrogen atom and the Lamb-Dicke effect in nonrelativisitc quantum electrodynamics. Ann. Henri Poincaré 9, no. 4, 743-773, 2008.
[25] H. Feshbach. Unified theory of nuclear reactions. Ann. Phys., 5:357-390, 1958.
[26] J. Fröhlich.
[27] J. Fröhlich, M. Griesemer and B. Schlein. Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field. Advances in Mathematics 164, Issue: 2, 349-398, 2001.
[28] J. Fröhlich, M. Griesemer and B. Schlein. Asymptotic completeness for Rayleigh scattering. Ann. Henri Poincar 3, no. 1, 107-170, 2002.
[29] J. Fröhlich, M. Griesemer and B. Schlein. Asymptotic completeness for Compton scattering. Comm. Math. Phys. 252, no. 1-3, 415-476, 2004.
[30] J. Fröhlich, M. Griesemer and I.M. Sigal. Spectral theory for the standard model of nonrelativisitc QED. Comm. Math. Phys., 283, no 3, 613-646, 2008.
[31] J. Fröhlich, M. Griesemer and I.M. Sigal. Spectral renormalization group analysis. Rev. Math. Phys., e-print, ArXiv, 2008.
[32] J. Fröhlich, M. Griesemer and I.M. Sigal. Spectral renormalization group and limiting absorption principle for the standard model of non-relativisitc QED. e-print.
[33] J. Fröhlich and A. Pizzo.
[34] V. Gergescu, C. Gérard, and J.S. Møller. Spectral Theory of massless Pauli-Fierz models. Commun. Math. Phys., 249:29-78, 2004.
[35] M. Griesemer. Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics. J. Funct. Anal., 210(2):321-340, 2004.
[36] M. Griesemer and D. Hasler. On the smooth Feshbach-Schur map. J. Funct. Anal., 254(9):23292335, 2008.
[37] M. Griesemer and D. Hasler. Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation. arXiv:0801.4458.
[38] M. Griesemer, E.H. Lieb and M. Loss. Ground states in non-relativistic quantum electrodynamics. Invent. Math. 145, no. 3, 557-595, 2001.
[39] Ch. Hainzl. One non-relativistic particle coupled to a photon field. Ann. Henri Poincaré 4, no. 2, 217-237, 2003.
[40] Ch. Hainzl, R. Seiringer, Mass renormalization and energy level shift in non-relativistic QED. Adv. Theor. Math. Phys. 6 (5) 847871, 2003.
[41] Ch. Hainzl, V. Vougalter, S. Vugalter. Enhanced binding in non-relativistic QED. Commun. Math. Phys. 233, no. 1, 13-26, 2003.
[42] D. Hasler and I. Herbst. Absence of ground states for a class of translation invariant models of non-relativistic QED. Commun. Math. Phys. 279, no. 3, 769-787, 2008. ArXiv
[43] D. Hasler, I. Herbst and M.Huber. On the lifetime of quasi-stationary states in non-relativistic QED. Ann. Henri Poincaré 9, no. 5, 1005-1028, 2008, ArXiv:0709.3856.
[44] D. Hasler and I. Herbst. Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation. 2008, ArXiv:0801.4458v1.
[45] D. Hasler and I. Herbst. Smoothness and analyticity of perturbation expansions in QED. 2010, ArXiv:1007.0969v1.
[46] D. Hasler and I. Herbst. Convergent expansions in non-relativistic QED. 2010, ArXiv:1005.3522v1.
[47] M. Hirokawa. Recent developments in mathematical methods for models in non-relativistic quantum electrodynamics. A garden of quanta, 209-242, World Sci. Publishing, River Edge, NJ, 2003.
[48] F. Hiroshima. Ground states of a model in nonrelativistic quantum electrodynamics. I. J. Math. Phys. 40 (1999), no. 12, 6209-6222.
[49] F. Hiroshima. Ground states of a model in nonrelativistic quantum electrodynamics. II. J. Math. Phys. 41 (2000), no. 2, 661-674.
[50] F. Hiroshima. Ground states and spectrum of quantum electrodynamics of nonrelativistic particles. Trans. Amer. Math. Soc. 353 (2001), no. 11, 4497-4528 (electronic).
[51] F. Hiroshima. Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants. Ann. Henri Poincaré 3 (2002), no. 1, 171-201.
[52] F. Hiroshima. Nonrelativistic QED at large momentum of photons. A garden of quanta, 167-196, World Sci. Publishing, River Edge, NJ, 2003.
[53] F. Hiroshima. Localization of the number of photons of ground states in nonrelativistic QED. Rev. Math. Phys. 15 (2003), no. 3, 271-312.
[54] F. Hiroshima. Analysis of ground states of atoms interacting with a quantized radiation field. Topics in the theory of Schrödinger operators, World Sci. Publishing, River Edge, NJ, 2004, 145-272.
[55] F. Hiroshima, and H. Spohn. Ground state degeneracy of the Pauli-Fierz Hamiltonian with spin. Adv. Theor. Math. Phys. 5 (2001), no. 6, 1091-1104.
[56] F. Hiroshima, and H. Spohn. Enhanced binding through coupling to a quantum field. Ann. Inst. Henri Poincaré 2(6) (2001), no. 6, 1159-1187.
[57] F. Hiroshima, H. Spohn, Mass renormalization in nonrelativistic quantum electrodynamics. J. Math. Phys. 46 (4), 2005.
[58] M. Hübner and H. Spohn. Radiative decay: nonperturbative approaches. Rev. Math. Phys. 7, no. 3, 363-387, 1995.
[59] W. Hunziker. Resonances, metastable states and exponential decay laws in perturbation theory. Comm. Math. Phys. 132, no. 1, 177-188, 1990.
[60] W. Hunziker and I.M. Sigal. The quantum $N$-body problem. J. Math. Phys. 41, no. 6, 34483510, 2000.
[61] Ch. King. Resonant decay of a two state atom interacting with a massless non-relativistic quantized scalar field. Comm. Math. Phys. 165(3), 569-594, (1994).
[62] E.H. Lieb and M. Loss. Existence of atoms and molecules in non-relativistic quantum electrodynamics. Adv. Theor. Math. Phys. 7, 667-710 (2003). arXiv math-ph/0307046.
[63] E.H. Lieb and M. Loss. Self-energy of electrons in non-perturbative QED, in: Conference Moshe Flato 1999, vol. I, Dijon, in: Math. Phys. Stud. vol. 21, Kluwer Acad. Publ., Dordrecht, 2000, pp. 327-344.
[64] E.H. Lieb and M. Loss. A bound on binding energies and mass renormalization in models of quantum electrodynamics. J. Statist. Phys. 108, 2002.
[65] M. Loss, T. Miyao and H. Spohn. Lowest energy states in nonrelativistic QED: Atoms and ions in motion J. Funct. Anal. 243, Issue 2, 353-393, 2007.
[66] M. Merkli, M. Mück, I.M. Sigal. Instability of Equilibrium States for Coupled Heat Reservoirs at Different Temperatures, J. Funct. Anal., 243, no. 1, 87-120 (2007).
[67] M. Merkli, M. Mück, I.M. Sigal. Nonequilibrium stationary states and their stability, Ann. Inst. H. Poincaré, 243, no. 1, 87-120 (2008).
[68] M. Mück. Thermal Relaxation for Particle Systems in Interaction with Several Bosonic Heat Reservoirs. Ph.D. Dissertation, Department of Mathematics, Johannes-Gutenberg University, Mainz, July 2004, ISBN 3-8334-1866-4
[69] M. Mück. Construction of metastable states in quantum electrodynamics. Rev. Math. Phys. 16, no. 1, 1-28, 2004.
[70] A. Pizzo. One-particle (improper) States in Nelson's massless model. Annales Henri Poincaré 4, Issue: 3, 439-486, June, 2003.
[71] A. Pizzo. Scattering of an infraparticle: The one particle sector in Nelson's massless model. Annales Henri Poincaré 6, Issue: 3, 553-606, 2005.
[72] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV, Ananlysis of Operators. Academic Press, 1978.
[73] J. Schur. Über Potenzreihen die im Inneren des Einheitskreises beschränkt sind. J. reine u. angewandte Mathematik, 205-232, 1917.
[74] I.M. Sigal. Ground state and resonances in the standard model of the non-relativistic quantum electrodynamics. J. Stat. Physics, 134 (2009), no. 5-6, 899-939. arXiv 0806.3297.
[75] H. Spohn. Dynamics of Charged Particles and their Radiation Field, Cambridge University Press, Cambridge, 2004.


[^0]:    ${ }^{1}$ The papers $[13,48,38, ?, 55]$ include the interaction of the spin with magnetic field in the Hamiltonian.

[^1]:    ${ }^{2}$ In the case of a molecule in the Born-Oppenheimer approximation, the resulting $V(x)$ also depends on the rescaled coordinates of the nuclei.

