Partial Differential Equations II  
MAT1061*  

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Abstract  

In these notes, we develop some basic techniques in solving partial differential equations and analyzing their solutions. The long term goal is to understand principal evolution equations. We establish an existence theory for the selected class of equations, describe their key properties, isolate their most important solutions and study stability or instability of these solutions. Some of non-evolution equations appear as static equations for the evolution ones.  

Given the time constrain, we have to be very selective about the equations we consider. The guiding principles are the importance of the equations in mathematics and in applications, relevance to the current research and the central role of the techniques needed to analyze these equations.  

In the appendices, we review some elements of modern analysis used in this course. For more details see [10], [19], [16] and [27].  

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1 Evolution Equations: Generalities

We would like to address a general theory of evolution PDEs, i.e. equations of the form

$$\partial_t u = F(u), \quad u|_{t=0} = u_0,$$

where $t \to u(t)$ is a path in some space, $Y$, which we assume to be a Banach space, $F$ is a map defined on the same space, $\partial_t u = \dot{u} = \frac{\partial u}{\partial t}$ and $u_0 \in Y$. We understand (1.1) as $\partial_t u(t) = F(u(t))$. $u_0$ is called the initial condition and (1.1), the initial value problem.

The situation we will be mostly interested in is when $F$ is a partial differential operator, linear or nonlinear. We consider a few examples.

Reaction-diffusion (or heat) equation. This is the equation of the form

$$\frac{\partial u}{\partial t} = \Delta u + g(u).$$

In this example, $u(t)$ is a vector-function of a spatial variable, say $x$, with values, say in $\mathbb{R}^n$ (we write $u(t)(x) \equiv u(x,t)$), $Y$ is some unfunctional space, say $C^2(\mathbb{R}^d, \mathbb{R}^n)$ or the Sobolev space $H^2(\mathbb{R}^d, \mathbb{R}^n)$ and $F(u) = \Delta u + g(u)$. Here $\Delta$ is the Laplace operator (the Laplacian):

$$\Delta u := \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}.$$
The term $\Delta$ is responsible for diffusion and the term, $g(u)$, describes the reaction in the system modelled.

**Linear and nonlinear Schrödinger equation.** The Schrödinger equation comes from Quantum Mechanics and is the equation of the form (in the dimensionless units)

$$i\frac{\partial \psi}{\partial t} = (-\Delta + V)\psi,$$

where $\psi$ is the complex function, $\psi: \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{C}$, and $V$ is operator $V: \psi(x,t) \to V(x)\psi(x,t)$ of multiplication by a real function $V(x)$. Here $-\Delta$ and $V(x)$ represent the kinetic and potential energies of the particle in question.

The nonlinear Schrödinger equation appears in quantum physics as well as in theory of classical waves (e.g. waves in plasmas or surface waves in fluids). It is the equation of the form (in the dimensionless units)

$$i\frac{\partial \psi}{\partial t} = (-\Delta + V)\psi + g(\psi),$$

where $g(\psi)$ is some appropriate function, often of the form $g(\psi) = f(|\psi|^2)\psi$. Again, often, one looks at the situations when $V = 0$.

**Wave equation.**

**Mean curvature flow.** The mean curvature flow, starting with a hypersurface $S_0$ in $\mathbb{R}^{n+1}$, is the family of hypersurfaces $S(t)$ given by immersions $x(\cdot, t)$ which satisfy the evolution equation

$$\begin{cases}
\frac{\partial x}{\partial t} = -H(x)\nu(x) \\
x|_{t=0} = x_0
\end{cases}$$

where $x_0$ is an immersion of $S_0$, $H(x)$ and $\nu(x)$ are mean curvature and the outward unit normal vector at $x \in S(t)$, respectively. The terms used above are explained in Appendix 5.10.

Sometimes (1.1) is called the dynamical system and $F$, the vector field (finite or infinite dimensional). Though we define $F$ on a vector space, it can be also defined on a manifold (again, finite or infinite dimensional).

To solve equation (1.1), we have to choose a space, say $Y$, to which the vector-function $t \to u(t)$ belongs for all $t$ considered. We have to make sure that $F$ is defined on this space. Depending on the problem at hand, we choose different spaces. For instance, for the examples above, $F$ maps the space $Y := C^k(\Omega)$ into the space $C^{k-2}(\Omega)$. More abstractly, $Y$ could be some space with a norm, or a Banach space. For the definition of Banach spaces see Appendix ??.

We also have to choose a space, say $X$, to which the function $t \to u(t)$ belongs. If $u(t) \in Y$ for $t$ in some interval $I$, then we can choose

- $X = C(I,Y)$, where, say, $I := [0,T]$.

Here $X = C(I,Y)$ is the space of continuous vector - functions $u: t \to u(t)$ on $I$ with values in $Y$ and with the norm

$$\|u\|_X := \sup_{t \in I} \|u(t)\|_Y.$$
Linear evolution equations and the exponential. Let $A : Y \to X$ be a linear operator between Banach spaces $Y$ and $X$ (for a discussion of linear operators, see Appendix B.1). We consider the linear equation

$$\frac{\partial u}{\partial t} = Au \quad \text{and} \quad u|_{t=0} = u_0. \tag{1.6}$$

An important example of the linear evolution equation is the linear (free) heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{and} \quad u|_{t=0} = u_0, \tag{1.7}$$

where $u : \mathbb{R}_+^n \times \mathbb{R}^+_t \to \mathbb{R}$, $\mathbb{R}^+_t := \{ t \in \mathbb{R} : t \geq 0 \}$, is an unknown function, and $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a given initial condition.

If for each initial condition $u_0$ from some Banach space, $Y$, the solution exists and is unique, then this defines the evolution operator, or more precisely, the evolution group, $U(t)$ as $U(t) : u_0 \to u(t)$, where $u(t)$ is the solution to (1.6). This evolution operator has the following properties

- $U(0) = 1$; $U(s) \circ U(t) = U(s + t)$; $\partial_t U(t) = AU(t), \, t \geq 0$.

The first and third properties follow from the definition and the second one, from the uniqueness of solutions of the initial problem (1.6). The families satisfying the first two properties are called the semi-groups and $A$ is called the generator (of $U(t)$). We usually write $U(t) = e^{tA}$, which can be considered as a definition of the exponential $e^{tA}$.

There are general situations where we can show the existence of the global (all $t$’s) solutions of the initial value problem (1.6) (for more details see Appendix C):

- $A$ is a bounded operator;
- $A$ is either self-adjoint ($A^* = A$) and bounded above, $A \leq C$ for some $C < \infty$ ($\langle u, Au \rangle \leq C\|u\|^2$) or anti-self-adjoint ($A^* = -A$);
- $A$ is a ‘constant coefficient pseudo-differential’ operator; more precisely, $A = a(-i\nabla_x)$, where $a$ is some decent function.

For a bounded operator $A$ the flow always exists since $e^{At}$ can be defined by

$$e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{1.8}$$

The series on the r.h.s. converges absolutely since

$$\left\| \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \left\| (tA)^n \right\| \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|^n = e^{\|A\|} < \infty.$$

For the second case we refer to [15]. In the third case $A$ and $e^{At}$ are defined using the Fourier transform as $(\hat{Au})(k) = a(k)\hat{u}(k)$ and $(e^{At}u)(k) = e^{a(k)t}\hat{u}(k)$, respectively (see Appendix A and [15] for more on the Fourier transform). Applying the inverse Fourier transform $\mathcal{F}^{-1} : f \mapsto \hat{f}$, so that $(\hat{f})^* = f = (\hat{f})^*$, and using that $\mathcal{F}^{-1} : g \mapsto (2\pi)^{-n/2} \hat{g} \ast \hat{f}$, we obtain

$$u = g_t * u_0, \tag{1.9}$$

where $g_t(x)$ is the inverse Fourier transform of the function $e^{a(k)t}$. As an illustrative example, we consider
The operator $e^{t\Delta}$ (heat kernel). Since $(e^{-|k|^2t})^{-1} = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ (see Appendix A), the equation (1.9), in this case, gives
\[ u = p_t * u_0, \] (1.10)
where $p_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$. In particular, $u \to u_0$ as $t \to 0$, as it should be.

Thus the operator $e^{t\Delta}$ has the integral kernel $p_t(x, y) = p_t(x - y) := (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$, $t > 0$, which satisfies $p_t(x, y) > 0$ and $\int p_t(x, y) dy = 1$.

Discussion. The semi-group $P_t = e^{t\Delta}$ has the following properties

(a) $P_t$ is positivity improving, i.e. if $f \geq 0$, then $P_t f > 0$,

(b) $P_t 1 = 1$.

Semi-groups with such properties are called stochastic semigroups. (The conditions $f \geq 0$ and $f > 0$ can be stated for abstract vector spaces in terms of closed and open cones.)

Remark. We can arrive at the formula $(e^{At}u)(k) = e^{a(k)t} \hat{u}(k)$, by first applying the Fourier transform the equation (1.6), which we write as $\frac{\partial u}{\partial t} = a(-i\nabla_x)u$ and $u|_{t=0} = u_0$, treating $t$ as a parameter and using the property $F : a(-i\nabla_x)f \mapsto a(k)f$. As a result, we obtain
\[ \frac{\partial \hat{u}(k, t)}{\partial t} = a(k)\hat{u}(k, t) \quad \text{and} \quad \hat{u}(k, t)|_{t=0} = \hat{u}_0(k). \]

We solve the latter equation (treating $k$ as a parameter) to find $\hat{u}(k, t) = e^{a(k)t} \hat{u}_0(k)$.

2 Local Existence for Evolution Equations

2.1 Reduction to a fixed point problem

Canonical form of evolution equations. We say the initial problem (1.1) is written in the canonical form if the map $F(u)$ is presented as $F(u) = Au + f(u)$, where $A$ is a linear operator and $f(u)$ satisfies $f(u) = o(\|u\|)$. Then (1.1) can be rewritten as
\[ \partial_t u = Au + f(u), \quad u|_{t=0} = u_0. \] (2.1)

where $A$ is a linear operator on $Y$ and $f$ is a nonlinear map, i.e. $f(u) = o(\|u\|)$ in some norm, or $f(0) = f'(0) = 0$. We will call $f$ the nonlinearity. We will see later that, if $F(0) = 0$ and $F(u)$ is once continuously (Gâteaux) differentiable, it can be always split into linear and nonlinear parts, $F(u) = Au + f(u)$, with $A$ and $f$ as above. (This is a general result, valid for all differentiable maps. It generalizes the Taylor theorem of Calculus.)

Duhamel principle. Consider the inhomogeneous initial value problem
\[ \partial_t u = Au + f, \quad u|_{t=0} = u_0, \] (2.2)

where $f = f(t)$ is a given function $f : [0, T) \to Y$. Assuming the homogeneous equation $\partial_t u = Au$, $u|_{t=0} = u_0$, has a unique solution, the solution, $u(t)$, of (2.2), is given by
\[ u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s) \, ds. \] (2.3)
Weak solutions. Consider the initial value problem an canonical form (2.1). We apply (2.3) to (2.1) to obtain
\[ u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(u(s)) \, ds. \] (2.4)
If \( u(t) \) solves (2.1), then it also solves the equation (2.4). Conversely, if \( u(t) \) solves (2.4) and is differentiable in \( t \), then it solves the equation (2.1).

If \( u(t) \) solves (2.4), but we do not know whether it is differentiable or not, we call \( u(t) \) a weak solution to (2.1).

Remark. There are several definitions of weak solutions. Another common definition states that \( u \) is a weak solution iff it satisfies the equation
\[ -\int \int u \partial_t gd^dxd t = \int \int (uA^*g + f(u)g) d^dxd t \] (2.5)
for all \( g \)'s of compact support and which are differentiable in \( t \) and are in the domain of (formally) adjoint operator \( A^* \).

Fixed point problem. Eq (2.4) can be written as the equation \( u = H(u) \), where
\[ H(u)(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}f(u(s)) \, ds, \] (2.6)
called the fixed point equation or the fixed point problem. A solution of such an equation is called a fixed point. In our next step we learn how to solve fixed point equations.

2.2 The contraction mapping principle

A key to dealing with a large class of equations is to reduce them to a fixed point problem,
\[ H(u) = u, \] (2.7)
for some map \( H \) and then use one of several fixed point theorem stating existence and uniqueness of solutions of the latter problem. (The equation (2.7) is called the fixed point equation and its solution, a fixed point of the map \( H \).) The most useful among these theorems is the Banach contraction mapping principle which we now formulate and prove.

Let \( X \) be a Banach space. Denote by \( d(u,v) = \| u - v \| \) the distance between the vectors \( u \) and \( v \). We remark that actually all we need for the next theorem is that \( X \) is a complete metric space (i.e. it does not have to have a norm). Let \( B \) be a closed set in \( X \). A map \( H : B \to B \) is called a strict contraction if and only if there is a number \( \alpha \in (0,1) \) s.t.
\[ d(H(u),H(v)) \leq \alpha \, d(u,v), \quad \forall u,v \in B. \]

Theorem 2.1 (the contraction mapping principle). If \( H \) is a strict contraction in \( B \), then \( H \) has a unique fixed point in \( B \).

Proof. We use the method of successive approximations to solve the equation \( u = H(u) \). Pick some \( u_0 \in B \) and define \( u_1 = H(u_0), \ldots, u_n = H(u_{n-1}) \). Since \( H \) is a contraction, \( u_n \in B \).

We claim that \( \{u_n\} \) is a Cauchy sequence in \( X \). Indeed, let \( n \geq m \), then
\[ d(u_n,u_m) \leq \alpha^m d(u_{n-m},u_0). \]
Taking here \( n = m + 1 \), we find \( d(u_{m+1}, u_m) \leq \alpha^m d(u_1, u_0) \). Next, by the triangle inequality (i.e. \( d(v, u) \leq d(v, w) + d(w, u), \forall w \in X \)), we obtain
\[
    d(u_k, u_0) \leq d(u_k, u_{k-1}) + d(u_{k-1}, u_{k-2}) + \cdots + d(u_1, u_0).
\]
Applying \( d(u_{m+1}, u_m) \leq \alpha^m d(u_1, u_0) \) to each term on the r.h.s. gives
\[
    d(u_k, u_0) \leq \left( \alpha^{k-1} + \alpha^{k-2} + \cdots + 1 \right) d(u_1, u_0) \leq \frac{1}{1 - \alpha} d(u_1, u_0).
\]
(If \( B \) is a bounded set in \( X \), then we do not need the step above, since in this case \( \|u_k\| \) are uniformly bounded.) The last two inequalities imply that
\[
    d(u_n, u_m) \leq \frac{\alpha^m}{1 - \alpha} d(u_1, u_0) \to 0 \quad \text{as} \ m, n \to \infty.
\]
Thus \( \{u_n\} \) is a Cauchy sequence in \( X \). Now since \( X \) is complete, \( u_n \in B \) and \( B \) is closed, there is a \( u_* \in B \) s.t. \( u_n \to u_* \). Since \( d(H(u_n), H(u_*)) \leq \alpha d(u_n, u_*) \to 0 \), we have also that \( H(u_n) \to H(u_*) \). This and the equation \( u_n = H(u_{n-1}) \) imply that the limit \( u_* \) satisfies the equation \( u_* = H(u_*) \). This demonstrates existence of a fixed point in \( B \), and we finish the proof by showing its uniqueness. Suppose that \( H(u_*) = u_* \), and \( H(v_*) = v_* \). Then we have \( d(v_*, u_*) = d(H(v_*), H(u_*)) \leq \alpha d(v_*, u_*) \). Hence, since \( \alpha \in (0, 1) \), \( d(v_*, u_*) = 0 \) and so \( v_* = u_* \).

**Remark.** As was mentioned above, there are many fixed point theorems and the Banach contraction mapping principle is one of these, albeit the most useful one.

In this subsection, we use the contraction mapping principle to prove local existence of solutions for several evolution PDEs, the reaction-diffusion (nonlinear heat), Hartree and nonlinear Schrödinger equations.

### 2.3 Local existence for reaction-diffusion (nonlinear heat) equations

We show local existence to the initial value problem for the reaction-diffusion (nonlinear heat) equation (or reaction-diffusion equation):
\[
    \frac{\partial u}{\partial t} = \Delta u + \lambda |u|^{p-1} u, \quad u|_{t=0} = u_0, \quad (2.8)
\]
on \( \mathbb{R}^n \), with \( p > 1 \) and bounded initial conditions \( u_0 \) on \( \mathbb{R}^n \). The special case of equation (2.8) without nonlinearity first appeared in the theory of heat diffusion. In that case, \( u_0(x) \) is a given distribution of temperature in a body at time \( t = 0 \), and \( u(x, t) \) is the unknown temperature-distribution at time \( t \). Presently, this equation appears in various fields of science, including the theory of chemical reactions and mathematical modeling of stock markets. Similar equations appear in the motion by mean curvature flow, vortex dynamics in superconductors, surface diffusion and chemotaxis.

We note that (2.8) is the initial value problem (1.1), \( \partial_t u = F(u), \ u|_{t=0} = u_0, \) in the canonical form, (2.1), with \( A = \Delta, \) a linear operator acting on \( H^2 \) and \( f(u) = \lambda |u|^{p-1} u \), a nonlinear map.

We will look for weak solutions in space \( X := C([0, T], Y) \), with \( T \) specified later and \( Y := L^\infty(\mathbb{R}^n) \). The space \( L^\infty \) is one of the standard \( L^p \)-spaces and is defined as
\[
    L^\infty(\Omega) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is measurable, and } \text{ess sup} \ |f| < \infty \}, \quad (2.9)
\]
where, recall that $\text{ess sup} |f| := \inf \{ \sup |g| : g = f \text{ a.e.} \}$, with the norm defined as
\[
\|f\|_{\infty} := \text{ess sup}|f|.
\] (Strictly speaking, elements of $L^p(\Omega)$ are equivalence classes of measurable functions; two functions define the same elements of $L^p(\Omega)$ if they differ only on a set of measure 0.) We often use the abbreviations $L^p$ and $\|v\|_p$ for $L^p(\Omega)$ and $\|v\|_{L^p}$.

**Theorem 2.2.** Let $u_0 \in L^\infty(\mathbb{R}^n)$ and $R > \|u_0\|_{\infty}$. Then, for $T < (pR^{p-1})^{-1}$, $(R-\|u_0\|_\infty)(R^p)^{-1}$, the reaction-diffusion equation (2.8) has a weak solution $u \in C([0, T], L^\infty)$, satisfying $\|u\|_{C([0, T], L^\infty)} \leq R$ and unique in the ball $\|u\|_{C([0, T], L^\infty)} \leq R$.

**Proof.** Using Duhamel’s principle, Eq (2.8) can be written as the fixed point equation $u = H(u)$, where
\[
H(u)(t) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}f(u)(s) \, ds
\] (2.11)
and we have written $f(u)(s)$ for $f(u(s))$. Let $Y := L^\infty(\mathbb{R}^n)$ and $X := C([0, T], Y)$, with $T$ specified later. The proof of existence and uniqueness will follow if we can show that the map $H$ has a unique fixed point in the ball
\[
B_R := \{ u \in X, \, \|u\|_X \leq R \},
\]
for some $R > 0$. We prove this statement via the contraction mapping principle.

We begin by proving that there is $R > 0$ s.t. $H$ is a well-defined map from $B_R$ to $B_R$. First, we show the estimate
\[
\|e^{t\Delta}u_0\|_X \leq \|u_0\|_Y
\] (2.12)
We have shown above that the operator $e^{t\Delta}$ has the integral kernel $p_t(x, y), t > 0$, i.e. $e^{t\Delta}u(x) = \int p_t(x, y)u(y) \, dy$, with the following properties: $p_t(x, y) > 0$ and $\int p_t(x, y) \, dy = 1$. Using these properties, we obtain the estimate (2.12).

Next, the elementary bound $\|u\|^{p-1}u \leq R^p$, for $|u| \leq R$, shows that, if $t < T$ and $u \in B_R$, then
\[
\sup_{\|w\|_Y \leq R} \|f(w)\|_Y = \sup_{|w| \leq R} |f(w)| \leq R^p
\] (2.13)
(remember that $Y := L^\infty(\mathbb{R}^n)$), which, together with the estimate (2.17), gives
\[
\left\| \int_0^t e^{(t-s)\Delta}f(u)(s) \, ds \right\|_X \leq \sup_{t \leq T} \int_0^t \|f(u)(s)\|_Y \, ds \leq TR^p.
\] (2.14)
Estimates (2.12) and (2.14) and the definition (2.11) of the map $H$ imply that $H : B_R \to B_R$, provided $\|u_0\|_Y + TR^p \leq R$.

Now, we prove that $H : B_R \to B_R$ is a strict contraction. Recalling the definition $f$ and using the elementary bound
\[
\|u_1 - u_2\|_{L^p} \leq pR^{p-1}\|u_1 - u_2\|,
\]
for $|u|, |u_1|, |u_2| \leq R$, we obtain, for $u_1, u_2 \in B_R$,
\[
\sup_{\|w_1\|_Y, \|u_2\|_Y \leq R} \|f(w_1) - f(w_2)\|_Y / \|w_1 - w_2\|_Y \leq pR^{p-1},
\] (2.15)
which, together with the definitions of \( H \) and \( \|u\|_X \) and estimate (2.12), gives
\[
\|H(u_1) - H(u_2)\|_X \leq \sup_{t \leq T} \int_0^t \|f(u_1)(s) - f(u_2)(s)\|_Y \, ds \\
\leq TpR^{p-1}\|u_1 - u_2\|_X.
\]

Therefore, if \( pR^{p-1}T < 1 \), then \( H \) is a strict contraction in \( B_R \). We see that the inequalities \( \|u_0\|_Y + TR^p \leq R \) and \( pR^{p-1}T < 1 \) are satisfied if \( R > \|u_0\|_Y \) and \( T < (pR^{p-1})^{-1}, (R - \|u_0\|_Y)/R^p \). This gives the existence and uniqueness for the stated conditions on \( R \) and \( T \).

This completes the proof of existence and uniqueness of the solution \( u \) and the estimate on it. \( \square \)

Abstract result on local well-posedness. Before proceeding to other equations we prove an abstract result on local well-posedness of the initial value problem (1.1), \( \partial_t u = F(u), \quad u|_{t=0} = u_0 \), in the canonical form, (2.1), which we reproduce here
\[
\partial_t u = Au + f(u), \quad (2.16)
\]
with the initial condition \( u|_{t=0} = u_0 \). Recall, that here \( A \) is a linear operator on \( Y \) and \( f \) is a nonlinear map, i.e. \( f(u) = o(\|u\|) \). For simplicity we assumed that \( f \) does not depend on \( t \) explicitly (and only through \( u \)). Recall, that we think of \( u \) as a path \( u : t \in I \to u(t) \in Y \) in \( Y \) and we are looking for weak solutions to (2.16). Specifically, we will consider solutions in the space \( C([0, T], Y) \), for some \( T > 0 \).

Definition 2.1. We say that (2.16) is well-posed (WP) in a Banach space \( X \) and on an interval \( [0, T) \) if for any initial condition \( \psi_0 \in X \) (2.16) has a unique solution in \( C([0, T), X) \) (i.e., for \( t \in [0, T) \)) and this solution depends continuously on the initial condition \( \psi_0 \).

If \( T < \infty \), then (2.16) is locally well-posed (LWP) and if \( T = \infty \), then (2.16) is globally well-posed (GWP).

We prove existence and uniqueness of weak solutions for (2.16). We have the following result:

Theorem 2.3. Consider the abstract nonlinear equation (2.16) and assume that
- A generates a semi-group \( e^{At} \), satisfying, for some constant \( K \),
\[
\sup_{t \geq 0} \|e^{tA}w\|_Y \leq K\|w\|_Y, \quad (2.17)
\]
- the nonlinearity \( f \) is a locally Lipschitz map, \( f : Y \to Y \), obeying, for some \( M_R, L_R < \infty \),
\[
\sup_{\|w\|_Y \leq R} \|f(w)\|_Y \leq M_R, \quad (2.18)
\]
and
\[
\sup_{\|w_1\|_Y, \|w_2\|_Y \leq R} \|f(w_1) - f(w_2)\|_Y / \|w_1 - w_2\|_Y \leq L_R. \quad (2.19)
\]

Let \( u_0 \in Y \) and \( R > K\|u_0\|_Y \). Then the equation (2.16) has a unique weak solution \( u \in C([0, T], Y) \), for
\[
T < (KL_R)^{-1}, (R - K\|u_0\|_Y)/KM_R,
\]
in the ball \( \|u\|_{C([0,T], Y)} \leq R \). The solution \( u \) depends continuously on the initial condition \( u_0 \). Furthermore, either the solution is global in time or blows up in \( Y \) in a finite time (i.e. either \( \|u(t)\|_Y < \infty, \forall t \), or \( \|u(t)\|_Y < \infty \) for \( t < t_* \) and \( \|u(t)\|_Y \to \infty \) as \( t \to t_* \) for some \( t_* < \infty \)).
Proof. Using Duhamel’s principle, Eq (2.16) can be written as the fixed point equation $u = H(u)$, where

$$H(u)(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}f(u(s)) \, ds$$

and we have written $f(u)(s)$ for $f(u(s))$. Let $X := C([0,T], Y)$, with $T < (KL_R)^{-1}$, $(R - K\|u_0\|_Y)/KM_R$ and $R > K\|u_0\|_Y$. The proof of existence and uniqueness will follow if we can show that the map $H$ has a unique fixed point in the ball

$$B_R := \{u \in X, \|u\|_X \leq R\}.$$

We prove this statement via the contraction mapping principle.

We begin by proving that $H$ is a well-defined map from $B_R$ to $B_R$. Using the assumptions (2.17) and (2.18), we obtain, for $u \in B_R$,

$$\|e^{tA}u_0\|_X \leq K\|u_0\|_Y$$

(2.21)

and, if $t < T$ and $u \in B_R$,

$$\left\|\int_0^t e^{(t-s)A}f(u(s)) \, ds\right\|_X \leq \sup_{t \leq T} \int_0^t K\|f(u(s))\|_Y \, ds \leq TKM_R.$$  

(2.22)

Estimates (2.21) and (2.22) imply that $H : B_R \to B_R$, provided $K\|u_0\|_Y + KTM_R \leq R$.

Now, we prove that $H : B_R \to B_R$ is a strict contraction. Recalling the definition of $\|u\|_X$ and using (2.17) and (2.19), we obtain, for $u_1, u_2 \in B_R$,

$$\|H(u_1) - H(u_2)\|_X \leq \sup_{t \leq T} \int_0^t K\|f(u_1(s)) - f(u_2(s))\|_Y \, ds$$

$$\leq KTL_R\|u_1 - u_2\|_X.$$

Therefore, if $L_RKT < 1$, then $H$ is a strict contraction in $B_R$. We see that the inequalities $K\|u_0\|_Y + KTM_R \leq R$ and $L_RKT < 1$ are satisfied if $R > K\|u_0\|_Y$ and $T < (KL_R)^{-1}$, $(R - K\|u_0\|_Y)/KM_R$. This completes the proof of existence and uniqueness of the solution $u$ and the estimate on it.

Now, we prove that the solution to the initial value problem is continuous with respect to changes in the initial condition $u_0$. Let $u$ and $v$ be the solutions with initial conditions $u_0$ and $v_0$. We estimate

$$\|u - v\|_X \leq \|e^{tA}(u_0 - v_0)\|_X + \int_0^t \|e^{(t-s)A}(f(u(s)) - f(v(s)))\|_X.$$

The estimate of these terms proceeds as above (take $u_1 = u$ and $u_2 = v$) and if $u, v \in B_R$, then

$$\|u - v\|_X \leq K\|u_0 - v_0\|_Y + KTL_R\|u - v\|_X.$$

Thus, if $T$ is as above, then $\|u - v\|_X \leq K(1 - KTL_R)^{-1}\|u_0 - v_0\|_Y$ completing the proof of continuity.

(expand) Finally, assume $[0, t_*)$ is the maximal interval of existence of $u$ and $\sup_{0 \leq t < t_*} \|u(t)\|_Y =: U < \infty$. Take $R = 2KU$ and let

$$\tau := \min((3KL_R)^{-1}, U(3M_R)^{-1}).$$

Then taking $u(t_* - \tau)$ as a new initial condition, we see that the solution exists in the interval $[0, t_* + \tau)$, a contradiction. This proves the dichotomy claimed in the theorem. \qed
Discussion: Generalize (2.17) to
\[ \sup_{t \geq 0} \rho(t) \left\| e^{itA}w \right\|_Y \leq K \left\| w \right\|_Y, \]
for some constant \( K \) and an appropriate positive function \( \rho(t) \).

2.4 Local existence for Hartree and Schrödinger equations

In this subsection we prove the local existence of solutions for Hartree equations. To this end we will use Theorem 2.3 proven in the previous subsection, whose conditions we will verify for the specific equations we consider.

Recall that under the conditions (2.17) and (2.19) for some Banach space \( Y \), the abstract nonlinear equation (2.16) has a weak unique solution \( u \in C([0,T],Y) \), for \( T < (KL_R)^{-1} \), \((R - K\|u_0\|_Y)/KM_R\), in the ball \( \|u\|_{C([0,T],Y)} \leq R \), with \( R > K\|u_0\|_Y \). For the space \( Y \), we will use one of the \( L^p \)-spaces:
\[ L^p(\Omega) := \{ f : \Omega \to \mathbb{C} \mid f \text{ is measurable, and } \int |f|^p dx < \infty \}. \]
(Strictly speaking, elements of \( L^p(\Omega) \) are equivalence classes of measurable functions: two functions define the same elements of \( L^p(\Omega) \) if they differ only on a set of measure 0; see [10, 15, 27].)

The norms on these spaces are defined as
\[ \|f\|_p := \left( \int |f|^p \right)^{1/p} \text{ if } 1 \leq p < \infty, \]
(2.25)

We often use the abbreviations \( L^p \) and \( \|v\|_p \) for \( L^p(\Omega) \) and \( \|v\|_{L^p} \).

Local existence for the Hartree equation. In this subsection we show local existence in \( L^2(\mathbb{R}^n) \) to the initial value problem for the Hartree equation
\[ i \frac{\partial u}{\partial t} = -\Delta u + (v \ast |u|^2)u, \quad u|_{t=0} = u_0, \]
on \( \mathbb{R}^n \), where, as usual \( v \ast f \) is the convolution of two functions, \( v \ast g(x) := \int v(x-y)g(y)dy \). We assume that \( v \) is a 'nice' function, i.e. sufficiently smooth and fast decaying at infinity. Equation (2.26) arises in the problem in quantum physics of many-body systems. In the treatment of this equation we will use \( L^p \)-spaces defined and discussed in [10, 15, 27].

**Theorem 2.4.** Assume \( v \in L^\infty \). Let \( u_0 \in L^2(\mathbb{R}^n) \) and \( R > \|u_0\|_2 \). Then, for \( T < (3R^2\|v\|_\infty)^{-1} \), \((R - \|u_0\|_2) (R^3\|v\|_\infty)^{-1}, \) the Hartree equation (2.26) has a solution \( u \in C([0,T],L^2) \), satisfying \( \|u\|_{C([0,T],L^2)} \leq R \) and unique in the ball \( \|u\|_{C([0,T],L^2)} \leq R \). The solution \( u \) depends continuously on the initial condition \( u_0 \). Furthermore, either the solution is global in time or blows up in \( L^2 \) in a finite time.

**Proof.** We check the conditions of Theorem 2.3 for (2.8) with \( p > 1 \) and \( Y = L^2(\mathbb{R}^n) \). Let \( w_t := e^{it\Delta}w \). Passing to the Fourier transform and using the Plancherel theorem, we obtain \( \|w_t\|_2 = \|\hat{w}_t\|_2 \). Next, since \( \hat{w}_t = e^{-i|\xi|^2t}\hat{w} \), we have \( \|\hat{w}_t\|_2 = \|\hat{w}\|_2 = \|w\|_2 \), which implies the estimate
\[ \|e^{it\Delta}w\|_2 = \|w\|_2, \]
(2.27)
uniformly in $t$. This gives the condition (2.17) with $K = 1$. Next, using the elementary inequality $\|v * g\|_p \leq \|v\|_p \|g\|_1$, $\forall p \geq 1$, we obtain

$$\|(v * |u|^2)u\|_2 \leq \|v\|_\infty \|u\|_3^2.$$ 

Moreover, using the triangle inequality $|(v * |u_1|^2)u_1 - (v * |u_2|^2)u_2| \leq |(v * (|u_1|^2 - |u_2|^2))u_1| + |(v * |u_2|^2)(u_1 - u_2)|$ and again the above inequality, we find

$$\|(v * |u_1|^2)u_1 - (v * |u_2|^2)u_2\|_2 \leq 3\|v\|_\infty (\max_i \|u_i\|_2^2)\|u_1 - u_2\|_2.$$ 

The conditions (2.18) and (2.19) hold, with $M_R = R^3\|v\|_\infty$ and $L_R = 3\|v\|_\infty R^2$. Applying the abstract result, Theorem 2.3, we arrive at the statement of the theorem. \qed

**Problem.** Extend the above theorem from $L^2(\mathbb{R}^n)$ to arbitrary Sobolev spaces $H^s(\mathbb{R}^n)$, $s > 0$, and (separately) from $-\Delta$ to an arbitrary self-adjoint operator $A$ on $L^2(\mathbb{R}^n)$. (For the definition of the Sobolev spaces $H^s(\mathbb{R}^n)$, $s > 0$, see [10, 15, 27].)

**Local existence for nonlinear Schrödinger equation.** Consider the initial value problem for the nonlinear Schrödinger equation (or reaction-diffusion equation):

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + \lambda |\psi|^{p-1}\psi, \quad \psi|_{t=0} = \psi_0,$$  \hfill (2.28)

with an initial condition $\psi_0 \in H^s(\mathbb{R}^n)$. We assume $p > 1$ and $s > 0$. Equation (2.28) arises in nonlinear optics, plasma physics, theory of water waves and in condensed matter physics.

**Problem.** Show local existence for (2.28) in the spaces $H^s(\mathbb{R}^n)$ with $s > n/2$ and $p$, an odd integer. (Hint: Use Sobolev embedding theorems, e.g. $H^s(\mathbb{R}^n) \subset L^{\infty}$ for $s > n/2$, so that $H^s$ is an algebra and so $M_R \leq |\lambda|R^p$ and $L_R \leq |\lambda|pR^{p-1}$.)

**Discussion.** Notice the difference in behaviour between the heat kernel $P_t := e^{t\Delta}$ and the propagator $U_t := e^{it\Delta}$. $e^{t\Delta}$ is a one parameter semi-group, which is well defined on the entire space, say $L^p$, only for $t \geq 0$, while $e^{it\Delta}$ is a one parameter group, defined for all $t$.

The characteristic property of the propagator $U_t$, on the other hand, is its unitarity. In particular it preserves the $L^2$ inner product, $\langle u, v \rangle := \int \bar{u}vdx: \langle u, U_t v \rangle = \langle u, v \rangle$.

More generally, we can consider the generalized nonlinear Schrödinger equation

$$\begin{cases}
i\partial_t \psi = -\Delta \psi + f(\psi), \\
\psi|_{t=0} = \psi_0,
\end{cases} \hfill (2.29)$$

where $\psi : \mathbb{R}^d_x \times \mathbb{R}_t \to \mathbb{C}$ and $f(\psi)$ satisfies $f(0) = 0$. In physical applications and in mathematical theory, we impose also the following condition (significance of which will become clear later)

$$f(e^{i\alpha}\psi) = e^{i\alpha}f(\psi).$$  \hfill (2.30)

A typical example of $f(\psi)$ is a function of the form

$$f(\psi) = g(|\psi|^2)\psi$$  \hfill (2.31)

or, more specifically, $f(\psi) = |\psi|^{p-1}\psi$, which gives (2.28).
3 Strichartz inequalities and well-posedness for the nonlinear Schrödinger equation

To prove the LWP for the generalized nonlinear Schrödinger equation (gNLS), (2.29), in the Sobolev spaces with $s \leq n/2$, we use more sophisticated estimates on the propagator $U(t) = e^{-iH_0 t}$. A class of such estimates are the Strichartz inequalities, which we now formulate and prove. We introduce the spaces

$$L^q L^r := L^q(\mathbb{R}, L^r(\mathbb{R}^d))$$

with the norms

$$\|u\|_{L^q L^r} := \left( \int \left( \int |u(x, t)|^r \, dx \right)^{1/r} \, dt \right)^{1/q}.$$  \hspace{1cm} (3.1)

**Definition 3.1.** The pair of real numbers $(q, r)$ is a $\sigma$-admissible pair iff

$$\frac{1}{2} \geq \frac{1}{r} > \frac{1}{2} - \frac{1}{2\sigma} \quad \text{and} \quad \frac{1}{q} = \sigma \left( \frac{1}{2} - \frac{1}{r} \right).$$  \hspace{1cm} (3.3)

For $1 \leq p \leq \infty$, we denote by $p'$ the index given by the equation $\frac{1}{p} + \frac{1}{p'} = 1$.

**Theorem 3.1** (Strichartz inequalities). If $(q, r)$ and $(\gamma, \rho)$ are two $d/2$-admissible pairs then

$$\|U(\cdot)\phi\|_{L^q L^r} \lesssim \|\phi\|_{L^2}$$

and

$$\|\int_{t_0}^{\infty} U(\cdot - s) f(s) \, ds\|_{L^q L^r} \lesssim \|f\|_{L_{\gamma}^\gamma L_{\rho}^\rho}.$$  \hspace{1cm} (3.4)

**Proof.** The proof is based on the $L^p$-estimates, $\|e^{i\Delta t} \psi_0\|_{p} \leq (4\pi t)^{-d(\frac{1}{2} - \frac{1}{p})} \|\psi_0\|_{p'}$ for the Schrödinger propagator $U(t)$, proven in Appendix A.3, Theorem A.3 and (A.24). Let $F(s, t) := \langle f(s), U(s - t)g(t) \rangle$. Then, these estimate imply

$$|F(s, t)| \lesssim |t - s|^{-\lambda(r)} \|f(s)\|_{L_{\gamma}^\gamma} \|g(t)\|_{L_{\rho}^\rho}$$

where $\lambda(r) = d(\frac{1}{2} - \frac{1}{r})$. Since $(r, q)$ is $\sigma$-admissible, $0 \leq \lambda(r) < 1$ and the Hardy-Littlewood-Sobolev inequality (see below) implies

$$\left| \int \int F(s, t) \, dsdt \right| \lesssim \|f\|_{L_{\gamma}^\gamma L_{\rho}^\rho} \|g\|_{L_{\gamma}^\gamma L_{\rho}^\rho}.$$  \hspace{1cm} (3.5)

Now, using the properties $U(t)$ of the propagator, we write $F(s, t) = \langle U(s)^* f(s), U(t)^* g(t) \rangle$ and therefore

$$\int \int F(s, t) \, dsdt = \langle \int U(s)^* f(s) \, ds, \int U(t)^* g(t) \, dt \rangle,$$

which, together with (3.7), gives the estimate

$$\|\int U(s)^* f(s) \, ds\|_{L^2} \lesssim \|f\|_{L_{\gamma}^\gamma L_{\rho}^\rho}.$$  \hspace{1cm} (3.6)

The latter estimate and (3.4) follow from each other by duality. Indeed, (3.4) and Hölder inequality imply that

$$|\langle \int U(t)^* f(t) \, dt, \phi \rangle| = \left| \int \langle f(t), U(t) \phi \rangle \, dt \right| \lesssim \|f\|_{L_{\gamma}^\gamma L_{\rho}^\rho} \|U(\cdot)\phi\|_{L^q L^r} \lesssim \|f\|_{L_{\gamma}^\gamma L_{\rho}^\rho} \|\phi\|_{L^2}.$$  \hspace{1cm} (3.7)
Due to the formula \( \|u\|_{L^2} = \sup_{\|\phi\|_{L^2}=1} |\langle u, \phi \rangle| \), this, in turn, implies (3.8). Proceeding in the opposite direction one shows that (3.8) implies (3.4). This proves (3.4).

Now we prove (3.5). By the duality this estimate is equivalent to the estimate

\[
\left| \int \int F(s,t) \, ds dt \right| \lesssim \|f\|_{L^\gamma} \|g\|_{L^r}.
\]

(3.10)

By the definition of \( F(s,t) \), we have

\[
\int \int F(s,t) \, ds dt = \int_{t_0}^\infty dt \left( \int_{t_0}^\infty ds \, U(s)^* f(s), U(t)^* g(t) \right)
\]

which gives

\[
\left| \int \int F(s,t) \, ds dt \right| \leq \| \int_{t_0}^\infty ds \, U(s)^* f(s) \|_{L^2} \| g \|_{L^1L^2}.
\]

This, together with (3.8), gives (3.10) for \((\gamma,\rho)\) admissible and \((q,r) = (\infty,2)\). Similarly one proves (3.10) for \((q,r)\) admissible and \((\gamma,\rho) = (\infty,2)\). Equation (3.7) gives (3.10) for \((q,r) = (\gamma,\rho)\). Interpolating between these three cases we arrive at (3.10). Hence (3.5) is proven.

**Hardy-Littlewood-Sobolev inequality.**

\[
\left| \int_{s<t} |t-s|^{-\lambda} f(s)g(t) \, ds dt \right| \lesssim \|f\|_{L^\ell} \|g\|_{L^m}.
\]

(3.11)

provided \( \frac{1}{\ell} + \frac{1}{m} + \lambda = 2, \, \lambda < 1, \, \ell, m > 1 \). Inequality (3.11) follows from the generalized Young inequality \( \| h * g \|_r \leq \|h\|_{w,p} \|f\|_q, \, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \, 1 < p, q, r < \infty \), where

\[
\|h\|_{w,p} = \sup_t (t^p \mu \{ x : |h(x)| > t \})^{1/p} < \infty.
\]

(3.12)

Indeed, we have \( \| |t|^{-\lambda} \|_{w,\frac{1}{\lambda}} < \infty \) and

\[
\left| \int |(t)^{-\lambda} * f)(g(t)) dt \right| \leq \| |t|^{-\lambda} * f \|_{m'} \|g\|_m \leq \| |t|^{-\lambda} \|_{w,\frac{1}{\lambda}} \| f \|_m \|g\|_m.
\]

(3.13)

**Remark.** We have \( \|h\|_{w,p} \leq \|h\|_p \). This follows from the following expression

\[
\|h\|_p = - \int_0^\infty \theta^p dm_f(t), \quad \text{where} \quad m_f(t) := \mu \{ x : |f(x)| > t \}.
\]

(3.14)

Define \( L^p_w := \{ h \text{ is measurable and } \|h\|_{w,p} < \infty \} \). Then, by the above, \( L^p \subset L^p_w \).

**Discussion.** As the proof of Theorem 3.1 shows, it can be generalized as follows.

**Theorem 3.2.** Assume a propagator \( U(t) \) satisfies the estimates

\[
\|U(t)f\|_{L^2} \leq c \|f\|_{L^2}
\]

(3.15)

and

\[
\|U(s)U(t)^* g\|_{L^\infty} \leq c |t-s|^{-\sigma} \|g\|_{L^1}.
\]

(3.16)

Then it satisfies Strichartz Inequalities (3.4) - (3.5) for \( \sigma \)-admissible pairs.
Now, we proceed to the existence result for the gNLS, (2.29).

**Theorem 3.3** (Local well-posedness for gNLS). Assume that the nonlinearity, \( f(\psi) \), satisfies the conditions

\[
|f^{(k)}(\psi)| \lesssim |\psi|^{p-k}, \quad \forall k = 0, \ldots, s + 1,
\]

with \( p < 1 + \frac{4}{d} \). Then (2.29) is LWP in \( H^s(\mathbb{R}) \), for \( s \geq 0 \), on \([0, T]\) for some \( T = T(\|\psi_0\|_{H^s}) > 0\), where \( T(s) \) is a non-increasing continuous function on \([0, \infty)\).

**Proof.** To fix ideas, we conduct the proof of \( s = 0 \). The proof for the general \( s \geq 0 \) is similar. Recall that we reduce (2.29) to the fixed point problem

\[
H(\psi) = \psi,
\]

where the map \( H \) is given by

\[
H(\psi)(x, t) := U(t)\psi_0(x) - i \int_0^t U(t - s)f(\psi(x, s))ds.
\]

Recall that a solution to (3.18) is called the \textit{weak solution} to (2.29). We define the map \( H \) on (balls in) the space \( L^q_T L^r := \mathcal{L}^q(I_T, L^r(\mathbb{R}^d)) \), where \( I_T = [0, T] \), see (3.1).

In what follows \( B_R(X) \) denotes the ball \( B_R(X) := \{u \in X : \|u\| \leq R\} \) in a Banach space \( X \).

**Proposition 3.1.** Under the assumptions of Theorem 3.3, there exist functions \( R = R(\|\psi_0\|_{L^2}) \), such that, for any \( d/2 \)-admissible pairs \( (q, r) \) and \( (a, b) \) satisfying

\[
\frac{1}{2p} \leq \frac{1}{b} < \frac{1}{p} \left( \frac{1}{2} + \frac{1}{d} \right) \quad \text{and} \quad \frac{1}{2} - \frac{1}{d} < \frac{1}{b} \leq \frac{1}{2},
\]

\( H \) is a contraction from \( B_R(L^q_T L^r) \) to \( B_R(L^a_T L^b) \), provided \( T < (cR^{p-1})^{-\frac{1}{p}} \) for an appropriate constant \( c \) given in the proof.

**Proof.** We use the Strichartz estimates (3.4) and (3.5) to obtain for any \( d/2 \)-admissible \( (q, r) \) and \( (\gamma, \rho) \)

\[
\|H(\psi)\|_{L^q_T L^r} \leq \|U(\cdot)\psi\|_{L^q_T L^r} + \left\| \int_0^t U(\cdot - s)f(\psi(s))ds \right\|_{L^q_T L^r} \lesssim \|\psi_0\|_{L^2} + \|f(\psi)\|_{L^a_{T} L^b}. \tag{3.21}
\]

Now we use our assumptions on the nonlinearity to obtain

\[
\|f(\psi)\|_{L^a_{T} L^b} \leq c\|\psi\|^{p}_{L^a_{T} L^{\rho'}} = c\|\psi\|^{p}_{L^a_{T} L^{\rho'}}, \tag{3.22}
\]

Next, we use the elementary inequality

\[
\|f\|_{L^\tilde{p}_{T} L^\tilde{s}} \leq T^{\frac{1}{p'} - \frac{1}{\tilde{p}}} \|f\|_{L^p_{T} L^s} \quad \forall \tilde{p} \geq p, \tag{3.23}
\]

to conclude that

\[
\|H(\psi)\|_{L^q_T L^r} \lesssim \|\psi_0\|_{L^2} + T^\alpha \|\psi(\cdot)\|^{p}_{L^a_{T} L^b}. \tag{3.24}
\]
where \( \alpha := \frac{1}{p'} - \frac{1}{a} > 0, a > p\gamma \) and \( b = p\rho' \). Due to the condition \( p < 1 + \frac{4}{d} \) on \( p \), we can find a \( d/2 \)-admissible pair \((\gamma, \rho)\) s.t. the pair \((a, b)\) defined above is \( d/2 \)-admissible. Indeed, the pairs \((\gamma, \rho)\) and \((a, b)\) should satisfy (3.3), which gives that \( \rho \) and \( p \) should satisfy \( \frac{1}{2} + \frac{1}{b} > \frac{1}{p'} \geq \frac{1}{2}, \frac{b}{2} \geq \frac{1}{p} > p(\frac{1}{2} - \frac{1}{\rho}) \) and \( p < 1 + \frac{4}{d} \), which is possible. Hence if \( \|\psi(\cdot)\|_{L^\gamma L^r} \leq R \), then \( \|H(\psi)\|_{L^\gamma L^r} \leq R \), provided that \( R \) is such that
\[
c\|\psi_0\|_{L^2} + cT^\alpha R^p \leq R. \tag{3.25}
\]

If \( \|\psi_0\|_{L^2} \) sufficiently small, then we can always choose \( R = R(\|\psi_0\|_{L^2}) \) so that (3.25) holds. This bound on \( H \) implies that \( H : B_R(L^\gamma_t L^r) \to B_R(L^\gamma_t L^b) \), provided (3.25) holds, for any \( d/2 \)-admissible pairs \((q, r)\) and \((a, b)\) satisfying (3.20).

Now we prove that \( H \) is a contraction. Proceeding as above and using that the first term on the r.h.s. of (3.19) is independent of \( \psi \), we find
\[
\|H(\psi) - H(\phi)\|_{L^\gamma L^r} = \left\| \int_0^T U(\cdot - s)(f(\psi(s)) - f(\phi(s)))ds \right\|_{L^\gamma L^r} \tag{3.26}
\]
Furthermore, using the condition (3.17) on the nonlinearity and the Hölder inequality \( \|g^{p-1}h\|_{L^{\gamma'} L^{\rho'}} \leq \|g\|_{L^{\gamma'} L^{\rho'}} \|h\|_{L^{\gamma'} L^{\rho'}} \), we obtain
\[
\|f(\psi) - f(\phi)\|_{L^{\gamma'} L^{\rho'}} \lesssim \|(|\psi|^{p-1} + |\phi|^{p-1})(\psi - \phi)\|_{L^{\gamma'} L^{\rho'}} \lesssim \left(\|\psi|^{p-1}_{L^{\gamma'} L^{p\rho'}} + \|\phi|^{p-1}_{L^{\gamma'} L^{p\rho'}} \right) \|\psi - \phi\|_{L^{\gamma'} L^{p\rho'}}. \tag{3.28}
\]
Using again (3.23) and picking \((a, b)\) as in the paragraph after (3.24) gives
\[
\|H(\psi) - H(\phi)\|_{L^\gamma L^b} \leq \gamma \|\psi - \phi\|_{L^\gamma L^b}, \tag{3.29}
\]
where \( \gamma = cT^\alpha R^{p-1} \). Choosing \( T < (cR^{p-1})^{-\frac{1}{\alpha}} \), we achieve \( \gamma < 1 \). This proves the proposition.

Taking \((a, b) = (q, r)\) in the proposition, we conclude that the map \( H \) has a fixed point in the ball \( B_R(L^q_t L^r) \) with \( R = R(\|\psi_0\|_{L^2}) \), provided the pair \((q, r)\) is \( d/2 \)-admissible and satisfies (3.20) (with \( b = r \) of course), and this fixed point depends continuously on the initial condition \( \psi_0 \in L^2(\mathbb{R}^d) \). Thus (2.29) has a weak solution on \( L^q_t L^r \). Since among different \((q, r)\), we can take \((q, r) = (\infty, 2)\), this implies the local well-posedness of (2.29) in \( L^2(\mathbb{R}^d) \).

Thus we have shown that (2.29) is locally well posed in \( L^2(\mathbb{R}^d) \). Now we show that it is locally well posed in \( H^1(\mathbb{R}^d) \). Take the derivative with respect to \( x_j \) of (2.29) to obtain
\[
\partial_{x_j} \psi = U(t)\partial_{x_j} \psi_0 + i \int_0^t U(t - s)f'(\psi(s))\partial_{x_j} \psi(s)ds, \tag{3.30}
\]
where we used here that \( \partial_{x_j} \) commutes with \( U(t) \), i.e., \( \partial_{x_j} U(t) = U(t)\partial_{x_j} \). The latter equation is linear in \( \partial_{x_j} \psi \). Estimating as before we obtain
\[
\|\partial_{x_j} \psi\|_{L^\gamma L^r} \leq c\|\partial_{x_j} \psi_0\|_{L^2} + cT^{\frac{1}{q} - \frac{1}{q'}} \|\psi\|_{L^{\gamma'} L^{r'}} \|\partial_{x_j} \psi\|_{L^{\gamma'} L^{r'}} \tag{3.31}
\]
where \( \hat{q} > q \). Now for \( \|\psi\|_{L^{\gamma'} L^{r'}} \leq R \) and for \( T \) such that \( cT^{\frac{1}{q} - \frac{1}{q'}} R^{q-1} < 1 \) we find that \( \|\partial_{x_j} \psi\|_{L^\gamma L^r} \leq c\|\partial_{x_j} \psi_0\| \). For \( q = \infty \) and \( r = 2 \) this leads to \( \sup_{0 \leq t \leq T} \|\psi\|_{H_1} \leq c\|\psi_0\|_{H_1} \). Hence (2.29) is locally WP in \( H^1(\mathbb{R}^d) \). Continuing in this fashion we can show that (2.29) is locally well-posed in any \( H^s(\mathbb{R}^d) \) for any non-negative integer \( s \) (and therefore for any nonnegative number).

\[\square\]
4 Global existence for the nonlinear Schrödinger equations

Consider the generalized Schrödinger equation (2.29) with \( f(\psi) = g(|\psi|^2)\psi \), which we reproduce here,
\[
i \partial_t \psi = -\Delta_x \psi + g(|\psi|^2)\psi, \tag{4.1}\]
where \( g(u) \) is a real function. We assume that \( f(\psi) = g(|\psi|^2)\psi \) satisfies the conditions (3.17). Then one can show that the energy,
\[
E(\psi) = \int_{\mathbb{R}^d} \{ |\nabla_x \psi|^2 + G(|\psi|^2) \} \, dx, \tag{4.2}
\]
where \( G \) is an anti-derivative of \( g \), \( g(u) = G'(u) \), and the ‘charge’ (or the number of particles),
\[
N(\psi) = \int |\psi|^2. \tag{4.3}
\]
are conserved, \( E(\psi) = \text{const} \) and \( N(\psi) = \text{const} \), during the evolution.

The conservation of the energy and the ‘charge’ are due to the time translation and gauge symmetries. We will discuss this deep relation later in the course. Meanwhile, we observe that, if the nonlinearity is a power one, \( g(|\psi|^2) = \kappa |\psi|^{p-1} \), then the generalized Schrödinger equation has also the scaling symmetry: it is invariant under the transformations, \( \psi(x,t) \to \lambda^{\frac{2}{p-1}} \psi(\lambda x,\lambda^2 t) \):

If \( \psi(x,t) \) is a solution to (2.29) (with \( g(|\psi|^2) = \kappa |\psi|^{p-1} \)), then so is
\[
\psi_\lambda(x,t) := \lambda^{\frac{2}{p-1}} \psi(\lambda x,\lambda^2 t), \tag{4.4}
\]
for any \( \lambda > 0 \). Under this scaling the energy and charge behave as
\[
E(\psi_\lambda) = \lambda^{\frac{4}{p-1}+2-d} E(\psi) \quad \text{and} \quad N(\psi_\lambda) = \lambda^2 N(\psi).
\]
We see that we have two exponents, \( p_* = 1 + \frac{4}{d-2} \) and \( p_{**} = 1 + \frac{4}{d} \), at which the scaling behaviour of the energy and charge changes. For \( p > p_* \), as \( \lambda \to \infty \), the energy and charge decrease \( E(\psi_\lambda) \to 0 \) and \( N(\psi_\lambda) \to 0 \), while the function \( \psi_\lambda \) becomes rougher and rougher, etc.

**Global existence of solutions.** We now turn to the question of global well-posedness i.e., well-posedness for all \( t \)'s. If \( G \geq 0 \) (the non-focusing nonlinearity), then the conservation of energy, implies \( \|\nabla_x \psi(t)\|_{L^2} \leq E(\psi) = E(\psi_0) \). Using, in addition, the conservation of charge, \( N(\psi) = N(\psi_0) \), we find \( \|\psi(t)\|_{H^1} \leq E(\psi_0) + N(\psi_0) \). Hence \( \psi(t) \) is uniformly bounded in the Sobolev norm. Thus if we have the local existence result of the type of Theorem 3.3 in the Sobolev space \( H^1(\mathbb{R}^n) \), then we also have the global one.

In the more difficult case \( G \leq 0 \) (the focusing nonlinearity), we have to impose additional conditions on the growth of nonlinearity. We follow [33].

**Theorem 4.1 (GWP).** Assume (3.17) with \( 1 \leq p < \frac{d+2}{d} \) and \( s = 1 \). Then (4.1) is GWP in \( H^1(\mathbb{R}^d) \).

On the interval \([0,T)\) on which problem (4.1) is well-posed we define the map
\[
\phi_t : H^1(\mathbb{R}^d) \mapsto H^1(\mathbb{R}^d) \tag{4.5}
\]
by
\[
\phi_t(\psi_0) = \psi(t) \tag{4.6}
\]
where \( \psi(t)(x) := \psi(x, t) \) is the unique solution to (4.1) with the initial condition \( \psi_0 \in H^1(\mathbb{R}^d) \). The family \( \{ \phi_t \}_{t \geq 0} \) is called the flow (for the nonlinear Schrödinger equation (2.29)). It has the following properties

\[
(a) \; \phi_0 = \mathbb{I}, \quad (b) \; \phi_t \circ \phi_s = \phi_{t+s},
\]

(4.7)

(with appropriate restrictions on \( t, s \) and \( t+s \)).

**Exercise 1.** Prove (a),(b). Hint: To prove property (b) show that the functions \( \phi_t(\phi_s(\psi(0))) \) and \( \phi_{s+t}(\psi(0)) \) satisfy (2.29) in \( t \) with the same initial condition \( \phi_s(\psi(0)) \).

**Proof of Theorem 4.1.** Idea: Use property (b) and an a priori bound on \( T(\| \phi \|_{H^1}) \). In the previous theorem we have shown that (2.29) is LWP on an interval \([0, T_0]\) where \( T_0 = T(\| \psi_0 \|_{H^1}) \).

Now consider (2.29) with the new initial condition \( \psi(T_0) \). It is WP on the interval \([0, T_1]\) where \( T_1 = T(\| \psi(T_0) \|) \). This solution can be written as

\[
\phi_t(\psi(T_0)) = \phi_t(\phi_{T_0}(\psi(0))).
\]

Motivated by property (b) of flows we construct the solution on \([0, T_0 + T_1]\) as

\[
\phi_t(\psi(0)) = \begin{cases}
\phi_t(\psi(0)) & \text{if } 0 \leq t \leq T_0 \\
\phi_{T_0-T_0}(\phi_{T_0}(\psi(0))) & \text{if } T_0 \leq t \leq T_0 + T_1
\end{cases}
\]

(4.9)

**Exercise 2.** Check that (4.9) solves (2.29) on \([0, T_0 + T_1]\) and is unique.

Thus, we found the solution \( \phi_t(\psi(0)) \) to (2.29) on the interval \([0, T_0 + T_1]\). We can continue in this way and obtain the unique solution to (2.29) with an initial condition \( \psi_0 \in H^1(\mathbb{R}^d) \) on the interval

\[
[0, \sum_{n=0}^n T_k], \quad \text{where } T_k = T(\| \psi(T_{k-1}) \|_{H^1}), \quad \forall k \geq 1
\]

(4.10)

for any \( n \). The question now is: does \( \sum_{k=0}^n T_k \to \infty \) or \( \sum_{k=0}^n T_k \to T < \infty \) for some \( T < \infty \) as \( n \to \infty \)? In the first case we will have the WP on the interval \([0, \infty)\), i.e., GWP, and in the second case we have the WP on the interval \([0, T]\), i.e., only a LWP. Thus our task is to estimate the interval lengths \( T_k \). To this end we use the conservation laws for the energy (4.2) and for the charge (4.3).

**Proposition 1.** Let \( p < 1 + \frac{4}{d} \). Define the function \( J(h, n) := C_{d,p}(h + n + n^{\frac{2d}{2d-2}}) \), where \( b := \frac{d(p-1)}{2} - 1 \) and \( c := p + 1 - \frac{d(p-1)}{2} \). Then

\[
\| \psi \|_{H^1}^2 \leq J(H(\psi), N(\psi)).
\]

(4.11)

**Proof.** For \( f(\psi) = g(|\psi|^2)\psi \) satisfying (3.17) and \( F(\psi) = G(|\psi|^2) \), we have

\[
|F(\psi)| \leq c|\psi|^{p+1}
\]

(4.12)

We have by the Gagliardo-Nirenberg’s inequality

\[
\left( \int |\psi|^{p+1} \right)^{\frac{1}{p+1}} \leq C\| \psi \|_{H^1}^a \| \psi \|_{L^2}^{1-a},
\]

(4.13)

where \( a = \frac{d(p-1)}{2} \frac{1}{p+1} \). Then

\[
\int |F(\psi)| \leq C\| \psi \|^b_{H^1} \| \psi \|^c_{L^2}
\]

(4.14)
where $b := \frac{d(p-1)}{2}$ and $c := p + 1 - \frac{d(p-1)}{2}$. This gives
\begin{equation}
E(\psi) \geq \frac{1}{2} \int \left( |\nabla \psi|^2 - C \|\psi\|_{H^1} \|\psi\|_{L^2}^c \right) = \frac{1}{2} \|\psi\|_{H^1}^2 - C \|\psi\|_{H^1}^b \|\psi\|_{L^2}^c - \frac{1}{2} \|\psi\|_{L^2}^2. \tag{4.15}
\end{equation}
Now, since $p < 1 + \frac{4}{d}$, we have that $b < 2$ and therefore
\begin{equation}
\frac{1}{4} \|\psi\|_{H^1}^2 - C \|\psi\|_{H^1}^b \|\psi\|_{L^2}^c \geq -C' \|\psi\|_{L^2}^{2r} \tag{4.16}
\end{equation}
The last two inequalities give (4.11).

Thus if $\psi$ is a solution to (2.29) with the initial condition $\psi_0$ we have by the conservation of energy and number of particles that
\begin{equation}
\|\psi\|_{H^1} \leq M_0, \text{ where } M_0 := J(H(\psi_0), N(\psi_0)). \tag{4.17}
\end{equation}
This means all $T_k$’s can be taken the same, $T_k = T(M_0)$, and therefore $\sum_{n=0}^n T_k = nT(M_0) \to \infty$ as $n \to \infty$. This implies the GWP of (2.29).

### 4.1 Relation between the global existence of evolution equations and functional inequalities

- the critical nonlinear Schrödinger equations vs the Nirenberg - Gagliardo inequality, $\|f\|_q \leq c_{p,n} \|\nabla f\|^2 \|f\|^p$, $n \geq 2$, $1 \leq p \leq q \leq r(n)$, $\frac{1}{q} = \frac{\theta}{r(n)} + \frac{1-\theta}{p}$ and $r(n) := \frac{2n}{n-2}$;
- the fast diffusion equation, $\partial_t u = \Delta u^m$, $0 < m < 1$, vs the Hardy-Littlewood-Sobolev inequality, $\int \int \frac{f(x)f(y)}{|x-y|} dxdy \leq \|f\|^2_p$;
- the (reduced) Keller-Segel or nonlinear Fokker-Planck equation,
\begin{align}
\frac{\partial \rho}{\partial t} &= \Delta \rho - \nabla \cdot (\rho \nabla c), \\
0 &= \Delta c + \rho, \tag{4.18}
\end{align}
with $\rho$ and $c$ describing the organism density and chemical concentration, vs the logarithmic Hardy-Littlewood-Sobolev inequality, $-\frac{2}{M} \int \int f(x)f(y) \log |x-y| dxdy \leq -\int f \log f + C(M)$, where $M := \int f$ (the dimension $n = 2$).

**UNDER CONSTRUCTION**

### 5 Elements of Variational Calculus

In this section, we introduce an important structure into the manifold of partial differential equations - the variational structure. It distinguishes a class of partial differential equations for which we can provide a uniform treatment of existence and stability problems.
5.1 Functionals

Functionals are maps which have \( \mathbb{R} \) as the target space. More precisely, let \( X \) be a vector space, and \( M \subset X \), a not necessarily open subset of \( X \). Then a functional is a map \( \mathcal{E} : M \to \mathbb{R} \). Usually, \( X \) is a space of functions. If \( X \) has a basis, then functionals on \( X \) can be represented as functions of an infinite number of coordinates along the basis. If \( X \) is a finite–dimensional space (which we are not concerned with here), then a functional on \( X \) is just a usual function of several variables. In the following list of examples, \( \Omega \) is a domain in \( \mathbb{R}^n \) and \( G : \mathbb{R} \to \mathbb{R} \), \( u : \Omega \to \mathbb{R} \), or \( u : \Omega \to \mathbb{C} \):

1) \( \mathcal{E}(u) = \int_{\Omega} G(u(x))d^n x \),

2) \( \mathcal{E}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 d^n x \),

3) \( \mathcal{E}(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + G(u) \right) d^n x \).

We also have to specify the spaces on which the functionals are defined. The (not necessarily linear or vector) spaces are chosen according to the specific functional and the problem at hand. Of course, we always try to choose the simplest possible space for a given problem.

For instance consider example 1). Assume that the domain \( \Omega \) is bounded, and that the function \( G \) satisfies the estimate

\[
|G(u)| \leq C |u|^p + C,
\]

for some constant \( C > 0 \). It is then natural to define \( \mathcal{E} \) on the space \( L^p(\Omega) \).

In example 2), we define \( \mathcal{E} \) on the Sobolev–space \( H^1(\Omega) \), defined in Section ??. In example 3), if \( \Omega \) is bounded, and \( G \) satisfies (5.1), then we define \( \mathcal{E} \) on \( H^1(\Omega) \cap L^p(\Omega) \).

An important example of functionals are linear functionals on a vector space \( X \). This is the special case of linear operators when the target space is \( Y = \mathbb{C} \). If \( X \) is a normed space, then we consider bounded or continuous linear functionals, for which the following norm is finite:

\[
||l|| = \sup_{||x||=1} |l(x)|.
\]

5.2 The Gâteaux derivative for functionals

Consider a functional \( \mathcal{E} : M \to \mathbb{R} \). If \( M \) is an open subset of a vector space \( X \), then the Gâteaux derivative \( d\mathcal{E}(u), u \in M \), is a linear functional on \( X \) defined, for every \( \xi \in X \), as

\[
d\mathcal{E}(u)\xi = \frac{\partial}{\partial \lambda} \mathcal{E}(u_\lambda)|_{\lambda=0},
\]

where \( u_\lambda := u + \lambda \xi \) if the latter derivative exists. We say that \( \mathcal{E} \) is \( C^1 \) in \( U \subset M \) if and only if \( \forall u \in U, d\mathcal{E}(u) \) is a bounded linear functional.

Let us now consider a simple example. Let \( G \) be a real differentiable function on \( \mathbb{R} \) satisfying the estimate

\[
|G(u)| + |G'(u)| \leq C |u|^2,
\]

where \( C \) is independent of \( u \in \mathbb{R} \), and let \( \Omega \subset \mathbb{R}^n \). Then the functional

\[
V(u) := \int_{\Omega} G(u(x))d^n x
\]
is defined on $L^2(\Omega)$. Using the definition, $dV(u)\xi = \frac{\partial}{\partial \lambda} V(u + \lambda \xi)|_{\lambda=0}$, we compute its Gâteaux derivative

$$
\left( d \int_{\Omega} G \circ u \, d^nx \right) \xi = \int_{\Omega} G'(u(x)) \xi(x) d^nx.
$$

Thus the Gâteaux derivative in this case is the linear functional standing on the r.h.s..

**Exercise 3.** Compute the Gâteaux derivatives in examples 1)–3).

### 5.3 Critical points and connection to PDEs

Let $M$ be an open set in a real vector space. Given a $C^1$–functional $E : M \to \mathbb{R}$, we say that $u_* \in M$ is a critical point (CP) of $E$ if and only if $dE(u_*) = 0$.

**Exercise 4.** Find the equations for the critical points in examples 1)–3) given at the beginning of this section.

The equation $dE(u_*) = 0$ (or, in detail, $dE(u_*)\xi = 0$, for every $\xi \in X$) for critical points of $E$ is sometimes called the Euler or Euler-Lagrange equation. Let us discuss very briefly the Euler-Lagrange equations for the functional

$$
E(u) = \int_{\mathbb{R}^n} \left( \frac{1}{2} |\nabla u|^2 + uf \right) d^n x,
$$

which is a modification of the Dirichlet functional in example 2). We consider this functional on the Sobolev space $H^1(\mathbb{R}^n)$. We compute

$$
dE(u)\xi = \frac{\partial}{\partial \lambda} E(u_\lambda)|_{\lambda=0} = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla \xi + f\xi) d^n x,
$$

where $u_\lambda \in H^1(\mathbb{R}^n)$ s.t. $u_0 = u$ and $\frac{\partial}{\partial \lambda} u_\lambda|_{\lambda=0} = \xi$. Hence the Euler-Lagrange equation, $dE(u)\xi = 0$, reads

$$
\int_{\mathbb{R}^n} (\nabla u \cdot \nabla \xi + f\xi) d^n x = 0, \forall \xi \in H^1(\mathbb{R}^n). \quad (5.5)
$$

If $u$ is twice differentiable, then we can integrate by parts to find

$$
dE(u)\xi = \int_{\mathbb{R}^n} (-\Delta u + f)\xi.
$$

Hence the Euler-Lagrange equation becomes $\int_{\mathbb{R}^n} (-\Delta u + f)\xi = 0, \forall \xi \in H^1(\mathbb{R}^n)$ and therefore

$$
\Delta u = f \quad \text{in } \mathbb{R}^n. \quad (5.6)
$$

This is the classical Poisson equation.

The above discussion motivates the following definition: We say that critical points of $E$ are called weak solutions to the Euler-Lagrange equation

$$
dE(u_*)\xi = 0, \forall \xi \in X. \quad (5.7)
$$

Thus solutions of the equation (5.5) are weak solutions to (5.6).
Discussion. If \( f \) is \( k \) times differentiable, then one can show that \( u \) is \( k+2 \) times differentiable. This property is called the elliptic regularity. One argues in the following way. Assume e.g. \( \Omega \) is a sufficiently regular domain and \( f \in H^k(\Omega) \). By the Poisson equation (5.6), we have that \( \Delta u \in H^k(\Omega) \) and therefore \( u \in H^{k+2}(\Omega) \). If \( k+2 > n/2 \), then by the Sobolev embedding theorem, \( u \in C^\alpha(\Omega) \), with \( \alpha < k + 2 - \frac{n}{2} \), where \( C^\alpha(\Omega) \) is the space of \( [\alpha] \)-times differentiable functions whose \( \lfloor \alpha \rfloor \) derivatives are Hölder continuous with the Hölder exponent \( \alpha \). (Here \( [\alpha] \) is the largest integer \( \leq \alpha \).) (The Sobolev embedding theorem states that \( H^2(\Omega) \subset C^\alpha(\Omega) \), with \( \alpha < k + 2 - \frac{n}{2} \).)

If we consider the similar functional on a domain \( \Omega \subset \mathbb{R}^n \), rather than the entire \( \mathbb{R}^n \), i.e.

\[
\mathcal{E}(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + uf \right) d^n x, \tag{5.8}
\]

then we have to be more careful. We first consider this functional on the Sobolev space \( H^1_0(\Omega) \) of functions in \( H^1(\Omega) \), which are equal to a given function \( g \) on \( \partial \Omega \). (Since the boundary \( \partial \Omega \) has \( n \)-dimensional Lebesgue-measure zero, we have to be careful about the meaning of “\( u = g \) on \( \partial \Omega \)”.)

We compute

\[
d\mathcal{E}(u)\xi = \frac{\partial}{\partial \lambda}\mathcal{E}(u_\lambda)|_{\lambda=0} = \int_{\Omega} (\nabla u \cdot \nabla \xi + f \xi) d^n x,
\]

where \( u_\lambda \in H^1(\Omega) \) s.t. \( u_0 = u \) and \( \frac{\partial}{\partial \lambda} u_\lambda|_{\lambda=0} = \xi \). Integrating by parts, we find

\[
d\mathcal{E}(u)\xi = \int_{\Omega} (-\Delta u + f)\xi + \int_{\partial \Omega} \frac{\partial u}{\partial \nu}\xi,
\]

where \( \nu \) is the outward unit normal vector to \( \partial \Omega \). Since our functions \( u \) are fixed (to \( g \)) on \( \partial \Omega \), we must take \( \xi \) vanishing on \( \partial \Omega \) (so that to have \( u_\lambda = u + \lambda \xi \) fixed at the boundary). Hence the Euler-Lagrange equation, \( d\mathcal{E}(u)\xi = 0 \), reads \( \int_{\Omega} (-\Delta u + f)\xi = 0 \), for every \( \xi \in H^1(\Omega) \) that vanishes on \( \partial \Omega \), and therefore

\[
\Delta u = f \quad \text{in } \Omega \\
u = g \quad \text{on } \partial \Omega. \tag{5.9}
\]

This is the Dirichlet boundary value problem.

If we consider the functional (5.4) on \( H^1(\Omega) \), then \( \xi \) is an arbitrary function in \( H^1(\Omega) \) and varying its values in \( \Omega \) and on \( \partial \Omega \) independently, we see that the Euler-Lagrange equation \( d\mathcal{E}(u)\xi = 0 \), \( \forall \xi \in H^1(\Omega) \), becomes

\[
\Delta u = f \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\]

This is the Neumann boundary value problem. Solutions of these two problems are rather different, so we see that the space on which a variational problem is considered plays an important role.

As another example we consider the problem of minimal surfaces from differential geometry. Let \( A(f) \) be the area of the hypersurface \( S \) in \( \mathbb{R}^{n+1} \) given as a graph of a function \( f : \Omega \rightarrow \mathbb{R} \), defined on an open subset \( \Omega \) of the hyperplane \( x_{n+1} \) in \( \mathbb{R}^{n+1} \): \( S = \text{graph} f := \{(x, f(x))|x \in \Omega \} \).

It is given by the formula \( A(f) = \int_{\Omega} \sqrt{1 + |\nabla f|^2} d^n x \) and (see Appendix to this section). We define \( A \) on \( C^2(\Omega) \). Critical points of the functional \( A(f) \) are called minimal surfaces. We have
Proposition 5.1. The equation for a minimal surface given locally as a graph of a function \( f \) is given by

\[
\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0.
\]  

(5.10)

Exercise 5. Show that the Euler-Lagrange equation for \( A(f) \) is given by (5.10).

The expression on the l.h.s. of (5.10) is the mean curvature of \( S \) at \( x = (x', f(x')) \),

\[
H(x) := \text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right).
\]  

(5.11)

Hence the equation for a minimal surface states that the mean curvature on it vanishes. Sometimes this equation is taken for a definition of the minimal surface.

One can also define surface in \( \mathbb{R}^{n+1} \) locally via a parametrization \( S = \text{Image}(\phi) \subset \mathbb{R}^{n+1} \), where \( U \) is an open subset of \( \mathbb{R}^n \) and \( \phi : U \to \mathbb{R}^n \) (parametrization). This is a parametric surface. Then the area \( A(\phi) \) of \( S \) is given by the functional \( A(\phi) = \int_U \left| \frac{\partial \phi}{\partial x_1} \wedge \frac{\partial \phi}{\partial x_2} \right| d^2 x \).

Action and Lagrange functionals. We give now more examples of functionals:

4) \( S(\varphi) = \int_0^T \left( \frac{1}{2} |\frac{\partial \varphi}{\partial t}|^2 - V(\varphi) \right) dt \), where \( \varphi : [0, T] \to \mathbb{R}^m \), and \( V : \mathbb{R}^m \to \mathbb{R} \);

5) \( S(\phi) = \int_0^T \int_{\Omega} \left( \frac{1}{2} |\frac{\partial \phi}{\partial t}|^2 - \frac{1}{2} |\nabla \phi|^2 - G(\phi) \right) d^n x dt \), where \( \phi : \Omega \times [0, T] \to \mathbb{R} \).

In these examples, the functionals come from expressions for an action.

Exercise 6. Compute the Gâteaux derivatives and find the Euler-Lagrange equation for in examples 4)-8).

The Euler-Lagrange equation in the 4) and 5) examples are Newton’s equation and the classical relativistic field theory. In the second case,

\[
\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi - G'(\phi) = 0.
\]

In conclusion, we give examples of functionals from geometry and image recognition:

6) \( L(\gamma) = \int_0^T |\gamma'(t)| dt \), where \( \gamma : [0, T] \to \mathbb{R}^m \);

7) \( \mathcal{E}(u) = \int_{\Omega} \left( |u - f|^2 + \lambda |\nabla u|^2 \right) d^n x \), where \( u, f : \Omega \to \mathbb{R} \) (\( f \) is fixed) and \( \lambda \geq 0 \),

8) \( \mathcal{E}(u, \gamma) = \int_{K \setminus \gamma} \left( |u - f|^2 + \lambda |\nabla u|^2 \right) d^2 x + \mu L(\gamma) \), where \( u, f : K \to \mathbb{R} \) (\( f \) is fixed), \( \lambda, \mu \geq 0 \), and \( \gamma \) is a closed curve in \( K \), i.e. \( \gamma : [0, 1] \to K \) and \( \gamma(0) = \gamma(1) \).

In these examples, \( L(\gamma) \), is the length of a curve \( \gamma \) given parametrically as \( \gamma : [0, T] \to \mathbb{R}^m \) (in example 6) and as \( \gamma : [0, 1] \to K \subset \mathbb{R}^2 \) (in example 8), \( \mathcal{E}(u) \) is an error functional measuring how much a smooth function \( u \) differs from a given function \( f \), and \( \mathcal{E}(u, \gamma) \) is the Mumford-Shah functional in image segmentation (\( f : K \to \mathbb{R} \) is a given image, \( u \) is a piecewise smooth approximation of \( f \), with jumps across \( \gamma \) are allowed, and \( \gamma \) is a segmentation of the image \( f \) into two regions: interior of \( \gamma \) and exterior of \( \gamma \)).

The following elementary but important result connects the main problem of variational calculus to the problem of existence of solutions of differential equations.
Theorem 5.1. Let $F$ be a functional on an open subset $M$ of a vector space $X$. If $u_0 \in M$ is a minimizer of $F$, then $u_0$ is a critical point of $F$.

**Proof.** Let $\xi$ be an arbitrary vector from $X$, and $\lambda$ sufficiently close to 0 so that there is a curve $u_\lambda$ s.t. $u_{\lambda=0} = u_0$ and $\frac{du_\lambda}{d\lambda}|_{\lambda=0} = \xi$. Then the function $f(\lambda) := F(u_\lambda)$ has a minimum at $\lambda = 0$, and therefore $\lambda = 0$ is a critical point of this function, $f'(0) = 0$. This is equivalent to $\frac{\partial}{\partial \lambda} F(u_\lambda)|_{\lambda=0} = 0$, which by the definition of the Gateaux derivative implies that $dF(u_0) \xi = 0$. This holds for every $\xi \in X$, and we conclude that $dF(u_0) = 0$ as a functional on $X$. \qed

A powerful method of proving existence of solutions of a large class of static equations is showing that these equations are Euler - Lagrange equations of certain functionals and then proving existence of minimizers (or saddle points) by the direct methods of variational calculus. This will be done in Section 6 below.

**Wave maps or the sigma model.** In this paragraph we describe the sigma model, which plays an important role in particle and condensed matter physics. In the former, it is used as a model for strong interactions and in the latter, in the theory of two-dimensional magnetism. In both cases, the model describes the low-energy excitations around broken symmetry ground state. Since uniform changes in the (classical) vacuum space costs nothing, the model involves only space-time derivatives of the excitation fields.

The sigma model is a classical field theory with a field, $\Phi$, which is a map from a $(d + 1)$-dimensional Minkowski space-time, $M^{d+1}$, with the Minkowski metric $\eta = \text{diag}(1, -1, \ldots, -1)$, to a Riemannian manifold, $N$, with a metric $(g_{ab})$, and the action functional, given as

$$S(\Phi) := \frac{1}{2} \int_{M^{d+1}} \langle \partial_\mu \Phi, \partial^\mu \Phi \rangle. \quad (5.12)$$

Here $\langle \partial_\mu \Phi, \partial^\mu \Phi \rangle$ is the Riemann scalar product in $N$, which, in local coordinates, is $g_{ab} \partial_\mu \Phi^a \partial^\mu \Phi^b$, as usual, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial^\mu := \eta^{\mu\nu} \partial_\nu$, and we assume the summation over repeated indices $\mu = 1, \ldots, d + 1$, $a, b, c = 1, \ldots, \text{dim}(N)$, $i, j = 1, \ldots, d$. Critical points of $S(\Phi)$ satisfy the Euler-Lagrange equation

$$\partial_\mu \partial^\mu \Phi^a + \Gamma^a_{be}(\Phi) \partial_\mu \Phi^b \partial^\mu \Phi^c = 0, \quad (5.13)$$

where $\Gamma^a_{be}(\Phi)$ is the Christoffel symbols on $N$. Solutions of this system of nonlinear PDEs, i.e. critical point of the action functional (5.12), are called wave maps.

**Hessians and local minimizers.**

5.4 Convexity and uniqueness

Recall that a set $A$ is said to be convex iff for every $u, v \in A$, we have $\lambda u + (1 - \lambda)v \in A$ and a functional $F : A \to \mathbb{R}$ is convex iff

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y)$$

and $f$ is said to be strictly convex iff

$$F(\lambda x + (1 - \lambda)y) < \lambda F(x) + (1 - \lambda)F(y).$$

We have the following standard result
Proposition 5.2. Let \( A \) be a convex set. Then

- \( F : A \to \mathbb{R} \) be a convex \( C^1 \)-function iff \( F(x) - F(y) \geq dF(y)(x - y) \)

and similarly, for the strictly convex functionals.

Proof. We give a proof in one direction. If \( F \) is convex, then so is \( g(s) := F(u + sv) \), \( \forall v \in X \). Hence the function \( g'(s) := \partial_s F(u + sv) \) is monotonically non-decreasing. Let \( u' = u' - u \). Then

\[
F(u') - F(u) = F(u + (u' - u)) - F(u) = \int_0^1 ds \partial_s F(u + sv) = \int_0^1 ds [\partial_s f(s) - \partial_s f(0)] + \partial_s f(0).
\]

By the monotonicity of \( \partial_s f(s) \), we have that \( F(u') \geq F(u) + \partial_s f(0) \). Now, by the definition of \( dF(u) \), we have \( \partial_s f(0) = dF(u)(u' - u) \). The last two relations give \( F(u') \geq F(u) + dF(u)(u' - u) \).

\[ \Box \]

Theorem 5.2. Let \( A \) be a convex set and \( F : A \to \mathbb{R} \) be a convex \( C^1 \)-function. We have

(i) If \( a \) is a critical point of \( F \), then \( a \) is a minimizer.

(ii) The set of all minimizers of \( F \) is a convex set.

(iii) If \( F \) is strictly convex, then it has at most one minimizer.

Proof. (i) and (iii) follow from Proposition 5.2 with \( y = a \) and the relation \( dF(a) = 0 \) and (ii), from the definition of the convexity.

Corollary 5.3. Weak solutions of the Laplace and Poisson equations with the Dirichlet boundary conditions are unique.

5.5 Constraints and Lagrange multipliers

Now we consider minimization of a functional \( E(u) \), on a set, \( M \), defined by side conditions or constraints, i.e., \( M \) is of the form

\[
M := \{ u \in X \mid J(u) = 0 \},
\]

where \( J \) is a functional on \( X \). Such a set can be thought of as an infinite dimensional hypersurface in \( X \). The restriction \( J(u) = 0 \) is called the constraint. Such problems appear naturally in applications.

Examples.

(1) Minimize the energy for a fixed entropy, or total mass, or total number of particles.

(2) Maximize output for a fixed cost.

Problems of minimizing functionals on sets of the form (5.14) also come up in proving existence of weak solutions of partial differential equations. This will be demonstrated later.

Since \( M \) is not open in \( X \), the definition of the Gâteaux derivative is more subtle, as we cannot in general take pieces of straight lines \( u_\lambda = u + \lambda \xi \) in the definition of \( dF(u) \). For \( M \) is not open in \( X \), to define the Gâteaux derivative, we take paths \( \lambda \mapsto u_\lambda \), s.t. \( u_0 = u \). Here \( \lambda \in [-\epsilon, \epsilon] \) for some \( \epsilon = \epsilon(u) \) sufficiently small. Then, for \( \xi \in X \) s.t. there is an \( \epsilon > 0 \), and a differentiable path \([ -\epsilon, \epsilon] \ni \lambda \mapsto u_\lambda \in M \) for which \( u_0 = u, \frac{du_\lambda}{d\lambda}|_{\lambda=0} = \xi \), we define

\[
dF(u)\xi := \frac{d}{d\lambda} F(u_\lambda)|_{\lambda=0}.
\]
The set of all the $\xi \in X$ s.t. there is an $\epsilon > 0$, and a differentiable path $[-\epsilon, \epsilon] \ni \lambda \mapsto u_\lambda \in M$ for which $u_0 = u, \frac{d u_\lambda}{d \lambda} |_{\lambda = 0} = \xi$ is called the tangent space to $M$ at $u, T_u M$. Thus, $dF(u) : T_u M \to \mathbb{R}$. By the definition, we have that $dF(u) \in (T_u M)'$. Here $X'$ is the space of linear functionals on $X$, called the dual space to $X$. The dual space to $T_u M$ is called the cotangent space to $M$ at $u$ and is denoted by $T_u^* M := (T_u M)'$

As before the points $u \in M$ for which $dF(u) = 0$ are called the critical points of $F$.

Now we consider critical points of a functional $E : X \to \mathbb{R}$ on the set (5.14) where $J$ is another functional on $X$. The key result here goes back to Lagrange and it is called the method of Lagrange multipliers:

**Theorem 5.4.** Let $E$ and $J$ be $C^1$ functionals, and let $u_0$ be a critical point of $E$ on $M$ defined by (5.14). Then there is a $\lambda \in \mathbb{R}$ (the Lagrange multiplier) s.t. $dE(u_0) - \lambda dJ(u_0) = 0$ on $X$.

**Proof.** Let $u_s$ be a differentiable path in $M$ (i.e. a differentiable path in $X$ satisfying $J(u_s) = 0$; existence of such paths is shown below in Proposition 5.3) starting at $u_0$, i.e. $u_{s=0} = u_0$ and let $\frac{du_s}{ds}|_{s=0} = \xi$. Differentiating $J(u_s) = 0$ in $s$ at $s = 0$ gives $d(J(u_0)\xi) = 0$, which implies that $\xi \in \text{null } J(u_0)$.

Moreover, the fact that $u_0$ is a minimizer of $E$ in $M$ implies that $s = 0$ is a minimizer of $E(u_s)$ in $s$, which means that $\frac{dE(u_s)}{ds}|_{s=0} = 0$. The latter equation gives $dE(u_0)\xi = 0, \forall \xi \in \text{null } J(u_0)$, (5.15)
i.e. $dE(u_0)$ is perpendicular to the subspace $dJ(u_0)^\perp := \text{null } J(u_0)$ and therefore $dE(u_0)$ is parallel to $dJ(u_0)$. In detail, for all $\eta, e \in X, e \notin \text{null } J(u_0)$, $\xi := \eta - \frac{dJ(u_0)}{dJ(u_0) e} e \in \text{null } dJ(u_0)$ and therefore

$$0 = dE(u_0)\xi = dE(u_0)\eta - \lambda dJ(u_0)\eta$$

with $\lambda = \frac{dE(u_0)\eta}{dJ(u_0)\eta}, \forall \eta \in X$. Hence $dE(u_0) - \lambda dJ(u_0) = 0$ on $X$ follows. (We showed that if $dE(u_0)$ is perpendicular to $dJ(u_0)$, then $dE(u_0)$ is parallel to $dJ(u_0)$).

**Example.** Below $\Omega$ is a domain in $\mathbb{R}^n$. Consider the Dirichlet functional

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 d^n x$$
on the $M = \{u \in H^1_0(\Omega) : J(u) = 1\}$, where $J(u) = \frac{1}{p} \int_{\Omega} |u|^p d^n x$. To find the equation of the critical points on the space $M$, we use the theorem. Since $dE(u) = -\Delta u$, and $dJ(u) = |u|^{p-2}u$, the theorem implies that $u$ satisfies

$$\Delta u + \lambda |u|^{p-2}u = 0 \text{ in } \Omega,$$

$$u = g \text{ on } \partial \Omega.$$

Compare this with the equation (6.4), with $f = 0$, for the problem on $H^1_0(\Omega)$.

**Exercise 5.1.** Find the equation for critical points of the following functionals:

1. $\frac{1}{2} \int_{\Omega} |\nabla u|^2$ on the space $M = \{u \in H^1_0(\Omega) : \int_{\Omega} |u|^2 = 1\}$;

2. $E(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + G(u) \right) d^n x$, on the space $M = \{u \in H^1(\Omega) : \int u f = a\}$ for some $a \in \mathbb{R}$ and $f \in C(\Omega)$;
(3) the mean energy \( E(f) = \int h |f| d^n x d^n p \) of a classical particle distribution \( f : \mathbb{R}^n \rightarrow \mathbb{R}^+ \), for entropy; \( S(f) = \int f \log f d^n x d^n p \), where \( h(x, p) \) is a classical Hamiltonian of the system.

There is a quantum analogue of the problem 3), with the classical particle distribution \( f \) replaced by the density operator \( \rho \), the mean energy given by \( E(\rho) = \text{Tr}(H\rho) \), where \( H \) is a quantum Hamiltonian, and the Boltzmann entropy \( S(f) \) is given by the Boltzmann entropy \( S(\rho) := -\text{Tr}(\rho \log \rho) \).

**Isoperimetric problem.** The celebrated isoperimetric problem is differential geometry requires to minimize the surface area of a closed surface for a given enclosed volume. Assume our (hyper)surface \( S \subset \mathbb{R}^{n+1} \) consists of the graph, \( \text{graph } f \), of a function \( f : \Omega \rightarrow \mathbb{R} \), defined on an open subset \( \Omega \) of the hyperplane \( x_{n+1} \) in \( \mathbb{R}^{n+1} \) and satisfying \( f \geq 0 \) and \( f = 0 \) on \( \partial \Omega \), and the subset \( \Omega \). Then the area of this surface is given by \( A(f) = \int_{\Omega} \sqrt{1 + |f|^2} d^nx \) plus the area of \( \Omega \). The latter is fixed and can be omitted. The enclosed volume by \( S \) is the volume under \( \text{graph } f \) and is equal to \( V(f) = \int_{\Omega} f d^nx \). Hence we have to minimize \( A(f) \) on the set \( M = \{ f \in H^1_0(\Omega) | \int_{\Omega} f d^nx = c \} \). Show that the Euler–Lagrange equation

\[
\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = h,
\]

where \( h \) is the Lagrange multiplier. This is a surface of a constant mean curvature.

### 5.6 Functionals on complex spaces

We consider functionals on open subsets, \( M \), of complex vector spaces, \( Z \). For such functionals, we define the complex Gâteaux derivatives as

\[
d\mathcal{E}(\psi) \equiv d_\psi \mathcal{E}(\psi) = (d_{\psi_1} - i d_{\psi_2}) \mathcal{E}(\psi) \quad \text{and} \quad \bar{d}\mathcal{E}(\psi) \equiv d_{\bar{\psi}} \mathcal{E}(\psi) = (d_{\psi_1} + i d_{\psi_2}) \mathcal{E}(\psi),
\]

where \( \psi = \psi_1 + i \psi_2 \), and similarly the complex gradients \( \partial_\psi \mathcal{E}(\psi) \) and \( \bar{\partial}_\psi \mathcal{E}(\psi) \). One way to compute \( d_\psi \mathcal{E}(\psi) \) and \( d_{\bar{\psi}} \mathcal{E}(\psi) \) is to treat \( \psi \) and \( \bar{\psi} \) as independent functions and compute the corresponding objects as partial Gâteaux derivatives.

To connect this to the real Banach theory considered above, we note that \( Z \) can be written as \( Z = V + iV \), for some real vector space. For example \( Z = H^1(\mathbb{R}^d, \mathbb{C}) = H^1(\mathbb{R}^d, \mathbb{R}) + iH^1(\mathbb{R}^d, \mathbb{R}) \). We associate with such a \( Z \), a real space \( \hat{Z} := V \oplus V \). There is one-to-one correspondence between \( Z \) and \( \hat{Z} \):

\[
\phi \leftrightarrow \tilde{\phi} := (\phi_1, \phi_2), \quad \phi_1 := \text{Re } \phi, \quad \phi_2 := \text{Im } \phi.
\]

Consider the map \( \text{compl} : \hat{Z} \rightarrow Z \), given by \( \tilde{\phi} := (\phi_1, \phi_2) \mapsto \phi = \phi_1 + i\phi_2 \), and its inverse \( \text{vect} : \phi = \phi_1 + i\phi_2 \mapsto \tilde{\phi} := (\phi_1, \phi_2) \). Using these maps identify a functional \( \mathcal{E}(\phi) \) on \( Z \) with a functional \( \mathcal{E}^{\text{real}}(\text{vect}(\phi)) = \mathcal{E}(\phi) \) on \( \hat{Z} \) and we can define the variational (or Gâteaux or Fréchet) differentiability and derivative, \( \partial_\phi \mathcal{E}(\phi) \), and partial derivatives, \( \partial_{\phi_1} \mathcal{E}(\phi) \) and \( \partial_{\phi_2} \mathcal{E}(\phi) \), with respect the real, \( \phi_1 \), and imaginary, \( \phi_2 \), parts of the field \( \phi \) for \( \mathcal{E}(\phi) \), by computing them for \( \mathcal{E}^{\text{real}}(\phi) \). After that we introduce the derivatives with respect \( \phi \) and \( \tilde{\phi} \) as follows

\[
\partial_\phi \mathcal{E}(\phi) := \partial_{\phi_1} \mathcal{E}(\phi) - i \partial_{\phi_2} \mathcal{E}(\phi), \quad \bar{\partial}_\phi \mathcal{E}(\phi) := \partial_{\phi_1} \mathcal{E}(\phi) + i \partial_{\phi_2} \mathcal{E}(\phi).
\]

Then, defining \( \partial_\tilde{\phi} \mathcal{E}(\phi) = \partial_{\phi} \mathcal{E}^{\text{real}}(\text{vect}(\phi)) \), we have the following relations

\[
\partial_{\phi} \mathcal{E}(\phi) = \text{compl}(\partial_{\tilde{\phi}} \mathcal{E}(\phi)), \quad \bar{\partial}_{\tilde{\phi}} \mathcal{E}(\tilde{\phi}) = \text{vect}(\partial_{\phi} \mathcal{E}(\phi)).
\]
Here are some examples (below $\Omega \subset \mathbb{R}^d$, $d = 2, 3$):

6) The Ginzburg-Landau energy functional $(\psi : \Omega \to \mathbb{C}, A : \Omega \to \mathbb{R}^d)$

$$E_\Omega(\psi, A) := \frac{1}{2} \int_\Omega \left\{ |\nabla A \psi|^2 + \frac{\kappa}{2} (|\psi|^2 - 1)^2 + |\text{curl} A|^2 \right\},$$

(5.20)

where $\nabla A = \nabla - iA$, the covariant derivative.

7) The action functional for nonlinear Schrödinger equation $(\psi : \Omega \times [0, T] \to \mathbb{C})$

$$S(\psi) = \frac{1}{2} \int_0^T \int_\Omega \left( -\text{Im}(\psi \overline{\psi}) + |\nabla \psi|^2 + G(|\psi|^2) \right) dxdt.$$  

(5.21)

**Exercise 7.** Compute the complex and real Gâteaux derivatives and the equations for the critical points in examples 6) – 7) above.

For complex vector spaces, we say that $u_0 \in M$ is a critical point (CP) of $E$ if and only if $\overline{dE}(u_0) = 0$.

### 5.7 Minimization problem and spectrum.

We would like to explain relation between the minimization (or in general critical point) theory and the spectral theory of operators. Consider an operator $A$ acting on a Banach space $X$ with a domain $\mathcal{D}(A)$ (i.e., $A : \mathcal{D}(A) \to X$). The spectrum, $\sigma(A)$, of an operator $A$ is the set in $\mathbb{C}$ defined by

$$\sigma(A) := \{ z \in \mathbb{C} : A - z \mathbb{I} \text{ is not invertible} \}.$$  

(5.22)

Clearly, eigenvalues of $A$ belong to $\sigma(A)$ (in fact, if $\lambda$ is an eigenvalue, then $Au_\lambda = \lambda u_\lambda$ for some nonzero $u_\lambda \in X$ ($u_\lambda$ is called an eigenvector), so $(A - \lambda)u_\lambda = 0$, and $A - \lambda$ is not invertible). In general, the spectrum can also contain continuous pieces and it can take very peculiar forms.

**Exercise 8.** The spectrum of the multiplication operator introduced in example 1) above is $\sigma(M_f) = \text{Ran} f$, the differentiation operator 2) has spectrum $\sigma\left( \frac{\partial}{\partial x_j} \right) = i\mathbb{R}$, and the identity operator 3) has the spectrum consisting of one point $\sigma(\mathbb{I}) = \{ 1 \}$.

This motivates the following definitions. We define the discrete spectrum of an operator $A$ as

$$\sigma_d = \{ \lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } A \text{ with finite multiplicity} \}.$$  

The part of the spectrum which complements the discrete spectrum is called the essential spectrum of an operator $A$:

$$\sigma_{\text{ess}}(A) := \sigma(A) \setminus \sigma_d(A).$$  

(Some authors use the term ”continuous spectrum” rather than ”essential spectrum”.) Hence we have $\sigma(A) = \sigma_d(A) \cup \sigma_{\text{ess}}(A)$.

If $A$ is a quantum hamiltonian (the Schrödinger operator), then the eigenfunctions corresponding to discrete eigenvalues are called the bound states. They describe the motion of quantum system localized essentially to a bounded domain of the physical space. The essential spectrum is related to the scattering states, describing the system broken into freely moving fragments.

Now, let $A$ be a self-adjoint operator on a complex Hilbert space $\mathcal{H}$. Consider the functional $\mathcal{E}(u) = \langle u, Au \rangle$, where $u \in \mathcal{H}$. In Quantum Mechanics, if $A$ is a quantum hamiltonian, then $\mathcal{E}$ is the expectation of the energy in the state $u$. 
There is a deep relation between existence of minimizers (or saddle points) for the functional \( \frac{1}{2} \langle u, Au \rangle \) on the set \( M \) (we can assume here that the operator \( A \) is bounded below, i.e., \( \langle u, Au \rangle \geq -C\|u\|^2 \) for some \( C < \infty \)) and spectral properties of the operator \( A \). In particular, we have

**Theorem 5.5.**  
(i) \( \inf_{u \in M} \langle u, Au \rangle = \inf_{\lambda \in \sigma(A)} \lambda \);  
(ii) \( \langle u, Au \rangle \) has a minimizer on \( M \) if and only if \( \inf_{\lambda \in \sigma(A)} \lambda \) is an eigenvalue of \( A \);  
(iii) If \( \inf_{u \in M} \langle u, Au \rangle < \inf_{\lambda \in \sigma_{\text{ess}}(A)} \lambda \), then \( \inf_{u \in M} \langle u, Au \rangle \) is an eigenvalue of \( A \).

A proof of this theorem can be found in [15] where one can also find a theorem on a relation between saddle points of the functional \( \langle u, Au \rangle \) and eigenvalues of the operator \( A \) greater than the smallest eigenvalue \( \inf_{\lambda \in \sigma(A)} \lambda \). Here we mention only that the Euler-Lagrange equation for the functional \( \langle u, Au \rangle \) on the space \( M := \{ u \in D(A) \mid \|u\| = 1 \} \) is

\[ Au = \lambda u \]

where \( \lambda \) is the Lagrange multiplier. This is the eigenvalue equation for \( A \).

In the case when \( A \) is a Schrödinger operator, the number \( \lambda_0 := \inf_{\lambda \in \sigma(A)} \lambda \) is called the ground state energy of \( A \) and the theorem above expresses the variational characterization of eigenvalues used frequently in Quantum Mechanics.

### 5.8 Tangent spaces.

Recall that we define the tangent space to \( M \) at \( u, T_u M \), to be the set of all the \( \xi \in X \) s.t. there is an \( \epsilon > 0 \), and a differentiable path \( [-\epsilon, \epsilon] \ni \lambda \mapsto u_\lambda \in M \) for which \( u_0 = u, \frac{du_\lambda}{d\lambda} |_{\lambda=0} = \xi \).

**Exercise 5.2.** Show that \( T_u X = X \) and therefore in particular, \( T_u H^s(\Omega) = H^s(\Omega) \).

Now, we describe tangent spaces to surfaces (5.14).

**Proposition 5.3.** If \( M = \{ u \in X : J(u) = 0 \} \), where \( J \) is a \( C^1 \) functional on \( X \), then for \( u \in M \), \( T_u M = \text{null} \, dJ(u) \).

**Proof.** If \( u \in M \) and \( \xi \in T_u M \), then, by the definition, there exists a path \( u_\lambda \in C^1((-\epsilon, \epsilon), M) \) with \( \xi = \partial_\lambda |_{\lambda=0} u_\lambda \) and \( u_{\lambda=0} = u \). Therefore,

\[ 0 = \partial_\lambda J(u_\lambda)|_{\lambda=0} = \langle dJ(u), \xi \rangle. \]

Hence \( \xi \in \text{null} \, dJ(u) \).

Now, let \( u \in M \) and \( \xi \in \text{null} \, dJ(u) \) and find \( u_\lambda \) such that \( u_{\lambda=0} = u \), \( \xi = \partial_\lambda |_{\lambda=0} u_\lambda \) and \( J(u_\lambda) = 0 \). Let \( \eta \notin \text{null} \, dJ(u) \) and define the path \( u_\lambda = u + \lambda \xi + \lambda \eta \), where \( a \) solves the equation \( f(a, \lambda) = 0 \), where

\[ f(a, \lambda) := \frac{1}{\lambda} J(u + \lambda \xi + \lambda \eta). \]

We write \( J(u + v) = J(u) + dJ(u)v + R_u(v) \), where \( R_u(v) \) is defined by this equation, i.e. \( R_u(v) := J(u + v) - J(u) - dJ(u)v \). Now, we take \( v = \lambda \xi + \lambda \eta \) in this equation and use \( J(u) = 0 \) and \( dJ(u) \xi = 0 \), to obtain \( f(a, \lambda) = adJ(u)\eta + \frac{1}{\lambda} R_u(\lambda \xi + \lambda \eta) \). Now, we estimate

\[ R_u(\lambda \xi + \lambda \eta) = J(u + \lambda \xi + \lambda \eta) - J(u) - dJ(u)(\lambda \xi + \lambda \eta) \]

\[ = \int_0^1 ds (dJ(u + s(\lambda \xi + \lambda \eta)) - dJ(u))(\lambda \xi + \lambda \eta) = o(\lambda). \]
Hence \( f(0,0) = 0 \). Now, compute \( \partial_a f(0,0) \). We have

\[
\partial_a R_u(\lambda \xi + \lambda \eta) = dJ(u + \lambda \xi + \lambda \eta) \lambda \eta - dJ(u) \lambda \eta = o(\lambda).
\]

Hence \( \partial_a f(a,\lambda) = dJ(u)(\lambda \eta) + o(\lambda) \) and, in particular, \( \partial_a f(0,0) = dJ(u) \eta \neq 0 \). Therefore, by the implicit function theorem, the equation \( f(a,\lambda) = 0 \) has a unique solution for \( a = a(\lambda) \) for \( \lambda \) sufficiently small and this solution satisfies \( a(\lambda) = o_\lambda(1) \).

By the definition of \( f(a,\lambda) \) the family \( u_\lambda = u + \lambda \xi + \lambda a \eta \), with \( a = a(\lambda) = o_\lambda(1) \) solving \( f(a,\lambda) = 0 \), has the properties mentioned above. Hence \( \xi \in \mathcal{T}_u M \).

5.9 Dual space

The set of all bounded linear functionals on \( X \) is called the dual space of \( X \) (or simply the dual (or adjoint space) of \( X \), and it is denoted as \( X' \). Thus \( dE(u) \in X' \). It is a vector space with the norm (5.2). In fact, since \( C \) is complete, one can show that \( X' \) is always a Banach space, whether \( X \) is complete or not.

If \( X \) is a space of functions, then \( X' \) can be identified with either a space of functions or a space of distributions or a space of measures. Here are some examples of dual spaces:

1) \( (L^p)' = L^q \), where \( 1/p + 1/q = 1 \), if \( 1 \leq p < \infty \) (space of functions),

2) \( (L^\infty)' \) is a space of measures which is much larger than \( L^1 \),

3) \( (H^s)' = H^{-s} \) (space of distributions if \( s > 0 \)).

Hartree, Hartree - Fock and Gross-Pitaevski equations (TODO)

5.10 Appendix I: Area and curvatures of a hypersurface

In this appendix we review some of the notions of the theory of surfaces in Euclidean spaces. Let \( S \) be a smooth \( n \)-dimensional surface in \( \mathbb{R}^{n+1} \). Such a surface is called a (smooth) hypersurface. Hypersurfaces appear as graphs or level sets of images of some functions.

Let \( \Omega \subset \mathbb{R}^n \) and \( f : \Omega \to \mathbb{R} \) be smooth. Then graph \( f := \{(x',f(x')) | x' \in \Omega \} \) is a hypersurface in \( \mathbb{R}^{n+1} \).

Let \( \varphi : \mathbb{R}^{n+1} \to \mathbb{R} \) be a smooth function and let \( 0 \in \text{Ran} \varphi \). Then the zero level set of \( \varphi \),

\[
\varphi^{-1}(0) := \{x \in \mathbb{R}^{n+1} | \varphi(x) = 0 \}
\]

is a smooth hypersurface.

Finally a map \( \psi : U \subset \mathbb{R}^n \to S \) is called a local parametrization of \( S \).

Lemma 5.1. Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( f : \Omega \to \mathbb{R} \). Then the area of the surface \( S := \text{graph } f \) is given by the formula

\[
A(f) := \int_\Omega \sqrt{1 + |\nabla f|^2} \, d^n x
\]

Proof. Let \( x = (x',x^{n+1}) \), where \( x' = (x^1,\ldots,x^n) \). The picture below shows the following formula for the area element of \( S \): \( \Delta A = \frac{dx'}{\cos \alpha} \), where \( \alpha \) is the angle between the \( x^{n+1} \)-axis (the unit vector \( e_{n+1} \)) and the normal \( \nu = \nu(x) \) to \( S \) at a given point \( x \in S \).
Let \( \varphi(x) = x^{n+1} - f(x') \) so that \( S = \{ x \in \Omega \times \mathbb{R} \mid \varphi(x) = 0 \} \). Then on \( S \):

\[
\nu(x) = \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|},
\]

and therefore for \( x \in S \)

\[
\cos \alpha = \frac{\nabla \varphi(x) \cdot e_{n+1}}{|\nabla \varphi(x)|} = \frac{1}{\sqrt{1 + |\nabla f|^2}}.
\]

Since \( \text{area}(S) = \int_S dA \), the result follows.

We give now the definition of different notions of curvatures at a point \( x_0 \in S \). We begin with a simple but coordinate dependant definition. Pick a coordinate system s.t. \( \nabla f(x'_0) = 0 \), where \( x_0 = (x'_0, x_{n+1}^0) \). Then we define

- the **principal curvatures at** \( x_0 \) **as the eigenvalues of** \( \text{Hess} f(x'_0) \),
- the **Gauss curvature at** \( x_0 \) **as** \( \det \text{Hess} f(x'_0) \),
- the **mean curvature at** \( x_0 \) **as** \( h(x_0) = \text{Tr Hess} f(x'_0) = \Delta f(x'_0) \).

**Lemma 5.2.** Let \( S = \text{graph} f \) for some \( f \) s.t. \( f(x') \neq 0 \). Then the mean curvature at \( x \) is given by (5.11).

**Proof.** Consider first an arbitrary coordinate system and a function \( f : \Omega \to \mathbb{R} \), s.t. \( S = \text{graph} f \). We denote as before \( x = (x', x^{n+1}) \in \mathbb{R}^{n+1}, x' = (x', \ldots, x^n) \in \Omega \subset \mathbb{R}^n \). As we have shown, the unit normal vector to \( S \) at \( x = (x', f(x')) \), \( \nu(x) \), can be expressed as

\[
\nu(x) = \frac{(-\nabla f(x'), 1)}{\sqrt{1 + |\nabla f(x')|^2}},
\]

(5.23)

Now for a given point \( x_0 \in S \), let \( x = (x', x_{n+1}^0) \in \mathbb{R}^{n+1} \) be a special coordinate system s.t. there is a domain \( \Omega \subset \mathbb{R}^n \) and a function \( \tilde{f} : \Omega \to \mathbb{R} \) s.t. \( S = \text{graph} \tilde{f} \) and \( \nabla \tilde{f}(x'_0) = 0 \). Then we can express the normal vector \( \nu(x) \) in terms of this function as

\[
\nu(x) = \frac{(-\nabla \tilde{f}(x'), 1)}{\sqrt{1 + |\nabla \tilde{f}(x')|^2}}.
\]

Now compute

\[
\text{div} \nu(x) = -\frac{\Delta \tilde{f}(x')}{(1 + |\nabla \tilde{f}(x')|^2)^{1/2}} + \frac{|\nabla \tilde{f}(x')|^2}{(1 + |\nabla \tilde{f}(x')|^2)^{3/2}}.
\]
and therefore we get \( \text{div} \nu(x_0) = -\Delta f(x_0) \). By the definition of the mean curvature at the point \( x_0 \), \( \Delta f(x_0) = -h(x_0) \), and therefore \( \text{div} \nu(x_0) = -h(x_0) \), which together with (5.23) implies the lemma.

Now we define the curvature in a coordinate-independent way. This way reveals the geometrical meaning of this notion and on the way deals with some important concepts in the theory of surfaces.

We define the tangent space, \( T_x S \), to \( S \) at \( x \) as

\[
T_x S := \{ \xi \in \mathbb{R}^{n+1} \mid \exists \text{ C}^1 \text{ path, } \gamma_s, \text{ in } S \text{ s.t } \gamma|_{s=0} = x, \text{ and } \partial_s \gamma|_{s=0} = \xi \}
\]

(i.e., \( T_x S \) is the space of initial velocities of curves on \( S \) starting at \( x \)). Let \( \nu(x) \) be an 'outward' unit normal to the surface \( S \) at a point \( x \in S \). Denote by \( S^n \) the unit \( n \)-dimensional sphere. The map \( \nu : S \to S^n \), given by

\[
\nu : x \to \nu(x),
\]

is called the Gauss map. The negative of its derivative

\[
A_x := -\partial \nu(x)
\]

at \( x \) is called the Weingarten map. The map \( A_x \) measures the rate of change in the direction of \( \nu(x) \) as it moves along \( S \). By definition,

\[
A_x : T_x S \to T_{\nu(x)} S^n.
\]

**Proposition 5.4.** 1) \( \text{Ran } A_x \subset T_x S \subset \mathbb{R}^{n+1} \); 2) \( A_x^* = A_x \).

**Proof.** 1) Differentiating the relation \( \langle \nu(x), \nu(x) \rangle = 1 \), we find \( \langle \partial \nu(x), \nu(x) \rangle = 0 \). Thus, \( A_x \xi \perp \nu(x) \), i.e., \( A_x \xi \in T_x S \).

2) Let \( \varphi_{st} \) be a two dimensional parameterized surface in \( S \), i.e., \( \varphi_{st} \in S \), such that \( \varphi_{st}|_{s=t=0} = x \), \( \partial_s \varphi_{st}|_{s=t=0} = \xi \) and \( \partial_t \varphi_{st}|_{s=t=0} = \eta \). Then

\[
\langle \xi, A_x \eta \rangle = -\langle \partial_s \varphi_{st}|_{s=t=0}, \partial_t \nu(\varphi_{st})|_{s=t=0} \rangle
\]

\[
= -\partial_t \langle \partial_s \varphi_{st}, \nu(\varphi_{st}) \rangle|_{s=t=0} + \langle \partial_t \partial_s \varphi_{st}, \nu(\varphi_{st}) \rangle|_{s=t=0}.
\]

Since \( \partial_s \varphi_{st} \perp \nu(\varphi_{st}) \), we have finally

\[
\langle \xi, A_x \eta \rangle = \langle \frac{\partial^2 \varphi}{\partial s \partial t}|_{s=t=0}, \nu(x) \rangle.
\]

(5.24)
Similarly, we obtain
\[\langle A_x \xi, \eta \rangle = \left\langle \frac{\partial^2 \varphi}{\partial s \partial t} \big|_{s=t=0}, \nu(x) \right\rangle\]
and therefore, \(\langle \xi, A_x \eta \rangle = \langle A_x \xi, \eta \rangle\), i.e., \(A_x^* = A_x\).

**Remark.** Let \(\psi : U \to S\) be a parametrization of \(S\) at \(x\). Take \(\xi = e_i\) and \(\eta = e_j\) in (5.24) where \(\{e_i\}\) is an orthonormal basis in \(\mathbb{R}^n \supset U\). Then
\[\begin{align*}
(A_x)_{ij} &= \left\langle \frac{\partial^2 \psi(u)}{\partial u_i \partial u_j}, \nu(x) \right\rangle
\end{align*}\]
where \(u = \psi^{-1}(x)\). The matrix \(A_x\) is called the 2nd fundamental form. (The 1st fundamental form of \(S\) is the metric on \(S\) induced by the Euclidean metric on \(\mathbb{R}^{n+1}\).)

**Definition 5.1.** The mean curvature of \(S\) at \(X\) is
\[H(x) := \text{Tr} \, A_x.\]
The Gauss curvature of \(S\) at \(x\) is
\[G(x) := \det A_x.\]
The principle curvatures of \(S\) at \(x\) are the eigenvalues of \(A_x\).

**Proposition 5.5.** In the level set representation, \(S = \varphi^{-1}(0)\), we have
\[\nu(x) = -\frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \quad \text{and} \quad H(x) = \text{div} \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right).\]

**Proof.** Let \(\gamma_s\) be a path on \(S\) starting at \(x\) with an initial velocity \(\xi: \gamma_s|_{s=0} = x\) and \(\partial_s \gamma_s|_{s=0} = \xi\). Differentiating \(\varphi(\gamma_s) = 0\) we find \(\nabla \varphi \cdot \xi = 0\). Hence \(\nabla \varphi \perp T_x S\) and therefore \(\nu(x)\) is equal to \(\frac{\nabla \varphi}{|\nabla \varphi|}\) up to a sign.

To prove the second formula we have by the definition of \(A_x\)
\[A_x \xi = \partial_s|_{s=0} \frac{\nabla \varphi(\gamma_s)}{|\nabla \varphi(\gamma_s)|} = \sum_i \partial_{x_i} \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right) \xi_i.\]

Place the coordinate system at the point \(x\) with \(e_{n+1} = \nu(x)\). Then \((A_x)_{ij} = \partial_{x_i} \left( \frac{\partial_{x_j} \varphi(x)}{|\nabla \varphi(x)|} \right)\) and therefore \(H(x) = \text{div} \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right)\) as claimed.

## 6 Minimization: direct methods

### 6.1 General result

The problem we address is the following: given a functional \(\mathcal{E}\) on a space \(M\), find a function \(u_0 \in M\) (if such exists) that minimizes \(\mathcal{E}\):
\[\mathcal{E}(u_0) = \inf_{u \in M} \mathcal{E}(u).\]
Such a function \( u_0 \) is called a minimizer for \( \mathcal{E} \). Thus to begin with, we want to assume that \( \mathcal{E} \) is bounded below, i.e.

\[
E_0 := \inf_{u \in M} \mathcal{E}(u) > -\infty.
\]

We also assume that \( M \) is a closed subset of a Banach space, \( X \). Let us first analyze the finite dimensional situation: \( X = \mathbb{R}^N \). How would we minimize a functional on \( M \)? We do this in three steps (that will be suitable to be generalized to the infinite dimensional case):

- **Step 1.** Since \( \mathcal{E} \) is bounded on \( M \) from below, \( E_0 := \inf_{u \in M} \mathcal{E}(u) > -\infty \), we can pick a sequence \( \{u_n\} \subset M \) s.t. \( \mathcal{E}(u_n) \to E_0 \), as \( n \to \infty \). Such a sequence is called a minimizing sequence. Clearly we can take the sequence \( \{u_n\} \) s.t. every element \( u_n \) satisfies \( \mathcal{E}(u_n) \leq E_0 + 1 \) (just throw out those \( u_n \) for which \( \mathcal{E}(u_n) > E_0 + 1 \)).

- **Step 2.** We hope that either such a sequence converges or at least contains a convergent subsequence. The limit of such a subsequence clearly is a candidate for a minimizer if the latter exists. How do we show that \( \{u_n\} \) has a convergent subsequence? Assume that

\[
\mathcal{E}(u) \to \infty \quad \text{as} \quad \|u\|_X \to \infty.
\]

Due to (6.1), we have that \( \|u_n\|_X \leq C, \forall n \), and for some \( C > 0 \). Hence by the Bolzano–Weierstrass theorem, \( \{u_n\} \) has a convergent subsequence, which for notational convenience we denote again by \( \{u_n\} \).

- **Step 3.** Let \( u_0 := \lim_{n \to \infty} u_n \). If \( \mathcal{E} \) is continuous, then

\[
\lim_{n \to \infty} \mathcal{E}(u_n) = \mathcal{E}(u_0).
\]

Since on the other hand we have \( \lim_{n \to \infty} \mathcal{E}(u_n) = E_0 \), we conclude that \( \mathcal{E}(u_0) = E_0 \), i.e. \( u_0 \) is a minimizer of \( \mathcal{E} \).

A functional \( \mathcal{E} \) on \( M \) is called coercive if and only if \( \mathcal{E}(u) \to \infty \) whenever \( \|u\|_X \to \infty \). Recall that a set \( K \) s.t. every infinite sequence of elements of \( K \) contains a convergent subsequence is called compact.

Let us look closer at the last two steps. The Bolzano–Weierstrass theorem states that a closed ball in \( \mathbb{R}^N \) is compact. This property does not hold in general in the infinite dimensional case. For instance, closed balls in \( L^2(\Omega) \) are not compact. As a concrete example, take for instance \( L^2(\mathbb{R}^n) \ni u_n(x) := u(x - n) \), for some \( u \in L^2(\mathbb{R}^n), \|u\|_2 = 1 \). Clearly, this sequence does not have a convergent subsequence.

We have however, the following weaker result (this result follows from the Banach-Alaoglu theorem): If \( X \) is a reflexive Banach space, then every uniformly bounded sequence \( \{u_n\} \) in \( X \) (i.e. \( \|u_n\|_X \leq C \)) has a weakly convergent subsequence \( \{u_{n_k}\} \). (In the previous example, \( u_n(x) := u(x - n) \) converges to 0 weakly. Here we recall some definitions. A Banach space is said to be reflexive, iff its second dual (the dual of the dual) is isometrically isomorphic to the space itself. Every Hilbert space is reflexive, see e.g., [24], Section 6.3, Theorem 2 and its Corollary. Furthermore, we say a sequence \( \{u_n\} \) in \( X \) is weakly convergent iff \exists u_0 \ s.t. \( \ell(u_{n_k}) \to \ell(u_0), \forall \ell \in X' \). The weak convergence is denoted by \( w \to \), as in \( u_n \xrightarrow{w} u_0 \). In the finite dimensional spaces, weak convergence is equivalent to strong convergence, and the above statement reduces to the Bolzano-Weierstrass theorem.)

Next, the continuity w.r.t. the weak convergence is a property hard to come by for functionals on infinite dimensional spaces. But there is a weaker property which often holds: the weak
This is a simple but powerful result. It says that in order to show that existence of minimizers. Let \( E \) be a reflexive Banach space (for simplicity, think of \( X \) as a Hilbert space), and let \( M \) be a subset of \( X \). We say that \( M \) is weakly closed in \( X \) if and only if \( u_n \rightharpoonup u_0 \) in \( X \) and \( u_n \in M \), imply \( u_0 \in M \). We consider a functional \( \mathcal{E} : M \to \mathbb{R} \). We have the following

**Theorem 6.1 (Key Theorem).** Assume that

1. \( M \) is weakly closed in \( X \),
2. \( \mathcal{E} \) is w.l.s.c.,
3. \( \mathcal{E} \) is coercive.

Then \( \mathcal{E} \) is bounded below and attains its minimum in \( M \) (i.e. there is a minimizer of \( \mathcal{E} \) on \( M \)).

**Proof.** Let \( E_0 = \inf_{u \in M} \mathcal{E}(u) \), and let \( u_n \) be a minimizing sequence for \( \mathcal{E} \), i.e.

\[
\lim_{n \to \infty} \mathcal{E}(u_n) = E_0. \tag{6.2}
\]

Clearly, we can assume that \( \mathcal{E}(u_n) \leq E_0 + 1 \) (we get rid of those \( u_n \)'s in the minimizing sequence for which \( \mathcal{E}(u_n) > E_0 + 1 \)). Then by the coercivity of \( \mathcal{E} \), there is a constant \( K \) s.t. \( \|u_n\| \leq K, \forall n \). Hence, by the Banach-Alaoglu theorem, \( \{u_n\} \) contains a weakly convergent subsequence, \( \{u_{n'}\}, u_{n'} \rightharpoonup u_0 \in X \). The element \( u_0 \) is a candidate for a minimizer. Since \( M \) is weakly closed, \( u_0 \in M \). Finally, w.l.s.c. of \( \mathcal{E} \) gives \( \lim \inf_{n' \to \infty} \mathcal{E}(u_{n'}) \geq \mathcal{E}(u_0) \). This together with equation (6.2) implies that \( E_0 \geq \mathcal{E}(u_0) \). On the other hand, \( E_0 = \inf_{u \in M} \mathcal{E}(u) \leq \mathcal{E}(u_0) \), and therefore \( \mathcal{E}(u_0) = E_0 \). This shows that \( u_0 \) is a minimizer and that \( \inf_{u \in M} \mathcal{E}(u) > -\infty \). \( \square \)

### 6.2 Applications

This is a simple but powerful result. It says that in order to show that \( \mathcal{E} \) has a minimizer, we have to check three conditions (\( \alpha \)- (\( \gamma \)). We begin with the discussion of these conditions. We begin with (\( \beta \)).

**Lemma 6.1.** The functional \( \mathcal{E}(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \) is w.l.s.c. on \( H^1(\Omega) \).

**Proof.** We have

\[
\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 = \frac{1}{2} \int_{\Omega} |\nabla (u + u_n - u)|^2 \\
\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \text{Re} \int_{\Omega} \nabla \bar{u} \cdot \nabla (u_n - u) + \frac{1}{2} \int_{\Omega} |\nabla (u_n - u)|^2.
\]
If \( u_n \rightharpoonup u \) in \( H^1(\Omega) \), then \( \int_\Omega \nabla u : \nabla (u_n - u) \to 0 \), and therefore

\[
\liminf_{n \to \infty} \frac{1}{2} \int_\Omega |\nabla u_n|^2 \geq \frac{1}{2} \int_\Omega |\nabla u|^2.
\]

\[\square\]

**Lemma 6.2.** The functional \( E(u) := \int_\Omega G(x, u) \), where \( G(x, u) \geq 0 \), l.s.c. in \( u \) and s.t. \( \int_\Omega G(x, u) \) is well defined for \( u \in H^1(\Omega) \), is w.l.s.c. on \( H^1(\Omega) \).

**Proof.** Indeed, let \( u_n \rightharpoonup u \) in \( H^1(\Omega) \) and let \( Q \) be a bounded subset of \( \Omega \) \((Q = \Omega \) if \( \Omega \) is bounded). Then by the Rellich-Kondrashov theorem, \( u_n \to u \) in \( L^2(\Omega) \) (up to picking a subsequence) and therefore \( u_n \to u \) a.e. in \( \Omega \). Since \( Q \) is arbitrary bounded subset of \( \Omega \), we have that \( u_n \to u \) a.e.. Therefore, by Fatou’s lemma, we get \( \liminf_{n \to \infty} \int_\Omega G(x, u_n(x)) \geq \int_\Omega \liminf_{n \to \infty} G(x, u_n(x)) \geq \int_\Omega G(x, u(x)) \), and \( E \) is w.l.s.c. on \( H^1(\Omega) \).

\[\square\]

Lemmas 6.1 and 6.2 imply

**Corollary 6.2.** If \( G(x, u) \geq 0 \), l.s.c. in \( u \) and s.t. \( \int_\Omega G(x, u) \) is well defined for \( u \in H^1(\Omega) \), then the functional

\[
E(u) = \int_\Omega \left( \frac{1}{2} |\nabla u(x)|^2 + G(x, u(x)) \right) dx.
\]

(6.3)
is w.l.s.c. on \( H^1(\Omega) \).

Now, we discuss (γ), the coercivity.

(c) The functional \( E(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + c|u|^2 \right) \), \( c > 0 \), is obviously coercive. Indeed, if \( c > 0 \), then \( E(u) \geq \delta \|u\|^2_{(1)} \), where \( \delta = \min(1/2, c) > 0 \), and, recall, \( \|u\|_{(1)} = \|u\|_{H^1(\Omega)} \), and therefore \( E \) is coercive on \( H^1(\Omega) \), i.e. \( E(u) \to \infty \) whenever \( \|u\|_{(1)} \to \infty \).

A more delicate situation is with the Dirichlet functional. We have

(d) The functional \( E(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \) is is coercive on \( H^1(\Omega) \), provided \( \Omega \) is bounded in one direction. Indeed, by the Poincaré inequality (see (6.11) below), we have \( \int_\Omega |u|^2 \leq D^2 \int_\Omega |\nabla u|^2 \), for any \( u \in H^1_0(\Omega) \), where \( D \) is the smallest diameter of \( \Omega \). So we get \( E(u) \geq \frac{1}{4} \min(1, D^{-2}) \|u\|^2_{(1)} \)

for every \( u \in H^1_0(\Omega) \). Therefore \( E \) is coercive on \( H^1_0(\Omega) \).

Collecting above statements we obtain

**Proposition 6.1.** Let \( G(x, u) \geq c|u|^2 \), for some \( c \geq 0 \). If \( c > 0 \), then the functional (6.3) is coercive on \( H^1(\Omega) \); if \( c = 0 \) and \( \Omega \) is bounded in one direction, then (6.3) is coercive on \( H^1_0(\Omega) \).

Now we give examples of weakly closed sets.

(f) \( M = X \) or \( M = g + X \) (for a fixed \( g \)), where \( X \) is a reflexive Banach space (in particular, a Hilbert space). Hence \( H^1_0(\Omega) \) is weakly closed in \( H^1(\Omega) \), if \( g \) has an extension, \( \tilde{g} \), from \( \partial \Omega \) to \( \Omega \). Indeed, in the latter case can write \( H^1_g = \tilde{g} + H^1_0(\Omega) \). Since \( H^1_0(\Omega) \) is a Hilbert space, the result follows.

(g) A convex, closed subset of a reflexive Banach space (by Mazur’s theorem).

(h) \( M = \{ u \in H^1_0(\Omega) : \int_\Omega |u|^p \, dx = 1 \} \), where \( \Omega \) is a bounded, smooth domain in \( \mathbb{R}^n \) and \( p < \frac{2n}{n-2} \) if \( n > 2 \) and \( p < \infty \) if \( n \leq 2 \).

**Proposition 6.2.** The set \( M \) defined above is weakly closed in \( H^1_0(\Omega) \).
Proof. By the Rellich-Kondrashov theorem, \( H^1_0(\Omega) \) is compactly embedded into \( L^p(\Omega) \), for \( p \) as in the proposition, and \( \Omega \) bounded. This means that any weakly convergent sequence \( u_n \rightharpoonup u_0 \) in \( H^1_0(\Omega) \) contains a subsequence \( \{u_{n'}\} \) s.t. \( u_{n'} \to u_0 \) in \( L^p(\Omega) \). Hence \( \|u_0\|_p = \lim_{n \to \infty} \|u_n\|_p = 1 \), and therefore \( u_0 \in M \). \qed

Before considering more examples, we give some applications.

**Dirichlet problem.** We use results above to show that the Dirichlet problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
 u &= g \quad \text{on } \partial \Omega.
\end{align*}
\]

(6.4)

has a solution \( u \in H^1_0(\Omega) \) if the domain \( \Omega \) is bounded in one direction. Indeed, the results (a), (d) and (f) and Theorem 6.1 imply that the functional \( \frac{1}{2} \int_\Omega |\nabla u|^2 \) on \( H^1_0(\Omega) \) has a minimizer. On the other hand, by Theorem 5.1 of Section 5, this minimizer is a critical point of and therefore satisfies the Euler-Lagrange equation (5.7), which in our case becomes the equation (5.5) with \( f = 0 \). Thus, by the definition above (5.7), the minimizers are weak solutions to (6.4). By the discussion below (5.7), they are in fact differentiable, i.e. they are classical solution to (6.4).

**The nonlinear eigenvalue problem.** Now we show how to use the Key Theorem in order to prove existence of solutions of differential equations. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \). For \( \lambda \in \mathbb{R} \) and \( p > 2 \), we consider the problem

\[
\begin{align*}
-\Delta u - |u|^{p-2}u &= \lambda u \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(6.5)

We want to prove existence of solutions of this boundary value problem. Denote \( \lambda_1 \) the lowest eigenvalue of \( -\Delta \) on \( \Omega \) with Dirichlet boundary conditions. We have the following

**Theorem 6.3.** Let \( 2 < p < \frac{2n}{n-2} \) if \( n > 2 \) and \( 2 < p < \infty \) if \( n \leq 2 \). Then for any \( \lambda < \lambda_1 \), there is a positive solution to the problem (6.5).

**Discussion.** Differential equation (6.5) is the Euler–Lagrange equation for the functional

\[
F(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p - \frac{\lambda}{2} |u|^2 \right) \, dx.
\]

However, for \( p > 2 \), this functional is not bounded from below. Indeed, take \( u_\mu = \mu u \) with a fixed function \( u \), and some \( \mu > 0 \). Then

\[
F(u_\mu) = \mu^2 \frac{1}{2} \int_\Omega (|\nabla u|^2 - \lambda |u|^2) \, dx - \mu^p \int_\Omega |u|^p \to -\infty, \quad \text{as } \mu \to \infty.
\]

Consequently, this functional does not have a minimizer. Taking \( u^{(\mu)} := \mu^\alpha u(\mu x) \) with a fixed function \( u \) and some \( \mu \), we find

\[
F(u^{(\mu)}) = \mu^{2+2\alpha-n} \frac{1}{2} \int_\Omega |\nabla u|^2 - \mu^{\alpha p-n} \frac{1}{p} \int_\Omega |u|^p - \mu^{2\alpha-n} \frac{\lambda}{2} \int_\Omega |u|^2.
\]

Take now \( \alpha \) so that \( 2 + 2\alpha - n > 0 \), and \( 2 + 2\alpha > \alpha p - n \), i.e. \( \frac{2}{p-2} > \alpha > \frac{n}{2} - 1 \). Since \( \frac{2}{p-2} > \frac{n-2}{2} \) (because \( \frac{2}{p} < \frac{2}{n-2} + 1 = \frac{n}{n-2} \)), this is possible. Then we get \( F(u^{(\mu)}) \to \infty \) as \( \mu \to \infty \), which shows that \( F \) is not bounded from above, and consequently, it has no maximizer either.
To get out of this dilemma, we consider the constraint problem: minimize the functional

\[ E(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 - \lambda |u|^2) \, dx, \]  

subject to the constraint \( J(u) = 1 \), where

\[ J(u) := \frac{1}{p} \int_{\Omega} |u|^p \, dx. \]  

If such a problem has a minimizer \( v_0 \), then by the Lagrange multiplier theorem, \( v_0 \) satisfies the equations

\[ -\Delta v_0 - \lambda v_0 - \mu |v_0|^{p-2} v_0 = 0, \]  

and

\[ \frac{1}{p} \int_{\Omega} |v_0|^p \, dx = 1, \]  

for some \( \mu \in \mathbb{R} \). But now we have the undesirable coefficient \( \mu \). (The parameter \( \mu \) is a function of \( \lambda \) determined through equation (6.9)). To get rid of this coefficient, we first show that \( \mu > 0 \). Indeed, multiplying (6.8) by \( v_0 \), integrating the result over \( \Omega \), and then integrating by parts, we obtain

\[ \mu \int_{\Omega} |v_0|^p \, dx = \int_{\Omega} (|\nabla v_0|^2 - \lambda |v_0|^2) \, dx. \]

The r.h.s. is \( \int_{\Omega} (v_0(-\Delta)v_0 - \lambda |v_0|^2) \geq (\lambda_1 - \lambda) \int_{\Omega} |v_0|^2 \), where, recall, \( \lambda_1 \) is the lowest eigenvalue of the negative Laplacian, \(-\Delta\), with Dirichlet boundary conditions. Since \( \lambda < \lambda_1 \), the r.h.s is positive and so \( \mu > 0 \). Now we rescale \( v_0 \) as

\[ u_0(x) := \mu^{\frac{1}{p-2}} v_0(x), \]

then clearly \( u_0 \) satisfies (6.5).

**Proof of Theorem 6.3.** By the above theorem, it suffices to show that the functional \( E \) defined in (6.6) has a minimizer in the set \( M = \{ u \in H_1^{(0)}(\Omega) : J(u) = 1 \} \).

To prove this, we have to show that

(a) \( M \) is weakly closed,  
(\( \beta \)) \( E \) is w.l.s.c.,  
(\( \gamma \)) \( E \) is coercive.

Properties (a)–(\( \gamma \)) were proved before as a part of our exercises. (Recall that for (\( \gamma \)) we need that \( \lambda < \lambda_1 \)). So by the key theorem, \( E \) has a minimizer \( v_0 \) in \( M \), which by the argument given in the discussion above leads to a (weak) solution of problem (6.6). For a definition of the weak solution, see below. A weak solution can be upgraded to a classical solution by the elliptic regularity argument also described below.
Discussion. i) There is one subtle issue here which we brushed under the rug: the argument above shows that $u_0 \in H^1_0(\Omega) \cap L^p(\Omega)$. Hence all we know is that $\Delta u_0 \in H^{-1}(\Omega)$ and therefore, we have to specify what we mean by saying that $u_0$ satisfies (6.5).

Note that if $u$ is a smooth function satisfying (6.5), then multiplying (6.5) by $v \in C_0^\infty(\Omega)$ and integrating by parts, we obtain

$$-\int_\Omega u \Delta v - \int_\Omega f(u)v = 0,$$

where $f(u) = |u|^{p-2}u - \lambda u$. A function $u \in H^1_0(\Omega)$ is called a weak solution to (6.5) if it satisfies (6.10) for any $v \in C_0^\infty(\Omega)$.

ii) One can show that $\mu \downarrow 0$ as $\lambda \uparrow \lambda_1$. This shows that a branch of non-trivial solutions of (6.5) bifurcates from the trivial solution $u \equiv 0$ at $\lambda = \lambda_1$.

We give the following general result about w.l.s.c.:

**Theorem 6.4.** If $E$ is convex, then $E$ is w.l.s.c.

**Proof.** By Proposition 5.2, we have $E(u_n) \geq E(u) + dE(u)(u_n - u)$. By the weak convergence we have $dE(u)(u_n - u) \to 0$, as $n \to \infty$, which, together with the previous relation, implies that $\liminf_{n \to \infty} E(u_n) \geq E(u)$. □

This result implies, in particular, that if $C \geq V(x) \geq 0$, then the functional $\int_\Omega V(x)|u|^2$ is w.l.s.c. in $L^2(\Omega)$.

**Exercise 9.** 1) Let $V(x) \geq 0$. Show that if $u_n \rightharpoonup u$ in $L^2(\Omega)$, then

$$\liminf_{n \to \infty} \int_\Omega V(x)|u_n|^2 \geq \int_\Omega V(x)|u|^2.$$

2) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $f \in L^2(\Omega)$. Show that the functional $\frac{1}{2} \int_\Omega |
abla u|^2 + \int_\Omega f u$ is coercive and w.l.s.c. on $H^1_0(\Omega)$.

3) Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $g(u) = (g_{ij}(u))$ be a family of $m \times m$ positive definite matrices satisfying $g(u) \geq \delta I$, for some $\delta > 0$. Show that the functional $E(u) = \frac{1}{2} \int_\Omega \sum_{i,j} g_{ij}(u) \nabla u^i \cdot \nabla u^j + \int_\Omega f u$ is coercive and w.l.s.c. on $H^1_g(\Omega, \mathbb{R}^m)$. Here, $u = (u_1, \ldots, u_m)$.

We give more examples of weakly closed sets.

(h) Let $\Omega \subset \mathbb{R}^n$ be bounded and

$$M := \{u \in L^1(\Omega) \mid \int_\Omega |\nabla u|^2 < \infty \text{ and } \int_\Omega u = c\}$$

for some $c \in \mathbb{R}$.

**Proposition 6.3.** $M \subset H^1(\Omega)$ and is weakly closed there.

**Proof.** By passing from $u(x)$ to $u(x) - c$ we reduce our problem to the case of $c = 0$. By the Poincaré inequality, Theorem 6.6, $\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2$, we have that $M \subset H^1(\Omega)$.

Now, if $u_n \to u$ weakly in $H^1(\Omega)$, then by the Kondrashov-Rellich theorem (see e.g. [24]) and Section 22, $u_n \to u$ strongly in $L^2(\Omega)$ and therefore in $L^1(\Omega)$ (remember that $\Omega$ is a bounded domain). Hence, if $u_n \in M$, i.e., $\int_\Omega u_n = 0$, then $\int u = 0$, i.e., $u \in M$. Thus $M$ is weakly closed in $H^1(\Omega)$.

□
(i) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $M \subset H^1(\Omega)$ be given by $M = \{ u \in H^1(\Omega) : u(x_0) = 0 \}$ for some $x_0 \in \Omega$ (say $x_0 = 0$).

**Proposition 6.4.** $M$ is weakly closed in $H^1(\Omega)$ for $n = 1$ and is not weakly closed for $n > 1$.

**Proof.** Let first $n = 1$. Since $\Omega$ is bounded, we have by the Rellich–Kondrashov theorem that if $u_n \rightarrow u_0$ in $H^1(\Omega)$, then there is a subsequence $\{ u_{n'} \}$ s.t. $u_{n'} \rightarrow u_0$ in $C(\Omega)$. If $u_n \in M$, then $u_n(0) = 0$, and therefore $u_0(0) = \lim_{n' \to 0} u_{n'}(0) = 0$.

Now let $n \geq 2$. Assume for simplicity that $\Omega$ is a ball and take $\varphi_n \in M$, $\varphi_n(x) = f(n|x|)$ with $f(r) = r$ for $r \leq 1$ and $f(r) = 1$ for $f \geq 1$. Then (see in the exercise below) $\varphi_n \to 1$ weakly in $H^1(\Omega)$ as $n \to \infty$. □

**Exercise 10.** Show the statement above.

The key theorem implies that the following functionals have minimizers on the specified sets:

1. $\int_\Omega (\frac{1}{2} |\nabla u|^2 + G(x,u))$ on $M$, if $G(x,u) \geq 0$, where $M$ is either $H^1_g$ with $\Omega$ bounded in one direction or

$$\{ u \in H^1_0 : \int_\Omega |u|^p d^n x = 1 \}$$

with $\Omega$ bounded and $2 \leq p \leq \frac{2n}{n-2}$, $m_+ = \begin{cases} m & \text{if } m > 0 \\ 0 & \text{if } m \leq 0 \end{cases}$.

2. $\frac{1}{2} \int_\Omega \sum_{ij} g_{ij}(u) \nabla u^i \cdot \nabla u^j$ on $H^1_g(\Omega, \mathbb{R}^n)$, provided $\Omega$ is bounded in one direction, and $g(u) = (g_{ij}(u)) \geq \delta I$, for some $\delta > 0$.

The Euler-Lagrange equations for the variational problems in the examples 1 and 2 above are

$$-\Delta u + G(x,u) + \lambda |u|^{p-2} u = 0$$

and

$$\sum_{ij} (g'_{ij}(u) \nabla u^i \cdot \nabla u^j + \text{div}(a_{ij}(u) \nabla u)) = 0,$$

respectively, where $\lambda$ is a Lagrange multiplier, $G'(x,u) = \partial_u G(x,u)$ and $g'_{ij}(u) = \partial_u g_{ij}(u)$.

Existence of minimizers yields existence of weak solutions of the above equations in a bounded $\Omega \subset \mathbb{R}^n$.

We have the following special cases for example 1:

i) If $G$ has a minimum at $u_0$ which is independent of $x$, then $u_0(x) \equiv u_0$ is a minimizer: $E(u_0) = 0$,

ii) $G(x,u) = V(x)|u|^2$ for some $V(x) \geq 0$, i.e. $G$ is quadratic (remember that in this case the equation for the critical points of $E$ is *linear*!). We can write $E(u)$ on $H^1_0(\Omega)$ as $E(u) = \int_\Omega \pi(-\frac{1}{2} \Delta + V(x)) u$, i.e. $E$ is the quadratic form of the Schrödinger operator $-\frac{1}{2} \Delta + V(x)$.

**Exercise 11.** Prove existence of (weak) solutions of the following boundary value problems (below, $\Omega$ is a bounded domain in $\mathbb{R}^n$):

---

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(a) the Dirichlet problem:
\[
\Delta u = f \quad \text{in } \Omega, \\
u = g \quad \text{on } \partial \Omega,
\]
for every \(f \in L^2(\Omega)\) and any smooth \(g : \partial \Omega \to \mathbb{R}\);

(b) the nonlinear eigenvalue problem:
\[
\Delta u + a(x)|u|^{p-1}u = \mu u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\]
where \(a(x)\) is a smooth and positive function on \(\Omega\), \(n \geq 3\), \(2 < p < \frac{2n}{n-2}\) and \(\mu \geq 0\);

(c) the nonlinear Dirichlet problem:
\[
\nabla \left( |\nabla u|^2 \nabla u \right) = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\]
for any \(f \in L^{4/3}(\Omega)\). (Hint: reduce this to a minimization problem on the Sobolev space
\[
W^{1,1}_0(\Omega) = \{u \in L^4(\Omega) : \int_\Omega |\nabla u|^4 < \infty \text{ and } u|_{\partial \Omega} = 0\}).
\]

How to gain smoothness: elliptic regularity. Assume we show that the following equation has a (weak) solution in \(H^1(\Omega)\):
\[
\Delta u = a(x)u^4 \quad \text{in } \Omega,
\]
and \(u = 0\) on \(\partial \Omega\) (Dirichlet boundary conditions). Here, \(a\) is smooth and \(\Omega \subset \mathbb{R}^3\) is bounded. This is not so good since \(\Delta u \in H^{-1}(\Omega)\). But it turns out that in fact \(u\) is smooth!

We can show this heuristically in the following way. By the Sobolev embedding theorem (i.e. \(H^1(\Omega) \subset L^\alpha(\Omega)\) with \(\alpha < \frac{2n}{n-2} = 6\) for \(n = 3\)), we have that \(u \in L^\alpha(\Omega)\) with \(\alpha < 6\). Hence \(u^4 \in L^\beta(\Omega)\) with \(\beta < 3/2\), so \(\Delta u \in L^\beta(\Omega)\) since \(a\) is smooth. Then by the Sobolev embedding theorem, \(u \in L^\alpha(\Omega)\) for any \(\alpha < \infty\). Iterating this we obtain that \(u\) is bounded and differentiable.

Poincaré inequality

Theorem 6.5 (Poincaré inequality). Let \(\Omega\) have a diameter \(d < \infty\) in some direction (i.e. it is possible to place \(\Omega\) between two parallel hyperplanes at a distance \(d\) from each other). Then for any \(u \in H^1_0(\Omega)\), we have
\[
\int_\Omega |u|^2 \leq (2d)^2 \int_\Omega |\nabla u|^2. \tag{6.11}
\]

Proof. We can assume that this hyperplanes are \(\{x_1 = 0\}\) and \(\{x_1 = d\}\). Assume \(u\) is real and estimate
\[
\|u\|_2^2 = \int_\Omega 1 \cdot |u|^2 = -\int_\Omega x_1 \frac{\partial}{\partial x_1} |u|^2 = -2\text{Re} \int_\Omega x_1 u_\ast \frac{\partial u}{\partial x_1} \leq 2d \int_\Omega |u| \left| \frac{\partial u}{\partial x_1} \right|.
\]

Applying now the Schwartz inequality to the integral on the r.h.s., we obtain
\[
\|u\|_2^2 \leq 2d \|u\|_2 \left\| \frac{\partial u}{\partial x_1} \right\|_2 \leq 2d \|u\|_2 \|\nabla u\|_2,
\]
where \(\|\nabla u\|_2^2 = \int_\Omega |\nabla u|^2 = \int_\Omega \sum_{n=1}^n |\frac{\partial u}{\partial x_n}|^2\). The latter inequality implies \(\|u\|_2 \leq 2d \|\nabla u\|_2\).
We mention without proof, the following variant of the result above.

**Theorem 6.6.** Let $\Omega$ be bounded and $\bar{u} := \frac{1}{|\Omega|} \int_\Omega u$. Then

$$\int_\Omega |u - \bar{u}|^2 \leq C \int_\Omega |\nabla u|^2.$$

### 6.3 Existence of ground state of nonlinear Schrödinger equation (without and with potential)

*(under construction)* (see [33], Introduction and Section II, and also [2], Sections 1 (in particular, Theorem 1 i) - ii)) and 3 and Appendix)
7 Interfaces, vortices, vortex lattices and harmonic maps

7.1 Allen-Cahn energy functional and interfaces

The difference in free energy of two phases of the same substance or of two substances in a domain $\Omega \subset \mathbb{R}^d$ can be often described by the following simple functional:

$$
E(u) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \lambda G(u) \right) d^d x,
$$

where $u : \Omega \rightarrow \mathbb{R}$, $\lambda > 0$, and $G(u) \geq 0$, has two strict, non-degenerate minima at $u = 1$ and $u = -1$, and $G(u) \rightarrow \infty$ as $|u| \rightarrow \infty$. In other words, $G$ is of the form of a double-well potential:

$$
G(u) = \frac{1}{4} (|u|^2 - 1)^2.
$$

This functional is called the Allen-Cahn (or Ginzburg-Landau) energy functional. It plays an important role in many areas of sciences and engineering. $\Omega$ is a large domain, say a box with sides of size $2L$ or a ball of radius $L$ or $\mathbb{R}^d$.

Note that $E(\varphi) \geq 0$ and has two absolute minimizers, $-1$ and $+1$, so that $E(-1) = E(+1) = 0$. These minimizers correspond to two homogeneous substance or phases, which we will call the $-1$ and $+1$ phase. We are interested in minimizers for which these phases co-exist. To obtain such minimizers we impose constraints and/or boundary conditions. For instance, if the total amount of the $-1$ phase is fixed and $\Omega = \mathbb{R}^d$, then imposing

$$
\int_{\Omega} (1 - u(x)) d^d x = \alpha,
$$

would guarantee that the total amount of the $-1$ phase is finite and determined by $\alpha$.

Before addressing these questions, we mention that the Euler-Lagrange equation for critical points of $E(u)$ is

$$
-\Delta u + \lambda G'(u) = \mu,
$$

where $\mu \neq 0$ if condition (7.2) is imposed. In the latter case $\mu$ is the corresponding Lagrange multiplier and is chosen so that (7.2) holds. This is the celebrated Allen-Cahn equation (also called sometimes Ginzburg-Landau equation).

Exercise 12. Find the Euler-Lagrange equation for the functional (7.1) with side condition (7.2).

Now we apply the direct method of variational calculus in order to find minimizers of the Allen-Cahn (or Ginzburg–Landau) energy functional (7.1), with the function $G(u)$ described in the paragraph after (7.1).
**Planar interface.** We want to consider the situation where the minimizers describe the planar interface. Assume this interface is the plane \( \{x_1 = 0\} \) (by a translation and a rotation, we can always reduce to this case). Then our unknown \( u \) depends on one variable only - on \( x = x_1 \) - and the problem becomes the one dimensional one and the functional (7.1) becomes

\[
\mathcal{E}(u) = \int_{\mathbb{R}} \left( \frac{1}{2} |\nabla u|^2 + \lambda G(u) \right) \, dx,
\]

and can interpreted as the energy per unit area of the interface \( \{x_1 = 0\} \).

We have four distinct boundary conditions (BC): \( u(x_1) \to \pm 1 \) as \( x_1 \to \infty \) and \( u(x_1) \to \pm 1 \), as \( x_1 \to -\infty \). Consequently, we are led to consider \( \mathcal{E} \) on the following four spaces:

\[
M_{\pm, \pm} = \{ u \in H^1(\Omega) : \text{one of the above BC holds} \}. 
\]

Clearly, \( \mathcal{E} \) attains its strict minimum on \( M_{+, +} \) and \( M_{-, -} \) at \( u_+(x_1) \equiv 1 \) and \( u_-(x_1) \equiv -1 \), respectively. In the phase separation model described by the Allen-Cahn (Ginzburg-Landau) functional, these minimizers describe homogeneous phases. Next, a minimizer on \( M_{+, -} \) is obtained from a minimizer on \( M_{-, +} \) by reflection \( u(x_1) \to u(-x_1) \). Thus it suffices to consider only minimizers on \( M_{-, +} \). Observe that since \( \mathcal{E} \) has the reflection symmetry, we can simplify our task by looking for odd minimizers, i.e. we pass from \( M_{-, +} \) to

\[
M_{-, +}^{\text{odd}} := \{ u \in M_{-, +} : u(-x_1) = -u(x_1) \}. 
\]

To fix ideas, we let \( G(u) = \frac{1}{4}(u^2 - 1)^2 \). Our result here is the following

**Theorem 7.1.** The Ginzburg-Landau energy functional (7.4), defined on \( M_{-, +}^{\text{odd}} \), has a minimizer.

**Proof.** We identify \( M_{-, +} \) with the space \( M_{-, +} = \chi + H^1_0(\mathbb{R}) \), where \( \chi \) is a smooth (and odd) function satisfying \( \chi(x) = 1 \) for \( x \geq \delta \) and \( \chi(x) = -1 \) for \( x \leq -\delta \). Then, since \( H^1_0(\mathbb{R}) \) is weakly closed, \( M_{-, +}^{\text{odd}} \) is weakly closed in \( H^1(\mathbb{R}) \).

Now, we would like to show that \( \mathcal{E} \) on \( H^1_0(\mathbb{R}) \) is w.l.s.c. and coercive, and therefore on \( M_{-, +}^{\text{odd}} \). Since \( G \geq 0 \), the w.l.s.c. follows from Corollary 6.2.

The coercivity of \( \mathcal{E}(u) \) on \( M_{+, c}^{(9)} \) is a more subtle question. Define \( v := \chi - u \in H^1_0(\mathbb{R}) \). We write

\[
G(u) = \frac{1}{4} (1 - \chi^2)^2 - 2 (1 - \chi^2)v \left( \chi - \frac{1}{2} v \right) + v^2 \left( \chi - \frac{1}{2} v \right)^2. 
\]

Assume \( |v| \leq 1 \) for \( |x| \geq \delta \). Then using that \( \chi(x) = \pm 1 \) for \( |x| \geq \delta \), we find \( G(u) \geq \frac{1}{4} v^2 \). This gives \( \int G(u) dx \geq \frac{1}{4} \int_{|x| \geq \delta} v^2 dx \geq \frac{1}{4} \int_{\mathbb{R}} v^2 dx - C\delta \). Therefore

\[
\mathcal{E}(1 - v) \geq \frac{1}{4} \int_{\mathbb{R}} (|\nabla v|^2 + v^2) dx - C \equiv \frac{1}{4} ||v||_{H^1}^2 - C. 
\]

This implies the coercivity of \( \mathcal{E} \) on \( H^1_0(\mathbb{R}) \).

Now, by the properties established above and the key theorem on minimization, \( \mathcal{E}(u) \) has a minimizer \( u_* \) on the set

\[
\tilde{M}_{-, +}^{\text{odd}} := \{ u \in M_{-, +}^{\text{odd}} : \|\chi - u\| \leq 1 \}. 
\]

Is this minimizer also a minimizer on \( M_{-, +}^{\text{odd}}? \) Observe that

\[
\mathcal{E}(u) \geq \mathcal{E}(\max(\min(u, 1), -1)). 
\]
Indeed, one can see that replacing the function \( u \) by the function \( \max(\min(u, 1), -1) \) decreases both gradient and potential term. Hence, if \( \{u_n\} \) is a minimizing sequence, then so is \( \{w_n := \max(\min(u_n, 1), -1)\} \) and the minimizer, \( w_* \), satisfies \( |w_*| \leq 1 \). Similarly, we can show that

\[
\mathcal{E}(u) \geq \mathcal{E}(w), \quad w := |u|\chi_{x \geq 0} - |u|\chi_{x < 0}.
\]

So we have \( w_* \geq 0 \) for \( x \geq 0 \) and \( w_* \leq 0 \) for \( x \leq 0 \). The last two properties imply \( |\chi - w_*| \leq 1 \). Hence a minimizer on \( M^\text{odd}_{-+,+} \) belongs to and therefore \( M^\text{odd}_{-+,+} \) is also a minimizer on \( M^\text{odd}_{-+,+} \).

The minimizers \( w_*(x) \) are called kinks and \( w_*(-x) \), anti-kinks. They describe planar interfaces. Of course, by shifting \( w_*(x) \) to \( w_*(x-h) \), we obtain a one–parameter family of minimizers, the kinks centred at different points of \( \mathbb{R} \).

**Lamellar phase.** In this situation, layers of oil and water coexist in a periodic array. To get a solution to (7.3), glue together a kink at \( z_1 \) and an antikink at \( z_2 \). There is no exact solution of this form: the kink and the antikink interact at any distance. They attract each other and as a result, they move toward each other and collapse. This means that if \( \varphi \) is a function consisting of a kink and an antikink glued together at a distance \( R \), then \( \mathcal{E}(\varphi) \) is monotonically increasing as a function of \( R \). Here, \( \varphi \) is a function consisting of a kink and an antikink glued together at a distance \( R \). However, presumably one can construct a periodic solution corresponding to an array of kinks and antikinks.

Idea of proving the existence of the lamellar solutions. Consider the variational problem of minimizing the functional

\[
\mathcal{E}(u) = \int_{-c}^{c} \left( \frac{1}{2} |\nabla u|^2 + \lambda G(u) \right) dx,
\]

(7.5)
under constraint that \( \int_{-c}^{c} u \, dx = \alpha \), i.e. on the space \( \{ u \in H^1([-c,c]) : u(\pm c) = 1, \int_{-c}^{c} u \, dx = \alpha \}, \) for \( \alpha \in (-2c,2c) \). For \( \alpha = 2c \), the minimizer is trivial, \( u_* = 1 \), and for \( \alpha = -2c \), there is no minimizer. The parameters \( a \) and \( b \) satisfy \( a + b = c \) and their ration is determined by \( \alpha \). Then the minimizer \( u_* \) is extended periodically to the entire real line.

**Spherical drops and cylinder solutions.** Here \( \Omega \) is either the ball, \( B_R \), of radius \( R \), centered at the origin, or the cylinder, \( C_R \) of radius \( R \) or \( \mathbb{R}^d, d = 2,3 \). We minimize the energy functional \( \mathcal{E} \), as was described above, on the set

\[
M^{(g)}_{\alpha} := \{ u \in H^1_g(\Omega) \mid \int_{\Omega} (1-u) = \alpha \}
\]

where \( g \equiv 1 \) on \( \partial B \), for some \( \alpha > 0 \).

**Theorem 7.2.** Let \( G(u) = \frac{1}{4}(u^2-1)^2 \) and \( \Omega \) be bounded. The Ginzburg-Landau energy functional \( \mathcal{E}(u) \) defined on \( M^{(g)}_{\alpha} \), \( \alpha > 0 \) and \( g \equiv 1 \), has a minimizer.

The proof of this theorem is similar to the one of Theorem 7.1.

**Exercise 13.** Check that the set \( M^{(g)}_{\alpha} \) is weakly closed and that the functional \( \mathcal{E}(u) \) defined on \( M^{(g)}_{\alpha} \) is w.l.s.c.

One can show that this minimizer, \( u_* \), is smooth. What is the shape of \( u_* \)? Is it spherically symmetric? One can show that the minimizer must have exactly one zero, i.e., it is of the second type as shown on the figure below.

The parameter \( R \) - the radius of the drop - is found from the condition (7.2).

One can also show that \( u_* \) is spherically symmetric. If one does not want to work hard, then one can look directly for spherically symmetric minimizers, i.e. minimizers of \( \mathcal{E}(u) \) of the form

\[
u(x) = \psi(|x|),
\]

subject to the side condition (7.2). But then one would have only minimizer among spherically symmetric functions.

**7.2 Vortices in Superfluids**

Macroscopically, equilibrium states of superfluids and Bose-Einstein condensates are described by a function \( \psi : \Omega \to \mathbb{C}, \Omega \subset \mathbb{R}^d \) called the order parameter, which satisfies the nonlinear differential equation

\[-\Delta \psi + (|\psi|^2 - 1)\psi = 0 \text{ in } \Omega \]
called the *Gross-Pitaevskii or Ginzburg-Landau* equation, with the boundary condition

\[ |\psi| \to 1 \text{ as } |x| \to \partial \Omega. \quad (7.8) \]

In 1958, V.L Ginzburg and L. Pitaevskii conjectured that for \( d = 2 \) and \( \Omega = \mathbb{R}^2 \) (the common ‘cylindrical’ geometry) this equation has solutions of the form

\[ \psi_n(x) = f_n(r)e^{in\theta} \quad (7.9) \]

where \((r, \theta)\) are polar coordinates of \( x \in \mathbb{R}^2 \) and \( n \) is an integer. It was conjectured that these solutions describe vortices observed in superfluids. (In 1947, L. Onsager has conjectured that these vortices are analogous to normal fluid vortices, except for the fact that the vortices of the superfluid were quantized while normal vortices were arbitrary.)

The rigorous result about existence and uniqueness of vortices came much later and is stated in the following

**Theorem 7.3.** Let \( d = 2 \) and \( \Omega \) be either \( \mathbb{B}_R \) or \( \mathbb{R}^2 \). Then, for all \( n \), there exists a solution of form (7.9) unique up to symmetry transformation. The function \( f_n(r) \) can be taken to be positive and it vanishes at \( r = 0 \) as

\[ f_n(r) = ar^n \]

for some \( a > 0 \) and is monotonically increasing to 1.

We prove this theorem by reformulating the b.v problem (7.7) - (7.8) as a variational problem of finding a minimizer of an appropriate functional. For a bounded domain \( \Omega \), (7.7) is the Euler-Lagrange equation for the functional

\[ \mathcal{E}(\psi) = \int_\Omega \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4}(|\psi|^2 - 1)^2, \]

called the *Gross-Pitaevskii or Ginzburg-Landau* energy functional, on the space

\[ M = \{ \psi \in H^1_g(\Omega) \mid \int_\Omega (|\psi|^2 - 1)^2 < \infty \} \]
for some function satisfying $|g| = 1$. For $\Omega = B_R$, we choose

$$g(x) = e^{in\theta}$$

**Renormalized Ginzburg-Landau functional.** For $d = 2$ and $\Omega$ unbounded, say, $\Omega = \mathbb{R}^2$, $\mathcal{E}(\psi) = \infty$ for functions $\psi$ of interest. We explain this more carefully for the most important case of $\Omega = \mathbb{R}^2$.

For $C^1$ complex functions (vector-fields) $\psi$ satisfying the boundary condition $|\psi| \to 1$ as $|x| \to \infty$, we define the degree by

$$\deg \psi = \frac{1}{2\pi} \int_{|x|=R} d(\arg \psi)$$

for $R$ sufficiently large. Here $\arg \psi = \frac{1}{i} \ln(|\psi|)$.

**Exercise 14.** Show that if $f(r) \to 1$ as $r \to \infty$, then $\deg(f(r)e^{in\theta}) = n$.

It is shown in [26] that if $\psi : \mathbb{R}^2 \to \mathbb{C}$ is a $C^1$ vector-field such that $|\psi(x)| \to 1$ as $|x| \to \infty$ and $\deg \psi \neq 0$, then $\mathcal{E}(\psi) = \infty$. Thus the Ginzburg-Landau energy functional $\mathcal{E}(\psi)$ is not defined in infinite domains in the most interesting cases (i.e., in the presence of vortices). To go around this problem, [26] have introduced the renormalized Ginzburg-Landau energy functional defined as follows

$$\mathcal{E}_{ren}(\psi) = \frac{1}{2} \int |\nabla \psi|^2 - \frac{(\deg \psi)^2}{r^2} \chi + \frac{1}{2} (|\psi|^2 - 1)^2$$

where $\chi$ is a cut-off function with the properties: $\chi \in C^\infty(\mathbb{R}^2)$ and

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \geq 2, \\ 0 & \text{for } |x| \leq 1. \end{cases}$$

It is shown in [26] that $\mathcal{E}_{ren}(\psi)$ is defined on a large class of functions which include $n$–vortices and their combinations and that

$$d\mathcal{E}_{ren}(\psi) = -\Delta \psi + (|\psi|^2 - 1)\psi,$$

i.e. $\mathcal{E}_{ren}(\psi)$ produce the correct Euler-Lagrange equations (7.7). However, as should be, the functional $\mathcal{E}_{ren}(\psi)$ is not bounded below for $|\deg \psi| \geq 2$. Indeed, consider a function $\psi$ describing $n := \deg \psi \geq 2$ single vortices. Then moving these vortices apart to infinity would decrease $\mathcal{E}_{ren}(\psi)$ indefinitely. The energy functional $\mathcal{E}_{ren}(\psi)$ is suitable and natural for the variational approach to the vortex problem in an unbounded domain.

Now, we can state the following variational problem: Minimize $\mathcal{E}_{ren}(\psi)$ among functions $\psi$ with a fixed degree $\deg \psi = n$. This the variational problem with the topological constraint.

For simplicity, we prove the theorem only in the case $\Omega = B_R$. A proof in the case $\Omega = \mathbb{R}^2$ uses the renormalized Ginzburg-Landau energy and can be found in [26]. We prove a somewhat stronger statement.

**Proof of Theorem 7.3 for $\Omega = B_R$.** We minimize the functional $\mathcal{E}(\psi)$ on the functions of the form $\psi = f(x)e^{in\theta}$ for some fixed $n$, where $f(x)$ is a real valued function satisfying the boundary condition $f(x) = 1$ for $|x| = R$. In other words, we minimize the functional

$$e_n(f) := \mathcal{E}(fe^{in\theta})$$
on a space of functions $f(r)$ satisfying $f(x) = 1$ for $|x| = R$.

In what follows, we abbreviate $B_R = B$. Write out this functional explicitly:

$$e_n(f) = \int_B \left( \frac{1}{2} |\nabla f|^2 + \frac{n^2}{2r^2} f^2 + \frac{1}{4} (f^2 - 1)^2 \right)$$

where we have used that

$$\nabla (fe^{in\theta}) = (\nabla f + in\nabla \theta)e^{in\theta},$$

and so for $f$ real

$$|\nabla (fe^{in\theta})|^2 = |\nabla f|^2 + n^2|\nabla \theta|^2 f^2,$$

and that $|\nabla \theta| = 1/r^n$.

**Lemma 7.1.** For all $n$, $e_n(f)$ has a minimizer on the set

$$M = \{ f \text{ real and } E(f) < \infty \}.$$

and all minimizers are radially symmetric.

**Proof.** Let $\xi = 1 - f$ and define the functional $F(\xi) = e_n(1 - \xi)$. Explicitly this functional is given as

$$F(\xi) = \int_B \frac{1}{2} |\nabla \xi|^2 + G(\xi)$$

where

$$G(\xi) = \frac{n^2}{r^2} (1 - \xi)^2 + \xi^2 (1 - \frac{1}{2} \xi)^2.$$

Observe that $\xi \to 0$ as $|x| \to \infty$. We consider $F(\xi)$ on the set

$$M = \{ \xi \in H_0^1(B) \mid \xi \text{ real, } E(\xi) < \infty \}.$$

Since $G(\xi) \geq 0$, we have that $F(\xi)$ is w.l.s.c. on $H_0^1(B)$. The functional $F(\xi)$ is also coercive by the difficult part of the Theorem. One can avoid using this difficult part (which follows if $B = \mathbb{R}^2$) proceeding as follows: we have

$$G(\xi) \geq \frac{1}{4} \xi^2 \text{ if } \xi \leq 1$$

and therefore $F(\xi)$ is coercive under the additional condition that $\xi \leq 1$. Now we can show that we can make this condition automatically satisfied for minimizing sequences. Indeed, let $\xi_n$ be a minimizing sequence for the functional $F(\xi)$. Define a new sequence $\{ \xi'_n(x) := \min(\xi_n(x), 1) \}$. Clearly,

$$|\nabla \xi_n| \geq |\nabla \xi_n'| \text{ and } G(\xi_n) \geq G(\xi'_n)$$
Exercise 15. Check these statements.

Hence $F(\xi_n) \geq F(\xi'_n)$ so that $\{\xi'_n\}$ is also a minimizing sequence. Of course, $\xi'_n \leq 1$ so $F(\xi)$ is coercive on this sequence and the proof of the Key Theorem can be modified so as to show that the functional $F(\xi)$ has a minimizer on the set $M$.

Let now $f_0$ be a minimizer of the functional $e_n(f)$. We show that $f_0$ must be radially symmetric. Introduce the function $u = (\overline{f_0}^2)^{\frac{1}{2}},$ where $\overline{g}(r) := \int_0^{2\pi} g(r, \theta) \frac{d\theta}{2\pi}$. Then $\overline{f_0}^2 = u^2$ and

$$\overline{f_0}^2 = u^4 \quad \text{and} \quad |\nabla_r f_0|^2 \geq |\nabla_r u|^2.$$  \hspace{1cm} (7.10)

Indeed, the first of these inequalities follows from the Cauchy-Schwartz inequality and the second is obtained as follows: for $r$ such that $u(r) \neq 0$,

$$\left| \nabla_r \left( \int_0^{2\pi} f_0^2(r, \theta) \frac{d\theta}{2\pi} \right)^{1/2} \right|^2 = \int_0^{2\pi} \left| \nabla_r f_0 \frac{d\theta}{2\pi} \right|^2 \leq \int_0^{2\pi} \frac{f_0^2 \frac{d\theta}{2\pi}}{\int_0^{2\pi} \frac{d\theta}{2\pi}} \int_0^{2\pi} |\nabla_r f_0|^2 \frac{d\theta}{2\pi},$$

by the Cauchy-Schwartz inequality again. Inequality (7.10) implies that

$$\int |\nabla f_0|^2 \, d^2x \geq \int_0^\infty |\nabla_r u|^2 \, r \, dr$$

with the equality taking place only if $f_0$ is independent of $\theta$. Hence

$$e_n(f_0) \geq e_n(u)$$

and the equality holds only if $f_0$ is radially symmetric. Since $f_0$ is a minimizer, we can conclude that it is radially symmetric. We omit the proof of monotonicity of $f_0$ and refer the reader to [26] for this proof. \hfill $\square$

Since $f_0$ is a minimizer of $e_n(f)$ and is radially symmetric, it satisfies the Euler-Lagrange equation

$$-\Delta_r f + \frac{n^2}{r^2} f + (f^2 - 1)f = 0 \quad \text{in} \quad \Omega,$$

where $\Delta_r$ is radial Laplacian in $\mathbb{R}^2: \Delta_r = \frac{1}{r} \partial_r (r \partial_r)$. Since $f_0$ is radially symmetric, we have that $\nabla f_0 \cdot \nabla \theta = 0$. This together with the relations $\Delta \theta = 0$ and $|\nabla \theta| = 1/r$ implies

$$\Delta (f_0 e^{i\theta}) = (\Delta_r f_0 - \frac{n^2}{r^2} f_0) e^{i\theta}.$$  

Therefore, the function $\psi_n(x) = f_0(r) e^{i\theta}$ satisfies Ginzburg-Landau equations (7.7) and (7.8). \hfill $\square$
7.3 The Ginzburg-Landau Equations

The Ginzburg-Landau theory gives a macroscopic description of superconducting materials and serves as an integral and paradigmatic part of particle physics. It is formulated in terms of a pair \((\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d, \ d = 1, 2, 3\), satisfying the system of nonlinear PDE called the Ginzburg-Landau equations:

\[
\begin{align*}
\Delta A \Psi &= \kappa^2(1 - |\Psi|^2) \Psi \\
\text{curl}^2 A &= \text{Im}(\bar{\Psi} \nabla A \Psi)
\end{align*}
\]  

(7.11)

where \(\nabla A = \nabla - i A\), and \(\Delta A = \nabla^2 A\), the covariant derivative and covariant Laplacian, respectively, and \(\kappa > 0\) is a parameter, coupling constant. For \(d = 2\), \(\text{curl} A := \partial_1 A_2 - \partial_2 A_1\) is a scalar, and for scalar \(B(x) \in \mathbb{R}\), \(\text{curl} B = (\partial_2 B, -\partial_1 B)\) is a vector.

Superconductivity. The complex-valued function \(\Psi(x)\) is called an order parameter, \(|\Psi(x)|^2\) gives the local density of (Cooper pairs of) superconducting electrons, and the vector field \(A(x)\) is the magnetic potential, so that \(B(x) := \text{curl} A(x)\) is the magnetic field. The parameter \(\kappa > 0\) is called the Ginzburg-Landau parameter, it depends on the material properties of the superconductor. The vector quantity \(J(x) := \text{Im}(\bar{\Psi} \nabla A \Psi)\) is the superconducting current. (See eg. [35, 34]).

Particle physics. In the Abelian-Higgs model, \(\psi\) and \(A\) are the Higgs and \(U(1)\) gauge (electromagnetic) fields, respectively. Geometrically, one can think of \(A\) as a connection on the principal \(U(1)\)-bundle \(\mathbb{R}^d \times U(1), \ d = 2, 3\).

Cylindrical geometry. In the commonly considered idealized situation of a superconductor occupying all space and homogeneous in one direction, one is led to a problem on \(\mathbb{R}^2\) and so may consider \(\Psi : \mathbb{R}^2 \to \mathbb{C}\) and \(A : \mathbb{R}^2 \to \mathbb{R}^2\). This is the case we deal with in this contribution.

Symmetries of the equations The Ginzburg-Landau equations (7.11) admit several symmetries, that is, transformations which map solutions to solutions.

**Gauge symmetry:** for any sufficiently regular function \(\gamma : \mathbb{R}^2 \to \mathbb{R}\),

\[
T_{gauge}^\gamma : (\Psi(x), A(x)) \mapsto (e^{i\gamma(x)} \Psi(x), A(x) + \nabla \gamma(x));
\]  

(7.12)

**Translation symmetry:** for any \(h \in \mathbb{R}^2\),

\[
T_h^{\text{trans}} : (\Psi(x), A(x)) \mapsto (\Psi(x + h), A(x + h));
\]  

(7.13)

**Rotation symmetry:** for any \(\rho \in SO(2)\),

\[
T_\rho^{\text{rot}} : (\Psi(x), A(x)) \mapsto (\Psi(\rho^{-1} x), \rho^{-1} A((\rho^{-1})^T x)),
\]  

(7.14)

One of the analytically interesting aspects of the Ginzburg-Landau theory is the fact that, because of the gauge transformations, the symmetry group is infinite-dimensional.

**Abrikosov lattices** In 1957, A. Abrikosov discovered a class of solutions, \((\Psi, A)\), to (7.11), presently known as Abrikosov lattice vortex states (or just Abrikosov lattices), whose physical characteristics, density of Cooper pairs, \(|\Psi|^2\), the magnetic field, curl \(A\), and the supercurrent, \(J_S = \text{Im}(\bar{\Psi} \nabla A \Psi)\), are double-periodic w.r. to a lattice \(\mathcal{L}\). By a lattice we understand here the Bravais lattice, i.e.

\[
\mathcal{L} = \{m\nu_1 + n\nu_2 : m, n \in \mathbb{Z}\},
\]

where \((\nu_1, \nu_2)\) is a basis of \(\mathcal{L}\).

Denote by \(\Omega\) a fundamental cell of the lattice \(\mathcal{L}\), say \(\{x\nu_1 + y\nu_2 : x, y \in [0, 1)\}\), where \((\nu_1, \nu_2)\) is a basis of \(\mathcal{L}\). For Abrikosov states, for \((\Psi, A)\), the magnetic flux, \(\int_{\Omega} \text{curl} A\), through a lattice cell, \(\Omega\), is quantized,
Lemma 2. \[
\frac{1}{2\pi} \int_{\Omega} \text{curl} \; A = \deg \Psi = n, \tag{7.15}
\]
for some integer \( n \).

Proof. The periodicity of \( n_s = |\Psi|^2 \) and \( J = \text{Im} (\bar{\Psi} \nabla A \Psi) \) imply that \( \nabla \varphi - A \), where \( \Psi = |\Psi| e^{i\varphi} \), is periodic, provided \( \Psi \neq 0 \) on \( \partial \Omega \). This, together with Stokes's theorem, \( \int_{\Omega} \text{curl} \; A = \int_{\partial \Omega} A = \oint_{\partial \Omega} \nabla \varphi \) and the single-valuedness of \( \Psi \), imply that \( \int_{\Omega} \text{curl} \; A = 2\pi n \) for some integer \( n \). \( \square \)

Using the reflection symmetry of the problem, one can easily check that we can always assume \( n \geq 0 \).

**Abrikosov lattices as gauge-equivariant states** We say a state \( (\Psi, A) \) is gauge - equivariant (with respect to a lattice \( L \), or \( L \)-equivariant) if there exists (possibly multivalued) function \( g_s : \mathbb{R}^2 \to \mathbb{R}, s \in L \), such that

\[
T_{s}^{\text{trans}}(\Psi, A) = T_{g_s}^{\text{gauge}}(\Psi, A). \tag{7.16}
\]

A key point in proving both theorems is to realize that a state \( (\Psi, A) \) is an Abrikosov lattice if and only if \( (\Psi, A) \) is gauge - equivariant.

**Lemma 3.** A state \( (\Psi, A) \) is an Abrikosov lattice if and only if \( (\Psi, A) \) is gauge-equivariant.

Proof. If state \( (\Psi, A) \) satisfies \( (7.16) \), then all associated physical quantities are \( L \)-periodic, i.e. \( (\Psi, A) \) is an Abrikosov lattice. In the opposite direction, if \( (\Psi, A) \) is an Abrikosov lattice, then \( \text{curl} \; A(x) \) is periodic w.r.to \( L \), and therefore \( A(x + s) = A(x) + \nabla g_s(x) \), for some functions \( g_s(x) \). Next, we write \( \Psi(x) = |\Psi(x)| e^{i\phi(x)} \). Since \( |\Psi(x)| \) and \( J(x) = |\Psi(x)|^2 (\nabla \phi(x) - A(x)) \) are periodic w.r.to \( L \), we have that so is \( \nabla \phi(x) - A(x) \), which, together with the relation \( A(x + s) = A(x) + \nabla g_s(x) \), gives that \( \nabla \phi(x + s) = \nabla \phi(x) + \nabla g_s(x) \), which implies that \( \phi(x + s) = \phi(x) + g_s(x) + c_s \), for some constants \( c_s \).

Since \( T_{s}^{\text{trans}} \) is a commutative group, we see that the family of functions \( g_s \) has the important cocycle property

\[
g_{s+t}(x) - g_s(x + t) - g_t(x) \in 2\pi \mathbb{Z}. \tag{7.17}
\]

This can be seen by evaluating the effect of translation by \( s + t \) in two different ways. We call \( g_s(x) \) the **gauge exponent**.

**Ginzburg-Landau energy.** The Ginzburg-Landau equations (7.11) are the Euler-Lagrange equations for critical points of the **Ginzburg-Landau energy functional** (written here for a domain \( Q \in \mathbb{R}^2 \))

\[
\mathcal{E}_Q(\Psi, A) := \int_Q \left\{ \left| \nabla A \Psi \right|^2 + (\text{curl} \; A)^2 + \frac{\kappa^2}{2} (|\Psi|^2 - 1)^2 \right\}. \tag{7.18}
\]

**Superconductivity:** In the case of superconductors, the functional \( \mathcal{E}(\psi, A) \) gives the difference in (Helmholtz) free energy (per unit length in the third direction) between the superconducting and normal states, near the transition temperature.

**Particle physics:** In the particle physics case, the functional \( \mathcal{E}(\Psi, A) \) gives the energy of a static configuration in the \( U(1) \) Yang-Mills-Higgs classical gauge theory.
Existence of Abrikosov lattices

**Theorem 4.** For any lattice $\mathcal{L}$, there exists a smooth Abrikosov lattice solution $u_s = (\Psi_s, A_s)$ (i.e. satisfying (7.16)) for this $\mathcal{L}$.

**Proof.** We want to solve the Ginzburg-Landau equations (7.11) for functions satisfying the condition (7.16). Notice that these equations are the Euler-Lagrange equations for the energy functional $\mathcal{E}_Q(\Psi, A)$, see (7.18), on the space $H^1(\Omega)$, satisfying (7.16). We say two states $u'$ and $u$ are gauge-equivalent, if there is differentiable real function $\chi$ s.t. $u' = T_{\chi}^{\text{gauge}} u$. We begin with the following statement whose proof can be found in [37]

**Lemma 5.** Any $\mathcal{L}$-equivariant state, $(\Psi', A')$, is gauge-equivalent to a $\mathcal{L}$-equivariant state, $(\Psi, A)$, satisfying $\text{div} A = 0$.

Now, we identify the quotient $\mathbb{R}^2/\mathcal{L}$ with a fundamental cell, $\Omega$, of the lattice $\mathcal{L}$ and introduce the space

$$H^1_{\text{equiv}}(\Omega) := \{u \in H^1_{\text{loc}}(\mathbb{R}^2) : u \text{ satisfies (7.16) and } \text{div} A = 0\},$$

equipped with the norm $\|u\|_{H^1_{\text{equiv}}(\Omega)} := \|u\|_{H^1(\Omega)}$. The space $H^1_{\text{equiv}}(\Omega)$ is weakly closed in $H^1(\Omega)$. Indeed, if $u_n \rightarrow u_s$ weakly in $H^1(\Omega)$, then

$$u_n \rightarrow u_s \text{ pointwise on } \Omega. \tag{7.19}$$

Using that $\mathbb{R}^2 = \cup_{s \in \mathcal{L}} (\Omega + s)$, we define $u_s$ on $\mathbb{R}^2$ by

$$u_s(x + s) := e_s(x)u_s(x), \quad \forall x \in \Omega, \quad s \in \mathcal{L}, \tag{7.20}$$

where $e_s(x) := T_{gs}^{\text{gauge}}$. By (7.16), $e_s(x)$ obeys $e_t(x + s)e_s(x) = e_{s+t}(x)$. Since

$$u_s(x + s + t) = e_{s+t}(x)u_s(x) = e_t(x + s)e_s(x)u_s(x) = e_t(x + s)u_s(x + s),$$

which can be rewritten as $u_s(y + t) = e_t(y)u_s(y), \quad \forall y \in \Omega + s, \quad s, t \in \mathcal{L}$, we see that $u_s$ satisfies (7.16). The equations (7.19) and (7.20) imply $u_n(x) \rightarrow u_s(x)$ pointwise on $\mathbb{R}^2$. Moreover, since $u_n \rightarrow u_s$ weakly in $H^1(\Omega)$ and $\text{div} A_n = 0$, we have that $\text{div} A_s = 0$. Thus $u_s \in H^1_{\text{equiv}}(\Omega)$.

Now, we analyze $\mathcal{E}_\Omega(\Psi, A)$. Since $\text{div} A = 0$, $(|\Psi|^2 - 1)^2 = |\Psi|^4 - 2|\Psi|^2 + 1 \geq \frac{1}{2} |\Psi|^4 - 1$ and $\Omega$ has a finite area, $|\Omega| < \infty$, the energy $\mathcal{E}_\Omega(\Psi, A)$ is obviously coercive,

$$\mathcal{E}_\Omega(\Psi, A) \geq \int_{\Omega} \left\{ |\nabla A|^2 + (\text{curl} A)^2 + \frac{\kappa^2}{4} |\Psi|^4 \right\} - \frac{\kappa^2}{2} |\Omega| \geq c\|\Psi, A\|_{H^1(\Omega)}^2 - C|\Omega|, \tag{7.21}$$

for some constants $c, C > 0$. Finally, since $G = \frac{\kappa^2}{4} (|\Psi|^2 - 1)^2 \geq 0$, the w.l.s.c. follows from Corollary 6.2. Hence $\mathcal{E}_\Omega(\Psi, A)$ has a minimizer on the space $H^1_{\text{equiv}}(\Omega)$ and this minimizer satisfies the Ginzburg-Landau equations (7.11). \qed
7.3.1 Appendix I. Relation to line bundles

The equivariant pairs \( u = (\Psi, A) \) are related to sections and connections on the line bundle, \( E \), defined by the automorphy exponents \( g_s(x) \) as

\[
E := \mathbb{R}^2 \times \mathbb{C}/\mathcal{L},
\]

where \( \mathcal{L} \) acts on \( \mathbb{R}^2 \times \mathbb{C} \) and on \( \mathbb{R}^2 \) as \( s : (x, \Psi) \to (x + s, e^{ig_s(x)}\Psi) \), and \( A \)'s, as connection one-forms on \( E \), with the connections \( \nabla_A \), defined on \( [\Psi(x)]'s \) as \( \nabla_A : X \to (\nabla_A)X \), where \( X \) is a vector field on the base manifold \( \mathbb{R}^2/\mathcal{L}, (\nabla_A)[\Psi] \) defined by the relation \( \nabla_A \Psi \sim \nabla'A'\Psi' \iff \exists_X : \mathbb{R}^2 \to G : (\Psi', A') = T^{\text{gauge}}_X(\Psi, A). \) (We will use the notation \( \nabla_A[\Psi] := [\nabla_A\Psi] = [(d - iA)\Psi] \).)

Line bundles have a topological invariant - the degree, which is defined as the Chern number,

\[
c(e) = c(g) \text{ of the automorphy exponents } g_s(x), \]

where \( \{\nu_1, \nu_2\} \) is a basis of \( \mathcal{L} \). (The degree of line bundles can be also defined through devisors or zero of sections; the expression (7.23) is convenient for our purposes.) The next result shows that \( c(g) \) are topological invariants and are related to the quantization of the magnetic flux.

**Proposition 6.**

- (characteristic class) The quantity (7.23) is independent of \( x \) and of the choice of the basis \( \{\nu_1, \nu_2\} \) and is an integer. Thus (7.23) is the topological invariant classifying Abrikosov lattice states.

- (magnetic flux quantization) The quantity \( \frac{1}{2\pi} \int_\Omega \text{curl} A \) (magnetic flux) is an integer and is equal to \( c(g) \) (magnetic flux quantization),

\[
\frac{1}{2\pi} \int_\Omega \text{curl} A = c(g),
\]

where \( A \) is the corresponding connection and \( \Omega \) is a fundamental lattice cell.

**Proof.** By the relation (7.17), \( g_{\nu_2}(x + \nu_1) + g_{\nu_1}(x) - g_{\nu_1 + \nu_2}(x) \in 2\pi\mathbb{Z} \) and \( g_{\nu_1}(x + \nu_2) + g_{\nu_2}(x) - g_{\nu_1 + \nu_2}(x) \in 2\pi\mathbb{Z} \). Subtracting the second relation from the first shows that \( c(g) \) is independent of \( x \) and is an integer.

To prove the second statement, we note that, by Stokes’ theorem, the magnetic flux through a lattice cell \( \Omega \) is \( \int_\Omega \text{curl} A = \int_{\partial\Omega} A \). Now, using the definition of the gauge transformation, (7.12), and the condition (7.16), we obtain

\[
\int_{\partial\Omega} A = \int_0^1 [\nu_1 \cdot (A(\nu_1 + \nu_2) - A(\nu_1)) - \nu_2 \cdot (A(\nu_2 + \nu_1) - A(\nu_2))] da \tag{7.25}
\]

\[
= \int_0^1 [(\nu_1 \cdot \nabla g_{\nu_2}(\nu_1)) - (\nu_2 \cdot \nabla g_{\nu_1}(\nu_2))] da. \tag{7.26}
\]

Next, the relation \( x \cdot \nabla g(ax) = \partial_ag(ax) \), gives \( \int_0^1 \nu \cdot \nabla g(\nu da) = \int_0^1 \partial_a g(\nu) da = g(\nu) - g(0) \), which yields

\[
\int_{\partial\Omega} A = g_{\nu_2}(\nu_1) - g_{\nu_2}(0) - g_{\nu_1}(\nu_2) + g_{\nu_1}(0),
\]

which, by (7.23), gives (7.24).
7.4 Harmonic maps

The harmonic map or the stationary sigma model (see (5.12)) is a map from a \(d\)-dimensional euclidean space-time, \(\mathbb{R}^d\), with the euclidean metric \(\eta = (\delta_{ij})\), to a Riemannian manifold, \(N\), with a metric \((g_{ab})\), which is a critical point of the energy functional, given by

\[
E(\Phi) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \partial_j \Phi, \partial^j \Phi \rangle.
\]

(7.27)

Here \(\langle \partial_j \Phi, \partial^j \Phi \rangle\) is the Riemann scalar product in \(N\), which, in local coordinates, is \(g_{ab} \partial_i \Phi^a \partial^i \Phi^b\), as usual, \(\partial_j = \frac{\partial}{\partial x^j}\) and \(\partial^i = \eta^{ij} \partial_j\), and we assume the summation over repeated indices \(i,j = 1,\ldots,d, a,b,c = 1,\ldots,\dim(N)\), \(i,j = 1,\ldots,d\). With this notation, we can write energy the energy functional as,

\[
E(\Phi) := \frac{1}{2} \int_{\mathbb{R}^d} g_{ab} \partial_i \Phi^a \partial^i \Phi^b.
\]

The Euler-Lagrange equation for critical points of \(E(\Phi)\), i.e. for harmonic maps is obtained from (5.13) by restricting the indices to \(i = 1,\ldots,d\), which gives

\[
\Delta \Phi^a + \Gamma^a_{bc}(\Phi) \partial_i \Phi^b \partial^i \Phi^c = 0,
\]

(7.28)

where \(\Delta = \partial_j \partial^j\) and \(\Gamma^a_{bc}(\Phi)\) is the Christoffel symbols on \(N\).

For a map, \(\Phi\), to have finite energy, it should converge to a constant at infinity. In this case for each moment of time, \(t\), \(\Phi\) can be extended to a continuous map from \(S^d\) to \(N\) taking the point at infinity to the limit of \(\Phi(x)\) at the spatial infinity. (In other words, we pass to the one-point compactification of \(\mathbb{R}^d\).) Then one can define the degree, \(\deg \Phi\), as the homotopy class of \(\Phi\) as a map from \(S^d\) to \(N\), i.e. a member of the homotopy group \(\pi_d(N)\). This degree is conserved under the dynamics generated by the Euler-Lagrange equations above.

In the most important case \(N = G/H\), where \(G\) is a compact Lie group and \(H\) is its subgroup, specifically, \(G = SO(n+1)\) and \(H = SO(n)\), so that \(N = S^n\). In the particle physics one takes \(d = 2, 3\) and \(n = 2\), and in the condensed matter physics one takes \(d = n = 2\), i.e. \(\Phi : \mathbb{R}^d \to S^2\), \(d = 2, 3\). In both cases, the degree of \(\Phi\) is an integer (the degree for maps from \(S^d\) to \(S^2\)), i.e. \(\pi_d(S^2) = \mathbb{Z}\), for \(d = 2, 3\).

Consider \(N = S^n\) and let \(S^n\) be embedded in \(\mathbb{R}^{n+1}\) in the standard way, so that \(\Phi\) can be thought of as a map from \(\mathbb{M}^{d+1}\) to \(\mathbb{R}^{n+1}\), satisfying \(|\Phi| = 1\). Then the energy can be written as

\[
E(\Phi) = \frac{1}{2} \int_{\mathbb{R}^d} \partial_1 \Phi \cdot \partial_2 \Phi dx^d.
\]

\((a \cdot b\) denotes the dot product in \(\mathbb{R}^{n+1}\).)

Let \(d = n = 2\). Then the degree is given by

\[
\deg \Phi = \frac{1}{8\pi} \int_{\mathbb{R}^2} \epsilon^{ij} \Phi \cdot (\partial_i \Phi \wedge \partial_j \Phi) dx^2 = \frac{1}{4\pi} \int_{S^2} \Phi^*(dS),
\]

where \(\epsilon^{ij}\) is the Levi-Cevita antisymmetric symbol with \(\epsilon^{12} = -\epsilon^{21} = 1, \epsilon^{11} = \epsilon^{22} = 0\). Explicitly, \(\deg \Phi\) is given by

\[
\deg \Phi = \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2 x \, j^0,
\]

(7.29)

where

\[
j^0 := \frac{1}{8\pi} \Phi \cdot (\partial_1 \Phi \wedge \partial_2 \Phi) dx^1 \wedge dx^2.
\]

(7.30)
A key point here is that one has the Bogomolnyi-type inequality
\[ E(\Phi) \geq 4\pi|\deg \Phi|. \]
Indeed, we have the Bogomolnyi-type identity
\[
E(\Phi) = \pm 4\pi \deg \Phi + \frac{1}{4} \int_{\mathbb{R}^2} |\partial_i \Phi \pm \epsilon_{ij} \Phi \wedge \partial_j \Phi|^2 d^2 x,
\]
which implies the inequality above. Moreover, the last relation yields that in every homotopy class solutions of the self-dual/anti-dual equations,
\[
\partial_i \Phi \pm \epsilon_{ij} \Phi \wedge \partial_j \Phi = 0
\]
are the minimizers of \( E(\Phi) \).

For any \( \deg \Phi = k \), these equations have explicit solutions (harmonic or anti-harmonic maps), \( \Phi_k^{\text{stat}} \), given by \( \Phi_k^{\text{stat}}(\rho, \phi) = (U_k(\rho), k\phi) \), where \( (\rho, \phi) \) are the polar coordinates in \( \mathbb{R}^2 \) and \( (\varphi, \theta) \) are the spherical coordinates in \( S^2 \) and \( U_k(\rho) = 2\arctan \rho^k \).

The solutions above can be found by going to the stereographic projection, \( \Phi \rightarrow W \), of \( S^2 \) to the complex plane \( \mathbb{C} \), as
\[
\Phi = \left( \frac{2 \Re W}{1 + |W|^2}, \frac{-2i \Im W}{1 + |W|^2}, \frac{1 - |W|^2}{1 + |W|^2} \right).
\]
Here we identified \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \), by \( z := x_1 + ix_2 \). In the new variables the energy is given by
\[
\mathcal{E}(W) := \int \frac{|\partial W|^2 + |\bar{\partial} W|^2}{(1 + |W|^2)^2} dz,
\]
where we used the complex derivatives \( \partial := \partial_1 - i\partial_2 \) and \( \bar{\partial} := \partial_1 + i\partial_2 \), while the degree is given by
\[
\deg \Phi \equiv \deg W = \frac{1}{4\pi} \int \frac{\partial W \bar{\partial} W - \bar{\partial} W \partial W}{(1 + |W|^2)^2} d\bar{z}.
\]
The Euler-Lagrange equation, \( d_{\bar{\partial}} E(W) = 0 \), is given by
\[
\Delta W = \frac{2W}{1 + |W|^2} (|\partial W|^2 + |\bar{\partial} W|^2).
\]
Clearly, \( \mathcal{E}(W) \) is minimized by \( W \) satisfying either \( \bar{\partial} W = 0 \) or \( \partial W = 0 \) (the Cauchy-Riemann equations), i.e. \( W \) is either a holomorphic or anti-holomorphic function. They are mapped into each other by the complex conjugation. We consider holomorphic solutions. One can show that they are of the form \( W(x) = \frac{Q(z)}{P(z)} \), where \( P(z) \) and \( Q(z) \) are polynomials with no common factors. The degree of denominator = \( \deg \Phi \) (the harmonic maps of degree \( n \)).

8 Gradient and Hamiltonian systems

8.1 Generalized gradient systems

In this section we introduce some additional structure for maps of Banach spaces, which isolate two key classes of evolution equations - gradient and hamiltonian systems. This structure isolates a class of maps, \( F : M \rightarrow X \), arising, in some sense, from functionals, \( \mathcal{E} : M \rightarrow \mathbb{R} \). In what
follows we use definitions of functional, the Gâteaux derivative and critical point from Section 5.

We begin with some general considerations. Let $X$ be a real Banach space, $M$, an open set in $X$ and $E : M \to \mathbb{R}$, a $C^1$ functional. Recall that the Gâteaux derivative, $dE(u)$, of $E$ at $u \in M$, is the linear functional on $X$ defined as

$$dE(u)\xi = \frac{\partial}{\partial \lambda} E(u_\lambda)\big|_{\lambda=0},$$

where $u_\lambda := u + \lambda \xi$, for $\xi \in X$, if the latter derivative exists. Thus, $dE(u)$ belongs to the dual space $X'$. (Recall that for a Banach space $V$, the dual space, $V'$, is the space of bounded linear functionals. $V'$ is also a Banach space in the norm $\|\alpha\| := \sup_{\|v\|=1} |\alpha(v)|$.)

We would like to associate with $dE(u)$ an element, $F(u) \in X$ of $X$ so that we define the evolution equation $\partial_t u = F(u)$. In other words, we would like to isolate a class of maps, $F : M \to X$, which originate from functionals $E : M \to \mathbb{R}$ (simpler objects).

Assume there is a linear invertible map $K : X \to X'$. Then, we map $dE(u) \in X'$ into an element of $X$ as

$$\nabla^K E(u) := K^{-1} dE(u) \in X.$$

There is a correspondence between linear maps $K : X \to X'$ and bi-linear forms $\beta$ on $X$, i.e. maps $\beta : X \times X \to \mathbb{R}$ linear in each argument, given as follows. With the map $K$, we can associate the bi-linear form, $\beta_K$, on $X$, defined as $\beta_K(v,w) := (Kv)(w)$ (remember that by the definition $Kv \in X'$). In opposite direction, given a bi-linear form $\beta : X \times X \to \mathbb{R}$ on $X$, we associate with it the map $K_\beta : X \to X'$ as $K_\beta : v \in X \to \beta_v(\cdot) := \beta(v,\cdot) \in X'$.

A form $\beta$ is non-degenerate, i.e. $\beta(v, w) = 0$, $\forall w \in Z, \Rightarrow v = 0$, iff the corresponding map $K_\beta : X \to X'$ is one-to-one. A form, $\beta$, is said to be strongly non-degenerate iff this map is onto. For $K = K_\beta$, we denote

$$\nabla^\beta E(u) := K^{-1}_\beta dE(u) \in X.$$  \hfill (8.2)

The map $u \in X \to \nabla^\beta E(u) \in X$ is our desired map. A map, $F : M \to X$, for which there is a functional, $E : M \to \mathbb{R}$, and a bi-linear form, $\beta : X \times X \to \mathbb{R}$, s.t. $F = \nabla^\beta E(u)$, will be called the $\beta$-gradient of $E$. Now we can write the corresponding evolution equation

$$\partial_t u = \nabla^\beta E(u).$$ 

Before moving on, we observe two important consequences for evolution equations of the corresponding maps being $\beta$-gradients:

(i) Static solutions to (8.3) are critical points of $E(u)$.

(ii) If $\beta(v, v) \leq 0$, then $E(u)$ is non-increasing under the evolution (8.3).

The first statement is obvious. To prove the second one, we first observe that the definition $K_\beta : v \in X \to \beta_v(\cdot) := \beta(v,\cdot) \in X'$ implies $(K_\beta v)(w) = \beta(v, w)$, which, in turn, gives

$$v'(w) = \beta(K^{-1}_\beta v', w).$$  \hfill (8.4)

Applying this to $v' = dE(u)$ and using the definition of the $\beta$-gradient, (8.2), gives furthermore

$$dE(u)w = \beta(\nabla^\beta E(u), w).$$  \hfill (8.5)
This implies \( \partial_t E(u) = dE(u)\partial_t u = \beta(\nabla^g E(u), \partial_t u) \), which, assuming \( u \) solves the gradient flow equation (8.3), implies

\[
\partial_t E(u) = \beta(\nabla^g E(u), \nabla^g E(u)) < 0. \tag{8.6}
\]

We will call a \( \beta \)-gradient equation (8.3) satisfying (ii) the generalized dissipative system.

The key examples of non-degenerate, bi-linear maps, \( \beta : X \times X \to \mathbb{R} \), are (the negative of) the inner products \( g(v, w) = \langle v, w \rangle \) (Riemannian metric) on a Hilbert space \( X \) and symplectic forms \( \omega(v, w) \) on a real Banach space, \( Z \). In the first case the map is symmetric w.r.t. switching the order of the arguments and positive definite and the second case, anti-symmetric. The corresponding maps (vector fields), \( \nabla^g E(u) \) and \( \nabla^\omega E(u) \), are called the gradient and hamiltonian vector fields, respectively. (In the second case, one uses the notation \( E \equiv H \) and \( \nabla^\omega E(u) \equiv X_H \).)

Moreover, the corresponding evolution equations, \( \partial_t u = -\nabla^g E(u) \) and \( \partial_t u = \nabla^\omega E(u) \) are called the gradient flow and hamiltonian equation, respectively.

**Energy dissipation vs energy conservation.** The second property of \( \beta \)-gradient equations, given above, and the relation \( -g(v, v) < 0 \) imply that for a gradient system the energy decreases under the flow (we say that a gradient system is dissipative). Namely, we have:

**Proposition 8.1.** If \( u \) solves the gradient flow equation (8.3), with \( \beta = -g \), then

\[
\partial_t E(u) = -\|\text{grad} E(u)\|^2_X < 0, \tag{8.7}
\]

where \( \langle \cdot, \cdot \rangle_X \equiv g(\cdot, \cdot) \) is the inner product on the Hilbert space \( X \).

For a hamiltonian system, since \( \omega(v, v) = 0 \), the energy, \( E(u) = H(u) \), remains constant under the flow (we say that a hamiltonian system is conservative). Namely, we have

**Proposition 8.2.** If \( u \) solves the gradient flow equation (8.3), with \( \beta = \omega \), then

\[
\partial_t E(u) = 0. \tag{8.8}
\]

### 8.2 Gradient systems

We give a few examples of the gradient systems or gradient flows,

\[
\partial_t u = -\nabla^g E(u). \tag{8.9}
\]

(Note that for each \( u \) both \( \partial_t u \) and \( \nabla^g E(u) \) are defined on the tangent space \( T_u M \).) As the first example, we consider the reaction-diffusion (or nonlinear heat) equation,

\[
\partial_t u = \Delta u + g(u),
\]

is a gradient system on the phase space \( X = H^2(\mathbb{R}^n) \) with the widely used functional

\[
E(u) = \int \left( \frac{1}{2} |\nabla u|^2 + G(u) \right) d^dx \tag{8.10}
\]

where \( G(u) \) is the anti-derivative of \( g(u) \), and \( L^2 \)-metric. Indeed, we compute (see Exercise ??)

\[
\text{grad} E(u) = -\Delta u + G'(u).
\]

The most widely used special case of this equation is the celebrated Allen-Cahn equation for which \( G(u) \) is a double-well potential, i.e. \( G(u) \) has two non-degenerate global minima at
the points ±1 (or any other two points) with the minimum value 0 (see Figure 1), for example, \( G(u) = \frac{1}{4}(u^2 - 1)^2 \) and therefore \( g(u) = u^3 - u \).

Another example is the Fisher-Kolmogorov-Petrovsky-Piscunov (FKPP) equation, which appears in population biology and combustion theory. For this equation, \( g(u) = u(u - 1) \) and \( G = \frac{1}{3}u^3 - \frac{1}{2}u^2 + \frac{1}{6} \). We will study the Allen - Cahn and Fisher-Kolmogorov-Petrovsky-Piscunov equations in more detail below.

As the second example, we consider the Cahn-Hilliard equation, which plays a central role in material science and presents a basic model with many generalizations and extensions,

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -\Delta (\epsilon^2 \Delta u - g(u)), \quad x \in \Omega \\
\frac{\partial u}{\partial n} &= 0, \quad x \in \partial \Omega
\end{aligned}
\]  
\hspace{1cm} (8.11)

where \( \Omega \) is a smooth domain in \( \mathbb{R}^n \) and \( g \) is the same as for the Allen-Cahn equation, i.e. a derivative of a double well potential, say \( g(u) = u^3 - u \). The Cahn-Hilliard equation equation (8.11) is derived from the conservation law of mass. It is a gradient system with the energy functional (8.10) and the metric

\[
\langle u, v \rangle_{H^{-1}_0} := -\int_\Omega u \Delta^{-1} v \, dx,
\]

that is

\[
\frac{du}{dt} = -\text{grad}_{H^{-1}_0} E_h(u).
\]

Here \( \Delta^{-1} \) is defined by the equations (A.16) - (A.17) of Appendix A.2:

\[
\begin{aligned}
\sigma_n f, \quad \text{where} \quad \sigma_n (x) &= \left\{ \begin{array}{ll}
\frac{(2 - n)\sigma_{n-1}}{2\pi} |x|^{-n+2} & \text{if } n \neq 2 \\
\frac{1}{2\pi} \ln |x| & \text{if } n = 2
\end{array} \right.
\end{aligned}
\]  
\hspace{1cm} (8.12)

The third example we consider is the mean curvature flow.

### 8.3 Hamilton equations

We begin with a few examples of the symplectic forms or symplectic structures, \( \omega \), and the corresponding Hamiltonian equations,

\[
\partial_t u = \nabla^\omega H(u).
\]  
\hspace{1cm} (8.13)

To begin with, a pair \((Z, \omega)\), a vector space \( Z \) and symplectic form \( \omega \), is called a symplectic space. \( Z \) is called the phase space and the function \( H \) (formerly \( E \)), the Hamiltonian. A Hamiltonian system is a pair: a symplectic space, \((Z, \omega)\), and a Hamiltonian function, \( H : Z \to \mathbb{R} \).

In what follows we assume that \( Z \) is a real Hilbert space with an inner product \( \langle \cdot, \cdot \rangle \) and denote the gradient corresponding to this inner product just by grad (or sometimes by \( \nabla \)) and similarly for partial gradients. Recall that for a bounded operator \( A \), the adjoint is defined by \( \langle A^* u, v \rangle = \langle u, Av \rangle \).
Proposition 8.3. Suppose there is a linear, invertible, bounded operator $J : Z^* \to Z$ such that $J^* = -J$ ($J$ is called a symplectic operator, or almost complex structure), then the bi-linear form

$$\omega_J(v, w) = \langle v, J^{-1}w \rangle$$

is a symplectic form on $Z$ and the hamiltonian vector field corresponding to a hamiltonian $H : Z \to \mathbb{R}$ is given by

$$X_H(u) = -J \, \text{grad} \, H(u).$$

Consequently the hamiltonian equation in this case is $\partial_t u = -J \, \text{grad} \, H(u)$.

Proof. Comparing the two definitions $\langle \text{grad} \, H(u), \xi \rangle = dH(u)\xi$ and $dH(u)\xi = \omega(X_H(u), \xi) = \langle X_H(u), J^{-1}\xi \rangle = \langle (J^{-1})^*X_H(u), \xi \rangle$, we conclude that $\langle \text{grad} \, H(u), \xi \rangle = \langle (J^{-1})^*X_H(u), \xi \rangle$. Since this is true for all $\xi \in Z$, we conclude that $\text{grad} \, H(u) = (J^{-1})^*X_H(u)$ and therefore $X_H(u) = J^* \, \text{grad} \, H(u) = -J \, \text{grad} \, H(u)$, which gives (8.15).

□

Examples:

1) The hamiltonian system of classical mechanics: the phase space and symplectic form are given by

$$Z = \mathbb{R}^3_x \times \mathbb{R}^3_k, \quad \omega(z, z') := x \cdot k' - x' \cdot k,$$

where $z := (x, k)$, and the hamiltonian $H : Z \to \mathbb{R}$. We can write the symplectic form in (8.16) as $\omega(z, z') = z \cdot J^{-1}z'$, where the symplectic operator $J$ is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

yielding the hamiltonian vector field $X_H(z) = (\partial_k H, -\partial_x H)$ and the equations

$$\dot{x} = \partial_k H \quad \text{and} \quad \dot{k} = -\partial_x H.$$

(These can also be computed directly using (8.5), with $\beta = \omega$.) For a classical particle of mass $m$ in a potential $V(x)$, we have $H(x, k) = \frac{1}{2m}|k|^2 + V(x)$, so that $X_H(z) = (\frac{1}{m}k, -\nabla V(x))$.

2) The hamiltonian system of classical field theory (CFT): the phase space and symplectic form are given by

$$Z = H^1(\mathbb{R}^d, \mathbb{R}^m) \times H^1(\mathbb{R}^d, \mathbb{R}^m), \quad \omega(v, v') := \int_{\mathbb{R}^d} (\xi \cdot \eta' - \xi' \cdot \eta),$$

where $v := (\xi, \eta)$. We can write the symplectic form in (8.19) as $\omega(v, v') = \langle v, J^{-1}v' \rangle$, where the symplectic operator $J$ is given by (8.17) (but defined on a different space). Given a hamiltonian $H(\phi, \pi)$, this yields the hamiltonian vector field $X_H(\phi, \pi) = (\partial_\pi H, -\partial_\phi H)$, which leads to the equations

$$\partial_t \phi = \partial_\pi H \quad \text{and} \quad \partial_t \pi = -\partial_\phi H.$$

Here $\partial_\pi$ and $\partial_\phi$ denote partial gradients. For the Klein-Gordon CFT,

$$H(\phi, \pi) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} |\nabla \phi|^2 + f(\phi) \right\} \, dx,$$

the hamiltonian vector field is $X_H(\phi, \pi) = (\pi, -\Delta \phi - f'(\phi))$ and the corresponding hamiltonian equation, which, after elimination of $\pi$, becomes the nonlinear wave (or Klein - Gordon) equation
\( \Box \phi - f'(\phi) = 0, \quad \Box := \partial_t^2 - \Delta_x. \) \hfill (8.22)

(\( \Box = \partial_\mu \partial^\mu \) in the Minkowski metric.)

3) More generally, let \( V \) be a Banach space and \( V' \), its dual. We consider \( Z = V \times V' \) and define a symplectic form on \( Z \) by \( \omega(v, \bar{v}) := \eta(\xi) - \bar{\eta}(\bar{\xi}) \), where \( v = (\xi, \eta) \), \( \bar{v} = (\bar{\xi}, \bar{\eta}) \in Z \). If \( V \) is reflexive, i.e. \( V'' = V \), and there is a linear invertible, bounded operator \( J : Z' \to Z \), which is anti-self-dual (i.e. \( J' = -J \), where the dual \( J' \) is defined by \( J'\alpha(v) = \alpha(Jv) \), or \( \langle J'u, v \rangle = \langle u, Jv \rangle \)), then again we can define a symplectic form on \( Z \) by (8.14), where \( \langle \eta, \xi \rangle \) is understood as a coupling between \( V \) and \( V' \) (i.e. linear functionals \( \eta(\xi) \) on \( V \) are written as \( \eta(\xi) = \langle \eta, \xi \rangle \)). In particular, the symplectic form \( \omega(v, \bar{v}) := \bar{\eta}(\bar{\xi}) - \eta(\xi) \) is given by (8.14) with \( J \) given by (8.17).

**Complex hamiltonian systems**

Now, we describe hamiltonian equations on complex Banach spaces. Let \( Z \) be a complex Banach space. As above we can identify it with the real space by writing it as \( Z = V + iV \), for some real vector space, and associating with it a real space \( \hat{Z} := V \oplus V \). After that we can transfer to it the hamiltonian theory we developed for real spaces.

Our goal is to find a natural complex expression for this theory. We assume that \( Z \) has an inner product (complex). Then we can define the symplectic form on \( Z \) as

\[
\omega(u, v) := \text{Im}\langle u, v \rangle = \text{Re}\langle u, -iv \rangle. \tag{8.23}
\]

Now, if we define the real inner product on \( Z \) by

\[
\text{Re}\langle u, v \rangle = \langle \bar{u}, v \rangle, \tag{8.24}
\]

then

\[
\omega(u, v) = \langle \bar{u}, J^{-1}v \rangle, \tag{8.25}
\]

where \( J \) is the standard symplectic operator (8.17), \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Indeed, under the map \( \text{vect}, i \) goes into \( J^{-1} \): \( \text{vect}(i\phi) = J^{-1}\phi \). This shows that, indeed, (8.23) gives the symplectic form on \( \hat{Z} \). We identify it as a symplectic form on the complex space \( Z \).

Recall that above (see Subsection 5) we defined the complex Gâteaux derivatives, \( d_\psi H(\psi) \) and \( d_{\overline{\psi}} H(\psi) \), and the complex gradients, \( \partial_\psi H(\psi) \) and \( \partial_{\overline{\psi}} H(\psi) \). One can easily show that the Hamilton equation in the complex case is

\[
\partial_t \psi = i^{-1}\partial \overline{\psi}. \tag{8.26}
\]

Now, we consider two specific complex hamiltonian theories: the complex Klein-Gordon and Schrödinger field theories.

**Complex Klein-Gordon equation (Classical field theory).**

Here the phase space is \( Z = H^1(\mathbb{R}^d, \mathbb{C}) \times H^1(\mathbb{R}^d, \mathbb{C}) \), identified with the real space

\[
Z = H^1(\mathbb{R}^d, \mathbb{R}^m) \times H^1(\mathbb{R}^d, \mathbb{R}^m), \tag{8.27}
\]

with \( m = 2 \), and the Hamiltonian is the same as in (8.21) and the symplectic form is given now by (8.19), or (8.14), with \( J \) given by (8.17).
Schrödinger Hamiltonian system. Here the phase space is the complex Hilbert space, $H^1(\mathbb{R}^d, \mathbb{C})$, the symplectic form $\omega(\xi, \eta) := \text{Im} \int \xi \eta$ and the Hamiltonian is given by

$$H(\psi, \psi^\ast) := \int_{\mathbb{R}^d} \left\{|\nabla_x \psi|^2 + V|\psi|^2 + G(|\psi|^2) \right\} dx.$$  

The symplectic form is given now by (8.25), i.e. $-\text{Im} \langle u, v \rangle = \langle \bar{u}, J^{-1}v \rangle$, where $\langle u, v \rangle$ is the complex $L^2-$ inner product. The Hamilton equation (see (8.26)) for this system becomes

$$i\partial_t \psi = (-\Delta_x + V)\psi + g(|\psi|^2)\psi,$$  

(8.28)

where $g(u) := G'(u)$.

Poisson brackets. With the symplectic form on $Z$, $\Omega(v, w)$, we associate the the bilinear map $(f, g) \rightarrow \{f, g\}$, from pairs of differentiable functions on $Z$ on to a function on $Z$, called the Poisson brackets, by

$$\{f, g\} := \omega(X_f, X_g).$$  

(8.29)

The map (8.29) has the following properties: for any functions $f$, $g$, and $h$ from $Z$ to $\mathbb{R}$,

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry);
2. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz rule);
3. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (the Jacobi identity).

Bilinear maps, $(f, g) \rightarrow \{f, g\}$, having these properties, are called the Poisson brackets. The map (8.29) also obeys $\{f, g\} = 0 \ \forall \ g \implies f = 0$. Poisson brackets with the latter property are said to be non-degenerate. Note that a space of smooth functions (or functionals), together with a Poisson bracket, has the structure of a Lie Algebra.

If the symplectic form is defined by (8.14), then the Poisson bracket is given by

$$\{f, g\} := \langle \partial f, J\partial g \rangle.$$  

(8.30)

Indeed, by the definition, we have that $\{f, g\} := \omega(X_f, X_g) = \langle X_f, J^{-1}X_g \rangle$. Now, using here that $X_f(u) = -J\partial f(u)$, etc., we see that $\{f, g\} = \langle J\partial f, J^{-1}J\partial g \rangle = \langle J\partial f, \partial g \rangle$.

Specifically, we have the following Poisson brackets:

1) One-particle Classical Mechanics with the phase space $Z = \mathbb{R}^3 \times \mathbb{R}^3$, with the Poisson bracket

$$\{f, g\} = \nabla_x f \cdot \nabla_k g - \nabla_k f \cdot \nabla_x g.$$  

(8.31)

(The symplectic operator is (8.17).)

2) For the phase space $Z = H^1(\mathbb{R}^d, \mathbb{R}^m) \times H^1(\mathbb{R}^d, \mathbb{R}^m)$, with $L^2-$inner product, we define the Poisson brackets as

$$\{f, g\} = \int \{\partial_x f \cdot \partial_\phi g - \partial_\phi f \cdot \partial_x g\} dx.$$  

(8.32)

(The symplectic operator is (8.17).)

3) Let $V$ be a real, inner-product vector space, called the space of velocities, $X$ is an open subset of $V$, called the position, or configuration, space. We assume that $V$ is reflexive, i.e. $V'' = V$. We define Poisson brackets on $Z = X \times V'$ by

$$\{f, g\} = \langle \partial_x f, \partial_\phi g \rangle - \langle \partial_\phi f, \partial_x g \rangle.$$  

(8.33)
The space $Z$ together with a Poisson bracket on $C^\infty(Z, \mathbb{R})$ is called a Poisson space.

For a complex symplectic space, as above, with the symplectic form (8.23), the Poisson bracket is given by
\[
\{f, g\} = \int \{\partial \psi f \partial \bar{\psi} g - \partial \bar{\psi} f \partial \psi g\} dx.
\]

The classical evolution of observables is given by
\[
f(u, t) = f(u_t),
\]
with the initial condition $f(u)$. Note that $f(u, t)$ solves the equation
\[
\frac{d}{dt} f(u, t) = \{f, H\}(u, t). \tag{8.35}
\]

In CFT, functions on a Poisson space, $(Z, \{\cdot, \cdot\})$, are called classical observables. The classical evolution of observables is given by $f(u, t) = f(u_t)$, where $u_t$ is the solution of (8.13) with the initial condition $u$. Note that $f(u, t)$ solves the equation
\[
\frac{d}{dt} f(u, t) = \{f, H\}(u, t).
\]

Exercise 1. Prove this.

If the Poisson bracket $\{f, g\}$ corresponds to the symplectic form $\omega$, then this equation is equivalent to the Hamilton equations (8.13).

**Conservation laws and symmetries.** We say that an observable $f(u)$ is conserved or is constant of motion, iff $f(u_t)$ is independent of $t$. The equation (8.35) implies

**Proposition 8.4.** $f(u)$ is conserved, if and only if its Poisson bracket with the Hamiltonian, $H$, vanishes: $\{f, H\} = 0$.

**Symmetries.** Conservation laws often arise from symmetries, that is, transformations which map solutions to solutions. Assume that a symmetry is realized by a one-parameter group $\tau_s$, $s \in \mathbb{R}$, of bounded, operators on $Z$: if $u$ is a solution then so is $\tau_s u$ for every $s$.

We list some examples of symmetries for the Schrödinger hamiltonian system with the phase space $Z = H^1(\mathbb{R}^d, \mathbb{C})$:

- Time translation invariance: $(\tau_s \psi)(x, t) := \psi(x, t + s)$, $s \in \mathbb{R}$.

- Space translation invariance: $(\tau_s \psi)(x, t) := \psi(x + s, t)$, $s \in \mathbb{R}^d$.

- Space rotation (and reflection) invariance: $(\tau_R \psi)(x, t) := \psi(R^{-1} x, t)$, $R \in O(\mathbb{R}^d)$.

- Gauge invariance: $(\tau_\gamma \psi)(x, t) := e^{i\gamma} \psi(x, t)$, $\gamma \in \mathbb{R}$. 

Theorem 8.1. Assume that the symplectic form is given by (8.14). Let \( \tau_s, s \in \mathbb{R} \), be a one-parameter group of bounded, unitary operators on \( Z \), commuting with \( J \). Then

- \( \tau_s, s \in \mathbb{R} \), is a symmetry of the Hamiltonian system, \( (Z, \omega, H) \), iff \( H \circ \tau_s = H \), \( \forall s \in \mathbb{R} \) (mod a constant).
- \( \tau_s, s \in \mathbb{R} \), is a symmetry \( \implies \) the ‘charge’ \( Q(u) := \frac{1}{2} \langle u, J^{-1}Au \rangle \) is conserved.

Proof. By the definition, \( \tau_s, s \in \mathbb{R} \), is a symmetry of the Hamiltonian system, \( (Z, \omega, H) \), iff \( \partial_t u = X_H(u) \) implies \( \partial_t \tau_s u = X_H(\tau_s u) \), for every \( s \in \mathbb{R} \). This can be rewritten as \( \tau_s X_H(\tau_s u) = X_H(u) \). Assuming that the equation \( \partial_t u = X_H(u) \) has a unique solution for every initial condition \( u_0 \in Z \), this relation and (8.15) give \( \tau_{-s} JdH(\tau_s u) = JdH(u) \), or \( \tau_{-s} dH(\tau_s u) = dH(u) \), which in turn is equivalent to \( dH(\tau_s u) = dH(u) \), which again is equivalent to \( H \circ \tau_s = H \), \( \forall s \in \mathbb{R} \). Indeed, by the definition of the gradient and the unitarity of \( \tau_s \), we have \( dH(\tau_s u)\xi = (dH)(\tau_s u)\tau_s \xi = dH(u) = \langle dH(\tau_s u), \tau_s \xi \rangle = dH(u) = \langle \tau_{-s}(dH)(\tau_s u), \xi \rangle = dH(u)\xi \). Hence \( dH(\tau_s u) = dH(u) \).

To prove the second statement, we differentiate the relation \( H(\tau_s u) = H(u) \) at \( s = 0 \) and use the definition of the Gâteaux derivative to obtain \( dH(u)Au = 0 \). Now, relating this expression to the gradient of \( H \) gives \( \langle dH(u), Au \rangle = 0 \). We write \( \langle dH(u), Au \rangle = \langle dH(u), JJ^{-1}Au \rangle \) and observe that \( J^{-1}Au \) is the gradient of \( Q(u) := \frac{1}{2} \langle u, J^{-1}Au \rangle \) to find that \( \langle dH(u), JdQ(u) \rangle = 0 \). Finally, using the formula (8.30), \( \{f, g\} := \langle df, Jdg \rangle \), relating the Poisson bracket to the inner product, \( \langle dH(u), JdQ(u) \rangle = \{H, Q\} \), we conclude that \( \{H, Q\} = 0 \), and therefore \( Q(u) \) is conserved. \( \square \)

We consider the Schrödinger Hamiltonian system, with the phase space \( Z = H^1(\mathbb{R}^d, \mathbb{C}) \) and symplectic form \( \omega(u, v) := -\text{Im} \langle u, v \rangle = \text{Re} \langle u, iv \rangle \), so that \( J = -i \). We list conservation laws for it, implied by the symmetries considered above:

- Time translation invariance \((\tau_s \psi)(x, t) := \psi(x, t + s), s \in \mathbb{R}\) \(\rightarrow\) conservation of energy, \(H(\psi, \tilde{\psi})\);
- Space translation invariance \((\tau_s \psi)(x, t) := \psi(x + s, t), s \in \mathbb{R}^d\) \(\rightarrow\) conservation of momentum \(P(\psi, \tilde{\psi}) := \int \psi(-i\nabla)\psi dx\);
- Space rotation (and reflection) invariance \((\tau_R \psi)(x, t) := \psi(R^{-1}x, t), R \in SO(\mathbb{R}^d)\) \(\rightarrow\) conservation of angular momentum \(L(\psi, \tilde{\psi}) := \sum \psi(x \wedge -i\nabla)\psi dx\);
- Gauge invariance \((\tau_\gamma \psi)(x, t) := e^{i\gamma}\psi(x, t), \gamma \in \mathbb{R}\) \(\rightarrow\) conservation of ‘charge’, \(\int |\psi|^2 dx\).
For the Hamiltonian system of classical field theory with the phase space and symplectic form given by (8.19) \((Z = H^1(\mathbb{R}^d, \mathbb{R}^m) \times H^1(\mathbb{R}^d, \mathbb{R}^m))\) and \(\omega(v, v') := \int_{\mathbb{R}^d} (\xi' - \xi) \cdot \eta \, dx\), where \(v := (\xi, \eta)\) and the symplectic operator \(J\) is given by (8.17), we have similar conservation laws, e.g. for the translation invariance we have conserved the field momentum (for \(m = 1\))

\[
P(\phi, \pi) := \int \pi \nabla \phi \, dx.
\]

(8.38)

Now, we consider an abstract real dynamical system with the phase space \(Z = \Omega \times V',\) where \(V\) is a reflexive (i.e. \(V'' = V\)) Banach space and \(\Omega \subset V\) with the symplectic form \(\omega(v, v') := \eta'(\xi) - \eta(\xi'),\) where \(v := (\xi, \eta)\), given by (8.14) with \(J\) given by (8.17). Symmetries \(\tau_s, s \in \mathbb{R},\) now act as

\[
\hat{\tau}_s : (\phi, \pi) \to (\tau_s \phi, \tau_s' \pi),
\]

with \(\tau_s'\) being the dual group action of \(\tau_s\) on \(V^*\), defined by \(\langle \tau_s' \pi, \phi \rangle = \langle \pi, \tau_s \phi \rangle\) (recall that \(\langle \cdot, \cdot \rangle\) is the coupling between \(X\) and \(X^*\)). The conserved classical observable corresponding to \(\tau_s\), is given by \(Q(\phi, \pi) := \frac{1}{2} \langle u, J^{-1} A u \rangle = \langle \pi, A \phi \rangle,\) where \(u = (\phi, \pi)\) and \(A\) is the generator of the group \(\tau_s, \partial_s \tau_s = A \tau_s,\) and \(\hat{A} := \text{diag}(A, -A')\). It has vanishing Poisson bracket ("commutes") with the Hamiltonian,

\[
\{H, Q\} = \partial_t H(\tau_s \phi, \tau_s' \pi)|_{s=0} = 0,
\]

and consequently is conserved under evolution: \(Q(\tau_t, \pi)\) is a constant in \(t\) where \(\Phi_t = (\phi_t, \pi_t)\) is a solution to (8.20). We formulate this as

\[
\tau_s \text{ is a symmetry of (8.20) } \rightarrow \langle \pi, A \phi \rangle \text{ is conserved}
\]

For the Schrödinger hamiltonian system, the one-parameter group \(\tau_s, s \in \mathbb{R},\) can be chosen to be unitary and the dual group is given by \(\tau'_s = C \tau_{-s} C\), where \(C\) is the complex conjugation, so that, since \(\pi = \bar{\psi},\) we have \(\tau_s' \pi = \tau_s \bar{\psi}\). Hence the operator \(T_s\) acts on \(H\) as \(T_s H(\psi, \bar{\psi}) := H(\tau_s \psi, \tau_s \bar{\psi})\). The conserved classical observable is defined as \(Q(\psi, \bar{\psi}) := \langle \psi, i A \psi \rangle\) where \(A\) is the (anti-self-adjoint) generator of the group \(\tau_s, \partial_s \tau_s = A \tau_s,\) and \(\langle \psi, \phi \rangle\) is the standard scalar product on \(L^2(\mathbb{R}^d, \mathbb{C})\). As above, it has vanishing Poisson bracket ("commutes") with the Hamiltonian,

\[
\{H, Q\} = \partial_s H(\tau_s \phi, \tau_s \bar{\psi})|_{s=0} = 0,
\]

and consequently is conserved under evolution: \(Q(\psi_t, \bar{\psi}_t)\) is a constant in \(t\) where \(\psi_t\) is a solution to (8.28).

**Hamiltonian structure for the sigma model.** We consider the sigma model, defined by the action functional (5.12) with the Euler-Lagrange equations given by (5.13). The system (5.13) is Hamiltonian, with the hamiltonian given by > ????. In particular, it has conserved energy,

\[
\mathcal{E}(\Phi) := \frac{1}{2} \int_{\mathbb{R}^d} g_{ab} \partial_i \Phi^a \partial^i \Phi^b.
\]

Critical points of \(\mathcal{E}(\Phi)\) are called harmonic maps. The Euler-Lagrange equation for harmonic maps is obtained from (5.13) by restricting the indices to \(i = 1, \ldots, d\), which gives

\[
\Delta \Phi^a + \Gamma^a_{bc}(\Phi) \partial_a \Phi^b \partial^c \Phi = 0,
\]

(8.39)

**Legendre transform.** (under construction)
9 Energy and stability

9.1 Stability: generalities

So far we studied mainly existence of static solutions. (Remember that at the same time this gives also the existence of stationary or standing waves and traveling wave solutions.) The next key question is, starting with initial conditions close to a static solution, how the solutions of the dynamical equation in question behave? This leads to the question of stability of static solutions of general dynamical systems,

\[
\frac{\partial u}{\partial t} = F(u).
\]  

(9.1)

Assume \( F : U \to Y \), where \( U \) is an open set of \( X \) and \( X \) and \( Y \) are Hilbert spaces and satisfy \( X \subset Y \). Let (9.1) have a static solution, \( u_* \), i.e., \( u_* \) is a time independent solution. (Such solutions are also called equilibria and sometimes stationary solutions.) Thus \( u_* \) is an equilibrium or static solution iff \( u_* \) is independent of time and satisfy the equation \( F(u_*) = 0 \). We would like to understand the behavior of solutions to equation (9.1) for initial conditions near an equilibrium point \( u_* \). There are the following general scenarios

- The solutions stay in a neighborhood of \( u_* \) (Lyapunov stability);
- The solutions converge to \( u_* \) as \( t \to +\infty \) (asymptotic stability);
- The solutions move away from \( u_* \) as \( t \to \infty \) (instability).

More precisely, we say that a static solution of (9.1) is Lyapunov stable if for any neighborhood, \( U \) of \( u_* \) have another neighborhood, \( V \subset U \), of \( u_* \) such that if an initial condition, \( u_0 \), is in \( V \), then the solution, \( u \), stays in \( U \). Otherwise, \( u_* \) is said to be unstable.

For systems with symmetry, the notion of stability/instability should be modified. Recall, that, if the dynamical system (9.1) with the symmetry group \( G \) has a static solution \( u_* \), then it has the manifold of static solutions

\[
\mathcal{M}_* = \{ T_g u_* : g \in G \}.
\]

We say that a static solution \( u_* \) (or more precisely, the manifold of static solutions \( \mathcal{M}_* \)) is orbitally stable if any solution to the equation (9.1), starting in a small neighborhood of \( u_* \) stays in a small neighborhood of the manifold \( \mathcal{M}_* \). In other words the solution \( u_t \) sticks very close to a possibly moving static solution \( T_{g(t)} u_* \).

9.2 Energy argument

The energy conservation or dissipation restricts severely possible dynamics of our system. We use this in investigating stability of static solutions, i.e. understanding the behavior of solutions to equation (9.1) near an equilibrium point \( u_* \). We assume that (9.1) is a generalized dissipative system in the sense of Subsection 8.1 (say a gradient or hamiltonian equation). Then there exist a functional \( E(u) \) (called energy, or entropy or Lyapunov functional), defined on an open set \( U \) in a Hilbert space \( X \), with the following properties:

(i) \( E(u) \) is non-increasing under the evolution equation (9.1) (a stronger statement: \( \partial_t E(u(t)) \leq 0 \), for any solution \( u(t) \) of (9.1))
(ii) Static solutions to (9.1) are critical points of $\mathcal{E}(u)$.

On a basis of energy considerations one expects the following behavior

1. if $u_*$ is a strict local minimizer then $u_*$ is a stable solution,

2. if $u_*$ is either a saddle point or a maximizer then $u_*$ is an unstable solution.

We will turn Possibility #1 into a rigorous statement below. The Possibility #2 is not always true. Sometimes conservation laws present in the equations can lead to stability in the cases of the second type. We will study this case below as well.

An important role below is played by the linear operator $\text{Hess } \mathcal{E}(u) := d\nabla \mathcal{E}(u)$, called the hessian of $\mathcal{E}$ at $u$. Note that $\langle \xi, \text{Hess } \mathcal{E}(u)\eta \rangle = d^2\mathcal{E}(u)\langle \xi, \eta \rangle$, where $d^2\mathcal{E}(u)\langle \xi, \eta \rangle$ is the hessian bilinear form defined as $d^2\mathcal{E}(u)(\xi, \eta) := d(d\mathcal{E}(u)\xi)\eta = \partial_t \partial_s|_{s=t=0}\mathcal{E}(u_{st})$, where $u_{st} := u + s\xi + t\eta$. Similarly to the finite dimensional case, we have the following statement

**Theorem 9.1.** If $u_*$ is a minimizer of $\mathcal{E}$, then $\text{Hess } \mathcal{E}(u_*) \geq 0$. If $\text{Hess } \mathcal{E}(u_*) \geq \theta$, for some $\theta > 0$, then $u_*$ is a minimizer of $\mathcal{E}$.

We say that $\mathcal{E}$ is coercive at $u_*$ if the following inequality satisfies

$$\text{Hess } \mathcal{E}(u_*) \geq \theta,$$

for some $\theta > 0$. (9.2)

In what follows we use the notation $\mathcal{E}'' \equiv \text{Hess } \mathcal{E}$ We make the following assumptions:

- (a) $\mathcal{E}(u)$ is is $C^3$,
- (b) $\mathcal{E}$ is coercive at $u_*$.

**Theorem 9.2.** Under the assumptions above, there is $\delta > 0$ s.t. for any $u$ satisfying $\|u - u_*\| \leq \delta$, we have the estimate

$$\|u - u_*\|^2 \leq \frac{4}{\theta} (\mathcal{E}(u) - \mathcal{E}(u_*)).$$

(9.3)

**Proof.** Using that $\mathcal{E} \in C^3$ and $d\mathcal{E}(u_*) = 0$ and writing $u = u_* + \xi$, we expand $\mathcal{E}(u)$ around $u_*$ to the third order as

$$\mathcal{E}(u) = \mathcal{E}(u_*) + \frac{1}{2} \langle \xi, \mathcal{E}''(u_*)\xi \rangle + R(\xi),$$

(9.4)

where $R(\xi)$ is the remainder term defined by this expression. By the assumption that $\mathcal{E}$ is $C^3$, it satisfies the estimate $R(\xi) = O(\|\xi\|^3)$. The vector $\xi$ is called a fluctuation of $u$ around $u_*$. Hence, for $\|\xi\| = \|u - u_*\| \leq \delta$, we have

$$\mathcal{E}(u) - \mathcal{E}(u_*) \geq \frac{1}{2} \theta \|\xi\|^2 - C\|\xi\|^3 \geq \frac{1}{2} \theta \|\xi\|^2 - C\delta \|\xi\|^2,$$

(9.5)

for some $C\infty$, independent of $\xi$. Choosing $\delta \leq \theta/2C$ gives (9.3).

**Theorem 9.3.** Under the assumptions (i), (ii), (a) and (b), the static solution $u_*$ of the evolution equation (9.1) is Lyapunov stable.

1st proof: Bootstrap. We write the solution $u_t$ of (9.1) as $u_t = u_* + \xi_t$. Let $\|\xi_0\| = \|u_0 - u_*\| \leq \delta/2$, where $u_0$ is the initial condition, $u_{t=0} = u_0$, and $\delta > 0$ is the same as in Theorem 9.2. The there is $T > 0$ s.t. $\|\xi_t\| \leq \delta$, for $0 \leq t \leq T$. Then by Theorem 9.2, $\|u_t - u_*\|^2 \leq \frac{4}{\theta} (\mathcal{E}(u_t) - \mathcal{E}(u_*))$, which, together with the assumption (i), that $\mathcal{E}(u_t)$ is a non-increasing function of time, $\mathcal{E}(u_t) \leq \mathcal{E}(u_0)$, gives

$$\|u_t - u_*\|^2 \leq \frac{4}{\theta} (\mathcal{E}(u_0) - \mathcal{E}(u_*)),$$

(9.6)

for $0 \leq t \leq T$. Choosing $\mathcal{E}(u_0) - \mathcal{E}(u_*) \leq \delta^2$, we see that the latter inequality holds for all times, which implies the Lyapunov stability of $u_*$. (details)
We have shown that (9.1) has a manifold of static solutions \(M\) which shows that by making

\[
\|\mu\|_\infty \langle 1
\]

\(2\)nd proof: Lyapunov functional. We introduce the functional (Lyapunov functional) \(\Lambda(\xi) := \frac{1}{2}(\xi, \mathcal{E}'(u_\ast)\xi)\). Using that, by the assumption, \(\|\xi\|^2 \leq \frac{1}{\theta}\Lambda(\xi)\) and \(R(\xi) \geq C\|\xi\|^3\), for some \(0 < C < \infty\), we have

\[
\mathcal{E}(u) - \mathcal{E}(u_\ast) \geq \Lambda(\xi) - \mu\Lambda(\xi)^{3/2},
\]

(9.7)

where \(\mu := C/\theta^{3/2}\). Denote \(\lambda = \Lambda(\xi)^{1/2}\) and \(\alpha = \mathcal{E}(u) - \mathcal{E}(u_\ast)\) and define the function \(F(\lambda)\) by

\[
F(\lambda) = \alpha - \lambda^2 + \mu\lambda^3.
\]

We have shown that \(\lambda\) satisfies \(F(\lambda) \geq 0\). Consider the graph of \(F(x)\), see Figure 2. For sufficiently small \(\alpha (540\alpha^2 < 1)\), the function \(F(\lambda)\) positive in two disjoint intervals \([0, \lambda_*]\) and \((\lambda_*, \infty)\), where \(\lambda_* > \lambda_* > 0\) are the zeros of \(F(\lambda), \lambda > 0\). For \(\alpha\) sufficiently small, \(\lambda_* \approx \sqrt{\alpha}\) and \(\lambda_* \approx 1/\mu\). Then if \(F(\lambda) \geq 0\) and \(\lambda < \lambda_*\), we must have \(\lambda \leq \lambda_*\). Taking \(\delta := \frac{1}{\theta}\lambda_*^2\), we conclude that if \(\|\xi\|^2 \leq \frac{1}{\theta}\Lambda(\xi) < \delta\), then

\[
\|\xi\|^2 \leq \frac{1}{\theta}\Lambda(\xi) \leq \frac{1}{\theta}\lambda_*^2 \approx \frac{1}{\theta}\alpha.
\]

(9.8)

By (9.7), the function \(\lambda_t := \Lambda(\xi_t)^{1/2}\) satisfies \(F(\lambda_t) \geq 0\), where the function \(F\) is defined there. By the (9.8) and the continuity of \(\Lambda(\xi_t)\), if \(\|\xi_0\|^2 \leq \frac{1}{\theta}\Lambda(\xi_0) < \delta\), then \(\|\xi_0\|^2 \leq \frac{1}{\theta}\Lambda(\xi_t) \leq \frac{1}{\theta}\lambda_*^2 \approx \frac{1}{\theta}(\mathcal{E}(u) - \mathcal{E}(u_*))\). Here \(\xi_0\) is the initial condition, \(\xi_{t=0} = \xi_0\). (If \(\lambda_t\) starts at \(t = 0\) at \(\lambda_0 \leq \lambda_*(0)\), then since \(f(\lambda_t) > 0\), \(\lambda_t\) must stay in the interval \([0, \lambda_*(t)]\).)

Next, by the assumption (i), \(E(u_t)\) is a non-increasing function of time: \(E(u_t) \leq E(u_0)\), where \(u_0\) is the initial condition, \(u_t |_{t=0} = u_0\). The last two results give

\[
\|\xi\| \leq \frac{4}{\delta}(\mathcal{E}(u_0) - \mathcal{E}(u_*)),
\]

(9.9)

which shows that by making \(u_0\) to be close to \(u_*\), we can make \(u_t\) to be arbitrary close to \(u_*\). Hence the static solution \(u_*\) is Lyapunov stable. (needs cleaning)

Assume the dynamical system (9.1) has a symmetry group \(G\), which means that the representation \(g \to T_g\) satisfies

\[
F(T_gu) = T_gF(u).
\]

(10.9)

This means that (9.1) has a manifold of static solutions \(M_* = \{T_gu_* : g \in G\}\). Indeed, if \(u_*\) is a static solution to (9.1), then, due to (10.10), so is \(T_gu_*\). Now, we have to address the stability
w.r.to this manifold, i.e. the orbital stability defined at the beginning of this section. Clearly, we have to modify our approach.

As far as the energy is concerned, the symmetry requirement means that
\[ E(T_g u) = E(u). \quad (9.11) \]
This implies that, if \( u_* \) is a critical point of \( E(u) \), then so is \( T_g u_* \). (This can be proved independently by differentiating the equality \( E(T_g(u_* + s\xi)) = E(u_* + s\xi) \) w.r. to \( s \).)

Note that the tangent space, \( T_{u_*}M_* \subset X \), to \( M_* \) at \( u_* \) is the vector space \( \tau(g)u_* \), where \( \tau(g) \) is the representation of the Lie algebra \( g \) of \( G \) acting on \( X \). (This can be seen by differentiating \( T_g(s)u_* \), where \( T_g(s) \) is any one-parameter subgroup of \( G \), w.r. to \( s \).)

In the case of symmetry, the inequality (9.2) never holds: the hessian \( E''(u) \) has always zero eigenvectors. Indeed, we have

**Proposition 9.1.** If \( A \in \tau(g) \) (one of generators of the algebra of the group \( G \)), then \( Au_* \) is an eigenfunction of \( E''(u) \) with eigenvalue 0,
\[ E''(u)Au_* = 0. \quad (9.12) \]

**Proof.** If \( u_* \) is a static solution to (9.1), then, due to (9.10), so is \( T_g u_* \), for any \( g \in G \). In particular, \( \nabla E(e^{sA}u_*) = 0 \), for any \( A \in \tau(g) \) and \( s \in \mathbb{R} \). Differentiate this equation with respect to \( s \) at \( s = 0 \) and use that \( \partial_s\nabla E(e^{sA}u_*)|_{s=0} = Au_* \) to obtain
\[ 0 = \partial_s\nabla E(e^{sA}u_*)|_{s=0} = E''(u)Au_* . \]
Hence (9.12) follows. \( \square \)

Thus in the case of symmetries, the assumption (b) fails. We replace it by the following weaker assumption
\[(b') \ E \text{ is coercive at } u_* \text{ in the sense that} \quad \langle \xi, E''(u)\xi \rangle \geq \theta \|\xi\|^2, \quad \text{for some } \theta > 0 \quad \text{and } \forall \xi \perp T_{u_*}M_* . \quad (9.13)\]

Now, we have the following strengthening of Theorem 9.3:

**Theorem 9.4.** Under the assumptions (a) and (b'), there is \( \delta > 0 \) s.t. for any \( u \) satisfying \( \text{dist}(u, M_*) \leq \delta \), we have the estimate
\[ \text{dist}(u, M_*) \leq \frac{2}{\theta}(E(u) - E(u_*)). \quad (9.14) \]

**Proof.** In the case of symmetry the argument above should be modified. In a neighborhood of the manifold \( M_* \) we decompose \( u \) as
\[ u = T_g u_* + \xi , \quad \langle \xi, \xi \rangle = 0 . \quad (9.15) \]
where \( g \) is the symmetry transformation \( g \). (This is a nonlinear, orthogonal decomposition with respect to the manifold \( M_* \), see Figure 3.) Note that \( a \) in (9.15) is determined by \( u \) uniquely. This follows from the implicit function theorem applied to the function \( F(g, u) := \langle u - T_g u_* , Au_* \rangle . \)

We introduce a neighbourhood of \( M_* \):
\[ U_\delta = \{ u \in X : \exists g \text{ s.t. } \|u - T_g u_*\|_X \leq \delta \} . \quad (9.16) \]
**Proposition 9.2.** There is $\delta > 0$ s.t. any $u \in U_\delta$ can be decomposed as follows
\[ u = T_g u_* + \xi, \quad \text{with} \quad g = g(u) \quad \text{and} \quad \xi \perp T_g u_* M_* . \] (9.17)

The latter condition can be written as
\[ \langle \xi, A u_* \rangle = 0, \quad \forall A \in g . \] (9.18)

**Proof.** Assume for simplicity that $G$ is a one-parameter (one-dimensional) group, parametrized by $a \in \mathbb{R}$ and write $T_a$ for $T_g(a)$. Let $A$ be the corresponding generator. We define the function $G(a, u) = \langle u - T_a u_* , AT_a u_* \rangle$ and rewrite the statement of the proposition as solving the equation
\[ G(a, T_a u_*) = 0 \] (9.19)
for $a$ as a function of $u$. To do this, we use the Implicit Function Theorem. Clearly, $G$ is $C^1$ in $u$ because is it a linear functional in $u$, and $C^1$ in $a$ since $T_a u_*$ is $C^1$ in $a$. Furthermore, $G(a, T_a u_*) = 0$. Finally, we show that $\partial_a G(a, u)|_{u = T_a u_*} \neq 0$:
\[ \partial_a G(a, u) = -\langle \partial_a T_a u_* , AT_a u_* \rangle + \langle u - T_a u_* , \partial_a AT_a u_* \rangle. \] (9.20)

Hence, using that $\partial_a T_a u_* = AT_a u_*$, we have, at $u = T_a u_* ,\quad \partial_a G(a, u)|_{u = T_a u_*} = -\|AT_a u_*\|^2 = -\|A u_*\|^2. \] (9.21)

Therefore, the conditions of the Implicit Function Theorem are satisfied and it implies that (9.19) has a unique solution for $a$ as a function of $u \in U_\delta$, which gives the statement of the proposition.

The importance of this proposition lies in the fact that for $\xi \perp T_{u_*} M_*$, we have the estimate (9.13).

Proceeding as in the proof of Theorem 9.3 and using (9.13) instead of (9.2) and using that $\mathcal{E}(T_g u_*) = \mathcal{E}(u_*)$, we find as in (9.3),
\[ \|u - T_g u_*\| \leq \frac{2}{\theta} (\mathcal{E}(u) - \mathcal{E}(u_*)), \] (9.22)
where $g = g(u)$ is the same as in (9.34). However, as little contemplation shows, $\|u - T_g u_*\| = \text{dist} ((, M_k) u, M_*)$, which completes our proof.

As above, the theorem above gives

**Corollary 9.5.** Under the assumptions (i), (ii), (a) and (b'), the static solution $u_*$ of the evolution equation (9.1) is orbitally stable.

We apply the analysis above on the problem of stability of kinks in the Allen - Cahn equation. This will demonstrate the concepts introduced above.
9.3 Orbital stability of kink solutions of the Allen - Cahn equation

Consider the question of stability of the kink solutions of the Allen - Cahn equation

\[
\frac{\partial u}{\partial t} = \Delta u - g(u),
\]

(9.23)

where \( g(u) = u^3 - u \), or, more generally, \( g : \mathbb{R} \to \mathbb{R} \) is the derivative, \( g = G' \), of a double-well potential: \( G(u) \geq 0 \) and has two non-degenerate global minima with the minimum value 0. \( G(u) \) is also called a bistable potential (see Subsection 7.1 and Figure 1). For \( g(u) = u^3 - u \), we take \( G(u) = \frac{1}{2}(u^2 - 1)^2 \).

To keep the notation simple, assume we are in the dimension 1, i.e. \( u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \). Recall that (9.23) has the kink static solution by \( \chi(x) \), see Figure 4; and Eq. (9.23) is translation invariant and therefore the functions \( \chi_a(x) := \chi(x + a) \forall a \in \mathbb{R} \) are also static solutions to (9.23). Thus we have an entire manifold of stationary solutions \( \mathcal{M}_{\text{kink}} = \{ \chi_a : a \in \mathbb{R} \} \). (In the multidimensional case, we have also rotations.)

![Figure 4: Hill-valley-hill structure for \(-G\), and the kink for \(u\).](image)

We now use the results above to derive orbital stability of the kinks. We have

**Theorem 9.6.** The kink solutions of (9.23) are orbitally stable w.r.to \( H^1(\mathbb{R}) \)-perturbations.

**Proof.** It is not difficult to see that (9.23) is an \( L^2 \)-gradient system with the energy functional (6.3), which we reproduce here

\[
\mathcal{E}(u) = \int \left( \frac{1}{2} |\nabla u|^2 + G(u) \right),
\]

(9.24)

defined on \( \chi + H^1(\mathbb{R}) \). Consequently, we have the properties (i) and (ii) of Subsection 9.2. Because of the translational symmetry mentioned above, we use Corollary 9.5. According to this corollary, to prove the theorem, it suffices to show that the energy (9.24) satisfies Conditions (a) and (b') on the space \( H^1(\mathbb{R}) \). Verifying Conditions (a) is straightforward. Conditions (b') requires more work. Let \( \mathcal{L} \) be the hessian of the energy functions \( \mathcal{E}(u) \) at \( \chi \): \( \mathcal{L} := \mathcal{E}''(\chi) \). We have Below we will use the following lower bound on the operator \( \mathcal{L} \):

**Proposition 9.3.** Say \( g'(-1) > 0 \). Let \( \rho \) be the second lowest eigenvalue of \( \mathcal{L} \). We have

\[
\langle \xi, \mathcal{L} \xi \rangle \geq \frac{1}{2} \rho \| \xi \|_{H^1}^2, \quad \forall \xi \perp \chi'.
\]

(9.25)

**Proof.** \( \mathcal{L} \) has the following properties:

1. the spectrum of \( \mathcal{L} \) in \((-\infty, \min\{g'(1), g'(-1)\})\) consists of isolated eigenvalues of finite multiplicities, which might accumulate only at \( \min\{g'(1), g'(-1)\} \);
2. 0 is the smallest eigenvalue of \( L \), it is simple with the eigenfunction \( \chi' \).

(For the second statement, cf. Proposition 9.1.)

For a Schrödinger type operator \( L \) on \( \mathbb{R}^n \), we define the number \( \mu(L) := \lim_{R \to \infty} \inf \{ \langle \psi, L\psi \rangle : \psi \in C_0^\infty(B_R) \} \), where \( B_R \) is the ball of the radius \( R > 0 \) in \( \mathbb{R}^n \). It is shown in Appendix ?? that the spectrum of \( L \) in \( (-\infty, \mu) \) consists of isolated eigenvalues of finite multiplicities, which might accumulate only at \( \mu(L) \). An easy computation shows that \( \mu(L) = \min\{g'(1), g'(-1)\} \).

This proves the first statement.

To prove the second statement, we observe that 0 is an eigenvalue of \( L \) with eigenfunction \( \zeta := \partial_x \chi \), \( L \chi' = 0 \).

Indeed, \( \chi \) solution to \( F(u) = 0 \), where we introduced the notation \( F(u) := -\Delta u + g(u) \).

Hence, for any \( a \in \mathbb{R} \), \( \chi_a \) is also a solution to \( F(\chi_a) = 0 \), where \( \chi_a = \bar{u}(x + a) \). Differentiate this equation with respect to \( a \):

\[ \partial_a F(\chi_a)|_{a=0} = \partial F(\chi_a) \partial_a \chi|_{a=0} = L \zeta, \]

where, recall \( L := \partial F(\chi) \). Hence we have \( L \zeta = 0 \) as claimed.

Now, since \( \chi' \) is positive, we have by the Perron - Frobenius theory (see Appendix B.4) that 0 is the smallest eigenvalue of \( L \) and it is simple.

The above properties and the spectral theorem imply that

\[ \langle L \xi, \xi \rangle \geq \rho \| \xi \|^2_{L^2} \]  

(9.28)

for all \( \xi \perp \chi' \), where \( \rho \) is the second lowest eigenvalue of \( L \). We combine this estimate together with \( \delta > 0 \) times the elementary estimate

\[ \langle L \xi, \xi \rangle \geq \| \nabla \xi \|^2_{L^2} - C \| \xi \|^2_{L^2}, \]

for \( C = -\min_{-1 \leq u \leq 1} g'(u) \), to obtain

\[ \langle L \xi, \xi \rangle \geq \delta \| \nabla \xi \|^2_{L^2} + (\rho - \delta C) \| \xi \|^2_{L^2}, \]

and optimizing over \( \delta \) (say taking \( \delta = \rho / 2C \)), completes the proof.

This proposition implies Condition (b') for the energy (9.24). Since satisfies Condition (a) \( (\mathcal{E}(u) \) is is \( C^3 \)), the statement of the theorem follows.

\[ \square \]

9.4 Asymptotic stability of kinks in the Allen-Cahn equation

We are interested in the asymptotic stability of the families of kink solutions of the non-homogeneous Allen-Cahn equation (9.23) on \( \mathbb{R} \), discussed in Subsection 9.3.

Our result below concerns asymptotic stability of the family (manifold) of the kink solutions

\[ \mathcal{M}_{\text{kink}} := \{ \chi(x-a) \mid a \in \mathbb{R} \}. \]

(9.29)

We assume that \( g \) is such that the initial value problem for (9.23) is locally well-posed in \( X \) in \( H^2(\mathbb{R}^n) \) for any initial condition \( u_0 \in X \), sufficiently close to \( \mathcal{M}_{\text{kink}} \). Moreover, we assume that the nonlinearity

\[ \mathcal{N}(\xi) := g(\chi + \xi) - g(\chi) - g'(\chi)\xi, \]

(9.30)
satisfies the following conditions:

\[ |N(\xi)| \lesssim |\xi|^2 + |\xi|^k, \ k \geq 2. \]  \hfill (9.31)

Then we have

**Theorem 9.7.** (Asymptotic stability of kinks) Under the assumptions above, the kink manifold \( \mathcal{M}_{\text{kink}} \) is asymptotically stable. More precisely, there are \( \delta > 0 \) and \( a(t) \) s.t. if \( \text{dist}(u_0, \mathcal{M}_{\text{kink}}) \leq \delta \), then the solution \( u \) to (9.23) with \( u|_{t=0} = u_0 \) satisfies

\[ \|u - \chi_{a(t)}\|_{H^1} \lesssim e^{-\frac{\epsilon}{2}t} \]  \hfill (9.32)

for all times \( t \geq 0 \). Moreover, there exists a real number \( a_\infty \) such that \( a(t) \to a_\infty \), exponentially fast, as \( t \to \infty \).

To prove the theorem we use, instead the energy dissipation property, the differential inequality for the Lyapunov functional \( \Lambda(\xi) := \frac{1}{2} \langle L\xi, \xi \rangle \), where, recall, \( L \) is the hessian of the energy functions \( E(u) \) at \( \chi \): \( L := E''(\chi) \), which is closely related to the energy ??.

### 9.4.1 Orthogonal Decomposition

Throughout, we will use the notation \( f_a(x) = f(x - a) \). Similarly to the general case, (9.16), we introduce a neighbourhood of \( \mathcal{M}_{\text{kink}} \):

\[ U_\delta = \{ u \in H^1(\Omega) : \exists a \text{ s.t. } \|u - \chi_a\|_{H^1(\Omega)} \leq \delta \}. \]  \hfill (9.33)

The following statement and its proof are specialization of Proposition 9.4 and its proof to the present situation.

**Proposition 9.4.** There is \( \delta > 0 \) s.t. any \( u \in U_\delta \) can be decomposed as follows

\[ u = \chi_a + \xi_a, \quad \text{with} \quad \int \xi(x)\chi'(x) \, dx = 0. \]  \hfill (9.34)

**Proof.** We define the function \( G(a, u) = \int (u - \chi_a)\chi_a' \, dx \) and rewrite the statement of the proposition as solving the equation

\[ G(a, \chi_a) = 0 \]  \hfill (9.35)

for \( a \) as a function of \( u \). To do this, we use the Implicit Function Theorem. Clearly, \( G \) is \( C^1 \) in \( u \) because it is a linear functional in \( u \), and \( C^1 \) in \( a \) since \( \chi_a \) is \( C^1 \) in \( a \). Furthermore, \( G(a, \chi_a) = 0 \). Finally, we show that \( \partial_a G(a, u)|_{u=\chi_a} \neq 0 \):

\[ \partial_a G(a, u) = -\int \partial_a \chi_a \chi_a' \, dx + \int (u - \chi_a) \partial_a \chi_a \, dx. \]  \hfill (9.36)

Hence, using that \( \partial_a \chi = -\frac{1}{\epsilon} \chi' \), we have, at \( u = \chi_a \),

\[ \partial_a G(a, u) = \frac{1}{\epsilon} \int (\chi_a')^2 \, dx. \]  \hfill (9.37)

Therefore, the conditions of the Implicit Function Theorem are satisfied and it implies that (9.35) has a unique solution for \( a \) as a function of \( u \in U_\delta \), which gives the statement of the proposition. \( \square \)
9.4.2 Evolution of Fluctuation $\xi$

**Proposition 9.5.** Assume that for every $t$, a solution $u$ of (9.23) is in $U_\delta$, with $\delta > 0$ given in Proposition 9.4. Then $a(t)$ and $\xi(x,t)$ given in that proposition ($u = \chi_a + \xi_a$) satisfy the following equations

$$-\partial_t \xi = \mathcal{L}\xi + \mathcal{N}(\xi) - \dot{a}\chi', \quad (9.38)$$

where $\mathcal{N}(\xi) := g(\chi + \xi) - g(\chi) - g'(\chi)\xi$, and $\mathcal{L} := dF(\chi)$, which is given explicitly by

$$\mathcal{L} := -\Delta + g'(\chi), \quad (9.39)$$

and

$$\dot{a}(t) \left[ \int (\chi')^2 dx + \int \partial_x \xi \chi' dx \right] = \int \mathcal{N}(\xi) \chi' dx. \quad (9.40)$$

**Proof.** For any $t$ decompose a solution $u$ of (9.23) according to Proposition 9.4, $u(x,t) = \chi_a(x,t) + \xi_a(x,t)$. Substituting the decomposition $u(x,t) = \chi_a(x,t) + \xi(x,t)$ into (9.23), we see that our equation becomes $-\partial_t \xi_a + \partial_x \xi_a \dot{a} = F(\chi_a + \xi_a)$, where, recall, we introduced the notation

$$F(u) := -\Delta u + g(u). \quad (9.41)$$

By changing the variables $x \rightarrow x + a$ in (9.44), we obtain $-\partial_t \xi + \partial_x \xi \dot{a} = F(\chi + \xi)$. We will need the Taylor expansion of $F$,

$$F(\chi + \xi) = F(\chi) + dF(\chi)\xi + \mathcal{N}(\xi), \quad (9.42)$$

where $\mathcal{N}_a(\xi) := g(\chi + \xi) - g(\chi) - g'(\chi)\xi$, and the observation that, by (9.8), $F(\chi) = 0, \forall a$, to obtain

$$F(\chi + \xi) = \mathcal{L}\xi + \mathcal{N}(\xi), \quad (9.43)$$

where $\mathcal{L} := dF(\chi)$. The last two equations give

$$-\partial_t \xi + \partial_x \xi \dot{a} = \mathcal{L}\xi + \mathcal{N}(\xi) - \chi'\dot{a}. \quad (9.44)$$

We compute $\mathcal{L} = -\Delta + g'(\chi)$. Thus, we obtain (9.38).

We have shown in the proof of Proposition 9.3 that since the kink $\chi$ breaks the translational symmetry, $\chi'$ is a zero eigenfunction of the operator $\mathcal{L}$:

$$\mathcal{L}\chi' = 0. \quad (9.45)$$

Equation (9.38) contains two unknowns: $a(t)$ and $\xi(x,t)$. To derive a separate equation for $a(t)$ we project equation (9.38) onto $\mathbb{C}\chi'$ by multiplying by $\chi'$ and integrating over $x$, to get

$$\dot{a}(t) \int \int (\chi')^2 dx + \dot{a}(t) \int \partial_x \xi \chi' dx = \int \partial_t \xi \chi' dx + \int \mathcal{L}\xi \chi' dx + \int \mathcal{N}(\xi) \chi' dx. \quad (9.46)$$

For the first term on the r.h.s. we have

$$\int \partial_t \xi \chi' dx = -\int \xi \partial_t (\chi') dx = 0. \quad (9.47)$$

For the second term on the r.h.s., we use (9.26) and the self-adjointness of $\mathcal{L}$ to obtain $\int \mathcal{L}\xi \chi' dx = 0$. The last relation together with (9.47) implies (9.40).
9.4.3 Bound on \( a(t) \)

**Proposition 9.6.** We assume that \( \| \xi \|_{L^2(dx)} \leq 1/2 \). Then, if \( u \) satisfies (9.23) and \( a(t) \) is defined by (9.34) then \( a(t) \) satisfies

\[
|\dot{a}(t)| \lesssim \|\xi\|_{L^2(dx)}^2. \tag{9.48}
\]

**Proof.** We use equation (9.40) and estimate the right-hand side of this equation. We have, by the assumption (9.31) and a Sobolev embedding theorem,

\[
\left| \int \mathcal{N}(\xi)\chi' dx \right| \leq \|\mathcal{N}(\xi)\|_{L^1(dx)}\|\chi'\|_{L^\infty(dx)} \lesssim \|\xi\|_{L^2(dx)}^2 + \|\xi\|_{L^k(dx)}^k \lesssim \|\xi\|_{L^2(dx)}^2 + \|\xi\|_{H^1(dx)}^k. \tag{9.49}
\]

We estimate the terms on the left-hand side of (9.40). Using integration by parts, we have

\[
|\int \partial_x \xi \chi' dx | = \int |\xi'' dx| \tag{9.50}
\]

\[
\leq \|\xi\|_{L^2(dx)}\|\chi''\|_{L^2(dx)}. \tag{9.51}
\]

Combining this estimate with (9.40) gives (9.48). \( \square \)

9.4.4 Upper Bound on \( \|\xi\|_{H^1} \)

**Proposition 9.7.** If \( u \) solves (9.23), and \( \xi \) is the fluctuation defined by (9.34), then

\[
\partial_t \langle \xi, \mathcal{L} \xi \rangle + \|\mathcal{L} \xi\|_{L^2}^2 + 2 \rho \langle \xi, \mathcal{L} \xi \rangle \leq C \langle \xi, \mathcal{L} \xi \rangle^2 + C \langle \xi, \mathcal{L} \xi \rangle^k \tag{9.52}
\]

where \( \rho \) is the constant from Proposition 9.3.

**Proof.** We define the Lyapunov functional \( \Lambda(\xi) := \frac{1}{2} \langle \mathcal{L} \xi, \xi \rangle \). We differentiate \( \Lambda(\xi) \) and use the self-adjointness of \( \mathcal{L} \) to find \( \partial_t \Lambda(\xi) = \langle \mathcal{L} \xi, \mathcal{L} \xi \rangle - \dot{a}(t) \langle \chi', \mathcal{L} \xi \rangle \). Using (9.38) on the r.h.s. of the latter expression gives

\[
\partial_t \Lambda(\xi) = -\langle \mathcal{L} \xi, \mathcal{L} \xi \rangle + \langle \mathcal{N}(\xi), \mathcal{L} \xi \rangle - \dot{a}(t) \langle \chi', \mathcal{L} \xi \rangle \tag{9.53}
\]

Using that \( \mathcal{L} \) is self-adjoint, we have that \( \langle \chi', \mathcal{L} \xi \rangle = 0 \). Therefore, by (9.53), we obtain

\[
\partial_t \Lambda(\xi) + \|\mathcal{L} \xi\|_{L^2}^2 = -\langle \mathcal{L} \xi, \mathcal{N}(\xi) \rangle. \tag{9.54}
\]

Now we prove the following estimates of the terms on the right-hand side:

\[
|\langle \mathcal{L} \xi, \mathcal{N}(\xi) \rangle| \leq C (\|\xi\|_{H^1}^4 + \|\xi\|_{H^1}^{2k}) + \frac{1}{2} \|\mathcal{L} \xi\|_{L^2}^2, \tag{9.55}
\]

\[
\|\mathcal{L} \xi\|_{L^2}^2 \geq \frac{1}{2} \rho \langle \mathcal{L} \xi, \xi \rangle. \tag{9.56}
\]

We start by using the bound on the nonlinearity, (9.31) to get

\[
|\langle \mathcal{L} \xi, \mathcal{N}(\xi) \rangle| \leq \|\mathcal{N}(\xi)\|_{L^2} \|\mathcal{L} \xi\|_{L^2} \tag{9.57}
\]

\[
\lesssim \left( \|\xi\|_{L^4}^2 + \|\xi\|_{L^{2k}}^k \right) \|\mathcal{L} \xi\|_{L^2} \tag{9.58}
\]

\[
\leq \frac{1}{4\delta} (\|\xi\|_{L^4}^2 + \|\xi\|_{L^{2k}}^{2k}) + \delta \|\mathcal{L} \xi\|_{L^2}^2. \tag{9.59}
\]
We use the Sobolev embedding inequality to find that we have (9.55).

Finally, for the inequality (9.56), we use the Cauchy-Schwarz inequality and the inequality (9.28) to find
\[
\frac{1}{2} \rho \|\xi\|^2 \langle L\xi, \xi \rangle \leq \langle L\xi, \xi \rangle \leq \|L\xi\|^2 \|\xi\|^2,
\]
which implies (9.56).

Writing \(\|L\xi\|^2_{L^2} = \frac{1}{2} \|L\xi\|^2_{L^2} + \frac{1}{2} \|L\xi\|^2_{L^2}\) and using (9.56) gives \(\|L\xi\|^2_{L^2} \geq \rho \Lambda(\xi) + \frac{1}{2} \|L\xi\|^2_{L^2}\).

Substituting this and the inequality (9.55) into (9.54), we obtain
\[
\partial_t \Lambda(\xi) + \frac{1}{2} \|L\xi\|^2_{L^2} + \rho \Lambda(\xi) \lesssim C(\|\xi\|^{4}_{H^1} + \|\xi\|^{2k}_{H^1}).
\]
Due to the inequality (9.25), this implies (9.52).

**Proposition 9.8.** If \(u\) solves (9.23) and \(u \in U_\delta\), where \(\delta > 0\) given in Proposition 9.4, so that (9.34) hold and if \(\xi\) is the fluctuation defined by (9.34), then
\[
\|\xi\|_{H^1} \lesssim e^{-\frac{\rho}{4}t} \|\xi_0\|_{H^1},
\]
where \(\rho\) is the constant from Proposition 9.3.

**Proof.** We set \(X(t) := e^{\frac{\rho}{4}t} \Lambda(\xi(t))\) and use (9.52) to find \(\dot{X} \leq Ce^{-\alpha t} X^2\), assuming for simplicity that \(X \lesssim 1\) and consequently dropping the term \(X^k\). Therefore,
\[
X(t) \leq X_0 + C \int_0^t e^{-\alpha s} X^2(s) ds.
\]
Setting \(M = \sup_{s \in [0,t]} \Lambda(\xi(s))\), we have
\[
M \lesssim e^{-\frac{\rho}{8}t} M_0 + CM^2.
\]
Therefore, \(M \leq e^{-\frac{\rho}{8}t} M_0\) provided \(t\) is sufficiently large. Since \(M \geq \frac{1}{2} \langle L\xi, \xi \rangle \gtrsim \|\xi\|^2_{H^1}\), we have (9.62).

Now, assume \(u_0 \in U_{\delta/2}\), where \(\delta > 0\) given in Proposition 9.4. Then by the local well-posedness, there is \(T > 0\), s.t. the solution \(u(t)\) satisfies \(u(t) \in U_\delta\) for \(0 \leq t \leq T\). Then Proposition 9.4 and therefore Proposition 9.8 hold for this \(u(t)\) for \(0 \leq t \leq T\). This gives the bound (9.62). This shows that, in fact, \(u(t) \in U_{\delta/2}\). Continuing in this fashion, we find that the solution \(u(t)\) exists for all \(t > 0\) and satisfies \(u(t) \in U_\delta\) and the bound (9.62) The bound (9.62) and the definition of \(\xi\) in Proposition 9.4 imply (9.32).

It remains to prove the convergence of \(a(t)\).

### 9.5 Asymptotic Behaviour

In the last subsection we proved that the solution \(u = \chi + \xi a\) approaches the manifold of kinks \(M_k\); however, this does not imply the solution approaches a particular kink on the manifold. We prove in this subsection that a limit does exist. More precisely, we prove that \(a(t) \to a_\infty\) as \(t \to \infty\).

As above, we assume \(u_0 \in U_{\delta/2}\), where \(\delta > 0\) given in Proposition 9.4. Then, as was shown above, the solution \(u(t)\) exists for all \(t > 0\) and satisfies \(u(t) \in U_\delta\) and the bound (9.62), which, together with the definition of \(\xi\) in Proposition 9.4, implies (9.32).
In this case, Proposition 9.6 is applicable and gives the estimate (9.48) on \(a\), which together with (9.62) implies that \(a(t)\) satisfies
\[
|\dot{a}(t)| \lesssim e^{-\frac{\rho}{2}t}||\xi_0||_{H^{1}}^2.
\] (9.65)
This shows that the function \(a(t) = a(0) + \int_0^t \dot{a}(s) \, ds\) converges as \(t \to \infty\) and
\[
|a(t) - a(\infty)| \leq \int_t^\infty |\dot{a}(s)| \, ds \lesssim e^{-\frac{\rho}{2}t}||\xi_0||_{H^{1}}^2.
\] (9.66)
We now complete the proof. The triangle inequality implies
\[
\|u(t) - \chi_{a,\infty}\|_{H^{1}} \lesssim \|u(t) - \chi_{a}\|_{H^{1}} + |a(t) - a_{\infty}|,
\]
which together with (9.32) and (9.66) gives the desired result. This proves Theorem 9.7.

10 Mean curvature flow (under construction)

11 Volume preserving mean curvature flow

11.1 Generalities

In this appendix we review some general properties of the volume preserving mean curvature flow. We follow [?], where one can find more details. Recall that the volume preserving mean curvature flow (VPF) is a family \(\{S_t; \, t \geq 0\}\) of smooth closed hypersurfaces in \(\mathbb{R}^{n+1}\), given say by immersions \(\psi : U \to \mathbb{R}^{n+1}\) (here \(U\) is either an open set in \(\mathbb{R}^n\) or a fixed hypersurface in \(\mathbb{R}^{n+1}\)), satisfying the following evolution equation:
\[
\partial_t \psi^N = \bar{H}(\psi) - H(\psi),
\] (11.1)
where \(\partial_t \psi^N\) denotes the normal velocity of \(S_t\) at time \(t\), \(\partial_t \psi^N = (\partial_t \psi, \nu)\), where \(\nu\) is the unit normal vector field on \(S_t\), \(H = H(t)\) stands for the mean curvature of \(S_t\) and \(\bar{H} = \bar{H}(t)\) is the average of the mean curvature on \(S_t\), i.e.,
\[
\bar{H} := \frac{\int_{S_t} H \, d\sigma}{\int_{S_t} d\sigma}, \, t \geq 0.
\] (11.2)
As an initial condition, we consider a simple hypersurface \(S_0\) in \(\mathbb{R}^{n+1}\), with no boundary in \(\Omega\) (e.g. either entirely \(\Omega\) or \(\partial S_0 \subset \partial \Omega\)) given by an immersion \(x_0\). (Another possibility \(\partial_t \psi^N = \bar{H}(\psi) - H(\psi)\) on a space of the volume preserving immersions?)

Like the MCF, the VPF is invariant under rigid motions (translations and rotations) and appropriate scaling. Moreover, it shrinks the area, \(A(\psi) = V(\psi)\), of the surfaces, but, unlike the MCF, the VPF preserves the enclosed volume, \(V_{\text{encl}}(\psi)\). As the result, it has stationary solutions - the Euclidean spheres (for closed surfaces) and cylinders for surfaces with flat boundaries. We summarize these properties as properties:

- (11.1) is invariant under rigid motions of the surface, i.e. \(\psi \mapsto R\psi + a\), where \(R \in O(n+1), \, a \in \mathbb{R}^{n+1}\) and \(\psi = \psi(u, t)\) is a parametrization of \(S_t\), is a symmetry of (11.1).
- (11.1) is invariant under the scaling \(x \mapsto \lambda x\) and \(t \mapsto \lambda^{-2}t\) for any \(\lambda > 0\).
• (11.1) is volume preserving.

• (11.1) is area shrinking. Actually, \( \frac{d}{dt}V(\psi(t)) = -\int_{S_t}(\bar{H} - H)^2d\sigma \leq 0 \).

• Static solutions of (11.1) are surfaces of constant mean curvature (CMC), \( H = h \).

The first two statements are proven as for the MCF. The third one follows from \( \frac{d}{dt}V_{\text{encl}}(\psi(t)) = \int_{S_t}(\partial_t\psi, \nu)d\sigma = \int_{S_t}(\bar{H} - H)d\sigma = 0 \). The fifth one is obvious.

Furthermore, we have the following important property

• (11.1) is a gradient flow for the area functional on closed surfaces with given enclosed volume.

Proof. To prove the last statement we use the formula (??), which we rewrite in the present notation as

\[
dV(\psi)\xi = \int_{\psi} H\nu \cdot \xi = \int_{U} H\nu \cdot \xi \sqrt{g}d^nu, \tag{11.3}
\]

and the fact that \( \xi \) is a vector field for deformations with a fixed enclosed volume, i.e. it is a tangent vector field to the manifold

\[
X_c := \{ \psi \in H^r(U, \mathbb{R}^{n+1}) : V_{\text{encl}}(\psi) = c \}, \tag{11.4}
\]

for \( r > 0 \) sufficiently large, and therefore it satisfies \( \int_{\psi}\xi \cdot \nu = 0 \). Indeed, let \( \psi_s \) be a family of constant enclosed volume surfaces deforming \( \psi \) and let \( \xi \) be the corresponding vector field at \( s = 0 \), i.e. \( \xi = \partial_s\psi_s |_{s=0} \). Since \( V_{\text{encl}}(\psi_s) = c \), we have \( dV_{\text{encl}}(\psi)\xi = \partial_s V_{\text{encl}}(\psi_s) |_{s=0} = 0 \). Hence the result follows from the formula

\[
dV_{\text{encl}}(\psi)\xi = \int_{\psi} \xi \cdot \nu. \tag{11.5}
\]

This formula can be proven by either considering an infinitesimal change in the enclosed volume under the variation of \( S \) or by using that, by the divergence theorem, \( V_{\text{encl}}(\psi) = \frac{1}{n+1} \int_{\Omega} \text{div } x = \frac{1}{n+1} \int_{S} x \cdot \nu \), where \( \Omega \) is domain enclosed by the surface \( S \), described by the immersion \( \psi \), and then differentiating the latter integral, see Appendix ?? (to be added). Conversely, if \( f \) satisfies \( \int_{S} f = 0 \), then there is a volume preserving normal variation with the vector field \( f\nu \). This implies that

\[
T_f X_c = \{ \xi : S \to \mathbb{R}^{n+1}, \int_{S} \xi \cdot \nu = 0 \}.
\]

Now, for an arbitrary normal vector field \( \eta = f\nu \), the vector field \( \xi := (f - \bar{f})\nu \), where \( \bar{f} := \frac{1}{|S|} \int_{S} f \), satisfies \( \int_{S} \xi \cdot \nu = 0 \) and therefore we have \( dV(\psi)\xi = \int_{\psi} H\nu \cdot \xi = \int_{\psi} H(f - \bar{f}) = \int_{\psi}(H - \bar{H})f \). This shows that the \( L^2 \)-gradient of \( V(\psi) \) on closed surfaces with given enclosed volume is \( H - \bar{H} \).

The third and forth properties suggests that, as \( t \to \infty \), the solution \( S_t \) converges to a closed surfaces with the smallest surface area for a given enclosed volume. The limiting surface must be a static solution to the VPF and therefore, by the fifth property, it is a surface of constant mean curvature (CMC). This leads to the isoperimetric problem:

• minimize the area \( V(\psi) \) given the enclosed volume \( V_{\text{encl}}(\psi) \).

Namely, we have
Proposition 11.1. (i) Minimizers of the area $V(\psi)$ for a given enclosed volume $V_{\text{encl}}(\psi)$ are critical points of the area functional $V(\psi)$ on space of immersions with the given enclosed volume $V_{\text{encl}}(\psi)$.

(ii) The (Euler-Lagrange) equation for these critical points is exactly the CMC equation $H = h$.

(iii) These critical points are critical points of the functional

$$V_h(\psi) := V(\psi) - h V_{\text{encl}}(\psi),$$

where $h$ is determined by $c = V_{\text{encl}}(\psi)$ and vice versa.

Proof. The first and third statements are standard results (the third statement is follows from the Lagrange multiplier theorem). The second statement follows from the relations $dV(\psi)\xi = \int_U H\nu \cdot \xi \sqrt{g} d^n u$ and $dV_{\text{encl}}(\psi)\xi = \int_S \langle \xi, \nu \rangle d\sigma$ and the definition of $V_h(\psi)$.

Thus finding stationary solutions to the VPF is the same as finding closed CMC surfaces. This leads to the following problems: (a) Find CMC surfaces and (b) determine their stability w.r. to the VPF.

However, we expect that the VPF converges to a minimal CMC surface (i.e. one solving the isoperimetric problem), not just to a CMC surface. For such we have an additional characterization, which is a standard fact from the variational calculus (see e.g. [15]):

Proposition 11.2. If $\varphi$ minimizes the area $V(\psi)$ for a given enclosed volume $V_{\text{encl}}(\psi)$, then $\text{Hess}^N V(\theta) \geq 0$ on $T_\theta X_c$.

Next, we review some spectral theory of the normal hessian, $\text{Hess}^N V(\varphi)$, which, recall, is defined as the hessian w.r.to normal variations, $\eta = f\nu$, i.e $\text{Hess}^N V(\varphi)f = \text{Hess} V(\varphi)(f\nu)$. For the details and proofs see [32].

Theorem 11.1. The normal hessian, $\text{Hess}^N V(\varphi)$, of the modified volume functional $V(\varphi)$, at a CMC $\varphi$, has the eigenvalue 0 of

- the multiplicity $n + 1 + \frac{1}{2}n(n - 1)$ (+1, if $h = 0$), with the eigenfunctions
  (a) $\nu^j(\varphi)$, $j = 1, \ldots, n + 1$, and
  (b) $\langle \sigma_j, \nu(\varphi) \rangle$, $j = 1, \ldots, \frac{1}{2}n(n - 1)$, where $\sigma_j$ are generators of the Lie algebra of $SO(n + 1)$ (Jacobi fields), if $\varphi$ is neither spherical nor axi-symmetric;

- the multiplicity $2n$, if $\varphi = \text{cylinder}$;

- the multiplicity $n + 1$, if $\varphi = \text{sphere}$.

Moreover, if $h \neq 0$ and $\int_S \langle \varphi, \nu(\varphi) \rangle > 0$, then the operator $d^N H(\varphi)$ has a negative spectrum. For a sphere, the negative spectrum consists of the single eigenvalue $-h$ with the eigenfunction $\equiv 1$.

The property of minimal CMC surfaces isolated in Proposition 11.2 plays an important role in their analysis and deserves the name. In geometric analysis, one says that a CMC surface $\theta$ is stable iff $\text{Hess}^N V(\theta) \geq 0$ on $T_\theta X_c$. In dynamical systems and partial differential equations, this notion is called (weak) linear or energetic stability.

An importance of the notion of stability is illustrated in the result of Barbosa and do Carmo ([?]) saying that a compact, stable CMC surface is a sphere.
Now, we define the notion of dynamic stability. We say that a CMC surface $\theta$ (which is a static solution to (11.8)) is \textit{asymptotically stable} iff there is $\epsilon$ s.t. for any initial condition $\psi_0 \in X_c$, with $c = V_{encl}(\theta)$, $\epsilon-$close to $\theta$ in the $H^s$ norm, the solution $\psi$ to the VPF (11.8), appropriately reparametrized, converges, as $t \to \infty$, to the manifold $M_c := \{ \phi \in X_c : H(\phi) = h(c) \}$ (i.e. there is a family $\alpha = \alpha(\cdot, t)$ of reparametrizations, s.t. $\text{dist}(\psi \circ \alpha, M_c) \to 0$).

There are also a weaker notion of the \textit{Lyapunov} or \textit{orbital} stability.

In $\mathbb{R}^{n+1}$ compact CMC surfaces are Euclidean spheres. Using techniques similar to those of the last lecture one can show the asymptotic stability of spheres in $\mathbb{R}^{n+1}$ under the volume preserving flow (J. Escher and G. Simonett ([?]) and D. Antonopoulou, G. Karali, I.M. Sigal ([?] )).

### 11.2 Trapped surfaces

We consider the volume preserving mean curvature flow (VPF) of closed hypersurfaces in the simplest non-euclidean space - the conformally euclidean one, $(\mathbb{R}^{n+1}, e^{-2\varphi(x)}ds_{\text{eucl}}^2)$. Moreover, we consider a slowly conformal factor $\varphi(x) = \varphi_c(x)$, depending on some small parameter $\epsilon > 0$ and satisfying the estimates

$$\varphi_c(x)|_{\epsilon=0} = 0, \quad \text{and} \quad |\partial_x^\alpha \varphi_c(x)| \lesssim \epsilon^{(|\alpha|)}.$$  \hspace{1cm} (11.6)

One can consider general, slowly varying ambient metric, but the problems we are interested in are interesting and rather non-trivial already for the conformal metric. We label all quantities pertaining to the metric

$$e^{-2\varphi_c(x)}ds_{\text{eucl}}^2$$  \hspace{1cm} (11.7)

by the subindex $\epsilon$.

Denote by $S_{\lambda,z}$ the Euclidean sphere centred at $z \in \mathbb{R}^{n+1}$ and of the radius $\lambda$. We define it by the immersion $\theta_{\lambda,z} : S^n \to \mathbb{R}^{n+1}$, given by

$$\theta_{\lambda,z}(\omega) := z + \lambda \omega.$$  

We also use the notation $\alpha := (\lambda, z) \in \mathbb{R}_+ \times \mathbb{R}^{n+1}$, so that, e.g. $\theta_{\lambda,z} \equiv \theta_{\alpha}$. The following result is shown in [?] (see also [?]).

**Theorem 11.2** ([?], see also [?]). Assume (11.7) and (11.6). Then for each non-degenerate critical point, $x_*$, of the scalar curvature $R_c$ of $(\mathbb{R}^{n+1}, e^{-2\varphi_c(x)}dx^2)$ and for each $\lambda > 0$, there exists a constant mean curvature (CMC) surface, close to the sphere $S_{\lambda,x_*}$.

Next, we have

**Theorem 11.3.** The CMC surfaces described above and corresponding to minima of $R_c$ are asymptotically stable and those corresponding to saddle points are unstable.

We indicate the problems in trying to prove Theorem 11.3 and ways to resolve them. We distinguish the quantities pertaining to the conformal metric (11.7) with the subindex $\epsilon$. Then the VPF becomes

$$\partial_t \psi^N = \mathcal{P}_\epsilon(\psi) - H_\epsilon(\psi),$$  \hspace{1cm} (11.8)

where $\partial_t \psi^N$ is the normal velocity, $\partial_t \psi^N := \langle \partial_t \psi, \nu_\epsilon \rangle_{\varphi}$, where $\nu_\epsilon$ is unit normal on $S_t$, $H_\epsilon(\psi)$ is the mean curvature of $\psi$ and $\mathcal{P}_\epsilon(\psi) := \frac{1}{|S_t|} \int_{S_t} H_\epsilon(\psi)$, the average of $H_\epsilon(\psi)$ over $S_t$. 
A closed surface $S$, given by an immersion $\theta$ and satisfying the static VPF equation,

$$H_\epsilon(\theta) - \bar{\mu}_\epsilon(\theta) = 0,$$  \hspace{1cm} (11.9)

is a constant minimal curvature surface, satisfying the equation

$$H_\epsilon(\theta) = h$$ \hspace{1cm} (11.10)

for some constant $h$ and vice versa.

If we write (11.10) as $F(\theta, h, \epsilon) = 0$, where $F(\theta, h, \epsilon) := H_\epsilon(\theta) - h$, then we see that any sphere $\theta_{\lambda, z}$, with the radius $\lambda = \sqrt{n}$, is a solution of this equation for $\epsilon = 0$. Now, we want to solve (11.9) near a sphere $\theta_\alpha$, where $\alpha = (\lambda, z)$. The first thing to do is to see whether we can apply the implicit function theorem. To this end we linearize the map

$$\hat{H}_\epsilon(\theta) := H_\epsilon(\theta) - \bar{\mu}_\epsilon(\theta),$$

the map on the l.h.s. of (11.9), at the immersion $\theta_\alpha$ (i.e. the sphere $S_\alpha$) in the normal direction. We show in Appendix ?? that the linearized map for $\epsilon = 0$ is

$$L_{\alpha,0}\phi = -\frac{1}{\lambda^2}[(\Delta + n)\phi - \frac{n}{|S^n|} \int_{S^n} \phi],$$ \hspace{1cm} (11.11)

where $\Delta$ is the Laplace - Beltrami operator on the sphere. (This expression can be either derived directly or deduced from the general formula for the normal hessian.) This operator is well understood. Considered on $L^2(S^n)$, it is self-adjoint, has purely discrete spectrum, with eigenvalues accumulating at $+\infty$. Its eigenvalues, their multiplicities and the corresponding eigenfunctions are well-known. In particular, $L_{\alpha,0}$ has

- the eigenvalue 0 of multiplicity $n + 2$ having the eigenfunctions $1, \omega^1, \ldots, \omega^{n+1}$; \hspace{1cm} (11.12)
- the remaining eigenvalues $\geq \frac{1}{\lambda^2}(n + 2)$. \hspace{1cm} (11.13)

The result above shows that the spectrum of the operator $L_{\alpha,0}$ has lots of zero eigenfunctions. Hence we expect that for $\epsilon$ small, the the linearized map $L_{\alpha,\epsilon}$ for $F(\theta, h, \epsilon) := H_\epsilon(\theta) - h$ will have zero or/and almost zero eigenvalues and therefore we will not be able to solve the equation (11.9) near arbitrary sphere $S_\alpha$. Hence we will not be able to use the implicit function theorem.

To overcome this difficulty, we split the equation (11.10) along the null space of the operator $L_{\alpha,0}$ and along the transversal space. In the transversal space, the implicit function theorem works and gives us the solution along this subspace. Substituting it into the equation along null $L_{\alpha,0}$, we reduce the original problem to one on the subspace null $L_{\alpha,0}$, which is finite dimensional. Then we tackle the latter problem. This method is called the Lyapunov - Schmidt decomposition. It generalizes the implicit function theorem.

### 11.3 Reduction

In this section we prove the following result giving, among other things, almost static solutions to (11.8), or put it differently, give an approximate solution to the static equation (11.9), near spheres, $S_\alpha$, i.e. near the sphere manifold

$$\mathcal{M}_0 = \{\theta_{\lambda, z} : (\lambda, z) \in \mathbb{R}^+ \times \mathbb{R}^{n+1}\}.$$
**Theorem 11.4.** There is a one-to-one map \( \Phi : \mathbb{R}^+ \times \mathbb{R}^{n+1} \rightarrow X_c \) s.t. \( \psi = \Phi(\alpha) \) solves (11.10) iff \( \alpha \) solves the equation \( \nabla_\alpha V_h(\Phi(\alpha)) \).

**Proof.** We look for solutions of (11.10) as shifted normal graphs over the \( n \)-dimensional unit sphere, \( S^n \),

\[
\theta_{pz}(\omega) := z + \rho(\omega)\omega. \tag{11.14}
\]

In this representation, the sphere of the radius \( \lambda \) and centered at \( z \) is given by \( \rho = \lambda \). We call \( z \) the center of the surface \( S \), if it is determined by the conditions

\[
\int_{\mathbb{S}^n} (\theta(\omega) - z) \cdot \omega^i = 0, \quad i = 1, \ldots, n + 1. \tag{11.15}
\]

These are \( n + 1 \) equations for the \( n + 1 \) unknowns \( z \in \mathbb{R}^{n+1} \). According to (11.15), the spheres \( \theta_{\lambda z} = \theta_{\rho = \lambda z}, \quad \lambda \in \mathbb{R}^+ \), have the centers \( z \), which justifies our interpretation of its solutions as the center of a surface.

Next, one can show that if \( \theta \) is sufficiently close to \( \mathcal{M} \), then it can be written as

\[
\theta = \theta_{pz}, \quad \text{with } z \text{ s.t } \rho \perp \omega^j, \quad j = 1, \ldots, n + 1. \tag{11.16}
\]

Here and in what follows the inner product and orthogonality relation is understood in the sense of \( L^2(S^n) \).

We look for the solution of (11.9) in the form (11.14), with

\[
\rho(\omega) = \lambda + \phi(\omega), \quad \text{where } \lambda = \langle \rho \rangle \quad \text{and} \quad \int_{\mathbb{S}^n} \phi = 0. \tag{11.17}
\]

Here \( \langle f \rangle \) denotes the average, \( \langle f \rangle := \frac{1}{|S^n|} \int_{\mathbb{S}^n} f \), of \( f \) over the standard \( n \)-sphere, \( S^n \) and \( \lambda \) is the leading term and represents the sphere centred at 0 of radius \( \lambda \). The property \( \int_{\mathbb{S}^n} \phi = 0 \) can be written as \( \phi \perp 1 \) in \( L^2(S^n) \). By (11.16), we have in addition that \( \phi \perp \omega^j, \quad j = 1, \ldots, n + 1 \), which together with the previous conditions can be written as

\[
\phi \perp \omega^j, \quad j = 0, \ldots, n + 1, \tag{11.18}
\]

where \( \omega^0 := 1 \), while \( \omega = (\omega_1, \ldots, \omega_{n+1}) \in S^n \).

We consider the linearized map \( L_{\lambda z, \epsilon} = d_{\rho} \hat{H}_\epsilon(\theta)|_{\rho = \lambda} \), where, recall, \( \hat{H}_\epsilon(\theta) := H_\epsilon(\theta_{pz}) - \overline{H}_\epsilon(\theta_{pz}) \), on the sphere \( \rho = \lambda \). We show in Appendix ?? below that \( L_{\lambda z, \epsilon} \) is a small perturbation of the operator \( L_{\alpha, 0} \), defined above in (11.11), namely \( L_{\alpha, \epsilon} = L_{\alpha, 0} + O(\epsilon) \).

\[
L_{\alpha, \epsilon} = L_{\alpha, 0} + M_{\alpha, \epsilon}, \tag{11.19}
\]

where, recall, \( \alpha = (\lambda, z) \) and the operator \( M_{\alpha, \epsilon} \) is defined by this relation and is given explicitly by

\[
M_{\alpha, \epsilon} = \frac{n}{\alpha} \nabla_\omega \varphi_\alpha \cdot \nabla - n(\omega \cdot \nabla_\omega)^2 \varphi_\alpha + h e^{-\varphi_\alpha} \nabla_\omega \varphi_\alpha \cdot \omega. \tag{11.20}
\]

Recall that \( L_{\alpha, 0} \) is self-adjoint and has purely discrete spectrum, described above. Since \( \varphi_\epsilon(x) := \varphi(\epsilon x) \) and \( \varphi_\epsilon(z) = \varphi(z + \lambda \omega) \), we have \( \nabla_\omega \varphi_\epsilon(z) = O(\epsilon) \), which, together with the above expression for the perturbation \( M_{\alpha, \epsilon} \), shows that this operator satisfies the estimate

\[
\|M_{\alpha, \epsilon}(L_{\alpha, 0} + \lambda^{-2 n + 1})^{-1/2}\| \lesssim \epsilon. \tag{11.21}
\]
In Appendix 11.6 we derive more precise estimates on $M_{\alpha,\epsilon}$.

Let $A_\epsilon(\rho,z) := H_\epsilon(\theta_{\rho z}) - \overline{\Pi}_{\epsilon}(\theta_{\rho z})$, where $\theta_{\rho z}$ is given in (11.14). Then the equation (11.10) can be rewritten as $A_\epsilon(\rho,z) = 0$. Plug the decomposition (11.17) into the latter equation to obtain an equation for $\phi, \lambda$ and $z$:

$$A_\epsilon(\lambda + \phi, z) = 0. \tag{11.22}$$

The explicit form of the map $A$ is given in (11.34) below. We want to construct approximate solutions to this equation. We split into the equation along the tangent space to the sphere manifold $\mathcal{M}$ and the orthogonal direction and solve it in the orthogonal direction.

Let $P_0$ be the orthogonal projection on $L^2(S^n)$ onto the eigenspaces of $\Delta + n$ corresponding to the non-positive eigenvalues (i.e. the eigenvalues 0 and $-n$) and $\bar{P}_0 = 1 - P_0$, the orthogonal projection onto $(\text{null}(\Delta + n))^\perp$. By the above, $\phi \in \text{Ran } P_0$. We consider the equation

$$\bar{P}_0 A_\epsilon(\lambda + \phi, z) = 0. \tag{11.23}$$

Our goal is to solve this equation for $\phi$. Now, it is easy to show that the implicit function theorem is applicable and gives a unique solution $\phi = \phi_{\alpha,\epsilon} \in \bar{P}_0 H^s$, satisfying $\phi_{\alpha,0} = 0$. More precisely, we have

**Lemma 7.** The equation (11.23) for $\phi$ has a unique solution $\phi = \phi_{\alpha,\epsilon} \in \bar{P}_0 H^s$, satisfying $\phi_{\alpha,0} = 0$.

**Proof.** We denote $F(\phi, \alpha, \epsilon) := \bar{P}_0 A_\epsilon(\lambda + \phi, z)$, where $\alpha = (\lambda, z)$, and use the implicit function theorem (IFT) for $F(\phi, \alpha, \epsilon) = 0$. We go through the checklist for IFT: we claim that

(a) $F : \bar{P}_0 H^2 \times \mathbb{R}^n \times \mathbb{R} \to \bar{P}_0 L^2$ and is $C^1$ in $\phi$ and $C$ in $\alpha$ and $\epsilon$;

(b) $F(0, \alpha, 0) = 0 \ \forall \alpha$;

(c) $\partial_\phi F(0, \alpha, 0)$ is invertible.

Properties (a) and (b) are straightforward: (a) follows from the explicit expression for the map $A$ given in in Appendix 11.5 and for (b), write $A_\epsilon(\alpha) = A_\epsilon(\lambda, z)$, then $F(0, \alpha, 0) = \bar{P}_0 A_0(\alpha) = 0$. Since $\partial_\phi G(\alpha, 0)$ is self-adjoint and has a purely discrete spectrum and null $\partial_\phi A_0(\alpha) = \text{Ran } \bar{P}_0$, we conclude, by the spectral theory (see [?]), that $\partial_\phi F(0, \alpha, 0) = \bar{P}_0 \partial_\phi A_0(\alpha) = \bar{P}_0(\Delta + n)$ is invertible on $\text{Ran } \bar{P}_0$. This gives a unique solution $\phi = \phi_{\alpha,\epsilon}$ of (11.23). By (b), we have that $\phi_{\alpha,0} = 0$.

Now, we define the adiabatic approximate solution as

$$\rho_{\alpha,\epsilon}(\omega) = \lambda + \phi_{\alpha,\epsilon}(\omega), \text{ where } \lambda = \langle \rho \rangle \text{ and } \phi_{\alpha,\epsilon} \in \bar{P}_0 H^s \text{ solves (11.23)}. \tag{11.24}$$

Hence $\rho_{\alpha,\epsilon} := \lambda + \phi_{\alpha,\epsilon}$ solves the equation $\bar{P}_0 A_\epsilon(\rho, z) = 0$ and $\theta_{\alpha,\epsilon}(\omega) := \theta_\alpha(\omega) + \phi_{\alpha,\epsilon}(\omega) \omega = z + \rho_{\alpha,\epsilon}(\omega) \omega$ solves the equation

$$\bar{P}_0 \dot{H}_\epsilon(\theta) = 0, \tag{11.25}$$

(or $\bar{P}_0 (H(\psi) - h) = 0$ with the specified $h$). Finally, we define the map $\Phi_{\epsilon}(\alpha)$ as $\Phi_{\epsilon}(\alpha) := \theta_{\alpha,\epsilon}$, where $\theta_{\alpha,\epsilon}(\omega)$ is given above.

We define the functions

$$v(\alpha) := V(\theta_{\alpha,\epsilon}) \ v_{\text{encl}}(\alpha) := V_{\text{encl}}(\theta_{\alpha,\epsilon}), \ v_h(\alpha) := V_h(\theta_{\alpha,\epsilon}). \tag{11.26}$$

Theorem 11.4 follows from the result below.
Theorem 11.5. $\partial_\alpha v_h(\alpha) = 0 \iff \partial_\alpha v(\alpha) = 0$ on $\{\alpha : v_{\text{encl}}(\alpha) = c\} \iff dV_h(\theta_{\alpha,\epsilon}) = 0 \iff P_0 H(\theta_{\alpha,\epsilon}) = 0$.

Proof. We compute $\partial_\alpha v_h(\alpha) = dV_h(\theta_{\alpha,\epsilon}) \partial_\alpha \theta_{\alpha,\epsilon}$. Hence, $dV_h(\theta_{\alpha,\epsilon}) = 0 \implies \partial_\alpha v_h(\alpha) = 0$. Next, let $\nabla V_h(\psi)$ denote the gradient of $V_h(\psi)$. By the definition, $dV_h(\theta_{\alpha,\epsilon}) \xi = \langle \nabla V_h(\theta_{\alpha,\epsilon}), \xi \rangle$, of $\nabla V_h(\psi)$, we have $\nabla V_h(\psi) = (H(\psi) - h)\nu(\psi)$ and therefore the normal gradient is $\nabla^N V_h(\psi) = H(\psi) - h$. Hence, the $P_0$-equation (11.23) implies that $P_0 \nabla^N V_h(\theta_{\alpha,\epsilon}) = 0$ and therefore

$$P_0 \nabla^N V(\theta_{\alpha,\epsilon}) = \nabla^N V(\theta_{\alpha,\epsilon}). \quad (11.27)$$

Note that $P_0$ is the projection onto $T_{\theta_{\alpha,\epsilon}} M_0$, the tangent space to $M_0 := \{\theta_{\alpha}\}$ at $\theta_{\alpha}$, w.r.t. normal variations. Indeed, the normal tangent vectors to $M_0$ are $\partial_{\alpha,\epsilon}^N \theta_{\alpha} := \omega \cdot \partial_\alpha \theta_{\alpha}$. Now, write $P_0 = \sum |\partial_{\alpha,\epsilon}^N \theta_{\alpha}| g^{ij}(\partial_{\alpha,\epsilon}^N \theta_{\alpha})$, where $g^{ij} := \langle \partial_{\alpha,\epsilon}^N \theta_{\alpha}, \partial_{\alpha,\epsilon}^N \theta_{\alpha} \rangle$. Let $P_\epsilon$ be the projection onto $T_{\theta_{\alpha,\epsilon}}^N M$, where $M_\epsilon := \{\theta_{\alpha,\epsilon}\}$. Then $P_\epsilon h = \sum g^{ij}(\partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon}) h(\partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon})$ where $\partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon} := \nu \cdot \partial_\alpha \theta_{\alpha,\epsilon}$, with $\nu$ being the unit normal vector field to the surface $\psi_\alpha$, and $g^{ij} := \langle \partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon}, \partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon} \rangle$. (check the projections) We have

Lemma 8. $\|P_0 - P_\epsilon\| \to 0$, as $\epsilon \to 0$.

Proof. It suffices to show that $\|\partial_{\alpha,\epsilon}^N \theta_{\alpha} - \partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon}\| \to 0$, as $\epsilon \to 0$. Since $\theta_{\alpha,\epsilon}(\omega) := \theta_{\alpha}(\omega) + \phi_{\alpha,\epsilon}(\omega)\omega$, the result follows from the estimates on $\phi_{\alpha,\epsilon}(\omega)$. (more details)

Using the relations $\partial_\alpha v_h(\alpha) = \langle \nabla V_h(\theta_{\alpha,\epsilon}), \partial_\alpha \theta_{\alpha,\epsilon} \rangle$ and $\nabla V_h(\theta_{\alpha,\epsilon}) = \nabla^N V_h(\theta_{\alpha,\epsilon}) \nu(\theta_{\alpha,\epsilon})$, we find $\partial_\alpha v_h(\alpha) = \langle \nabla^N V_h(\theta_{\alpha,\epsilon}), \partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon} \rangle$. Using this and the basis $\{\partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon}\}$, we obtain

$$P_\epsilon \nabla^N V_h(\theta_{\alpha,\epsilon}) = \sum g^{ij}(\partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon}) \partial_{\alpha,\epsilon}^N \theta_{\alpha,\epsilon}. \quad (11.28)$$

This gives $\partial_\alpha v_h(\alpha) = 0 \implies P_\epsilon V_h(\theta_{\alpha,\epsilon}) = 0$. Using the last relation, together with $P_0 \nabla^N V_h(\psi) = \nabla^N V_h(\psi)$, we find $\nabla^N V_h(\psi) = P_0 \nabla^N V_h(\theta_{\alpha,\epsilon}) - P_\epsilon \nabla^N V_h(\theta_{\alpha,\epsilon}) = (P_0 - P_\epsilon) \nabla^N V_h(\theta_{\alpha,\epsilon})$, which gives $\|\nabla^N V_h(\theta_{\alpha,\epsilon})\| \leq \|(P_0 - P_\epsilon) \nabla^N V_h(\theta_{\alpha,\epsilon})\|$. Since, $\|P_0 - P_\epsilon\| < 1$, for $\epsilon$ sufficiently small, we have $\nabla^N V_h(\theta_{\alpha,\epsilon}) = 0$, or $dV_h(\theta_{\alpha,\epsilon}) = 0$, as required.

(the equivalence with the constrained variational problem for functionals $V(\psi)$ and $v(\alpha)$ to be proved)

Stability. Let $\alpha_*$ be a critical point of $v_h(\alpha)$, so that, by Theorem 11.5, $\psi_{\alpha_*}$ is a critical point of $V_h(\psi)$. In this paragraph, the stability is understood in the sense of geometric analysis, see the end of Section 11.2. We have

Theorem 11.6. The surface $\psi_{\alpha_*}$ is stable (linearly, or in the sense of geometric analysis) $\iff$ the hessian of $v_h(\alpha)$ ($\iff$ of $v(\alpha) = 0$ on $\{\alpha : v_{\text{encl}}(\alpha) = c\}$) at $\alpha_*$ is positive definite (i.e. $\alpha_*$ is a minimum of $v_h(\alpha)$).

Proof. Differentiating the equation the relation $\partial_\alpha v_h(\alpha) = \langle V''_h(\theta_{\alpha,\epsilon}), \partial_\alpha \theta_{\alpha,\epsilon} \rangle$ and using that $\partial_\alpha \nabla V_h(\theta_{\alpha,\epsilon}) = V''_h(\psi_{\alpha_*}) \partial_\alpha \theta_{\alpha,\epsilon}$, where $V''_h(\psi) = d\nabla V_h(\psi)$ is the hessian of $V_h(\psi)$, we find $\partial_\alpha \partial_\alpha \partial_\alpha v_h(\alpha) = \langle V''_h(\theta_{\alpha,\epsilon}), \partial_\alpha \theta_{\alpha,\epsilon} \partial_\alpha \theta_{\alpha,\epsilon} \partial_\alpha \theta_{\alpha,\epsilon} \rangle + \langle \nabla V_h(\psi_{\alpha_*}), \partial_\alpha \partial_\alpha \partial_\alpha \psi_{\alpha_*} \partial_\alpha \theta_{\alpha,\epsilon} \rangle$. At a critical point $\alpha_*$ of $v_h(\alpha)$, we have $\nabla V_h(\theta_{\alpha_*}) = 0$ and therefore the previous relation gives

$$\partial_\alpha \partial_\alpha \partial_\alpha v_h(\alpha_*) = \langle V''_h(\theta_{\alpha_*}), \partial_\alpha \theta_{\alpha_*}, \partial_\alpha \theta_{\alpha_*}, \partial_\alpha \theta_{\alpha_*} \rangle. \quad (11.29)$$

It follows from (11.12) - (11.13) and the perturbation theory that the spectrum of the operator $V''_h(\theta_{\alpha_*})$ restricted to $(TM)^{-}$ is bounded below by $n + 2 - O(\epsilon) > 0$. Hence the smallest
eigenvalues of $V_h''(\theta_{\alpha,\epsilon})$ are coming from its restriction to $T\mathcal{M}$ and these eigenvalues determine the stability of $\theta_{\alpha,\epsilon}$. By the equation (11.29), these eigenvalues are given by the eigenvalues of the hessian $v_h''(\alpha)$ of $v_h(\alpha)$.

\[\square\]

**Remark 1.** Similarly, we could have reformulated the statement of Theorem 11.3 in terms of the constrained variational problem for functionals $V(\psi)$ and $v(\alpha)$.

### 11.4 Expansions. Completion of the proof of Theorem 11.2

We assume for simplicity of notation that $\varphi_\epsilon(x)$ in (11.7) can be written as

$$
\varphi_\epsilon(x) = \varphi_0(z_\epsilon + \epsilon(x - z_\epsilon)). \tag{11.30}
$$

We denote by rem$_s$ a generic remainder which may depend on the parameters $\alpha$ and, in the $H^2$ norm (as a function of the relevant variables) satisfies the estimates

$$
\partial^k_\alpha \text{rem}_s = O(\epsilon^k). \tag{11.31}
$$

Now, expanding $\lambda, z, \phi$ in $\epsilon$ and using results of [?, ?], one arrives at the following expansions

$$
v(\alpha, \epsilon) = V(\theta_\alpha) - \frac{n^2}{2(n + 3)\hbar\varepsilon^2} R(z_\epsilon)\epsilon^2 + \text{rem}_3, \quad v_{\text{encl}}(\alpha, \epsilon) = \ldots. \tag{11.32}
$$

(see also [?, ?] for earlier similar results and [?] for direct proofs).

We can derive Theorem 11.2 from the equation (11.32) and Theorem 11.5, as follows. The equation (11.32) implies that for the equation $\partial_\alpha v(\alpha) = 0$ on $\{\alpha : v_{\text{encl}}(\alpha) = c\}$ ($\iff \partial_\alpha v_h(\alpha) = 0$), and therefore the equation $P_0 \hat{H}_\epsilon(\psi_\alpha) = 0 \iff dV_h(\psi_\alpha) = 0$, see Theorem 11.5), to have a solution, the origin should be a critical point of the scalar curvature,

$$
\nabla^0 R(z_\epsilon) = 0. \tag{11.33}
$$

On the other hand, if $\nabla^0 \nabla^0 R_\epsilon(z_\epsilon)$ is non-degenerate, then the equations $\partial_\alpha v_h(\alpha) = 0$, $j = 1, \ldots, n + 1$, have a unique solution for $z$ near 0 and this solution satisfies $z_\epsilon = O(\epsilon)$ (or going back to the original critical point, $z^{(\epsilon)} = z_\epsilon + O(\epsilon)$). (If we write $z^{(\epsilon)} = z_\epsilon + z_1\epsilon + O(\epsilon^2)$, then $z_1$ satisfies the equation of the form $\nabla^0 \nabla^0 R_\epsilon(0)z_1 = O(\epsilon^3)$.) The solution $\lambda^{(\epsilon)}$ can be found from either $\partial_{\alpha_0} v_h(\alpha) = 0$, or $v_{\text{encl}}(\alpha) = c$. This solution obeys the estimate $\lambda^{(\epsilon)} = \frac{\lambda}{n} + O(\epsilon^2)$.

Now, since $\partial_\alpha$ solves (11.23) and $\alpha^{(\epsilon)}$ solves the equation $P_0 A_\epsilon(\lambda^{(\epsilon)} + \phi^{(\epsilon)}, z) = 0$, we conclude that $\rho^{(\epsilon)} := \lambda^{(\epsilon)} + \phi^{(\epsilon)}$ solves the equation (11.22). Since $\rho, z$ solving the equation (11.22) is equivalent to $z + \theta_\rho$ solving the equation (11.10), this implies that $\theta^{(\epsilon)} := \theta_\rho^{(\epsilon)}, z^{(\epsilon)} := z^{(\epsilon)} + \rho^{(\epsilon)}(\omega)\omega$ solves (11.10). This finishes the thumbnail sketch of Theorem 11.2.

The weak form of Theorem 11.3 (i.e. linear, or in the sense of geometric analysis, stability) follows from the equation (11.32) and Theorem 11.6. \[\square\]

### 11.5 Appendix. The explicit form of the map $A$

We show now that the map $A_\epsilon(\rho, z) := H_\epsilon(\theta_{\rho,\omega})$, where $\theta_{\rho,\omega}(\omega) := z + \rho(\omega)\omega$, has the following explicit form

$$
A_\epsilon(\rho, z) := e^{\phi_{\rho,\omega}} \left[ h_0(\rho) - n\mu(\rho)^{-1}(\omega - \rho^{-1}\nabla\rho) \cdot \nabla z \phi_{\rho,\omega} \right], \tag{11.34}
$$
where $\mu(\rho) := \sqrt{1 + \rho^{-2} |\nabla \rho|^2}$, where we denoted
\[
\varphi_{\rho z}(\omega) := \varphi(\theta_{\rho z}) = \varphi(z + \rho(\omega) \omega)
\]
and the map $h_0(\rho)$ is given by
\[
h_0(\rho) := -\mu(\rho)^{-1}\left\{ \frac{1}{\rho^2} \Delta \rho - \frac{n}{\rho} - \frac{1}{\rho^3 \mu(\rho)^2} \left[ \frac{1}{\rho} (\nabla \rho \cdot \text{Hess}(\rho) \nabla \rho) + |\nabla \rho|^2 \right] \right\}.
\] (11.35)

Here we used the following notation: $\nabla_i \rho = \frac{\partial \rho}{\partial x_i}$ and $(\nabla_i w)_j = \frac{\partial w_j}{\partial x_i} - \Gamma^k_{ij} w_k$, where $\Gamma^k_{ij}$ are the Christoffel symbols (here and in what follows the summation over the repeated indices is assumed; see Appendix 5.10), $\Delta$, the Laplace-Beltrami operator on $\mathbb{S}^n$ in the standard metric and $(\text{Hess})_{ij} = \nabla_i \nabla_j$.

To derive (11.34) - (11.35), we recall that the mean curvature for a hypersurface $S$ immersed in $(\mathbb{R}^{n+1}, e^{-2\varphi} dx^2)$, is given by (see [?])
\[
H_\epsilon(x) = e^{\varphi_\epsilon}(H_0 - n \nabla \varphi_\epsilon),
\] (11.36)
where $H_0(x) = H_{\text{eucl}}(x)$, $\nu$ is the euclidean unit normal vector field on $S$ and $\nabla_\nu f = \nu f$ is the Euclidean directional derivative in the direction $\nu$. Note that $\nu$ is related to the conformal unit normal vector field, $\nu_\epsilon(x)$, on $S$ by $\nu_\epsilon(x) = e^{\varphi_\epsilon} \nu(x)$. We plug the expression (11.36) into the definition $A_\epsilon(\rho, z) := H_\epsilon(\theta_{\rho z})$, to obtain
\[
A_\epsilon(\rho, z) = e^{\varphi_{\rho z}} [H_0(\theta_{\rho z}) - n \nabla_\nu \varphi_{\rho z}].
\] (11.37)

Now, we express the first two terms on the r.h.s. in (11.37) explicitly in terms of $\rho$ and $z$. Using the relations $H_0(\theta_\rho + z) = H_0(\theta_\rho)$, and $\nu(\theta_\rho + z) = \nu(\theta_\rho)$, and using Proposition 11.3 of Appendix 11.7 which gives an explicit expression of $H_0(\theta_\rho)$ in terms of $\rho$, (or the level set representation, as in [?], Appendix ??) and the relation (11.50) for $\nu$ ($\nu = (1 + \rho^{-2} |\nabla \rho|^2)^{-1/2}(\omega - \rho^{-1} \nabla \rho)$), we obtain the expression (11.34) for the map $A_\epsilon(\rho, z)$.

### 11.6 Appendix. The operator $M_{\alpha, \epsilon}$

In this appendix, we address the operator $M_{\alpha, \epsilon}$. We use that $\varphi(x) = \varphi_0(\varepsilon x)$ (see (11.30)), $\varphi_0(0) = 0$ and the Taylor expansion
\[
\varphi_\alpha(\omega) = \varepsilon \varphi_\alpha'(0) w + \varepsilon^2 \frac{1}{2} \varphi_\alpha''(0) (w, w) + \text{rem}_3,
\] (11.38)
where $w := z + \lambda \omega$ and $\text{rem}_3$ satisfies the estimates (11.31).

We use the equations ?? and $\varphi(x) = \varphi_0(\varepsilon x)$ (see (11.30)), $\varphi_0(0) = 0$ and the Taylor expansion expand
\[
\varphi_\alpha(\omega) = \sum_{k=1}^s \varepsilon^k \frac{1}{k!} \varphi_0^{(k)}(0) w^k + \text{rem}_{s+1},
\] (11.39)
where $w := z + \lambda \omega$, and $\text{rem}_s$ satisfies the estimates (11.31). Here and in what follows we will use the notation $\varphi_0^{(k)}(x)$ for the $k$-linear function defined as,
\[
\varphi_0^{(k)}(x)(w_1, \ldots, w_k) = \sum_{i_1, \ldots, i_k} \varphi_0(k)(x)(e_{i_1}, \ldots, e_{i_k}) w_i^1, \ldots, w_i^k
\]
\[
= \sum_{i_1, \ldots, i_k} \partial_{e_{i_1}} \ldots \partial_{e_{i_k}} \varphi(x) w_i^1, \ldots, w_i^k
\] (11.40)
and the shorthand \((\text{below } k - \ell \text{ and } \ell \text{ are the powers, not to be not to confused with the upper indices})\)

\[
\varphi^{(k)}_0(0)w_1^\ell w_{k-\ell}^\ell = \varphi^{(k)}_0(0)(w_1, \ldots, w_1, w_2, \ldots, w_2).
\]

(11.41)

(For \(k = 1, 2, 3\), we write \(\varphi^{(1)}_0(x) = \varphi_0(x), \varphi^{(2)}_0(x) = \varphi''_0(x), \varphi^{(3)}_0(x) = \varphi''''_0(x)\).)

We use (11.20) and (11.38) or (11.39), to obtain (recall that \(w = z + \lambda \omega\))

\[
M_{\alpha,\epsilon} = \epsilon M^{(1)}_{\alpha,\epsilon} + \epsilon^2 M^{(2)}_{\alpha,\epsilon} + \text{rem}_3,
\]

(11.42)

where \(M^{(1)}_{\alpha,\epsilon}\) and \(M^{(2)}_{\alpha,\epsilon}\) are given by

\[
M^{(1)}_{\alpha,\epsilon} := n\lambda^{-1} \nabla \varphi_0(0) \cdot \nabla + h \nabla \varphi_0(0) \cdot \omega,
\]

(11.43)

\[
M^{(2)}_{\alpha,\epsilon} := n\lambda^{-1} \varphi''_0(0) (\nabla, w) - n \varphi''_0(0) (\omega, \omega) + h \varphi''_0(0) (w, \omega)
= n\lambda^{-1} \varphi''_0(0) (\nabla, w) - (n - \lambda h) \varphi''_0(0) (\omega, \omega) + h \varphi''_0(0) (z, \omega).
\]

(11.44)

### 11.7 Appendix. Mean curvature of normal graphs over the round sphere \(S^n\)

Let \(\hat{S}\) be a fixed convex \(n\)-dimensional hypersurface in \(\mathbb{R}^{n+1}\). We consider hypersurfaces \(S\) given as normal graphs \(\theta(\hat{x}, t) = u(\hat{x}) \hat{v}(\hat{x})\), over a given hypersurface \(\hat{S}\). Here \(\hat{v}(\hat{x})\) is the outward unit normal vector to \(\hat{S}\) at \(\hat{x} \in \hat{S}\). We would like to derive an expression for the mean curvature of \(S\) in terms of the graph function \(\rho\).

To be more specific, we are interested in \(\hat{S}\) being the round sphere \(S^n\). Then a normal graph is given by

\[
\theta : \omega \mapsto \rho(\omega) \omega.
\]

(11.45)

The result is given in the next proposition.

**Proposition 11.3.** If a surface \(S\), defined by an immersion \(\theta : S^n \to \mathbb{R}^{n+1}\), is a graph (11.45) over the round sphere \(S^n\), with the graph function \(\rho : S^n \to \mathbb{R}_+\), then the mean curvature, \(H\), of \(S\) is given in terms of the graph function \(u\) by the equation

\[
H = p^{-1/2}(n - \rho^{-1} \Delta \rho) + p^{-3/2}\left[|\nabla \rho|^2 + \rho^{-1} g^{ik} g^{jk} \nabla_i \nabla_j \rho \nabla_k \rho \nabla_\ell \rho\right],
\]

(11.46)

where \(\nabla\) denotes covariant differentiation on \(S^n\), with the components \(\nabla_i\) in some local coordinates and

\[
p := \rho^2 + |\nabla \rho|^2.
\]

(11.47)

**Proof.** We do computations locally and therefore it is convenient to fix a point \(\omega \in S^n\) and choose normal coordinates at this point. Let \((e_1, \ldots, e_n)\) denote an orthonormal basis of \(T_\omega S^n \subset T_\omega \mathbb{R}^{n+1}\) in this coordinates, i.e. the metric on \(S^n\) at \(\omega \in S^n\) is \(g^\text{sphere}_{ij} = g^\text{sphere}(e_i, e_j) = \delta_{ij}\) and the Christphel symbols vanish, \(\Gamma^k_{ij} = 0\) at \(\omega \in S^n\). By \(\nabla_i\) we denote covariant differentiation on \(S^n\) w.r.to this basis (frame). First, we compute the metric \(g^{ij}\) on \(S_t\). To this end, we note that, for the map (11.45), one has

\[
\nabla_i \theta = (\nabla_i \rho) \omega + \omega e_i, \quad i = 1, \ldots, n,
\]

(11.48)
where we used the relation $W \frac{\partial \psi}{\partial \nu} = g^{ij} b_{jk} \frac{\partial \psi}{\partial \nu}$ (see (??) of Appendix ??), where $W$ is the Weigarten map, which in our situation (i.e. $b^{\text{sphere}}_{ij} = g^{\text{sphere}}_{ij} = \delta_{ij}$ and $\nu(\omega) = \omega$) gives $\nabla_i \omega = e_i$. It follows that the Riemannian metric $g$ induced on $S$ has components

$$g_{ij} = \langle \nabla_i \theta, \nabla_j \theta \rangle = \rho^2 \delta_{ij} + \nabla_i \rho \nabla_j \rho.$$

Recall that the symbol $\langle \cdot, \cdot \rangle$ denotes the pointwise Euclidean inner product of $\mathbb{R}^{n+1}$. As can be easily checked directly, the inverse matrix is

$$g^{ij} = \rho^{-2} (\delta_{ij} - \rho^{-1} \nabla_i \rho \nabla_j \rho). \quad (11.49)$$

It is easy to see that the following normalized vector field on $S$ is orthogonal to all vectors (11.48) and therefore is the outward unit normal to $M$ is

$$\nu = p^{-1/2} (\rho \omega - e_i \nabla_i \rho) = p^{-1/2} (\rho \omega - \nabla \rho). \quad (11.50)$$

Straightforward calculations show that

$$p^{1/2} \nabla_j \nu = 2 (\nabla_j \rho) \omega + \rho e_j - \nabla_j \nabla \rho + \{ \cdots \} \nu,$$

where the terms in braces are easy to compute but irrelevant for what follows. One thus finds that the matrix elements $b_{ij} = \langle \nabla_i \theta, \nabla_j \nu \rangle$ of the second fundamental form are

$$b_{ij} = p^{-1/2} (\rho^2 \delta_{ij} + 2 \nabla_i \rho \nabla_j \rho - \rho \nabla_i \nabla_j \rho). \quad (11.51)$$

Since $H = \text{tr}_g(h) = g^{ij} b_{ij}$, we use (11.49) and (11.51) and $\delta_i \delta_i = n$ to compute:

\[
H = p^{-1/2} [\delta_{ij} \delta_{ij} + 2 \rho^2 \delta_{ij} \nabla_i \rho \nabla_j \rho - \rho^{-2} \delta_{ij} \rho \nabla_i \nabla_j \rho - p^{-1} \nabla_i \rho \nabla_j \rho \delta_{ij} - 2 (\rho^2 p)^{-1} \nabla_i \rho \nabla_j \rho \nabla_i \rho \nabla_j \rho + (\rho p)^{-1} \nabla_i \rho \nabla_j \rho \nabla_i \nabla_j \rho] \\
= p^{-1/2} [n + 2 \rho^{-2} \nabla_i \rho \nabla_j \rho - \rho^{-1} \nabla_i \nabla_j \rho - p^{-1} \nabla_i \rho \nabla_j \rho - 2 (\rho^2 p)^{-1} \nabla \rho |^4 + (\rho p)^{-1} \nabla \rho |^4] \\
= p^{-1/2} [n - \rho^{-1} \Delta \rho + (\rho p)^{-1} \nabla_i \rho \nabla_j \rho \nabla_i \rho \nabla_j \rho + p^{-1} \rho^{-2} (2 p | \nabla \rho |^2 - \rho^2 | \nabla \rho |^2)] \\
= p^{-1/2} [n - \rho^{-1} \Delta \rho + \rho^{-1} p^{-1} \nabla_i \rho \nabla_j \rho \nabla_i \rho \nabla_j \rho + p^{-1} | \nabla \rho |^2].
\]

which implies (11.46).

## 12 PDEs of quantum statistics

### 12.1 Quantum statistics

We formalize the theory of quantum statistics by making the following postulates (for details see [15]):

- States: positive trace-class operators on a Hilbert space $\mathcal{H}$ (as usual, up to normalization);
- Evolution equation : $i h \frac{\partial \rho}{\partial t} = [h, \rho]$, where $h$ is a self-adjoint operator on $\mathcal{H}$;
- Observables : self-adjoint operators on $\mathcal{H}$;
- Averages : $\langle A \rangle_{\rho} := \text{Tr}(A \rho)$. 

We call the theory described above quantum statistics. Assuming $\mathcal{H} = L^2(\mathbb{R}^n)$, the last two items lead to the following expression for the probability density for the coordinates:

- $\rho(x; x)$ - probability density for coordinate $x$;

and similarly for the momenta. In particular, if $\rho = P_\psi$, then

$$\rho(x; x) = |\psi(x)|^2,$$

as should be the case according to our interpretation.

Note that the state space here is not a linear space but a positive cone in a linear space. It can be identified with the space of all positive (normalized) linear functionals $A \to \omega(A) := \text{Tr}(A\rho)$ on the space of bounded observables. Denote the spaces of bounded observables and of trace class operators on $\mathcal{H}$ as $L^\infty(\mathcal{H})$ and $L^1(\mathcal{H})$, respectively. There is a duality between density matrices and observables $\langle \rho, A \rangle = \text{Tr}(A\rho)$ for $A \in L^\infty(\mathcal{H})$ and $\rho \in L^1(\mathcal{H})$.

Quantum mechanics is a special case of this theory, and is obtained by restricting the density operators to be rank-one orthogonal projections.

Another special case of quantum statistics is probability.

### 12.2 Self-consistent approximation

Consider a system of $n$ particles in an external potential $V(x)$, interacting via a pair potential $v(x)$. Assuming the particles are identical and in the same state, in the self-consistent approximation, one replace the many body potential, $\sum_j v(x-x_j) = \int v(x-y)n_{\text{exact}}(y)dy = (v\ast n_{\text{exact}})(x)$, where $n_{\text{exact}}(y) := \sum_j \delta(x-x_j)$, created by $n-1$ particles at $x$ and affecting the remaining one, by the 'mean-field' potential $\sum_j \int v(x-y)n_\gamma(y)dy = (v\ast n_\gamma)(x)$. This leads to the time-dependent generalized Hartree or the Hartree-von Neumann equation,

$$i\frac{\partial \gamma}{\partial t} = -[h_\gamma, \gamma]$$

with

$$h_\gamma = h + v \ast n_\gamma \quad \text{and} \quad n_\gamma(x) = \gamma(x, x).$$

(12.2)

Here $h = -\Delta + V$ acting $\mathcal{H} = L^2(\mathbb{R}^n)$. We describe some basic properties of this equation.

If the external potential $V$ is zero (or independent of time), then the equation (12.1) is invariant, under spatial (or time) translations

$$T^\text{trans}_h : \gamma \mapsto U_h \gamma U_h^{-1},$$

for any $h \in \mathbb{R}^d$ (space translations) or $h \in \mathbb{R}$ (time translations), and rotations

$$T^\text{rot}_\rho : \gamma \mapsto U_\rho \gamma U_\rho^{-1},$$

(12.4)

for any $\rho \in SO(d)$. Here $U^\text{trans}_h$ and $U^\text{rot}_\rho$ are the standard translation and rotation transforms $U^\text{trans}_h : \phi(x) \mapsto \phi(x + h)$ and $U^\text{rot}_\rho : \phi(x) \mapsto \phi(\rho^{-1}x)$.

Since the equation (12.1) is a hamiltonian system (see an appendix to this section), these symmetries lead to the conservation laws. In particular, we have
• Time translation invariance $\rightarrow$ conservation of energy,

$$E(\gamma) := \text{Tr} \left( (h + \frac{1}{2} v * n_\gamma) \gamma \right)$$

(12.5)

$$= \text{Tr}(h \gamma) + \frac{1}{2} \int n_\gamma v * n_\gamma dx.$$  

(12.6)

(The half compensates for the differentiating the quadratic term.)

• Unitarity of the Schrödinger evolution (gauge invariance) and the cyclicity of the trace $\rightarrow$ conservation of number of particles (total charge),

$$\text{Tr} \gamma = \text{const.}$$  

(12.7)

• Unitarity of the Schrödinger evolution $\rightarrow$ conservation of positivity: if $\gamma_0 \geq 0$, then for all times $\gamma \geq 0$,

$$\gamma_0 \geq 0 \quad \Rightarrow \quad \gamma \geq 0.$$  

(12.8)

• Unitarity of the Schrödinger evolution $\rightarrow$ conservation of the eigenvalues $\gamma_j$'s of $\gamma$,

$$\partial_t \gamma_j = 0.$$  

(12.9)

Indeed, for (12.7), we have

$$-i\partial_t \text{Tr} \gamma = \text{Tr} \left( [h_g, \gamma] \right) = 0.$$  

(12.10)

To prove (12.8), we let $U(\gamma)$ be the evolution generated by the self-adjoint, time-dependent operator $h_\gamma$. We can rewrite the equation (12.1) as

$$\gamma = U(\gamma)^* \gamma_0 U(\gamma).$$  

(12.11)

Then the positivity of $\gamma_0$ implies the positivity of $\gamma$. ((12.11) implies also (12.7).)

Eq. (12.9) follows from the isospectral properties of (12.1): $\gamma$ can be written in the form (12.11). (A different proof is given as follows. Using that $\gamma_j = \langle \phi_j, \gamma \phi_j \rangle$, we find $\partial_t \gamma_j = \langle \partial_t \phi_j, \gamma \phi_j \rangle + \langle \phi_j, \gamma \partial_t \phi_j \rangle + \langle \phi_j, [\partial_t \gamma, \phi_j] \rangle$. The first two terms give $\gamma_j \langle \partial_t \phi_j, \phi_j \rangle + \gamma_j \langle \phi_j, \partial_t \phi_j \rangle = \gamma_j \partial_t \langle \phi_j, \phi_j \rangle = 0$, while the third terms gives $\langle \phi_j, i[h_\gamma, \gamma] \phi_j \rangle = 0$. This proves (12.9).)

**Proposition 9.** $\gamma$ satisfies the equation (12.1) iff the eigenfunctions $\phi_j$ of $\gamma$ satisfy the equations

$$i\partial_t \phi_j = h_\gamma \phi_j,$$  

(12.12)

where $h_\gamma$ is given in (12.2) and can be expressed in terms of $\phi_j$'s using

$$n_\gamma(x) = \gamma(x, x) = \sum_j |\phi_j(x)|^2.$$  

(12.13)

**Proof.** To prove (12.12), we note that, since the spectrum of $\gamma$ is discrete, we can write it in terms of the projections on the eigenfunctions its $\phi_j$ (associated with the eigenvalues $\gamma_j$):

$$\gamma = \sum_j \gamma_j P_{\phi_j}.$$  

(12.14)
We plug (12.14) into \( i \frac{\partial \gamma}{\partial t} + [h, \gamma] \) and use (12.9), to obtain
\[
i \frac{\partial \gamma}{\partial t} + [h, \gamma] = \sum \gamma_j \langle (i \partial_t - h) \phi_j | \phi_j \rangle - \langle |\phi_j \rangle (i \partial_t - h) \phi_j \rangle.
\] (12.15)

Hence, if (12.12) are satisfied then so is (12.1). On the other hand, if (12.1) is satisfied, then multiplying (12.15) scalarly by \( \phi_k \) we obtain
\[
\gamma_k (i \partial_t - h) \phi_k = \sum \gamma_j \phi_j \langle (i \partial_t - h) \phi_j, \phi_k \rangle
= \sum \gamma_j \phi_j \langle (i \partial_t - h) \phi_k \rangle = \gamma (i \partial_t - h) \phi_k.
\]

Assuming the eigenvalues \( \gamma_k \) are non-degenerate, this implies that there are real numbers \( \mu_k \) s.t. \( (i \partial_t - h) \phi_k = \mu_k \phi_k \) and therefore \( (i \partial_t - h)(e^{\mu_k t} \phi_k) = 0 \).

**Remark.** Solutions to the equations (12.12) are parametrized by the numbers \( \gamma_j \)'s.

### 12.3 Equilibrium states and entropy

Clearly, if, for some function \( f \), the operator \( \gamma \) satisfies the equation \( \gamma = f(h, \gamma) \), then it also satisfies (12.1), namely it is a static solution to (12.1). However, not all such solutions are of physical interest. We select those which are using the entropy principle, namely, requiring that they minimize the energy for the fixed entropy (and the number of particles).

We have already defined the energy functional \( E(\gamma) \) on trace class operators \( \gamma \), see (12.5). Now, we take the entropy of \( \gamma \) to be of the form
\[
S(\gamma) = \text{Tr} g(\gamma),
\] (12.16)
where
\[
g(x) := -\frac{1}{2} (x \ln x + (1 - x) \ln(1 - x)).
\] (12.17)

We are interested in minimizing the internal energy \( E(\gamma) \) on the set \( S_\star := \{0 \leq \gamma \leq 1\} \cap \{\text{the entropy, } S(\gamma), \text{ and the number of particles, } N(\gamma) := \text{Tr } \gamma, \text{ are fixed}\} \). As usual, we define the free energy on the convex set \( S := \{0 \leq \gamma \leq 1\} \) as
\[
F_{T \mu}(\gamma) = E(\gamma) - TS(\gamma) - \mu N(\gamma),
\] (12.18)
fixed the chemical potential, \( \mu \), rather than the number of particles, \( N(\gamma) \). We define \( F_{T \mu}(\gamma) \) on the Sobolev space space \( H^{1,1}(H) \) defined as follows. Let \( b := (c \mathbf{1} + h)^{1/2} \), where \( c > 0 \) is such that \( c \mathbf{1} + h \geq 1 \). Define the Sobolev space \( H^{s,1}(\mathcal{H}) \) as
\[
H^{s,1}(\mathcal{H}) := \{ \gamma \in L^1(\mathcal{H}) : b^{s/2} \gamma b^{s/2} \in L^1(\mathcal{H}) \}.
\] (12.19)

Define \( h_{\gamma \mu} := h_\gamma - \mu \). We begin with the following lemma proven in Appendix 12.6,

**Lemma 10.** Minimizers, \( \gamma \), of the internal energy \( E(\gamma) \) on the convex set \( S \), with \( S(\gamma) \) and \( N(\gamma) \) fixed, are critical points of \( F_{T \mu}(\gamma) \), i.e. they satisfy the Euler-Lagrange equations
\[
d_\gamma F_{T \mu}(\gamma) = 0,
\] (12.20)
for some \( T \) and \( \mu \) (the latter are determined by fixing \( S(\gamma) \) and \( \text{Tr}(\gamma) \)).
For $S$ given in (12.16), the equation (12.20) becomes
\[ \gamma = g^\# \left( \frac{1}{T} h_{\gamma \mu} \right), \] (12.21)
where $g^\#(h) := (g')^{-1}(h)$. Indeed, the Gâteaux derivatives, $\partial_{\gamma} F_T(\gamma)$, is given by (see Appendix 12.6)
\[ \partial_{\gamma} F_T(\gamma) = h_{\gamma \mu} - T g'(\gamma), \] (12.22)
where $h_{\gamma \mu} = h_{\gamma} - \mu$. This together with (12.20) gives
\[ h_{\gamma \mu} = T g'(\gamma), \] (12.23)
which upon inverting the function $g'$ gives (12.21).

Note that for $g(x) := -\frac{1}{2} (x \ln x + (1 - x) \ln(1 - x))$, we have
\[ g'(x) = -\frac{1}{2} \ln \frac{x}{1 - x} \quad \text{and} \quad g^\#(h) = (1 + e^{2h})^{-1}. \] (12.24)

12.4 Local and global existence

Let $Y := H^{1,1}(\mathcal{H})$, where $H^{s,1}(\mathcal{H})$ is the Sobolev space space defined in (12.19). Recall, that we think of $\gamma$ as a path $\gamma : t \in I \rightarrow u(t) \in Y$ and we are looking for weak solutions to (12.1), i.e. for solutions of this equation in the integral form, in the space $C([0, T], Y)$, for some $T > 0$. Define the ball $B_{R,T} := \{ \gamma \in C([0, T], Y) : \|\gamma\|_{C([0, T], Y)} \leq R \}$. We have the following result (cf. J. M. Chadam, J. M. Chadam and R. T. Glasssey, A. Bove, G. Da Prato and G. Fano ([5], [6], [4]).

**Theorem 12.1.** (i) Assume $v \in L^2 \cap L^1$. There are functions $L_R$ and $M_R$, $R > 0$, s.t. for $\gamma_0 \in Y$, $R > 2K \|\gamma_0\|_Y$ and $T < (K L_R)^{-1}$, $R/2K M_R$, the equation (12.1) has a unique weak solution $\gamma \in B_{R,T}$. The solution $\gamma$ depends continuously on the initial condition $\gamma_0$. Furthermore, either the solution is global in time or blows up in $Y$ in a finite time (i.e. either $\|\gamma(t)\|_Y < \infty$, $\forall t$, or $\|\gamma(t)\|_Y < \infty$ for $t < t_*$ and $\|\gamma(t)\|_Y \rightarrow \infty$ as $t \rightarrow t_*$ for some $t_* < \infty$).

(ii) If in addition $v*$ is positive definite, then the solutions are global.

**Proof.** We will use Theorem 2.3. Note that the equation (12.1) is in the form (2.16) with the linear operator $A$ acting on density operators as $A(\rho) := i[h, \rho]$ and $f(\gamma) := [v* n_\gamma, \gamma]$. We have to check that the conditions (2.17) - (2.19) are satisfied.

The operator $A$ is an operator of adjoint representation and it generates the one parameter group $e^{At} = e^{ith} e^{-ith}$, which, by the cyclicity of the trace, unitarity of $e^{-ith}$ and the fact that $b$ commutes with $e^{-ith}$, satisfies (2.17), for $K = 1$.

Next, it is straightforward to show that $f(\gamma)$ satisfies (2.18) - (2.19) for some $M_R, L_R < \infty$. Then Theorem 2.3 implies the first statement of the theorem.

To prove that the solutions are global one uses the conservation of the energy and number of particles.

12.5 Existence of ground states and Gibbs states

(needs checking and filling in details)
Theorem 12.2. Assume \( v \in \mathbb{R}^{k} \) and \( g \) is strictly convex and satisfies \( \gamma \) and \( g_{*} \geq 0 \) \((g_{*} \text{ is the Legandre transform of } g)\). Then the equation (12.1) has equilibrium states satisfying (12.21) (see Lemma 10).

Proof. We can minimize the free energy functional (12.25) on the convex set \( S := \{ \gamma \in H^{1,1}(\mathcal{H}) : 0 \leq \gamma \leq 1 \} \), where \( H^{1,1}(\mathcal{H}) \) is the Sobolev space space defined in (12.19), directly. However, we prefer to follow, under additional condition that

\[
v_{*} = \bar{v}(-i\nabla) = (-\Delta)^{-1},\]

an elegant proof of Markowich, Rein and Wolansky ([21]) In this proof we pass from the functional \( F_{Tg}(\gamma) \) to the dual one

\[
\Phi_{Tg}(V) = -\inf_{\gamma} F_{Tg}(\gamma, V),
\]

where

\[
F_{Tg}(\gamma, V) := \text{Tr} \left( hV\gamma - T \text{Tr} g(\gamma) - \mu \text{Tr} \gamma - \frac{1}{2} \int |\nabla V|^2 \right),
\]

with \( hV := h + V \). Notice that \( F(\gamma, v_{\gamma}) = F_{Tg}(\gamma) \), where \( v_{\gamma} := v_{*} n_{\gamma} \). Using that \( d_{\gamma} F_{Tg}(\gamma, V) = hV - Tg'(\gamma) - \mu \) and \( \left[ \text{Tr} \left( (hV - \mu)\gamma - T \text{Tr} g(\gamma) \right) \right]_{hV - \mu - Tg'(\gamma) = 0} = Tg_{*}(\frac{1}{T}(hV - \mu)) \), we compute

\[
\Phi_{Tg}(V) = \frac{1}{2} \int |\nabla V|^2 - T \text{Tr} g_{*}(\frac{1}{T}(hV - \mu)).
\]

Now, one can show easily that

- If \( V_{*} \) is a critical point of \( \Phi_{Tg}(V) \), then \( g_{*}(\frac{1}{T}(hV_{*} - \mu)) \) is a critical point of \( F_{Tg}(\gamma) \). (Indeed, since \( d_{V}\Phi_{Tg}(V) = -\Delta V - n_{\gamma} \), where \( \gamma := g_{*}(\frac{1}{T}(hV - \mu)) \), a critical point of \( \Phi_{Tg}(V) \) satisfies \( V = (-\Delta)^{-1} n_{\gamma} =: V_{\gamma} \), where \( \gamma := g_{*}(\frac{1}{T}(hV - \mu)) \), or \( \gamma := g_{*}(\frac{1}{T}(hV - \mu)) \) satisfies \( \gamma := g_{*}(\frac{1}{T}(hV_{*} - \mu)) \), which is (12.21).)

- The dual functional \( \Phi_{Tg}(V) \) is coercive and weakly lower semi-continuous and therefore has a minimizer. Moreover, it is strictly convex and so the minimizer is unique.

The last two statements imply the desired result. \( \square \)

12.6 Appendix: Proof of Lemma 10

Proof of Lemma 10. To prove (12.21), we observe that if \( \gamma \in \mathcal{S} \) is a minimizer, then for any \( \gamma' \), we have \( E((1 - s)\gamma + s\gamma') - E(\gamma) \geq 0 \), provided \( S((1 - s)\gamma + s\gamma') - S(\gamma) = 0 \) and \( \text{Tr}((1 - s)\gamma + s\gamma') - \text{Tr}(\gamma) = 0 \). Dividing this by \( s > 0 \) and taking \( s \to 0 \), we obtain \( E'(\gamma)(\gamma' - \gamma) \geq 0 \) for any \( \gamma' \), satisfying \( S'(\gamma)(\gamma' - \gamma) = 0 \) and \( \text{Tr}(\gamma' - \gamma) = 0 \). Hence \( \gamma' = \gamma \) minimizes \( E'(\gamma)(\gamma' - \gamma) \) on the set \( \mathcal{S} \), provided \( S'(\gamma)(\gamma' - \gamma) = 0 \) and \( \text{Tr}(\gamma' - \gamma) = 0 \). Since \( E'(\gamma)(\gamma' - \gamma) \) is a linear function of \( \gamma' \), this is possible iff \( \gamma \) satisfies

\[
E'(\gamma) - T S'(\gamma) - \mu N'(\gamma) = 0,
\]

for some \( T \) and \( \mu \). On the other hand, by the definition (12.25), we have

\[
F'_{T}(\gamma, a)\Phi = [E'(\gamma, a) - T S'(\gamma) - \mu N'(\gamma)]\Phi,
\]

where
which together with (12.28) gives (12.20).

To prove (12.20), we first compute $\partial_s g(\gamma + s\Phi)|_{s=0}$. To this end, we recall that $S(\gamma) = \text{Tr} g(\gamma)$ and write $g(\gamma)$ as the Cauchy integral $g(\gamma) = \frac{1}{2\pi} \int dz g(z)(\gamma - z)^{-1}$. It is straightforward to compute
\[ \partial_s g(\gamma + s\Phi)|_{s=0} = -\frac{1}{2\pi} \int dz g(z)(\gamma - z)^{-1}\Phi(\gamma - z)^{-1}, \tag{12.30} \]
which gives
\[ \partial_s S(\gamma + s\Phi)|_{s=0} = -\frac{1}{2\pi} \int dz g(z) \text{Tr}((\gamma - z)^{-1}\Phi(\gamma - z)^{-1}) \]
\[ = -\frac{1}{2\pi} \int dz f(z) \text{Tr}((\gamma - z)^{-2}\Phi), \]
which implies
\[ S'(\gamma)\Phi = \text{Tr}(g'(\gamma)\Phi). \tag{12.31} \]
Next, the relation (12.5) implies that
\[ E'(\gamma)\Phi = \text{Tr}((h_{\gamma\mu}|_{\mu=0})\Phi), \tag{12.32} \]
where $h_{\gamma\mu}$ was defined above. Eq. (12.29), together with (12.31) and (12.32) and the computation $N'(\gamma) = 1$, gives (12.22). \qed

12.7 Appendix: Hamiltonian formulation

Following [11], as canonical variables, we take $\kappa$ and $\kappa^*$, where $\kappa\kappa^* = \gamma$. Then the Hamiltonian and Poisson bracket are defined as
\[ H(\kappa, \kappa^*) := \text{Tr}(\kappa^* (h + \frac{1}{2} v * n_\kappa)\kappa), \tag{12.33} \]
with $n_\kappa(x) := n_{\kappa^*}\kappa(x) := \int |\kappa(x, y)|^2 dy$, and
\[ \{A, B\}(\kappa, \kappa^*) = -i \text{Tr}(\partial_\kappa A \partial_{\kappa^*} B - \partial_{\kappa^*} B \partial_\kappa A) (\kappa, \kappa^*). \tag{12.34} \]

Another way to introduce a hamiltonian structure is as follows. For a functional $A(\rho)$ we define the gradient operator, $\partial_\rho A(\rho)$, in the trace metric by the equation $\text{Tr}(\partial_\rho A(\rho)\xi) = \partial_\rho A(\rho + s\xi)|_{s=0}$. On the space of classical field observables we define the Poisson bracket by
\[ \{A(\rho), B(\rho)\} = -i \text{Tr}(\partial_\rho A(\rho)\partial_\rho B(\rho) - \partial_\rho B(\rho)\partial_\rho A(\rho)). \tag{12.35} \]
The Jacobi identity for (12.35) is proven in Appendix ??.

We observe that
\[ \{A(\rho), B(\rho)\}|_{\rho=\psi} = i \int (\partial_\psi A(\psi)A\partial_\psi B(\psi) - \partial_\psi A(\psi)A\partial_\psi B(\psi))(\psi, \overline{\psi}) d\psi. \tag{12.36} \]
The r.h.s. is the standard Poisson bracket, $\{A, B\}(\psi, \overline{\psi})$, for the Hartree equation (see [?]). Indeed, $\partial_\rho A(\rho)$ and $\partial_\rho B(\rho)$ are operators on $L^2(\mathbb{R}^3)$ (1-particle observables) and therefore
\[ \text{Tr}(\partial_\rho B(\psi)P_\psi A(\rho)P_\psi) = \langle (\partial_\rho A(\psi))^* \psi, (\partial_\rho B(\psi)) \psi \rangle. \]
Since, as it is easy to see, \((\partial_\rho B)(P_\psi)\psi = \partial_{\psi(x)} B(\psi, \bar{\psi})\) and \((\partial_\rho A)(P_\psi)^*\psi = \partial_{\psi(x)} A(\psi, \bar{\psi})\), this gives \(\text{Tr}((\partial_\rho B)(P_\psi)P_\psi(\partial_\rho A)(P_\psi)) = \int \partial_{\psi(x)} A_{\psi(x)} B\), which implies the desired relation.

We define the Hamiltonian functional on \(L^1(\mathcal{H})\) as

\[ H(\rho) := \text{Tr}(h\rho\rho) + \frac{1}{2} \int n_\rho v \ast n_\rho dx, \quad (12.37) \]

The resulting Hamilton equation,

\[ \partial_t \rho = \{H(\rho), \rho\}, \quad (12.38) \]

is exactly the Hartree-von Neumann equation considered above. Indeed, this fact follows from the equation

\[ \{H(\rho), \rho\} = -i [h_\rho, \rho], \quad (12.39) \]

where \(h_\rho := -\Delta + V(x) + (v \ast n_\rho)\). To show the latter equation we use the definition of the Poisson bracket and the relation \(\text{Tr}(\partial_{\rho\psi} \xi) = \xi\), which follows from the definition of \(\partial_\rho A(\rho)\) above, to obtain \(\{H, \rho\} = -i (\partial_\rho H(\rho)\rho - \rho \partial_\rho H(\rho))\). Next, computing \(\partial_\rho H(\rho)\), we conclude that \(\{H, \rho\}\) is equal to the r.h.s. of (12.39).

**Remark 2.** Equation (12.38) should be understood in the weak sense: for all \(a \in A_1\),

\[ \partial_t \text{Tr}(a\rho) = \text{Tr}(a \{H(\rho), \rho\}). \quad (12.40) \]

Using the linearity of the Poisson brackets in the second factor, we obtain

\[ \text{Tr}(a\{H, \rho\}) = \{H(\rho), \text{Tr}(a\rho)\}. \quad (12.41) \]

Thus equation (12.38) is equivalent to the equation

\[ \partial_t \text{Tr}(a\rho) = \{H(\rho), \text{Tr}(a\rho)\}. \quad (12.42) \]

The Hamiltonian (12.37) and the Poisson brackets (12.35) generate the Hartree-von Neumann flow \(\varphi_t\) on one-particle density matrices. This flow induces the flow on generalized classical observables:

\[ A(\rho) \to A(\varphi_t(\rho)). \quad (12.43) \]

### 12.8 Appendix: Hilbert Space Approach

Quantum statistical dynamics can be put into a Hilbert space framework as follows. Consider the space \(\mathcal{H}_{HS}\) of Hilbert-Schmidt operators acting on the Hilbert space \(\mathcal{H}\). These are the bounded operators, \(K\), such that \(K^*K\) is trace-class (see Section ??). There is an inner-product on \(\mathcal{H}_{HS}\), defined by

\[ \langle F, K \rangle := \text{Tr}(F^*K). \quad (12.44) \]

**Exercise 2.** Show that (12.44) defines an inner-product.

This inner-product makes \(\mathcal{H}_{HS}\) into a Hilbert space (see [? 27]). On the space \(\mathcal{H}_{HS}\), we define an operator \(L\) via

\[ LK = \frac{1}{\hbar}[H, K], \]
where $H$ is the Schrödinger operator of interest. The operator $L$ is symmetric. Indeed,

$$h\langle F, LK \rangle = \text{Tr}(F^*[H, K]).$$

Using the cyclicity of the trace, the right hand side can be written as

$$\text{Tr}(F^*HK - F^*KH) = \text{Tr}(F^*HK - HF^*K) = \text{Tr}([F^*, H]K) = \text{Tr}([H, F]^*K) = h\langle LF, K \rangle$$

and so $\langle F, LK \rangle = \langle LF, K \rangle$ as claimed. In fact, for self-adjoint Schrödinger operators, $H$, of interest, $L$ is also self-adjoint.

Now consider the Landau-von Neumann equation

$$i\frac{\partial k}{\partial t} = Lk \quad (12.45)$$

where $k = k(t) \in \mathcal{H}_{HS}$. Since $k(t)$ is a family of Hilbert-Schmidt operators, the operators $\rho(t) = k^*(t)k(t)$ are trace-class, positive operators. Because $k(t)$ satisfies (12.45), the operators $\rho(t)$ obey the equation

$$i\frac{\partial \rho}{\partial t} = L\rho = \frac{1}{\hbar}[H, \rho]. \quad (12.46)$$

If $\rho$ is normalized – i.e., $\text{Tr} \rho = 1$ – then $\rho$ is a density matrix satisfying the Landau-von Neumann equation (12.46). The stationary solutions to (12.45) are just eigenvectors of the operator $L$ with eigenvalue zero.

To conclude, we have shown that instead of density matrices, we can consider Hilbert-Schmidt operators, which belong to a Hilbert space, and dynamical equations which are of the same form as for density matrices. Moreover, these equations can be written in the Schrödinger-type form (12.45), with self-adjoint operator $L$, sometimes called the Liouville operator.

## A Fourier transform

In this section, we describe one of the most powerful tools in analysis – the Fourier transform. This transform allows us to analyze a fine structure of functions and to solve differential equations. The Fourier transform takes functions of time to functions of frequencies, functions of coordinates to functions of momenta, and vice versa.

### A.1 Definitions and properties

Initially, we define the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}$:

$$\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) : \langle x \rangle^N|\partial^\alpha f(x)| \text{ is bounded} \ \forall N \text{ and } \forall \alpha \}, \quad (A.1)$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_j$ non-negative integers, $\partial^\alpha := \prod_{j=1}^n \partial_{x_j}^{\alpha_j}$ and $|\alpha| = \sum_{i=1}^n \alpha_j$. On $\mathcal{S}$, we define the Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ by

$$\hat{f}(k) := (2\pi)^{-n/2} \int f(x)e^{-ik \cdot x}dx. \quad (A.2)$$

Define also the inverse Fourier transform of $f(k)$ as

$$\hat{f}(x) := (2\pi)^{-n/2} \int f(k)e^{ik \cdot x}dk. \quad (A.3)$$

Some key properties of the Fourier transform are collected in the following
**Theorem A.1.** Assume $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then we have:

(a) $(-i\partial)^\alpha f \mapsto k^\alpha \hat{f}$, and $x^\alpha f \mapsto (-i\partial)^\alpha \hat{f}$,

(b) $fg \mapsto (2\pi)^{-n/2} \hat{f} \ast \hat{g}$, and $f \ast g \mapsto (2\pi)^{n/2} \hat{f} \hat{g}$,

(c) $(\hat{f})^\sim = f = (\hat{f})^\sim$,

(d) $\hat{f} = \hat{\bar{f}}$,

(e) $\int \hat{f} g = \int f \hat{g}$,

(f) $\int \hat{f} g = \int \hat{f} g$.

Properties (a) - (f) hold (possibly, with signs changed in (a)) also when $^\sim$ is replaced by $^\ast$.

We give a formal proof. Integrating by parts, we compute

$$
-i(\partial_x f)^\sim(k) = (2\pi)^{-n/2} \int (-i) \partial_x f(x) e^{-ik \cdot x} dx
$$

$$
= (2\pi)^{-n/2} \int f(x) i \partial_x e^{-ik \cdot x} dx
$$

$$
= k_j \hat{f}(k).
$$

**Exercise 16.** Prove the remaining relations in (a).

Now we prove the second relation in (b). Using $e^{-ik \cdot x} = e^{-ik \cdot (x-y)} e^{-ik \cdot y}$ and changing the variable of integration as $x' = x - y$, we obtain

$$
\hat{f} \ast g(k) := (2\pi)^{-n/2} \int e^{-ik \cdot x} \left( \int f(x-y) g(y) dy dx \right)
$$

$$
= (2\pi)^{-n/2} \int \left( \int e^{-ik \cdot (x-y)} f(x-y) dx \right) e^{-ik \cdot y} g(y) dy
$$

$$
= (2\pi)^{n/2} \hat{f}(k') \hat{g}(k).
$$

**Exercise 17.** Prove the first relation in (b) from the second one and (c).

The proof of (c) is more subtle. We use an approximation of unity $\varphi_t(x) = t^{-n} \varphi(x/t)$ and compute $\varphi_t \ast (\hat{f})^\sim$. Let us define $\varphi^x(y) := \varphi(x-y)$. Using property (b), we find

$$
\varphi_t \ast (\hat{f}) = \int \varphi_t^y \cdot (\hat{f}) dy = \int (\varphi_t^y)^\sim \hat{f} dy = \int ((\varphi_t^y)^\sim)^\sim f dy.
$$

**Exercise 18.** Show (formally, without justification of the interchange of the order of integration etc.) that

$$
((\varphi_t^x)^\sim)^\sim = (\varphi_t^x)^\sim = t^{-n} (\varphi_t^x)^\sim \left( \frac{x-y}{t} \right).
$$

Thus we have

$$
\varphi_t \ast (\hat{f})^\sim = ((\varphi_t^x)^\sim)_{t \ast f}
$$

We can choose $\varphi$ such that $(\varphi_t^x)^\sim \in L^1$, and $\int (\varphi_t^x)^\sim dx = 1$. Indeed, take e.g. $\varphi(x) = (4\pi)^{-n/2} e^{-|x|^2}$ and use the fact that $((e^{-|x|^2})^\sim) = e^{-|x|^2}$. With this in mind, we take the limit $t \to 0$ in (A.4) and use the properties of the approximation of identity to get

$$
\varphi_t \ast (\hat{f})^\sim \to (\hat{f})^\sim \quad \text{and} \quad ((\varphi_t^x)^\sim)_{t \ast f} \to f \quad \text{as} \ t \to 0
$$

to obtain $(\hat{f})^\sim = f$. Similarly one shows that $(\hat{f})^\sim = f.$
Exercise 19. Prove the relations in (d) – (f).

By definition of the Dirac $\delta$–function, we obtain
\[
\mathcal{F} : \delta(x - x_0) \to (2\pi)^{-n/2} e^{-ikx_0}.
\]
Hence the property (c) implies that $\mathcal{F}^{-1} : (2\pi)^{-n/2} e^{-ikx_0} \to \delta(x - x_0)$, and, by taking the complex conjugate (remember that $\mathcal{F}(f) = \mathcal{F}^*(f) = \mathcal{F}^{-1}(\hat{f})$), we arrive at
\[
\mathcal{F} : (2\pi)^{-n/2} e^{-ikx_0} \mapsto \delta(k - k_0).
\] (A.5)

Exercise 20. Using (A.5), prove formally that $(\hat{f})^\sim = f (\hat{f})^\sim$, and that $(fg)^\sim = (2\pi)^{-n/2} \hat{f} \ast \hat{g}$.

Statement (f) is called the Plancherel Theorem. The *adjoint* $\mathcal{F}^*$ of the Fourier transform is defined by $\langle \mathcal{F}^*u, v \rangle = \langle u, \mathcal{F}v \rangle$ for all $u, v \in \mathcal{S}(\mathbb{R}^n)$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $L^2(\mathbb{R}^n)$. Then (d) and (e) show that $\mathcal{F}^* = \mathcal{F}^{-1}$. This together with (e) implies that $\mathcal{F}\mathcal{F}^* = \text{id} = \mathcal{F}^*\mathcal{F}$ on $\mathcal{S}$, which is a restatement of the Plancherel theorem.

Corollary 11. $\mathcal{F}$ extends to a unitary operator on $L^2$, i.e. to a bounded operator satisfying $\mathcal{F}^* = \mathcal{F}^{-1}$.

The next theorem gives the important example of the Fourier transform - the Fourier transform of a Gaussian:

Theorem A.2. Let $A$ be a $n \times n$ matrix s.t. $\text{Re}A := (A + A^*)/2$ is positive definite (i.e. $x \cdot \text{Re}A x > 0$ if $x \neq 0$). Then we have
\[
\mathcal{F} : e^{-x \cdot Ax/2} \mapsto (\det A)^{-1/2} e^{-k \cdot A^{-1}k/2}
\] (A.6)

Proof. We prove the theorem only for positive definite matrices. If $A$ is positive definite (i.e. if $x \cdot Ax > 0$ for $x \neq 0$), then there is an orthogonal matrix $U$ (i.e. $U$ is real and $UU^T = U^T U = \text{id}$) s.t. $\overline{A} := U^T AU$ is diagonal, say $\overline{A} = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Letting $x = Uy$ and noticing that $x \cdot Ax = y \cdot U^T AU y$, and that det $U = 1$, we get
\[
\int e^{-x \cdot Ax/2} e^{-ik \cdot x} \, dx = \int e^{-y \overline{A} y/2} e^{ik' \cdot y} \, dy = \prod_{1}^{n} \int e^{-\lambda_j y_j^2/2} e^{ik_j y_j} \, dy_j,
\]
where $k' = U^T k$, and we have used $k \cdot U y = U^T k \cdot x$. It is left as an exercise to show that for $n = 1$,
\[
\mathcal{F} : e^{-\lambda x^2/2} \mapsto \chi^{-1/2} e^{-k^2/2\lambda}.
\] (A.7)

The last two relations imply the desired statement. \hfill \square

Exercise 21. Show (A.7).

The function $e^{-x \cdot Ax}$ is called a Gaussian. It is one of the most common functions in applications. There is another important function whose Fourier transform can be explicitly computed:
\[
\mathcal{F} : |x|^{-\alpha} \mapsto \left\{
\begin{array}{ll}
C_{n,\alpha} |k|^{-n+\alpha} & \text{if } \alpha \neq n, \\
C_{n,\alpha} \ln |k| & \text{if } \alpha = n.
\end{array}
\right.
\] (A.8)

The coefficients are given for $\alpha = 2$ by
\[
C_{n,2} = \left\{
\begin{array}{ll}
((2 - n)\sigma_{n-1})^{-1} & \text{for } n \neq 2, \\
-(\sigma_{n-1})^{-1} = -(2\pi)^{-1} & \text{for } n = 2,
\end{array}
\right.
\] (A.9)
where $\sigma_n$ is the volume of the $n$-dimensional unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. One can easily deduce formula (A.8) modulo the constants (A.9). Indeed, since $|x|^{-\alpha}$ is rotationally invariant, then so is its Fourier transform. Also, since $|x|^{-\alpha}$ is homogeneous of degree $-\alpha$, then its Fourier transform is homogeneous of degree $-n + \alpha$. Hence (A.8) follows. Though it is easy to compute the Fourier transform of $|x|^{-\alpha}$, it is not easy to justify it. Indeed, the function $|x|^{-\alpha}$ is rather singular and definitely does not belong to $S(\mathbb{R}^n)$.

**Exercise 22.** For $n = 1$, compute the Fourier transform of the characteristic function $\chi_{(-a,a)}(x)$, using definition (A.2).

As an example we show the following relation

$$((|k|^2 + \mu^2)^{-1}) = \frac{e^{-\mu|x|}}{4\pi|x|}, \quad \text{for } n = 3, \tag{A.10}$$

which appears often in applications. Indeed, let $f(k) = (|k|^2 + \mu^2)^{-1}$ and $n = 3$. We have

$$\hat{f}(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \frac{e^{ik \cdot x}}{|k|^2 + \mu^2} \, dk = \lim_{R \to \infty} (2\pi)^{-1/2} \int_{-R}^{R} \frac{e^{ir|x|}}{r^2 + \mu^2} \, dr \, dv$$

$$= \lim_{R \to \infty} (2\pi)^{-1/2} \int_{-R}^{R} \frac{e^{ir|x|}}{r^2 + \mu^2} \, dr.$$

In the second equality, we change to spherical coordinates $v = \cos \phi = \frac{k \cdot x}{|k| |x|}$ with $r = |k|$. Now, the last integral can be computed by changing to a contour integral over a rectangle in the upper half complex plane with top vertices at $-R + i\sqrt{R}$ and $R + i\sqrt{R}$ and taking the limit as $R \to \infty$. By the Residue theory we have (A.10) (show this).

**Exercise 23.** Let $f_h(x) := f(x - h)$ for $h \in \mathbb{R}^n$ and $f^{(\lambda)}(x) := \lambda^{n/2} f(\lambda x)$ for $\lambda \in \mathbb{R}_+$ be the translation and dilation of $f(x)$. Show that

$$\mathcal{F} : f_h(x) \mapsto e^{-ik \cdot h} \hat{f}(k), \quad \tag{A.11}$$

and

$$\mathcal{F} : f^{(\lambda)}(x) \mapsto \hat{f}^{(1/\lambda)}(k). \quad \tag{A.12}$$

Define the unitary operators $T_h : f(x) \mapsto f(x - h)$ (the operator of translation by $h$) and $S_{\lambda} : f(x) \mapsto \lambda^{n/2} f(\lambda x)$ (the operator of dilation by $\lambda$). Then (A.11) and (A.12) imply

$$\mathcal{F} \circ T_h = M_{e^{-ik \cdot h}} \circ \mathcal{F},$$

where we recall that $M_{e^{-ik \cdot h}}$ is the operator of multiplication by $e^{-ik \cdot h}$, and

$$\mathcal{F} \circ S_{\lambda} = S_{1/\lambda} \circ \mathcal{F}.$$
Exercise 24. Prove inequality (A.13) (Hint: in the case \( n = 1 \), notice that
\[
\langle f, \{(−i\partial_x)x−x(−i\partial_x)\}f \rangle = −i\|f\|^2,
\]
where the scalar product and the norm are in \( L^2(\mathbb{R}) \). On the other hand,
\[
\langle f, \{(−i\partial_x)x−x(−i\partial_x)\}f \rangle = 2i\Re \langle (−i\partial_x)f, xf \rangle.
\]
Use these two observations to show that
\[
\|f\|^2 ≤ 2\|(−i\partial_x)f\|\|xf\|,
\]
and finish the proof by invoking Plancherel’s theorem).

Let \( w \in L^2(\mathbb{R}^n) \) be a given function. Then the integral
\[
(2\pi)^{-n/2} \int w(x−y)f(x)e^{-ik·x}d^n x \tag{A.14}
\]
is called the windowed Fourier transform of \( f \) (with the window function \( w \)). In signal analysis, i.e. for \( n = 1 \), one often uses the Gaussian \((2\pi)^{-1/2}e^{-x^2/(2α)} \) for \( w \). In this case, the transform (A.14) is called the Gabor transform.

For more details see [19], Chapter 5.

A.2 Applications of Fourier transform to partial differential equations

Our goal in this section is to apply the Fourier transform in order to solve elementary but very basic partial differential equations (PDE’s).

The Poisson equation on \( \mathbb{R}^n \):
\[
−\Delta u = f, \tag{A.15}
\]
where \( u : \mathbb{R}^n → \mathbb{R} \) is an unknown function, \( f : \mathbb{R}^n → \mathbb{R} \) is a given function, and \( \Delta \) is the Laplace operator (the Laplacian):
\[
\Delta u := \sum_{j=1}^n \frac{∂^2 u}{∂x_j^2}.
\]
The Poisson equation first appeared in the problem of determining the electric potential \( u(x) \), created by a given charge distribution \( ρ(x) = f(x)/(4π) \). Since then, it came up in various fields of mathematics, physics, engineering, chemistry, biology and economics.

In order to solve the Poisson equation, we apply the Fourier transform to both sides of (A.15) to obtain:
\[
|k|^2 \hat{u}(k) = \hat{f}(k).
\]
This equation can be easily solved: \( \hat{u} = \hat{f}/|k|^2 \). We can now apply the inverse Fourier transform to the last equality to get
\[
u = \hat{g} * f, \quad \text{where} \quad g(k) = |k|^{-2}. \tag{A.16}
\]
But the inverse Fourier transform of \( g(k) = |k|^{-2} \) is known:
\[
\hat{g}(x) = \begin{cases} 
(2−n)σ_{n−1}|x|^{−n+2} & \text{if } n \neq 2 \\
(2π)^{−1} \ln |x| & \text{if } n = 2,
\end{cases}
\tag{A.17}
\]
where \( σ_n \) is the volume of the unit–sphere \( S_n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \) in dimension \( n \).
Explicitly, (A.16) can be written as
\[ u(x) = [(2 - n)\sigma_{n-1}]^{-1} \int \frac{f(y)}{|x - y|^{n-2}} dy, \]
for \( n \neq 2 \), and similarly for \( n = 2 \). In particular, for \( n = 3 \), we have the celebrated Newton formula
\[ u(x) = -\frac{1}{4\pi} \int |x - y| dy. \]
Of course, the functions appearing in the above derivation are not necessarily from the Schwartz space \( S \) and therefore these manipulations must be justified. We leave this as an exercise, while proceeding in a similar fashion with other equations.

The heat equation on \( \mathbb{R}^n \):
\[ \frac{\partial u}{\partial t} = \Delta u \quad \text{and} \quad u|_{t=0} = u_0, \tag{A.18} \]
where \( u : \mathbb{R}_+^n \times \mathbb{R}_+^* \to \mathbb{R} \) is an unknown function, and \( u_0 : \mathbb{R}^n \to \mathbb{R} \) is a given initial condition. Problem (A.18) is called an initial value problem. It first appeared in the theory of heat diffusion. In that case, \( u_0(x) \) is a given distribution of temperature in a body at time \( t = 0 \), and \( u(x,t) \) is the unknown temperature–distribution at time \( t \). Presently, this equation appears in various fields of science, including mathematical modeling of stock markets.

As before, we apply the Fourier transform to (A.18) and solve the resulting equation
\[ \frac{\partial \hat{u}}{\partial t} = -|k|^2 \hat{u} \quad \text{and} \quad \hat{u}|_{t=0} = \hat{u}_0 \]
to get \( \hat{u} = e^{-|k|^2 t} \hat{u}_0 \). Applying the inverse Fourier transform, and using that \( (e^{-|k|^2 t})^{-} = (4\pi t)^{-n/2} e^{-|x|^2/(4t)} \), we obtain
\[ u = g_{-\pi} * u_0, \tag{A.19} \]
where \( g_s(x) = s^{-n}g(x/s) \) with \( g(x) = (2\pi)^{-n/2} e^{-|x|^2/2} \). In particular, \( u \to u_0 \) as \( t \to 0 \), as it should be.

The Schrödinger equation on \( \mathbb{R}^n \):
\[ i\frac{\partial \psi}{\partial t} = -\Delta \psi \quad \text{and} \quad \psi|_{t=0} = \psi_0. \tag{A.20} \]
This is an initial value problem for the unknown function \( \psi : \mathbb{R}_+^n \times \mathbb{R}_+^* \to \mathbb{C} \). Equation (A.20) describes the motion of a free quantum particle. Proceeding as with the heat equation, we obtain
\[ u = \phi_{-\pi} * \psi_0, \tag{A.21} \]
where \( \phi_s(x) = s^{-n}\varphi(x/s) \) with \( \phi(x) = (2\pi i)^{-n/2} e^{i|x|^2/(2t)} \). Observe that this formula can be obtained from (A.19) by performing the substitution \( t \to t/i \).

**Exercise 25.** Derive equation (A.21) using the Fourier transform.
The wave equation on $\mathbb{R}^n$:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \quad \text{with} \quad u|_{t=0} = u_0 \quad \text{and} \quad \partial_t u|_{t=0} = u_1. \quad (A.22)$$

This is a second order equation in time and consequently, it has two initial conditions $u_0$ and $u_1$. The wave equation (A.22) describes various wave phenomena: propagation of light and sound, oscillations of strings, etc. Proceeding as with the heat equation, we find

$$u = \partial_t W_t * u_0 + W_t * u_1, \quad (A.23)$$

where $W_t(x)$ is the inverse Fourier transform of the function $\frac{\sin(|k|t)}{|k|}$. The latter can be computed explicitly for $n = 1, 2, 3$:

$$W_t(x) = \begin{cases} \frac{1}{2} \chi_{\rho^2 \geq 0} & \text{for } n = 1, \\ \frac{1}{(2\pi)^{-\frac{1}{2}}} \rho^{-1} \chi_{\rho^2 \geq 0} & \text{for } n = 2, \\ \frac{1}{(2\pi)^{-\frac{1}{2}}} \delta(\rho^2) & \text{for } n = 3, \end{cases}$$

where $\rho^2 := t^2 - |x|^2$, and $\chi_{\rho^2 \geq 0}$ stands for the characteristic function of the set $\{(x, t) \in \mathbb{R}^{3+1} : \rho^2 \geq 0\}$, i.e.

$$\chi_{\rho^2 \geq 0} = \begin{cases} 1 & \text{if } \rho^2 \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\delta(x)$ is the Dirac $\delta$-function, a generalized function, or distribution.

Thus the dependence of $W$ on $x$ and $t$ comes through the combination $\rho^2 = t^2 - |x|^2$, which is the Minkowski–distance in space–time, playing a crucial role in relativity. Observe that $\chi_{\rho^2 \geq 0} = \chi_{|x| \leq t}$ and $\delta(\rho^2) = (2t)^{-\frac{1}{2}} \delta(t - |x|)$.

**Exercise 26.** Prove (A.23), and find $W_t(x)$ for $n = 1$.

We examine closer the special case when $n = 3$ and $u_0 = 0$. Then we get

$$u = W_t * u_1 = \frac{1}{4\pi t} \int \delta(|x - y| - t)u_1(y)dy$$

$$= \frac{1}{4\pi t} \int_{S(x,t)} u_1(y)dS(y)$$

$$= \frac{t}{4\pi} \int_{S(0,1)} u_1(x + tz)dS(z),$$

where $S(x, t) = \{y \in \mathbb{R}^3 : |y - x| = t\}$ is a sphere of radius $t$ centered at $x$. We see that only the initial condition evaluated on the sphere $S(x, t)$ matters in order to determine the solution at time $t$ and at position $x$. This is called the Huygens’ principle.

### A.3 Estimates on propagators

In this subsection we derive estimates on solution of the Schrödinger equation (A.20) which play an important role in applications.

**Theorem A.3.** Let $\psi := e^{i\Delta t} \psi_0$ be the solution to the Schrödinger initial value problem (A.20). Then we have the estimate

$$\|e^{i\Delta t} \psi_0\|_p \leq (4\pi t)^{-\frac{n}{2} - \frac{1}{p}} \|\psi_0\|_{p'} \quad (A.24)$$

where $p$ and $p'$ are indices satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ and $2 \leq p \leq \infty$. 

Proof. We use representation (A.21) of the solution $\psi$:

$$
\psi(x,t) = (4\pi it)^{-n/2} \int e^{i|x-y|^2/4t} \psi_0(y) \, dy.
$$

A simple estimate gives $|\psi(x,t)| \leq (4\pi t)^{-n/2} \int |\psi_0(y)| \, dy$, i.e.,

$$
\|\psi\|_\infty \leq (4\pi t)^{-n/2} \|\psi_0(y)\|_{L^1}.
$$

This is estimate (A.24) for $p = \infty$ and $p' = 1$. In the general case expanding $|x-y|^2 = |x|^2 - 2x \cdot y + |y|^2$ we write

$$
\psi(x,t) = (4\pi t)^{-n/2} e^{i|x|^2/4t} \int e^{-ix \cdot y/2t} e^{-|y|^2/4t} \, dy.
$$

We rewrite this expression as

$$
\psi = (2i)^{-n/2} M_f S_t F M_f \psi_0, \quad \text{(A.25)}
$$

where $f = e^{i|x|^2/4t}$, $M_f$ is the multiplication operator by the function $f$, $F$ is the Fourier transform and $S_t$ is the rescaling operator

$$(S_t u)(x) = t^{-n/2} u(xt).$$

Since $\|M_f u\|_p \leq \|f\|_\infty \|u\|_p$ for any $1 \leq p \leq \infty$, we have by the Hausdorff–Young inequality (see Section A)

$$
\|F u\|_p \leq \|u\|_{p'}
$$

for any $p$ and $p'$ as in the theorem. Finally, we use that, for any $1 \leq p \leq \infty$,

$$
\|S_t u\|_p \leq t^{-\frac{n}{2} + \frac{2}{p}} \|u\|_p.
$$

(Prove the above inequality.) Recalling equation (A.25) and applying the three inequalities we obtain the desired estimate (A.24).

\[\square\]

## B Linear operators and their spectra

### B.1 Linear operators

Linear operators or simply operators are linear maps from one vector space $Y$ into another vector space $X$. We denote linear operators usually by capital roman letters, $A, B, \ldots$ and use the notation

$$
A : Y \to X
$$

and $Au$ to denote an operator $A$ mapping $Y$ into $X$ and application of $A$ to a vector $u \in Y$, respectively. To define an operator $A$, we have to give a rule that prescribes to each element of $Y$ an element of $X$ (the image of $u$). We require this rule to be linear, i.e. $\forall u, v \in Y$ and $\alpha, \beta \in \mathbb{C}$:

$$
A(\alpha u + \beta v) = \alpha Au + \beta Av. \quad \text{(B.2)}
$$

To fix ideas here and in what follows, we consider vector spaces over the complex numbers $\mathbb{C}$, i.e. complex vector spaces. All the material of this section, except for spectral theory, remains unchanged if we substitute $\mathbb{R}$ for $\mathbb{C}$.
If the space $Y$ in (B.1) is a subset of the space $X$ then sometimes one calls $Y$ the \textit{domain} of $A$ (in $X$) and denotes it $\mathcal{D}(A) \equiv Y$. In this case we say that $A$ is defined in $X$ with domain $\mathcal{D}(A)(= Y)$. The range (or image) of $A$ is defined as

$$\text{Ran} \,(A) := \{ Au : u \in Y \} \equiv \text{AY}.$$  

Ran $(A)$ is a vector space (show this). We may assume that $\mathcal{D}(A)$ is \textit{dense} in $X$, i.e. for any $u \in X$, there is a sequence $\{ u_n \} \subset \mathcal{D}(A)$ s.t. $u_n \to u$ as $n \to \infty$. Indeed, if $\mathcal{D}(A)$ is not dense to begin with, we consider instead of the space $X$ simply the space $Y := \overline{\mathcal{D}(A)}$, the closure of $\mathcal{D}(A)$, which is obtained by adding to $Y$ limits of all possible sequences $\{ u_n \}$ convergent in $X$.

\textbf{Examples.}

1) The identity operator $\mathbbm{1} : L^p \to L^p$.
2) The multiplication operator $M_f : L^p \to L^p$, $u \mapsto f u$ for a fixed $f \in L^\infty$.
3) The differentiation operator $\partial_j \equiv \frac{\partial}{\partial x_j}$ in $L^2(\mathbb{R}^n)$ with the domain $\mathcal{D}(\partial_j) = H^1(\mathbb{R}^n)$.
4) The Laplacian $\Delta := \sum_1^n \frac{\partial^2}{\partial x_j^2}$ in $L^2(\Omega)$ with the domain $\mathcal{D}(\Delta) = H^2(\Omega)$.
5) The Schrödinger operator $-\Delta + V$, where $V(x)$ is the operator of multiplication by $V(x)$, i.e. $V = M_V$, called the potential. (The operator $-\Delta + V$ acts as $\psi(x) \to -(\Delta \psi)(x) + V(x) \psi(x)$.)
6) Integral operators, i.e., operators of the form

$$(Ku)(x) = \int K(x,y)u(y)dy,$$

for some function $K(x, y)$ (called the \textit{kernel} or \textit{integral kernel}). The domain and range of the integral operator $K$ depend on the properties of the kernel $K(x, y)$.

6) The Fourier transform $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$,

$$\mathcal{F} : u(x) \to (2\pi)^{-n/2} \int e^{-ikx} u(x) dx.$$  

7) Convolution operator $C_f : L^p \to L^p$, $C_f : u \to f * u$ for fixed $f \in L^1$, where

$$(f * u)(x) = \int f(x - y)u(y)dy.$$  

8) The wavelet transform

$$W_\psi : f \to \int \overline{\psi_{ab}} f dx,$$

where $\psi_{ab} = \frac{1}{|a|} \psi \left( \frac{x - b}{a} \right)$ for a fixed function $\psi$ satisfying $\int \frac{\hat{\psi}(k)^2}{k} dk < \infty$.

Note that the Fourier transform and convolution are integral operators with the integral kernels $(2\pi)^{-n/2}e^{ikx}$ and $(x - y)$, respectively.

In fact also in examples 1)-4), the operators can be represented as integral operators, but with distributional kernels, e.g. $K(x, y) = f(x)\delta(x - y)$ for $M_f$, and $K(x, y) = -\delta(x_j - y_j) \prod_{i \neq j} \delta(x_i - y_i)$ for $\partial_j$.

For an operator $A : Y \to X$ is said to be \textit{bounded} if the norm

$$\| A \| \equiv \| A \|_{Y \to X} = \sup_{\| u \|_Y = 1} \| Au \|_X$$  

(B.3)
is finite, $\|A\|_{Y \rightarrow X} < \infty$. (This is equivalent to the condition that the graph, $\{(u, Au) : u \in Y\}$, of $A$ is closed, as a subset of $Y \times X$. This is the content of the closed graph theorem, see [27], Theorem III.12). Observe that the definition of the norm implies that

$$\|Au\| \leq \|A\| \|u\| \quad (B.4)$$

for all $u \in Y$.

If $Y$ is a dense subset of $X$ and $\|Au\| \leq C\|u\|$, for all $u \in Y$ ($C$ is independent of $u$), and if the space $X$ is complete (i.e. a Banach space), then $A$ can be extended to the bounded operator on $X$ with the norm $\|A\|_{X \rightarrow X} = \|A\|_{Y \rightarrow X} = \sup_{\|u\|_Y = 1} \|Au\|$. If $Y$ is a subset of $X$ and $\|Au\|$ is not uniformly bounded in $u \in Y$, then we say that $A$ is unbounded (in $X$). In this case, we call $Y$ the domain of $A$ and write $Y = D(A)$. We also assume that domains are dense the corresponding spaces.

Examples of bounded operators are the identity operator, $1$, of example 1) above (in fact, $\|1\| = 1$), the multiplication operator, $M_f$, of example 2), the integral operator, $K$, of example 6) with a kernel $K(\cdot, \cdot) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, as an operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

**Exercise B.1.** Show that

1. $\|AB\| \leq \|A\|\|B\|$ (so bounded operators form a Banach algebra);
2. $\|w\| = \sup_{\|v\| = 1} |\langle w, v \rangle|$, and therefore $\|A\| = \sup_{\|u\|,\|v\| = 1} |\langle Au, v \rangle|$;
3. $\|M_f\| = \|f\|_\infty$ (Example 2 above);
4. for integral operators (Example 5 above)

$$\|K\|_{L^2 \rightarrow L^2} \leq \left( \int |K(x, y)|^2 dx dy \right)^{1/2}$$

provided the r.h.s. is finite.

It is easy to see that $\mathcal{F} : L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$,

$$\left| (2\pi)^{-n/2} \int e^{ix \cdot k} f(x) dx \right| \leq (2\pi)^{-n/2} \int |f(x)| dx.$$

It is considerably more difficult to show that $\mathcal{F}$ extends from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to a bounded operator on $L^2(\mathbb{R}^n)$. In fact $\mathcal{F}$ is an isometry in the sense that $\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}$ (the Plancherel theorem, see Appendix A).

**Exercise B.2.** Show that the differentiation operator in example 3) is not bounded in $L^2(\mathbb{R}^n)$ by finding a sequence $f_n$ of functions from $\mathcal{D}(\frac{\partial}{\partial x_j})$ such that $\|f_n\| \leq 1$, $\forall n$, and $\|\frac{\partial}{\partial x_j} f_n\|_2 \rightarrow \infty$, as $n \rightarrow \infty$.

We say an operator $A : Y \rightarrow X$ is invertible if and only if there is an operator $A^{-1} : X \rightarrow Y$ such that $A^{-1} A = \mathbb{1}_Y$ and $A A^{-1} = \mathbb{1}_X$, where $\mathbb{1}_X$ and $\mathbb{1}_Y$ are the identity operators in $X$ and $Y$ respectively.
Proposition B.1. An operator $A$ is invertible if and only if for every $f \in X$, the equation
\[ Au = f \tag{B.5} \]
has a unique solution $u(= A^{-1}f) \in Y$, i.e. if and only if $A$ is one-to-one ($Au = 0 \Rightarrow u = 0$) and onto (Ran $A = X$).

Exercise 27. Prove the proposition above. Show also that
1) if $A$ is just one-to-one (i.e. not necessarily onto), then $A$ is invertible as an operator from $Y$ to Ran $A \subset X$, i.e. the equation $Au = f$ has a unique solution $u = A^{-1}f$ for any $f \in \text{Ran } A$.
2) if $A$ and $B$ are invertible then so is $AB$ and
\[ (AB)^{-1} = B^{-1}A^{-1} \]
(generalize to an arbitrary number of factors).

Example B.1. We show invertibility of several important operators we encountered above.

1. $1$ is clearly invertible.
2. $M_f$ is invertible for $\text{ess inf } |f| > 0$ and $M_f^{-1} = M_{1/f}$.
3. $\frac{d}{dx}$ (see discussion below).
4. $F$ is invertible with $F^{-1}$ given in (A.3) (see Theorem A.1(f)).
5. $-\Delta + 1$ is invertible with $(-\Delta + 1)^{-1}f = G * f$ where $G := F^{-1}(\frac{1}{|x|^2+1})$ and $*$ denotes convolution.
6. curl is invertible with $\text{curl}^{-1}w = \int \frac{(x-y)^2}{|x-y|^4}w(y)dy$ in $2d$ (Biot-Savart formula).

Consider the operator $A = \frac{d}{dx} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$. Then $A$ is one-to-one: $Au = 0$ and $u \in H_1(\mathbb{R})$ imply $u \equiv 0$. Thus $A$ has the right inverse $B$: $f(x) \to \int_{x_0}^{x} f(y)dy$ for some fixed $x_0 \in \mathbb{R}$: $AB = \mathbb{I}$. However, $B$ does not map $L^2(\mathbb{R})$ into $H^1(\mathbb{R})$. In fact it is not defined on the entire $L^2(\mathbb{R})$ but only on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and maps this space into $L^\infty$. Put differently, the operator $A$ is not onto: Ran $A = \{f \in L^2(\mathbb{R}) | \int_{-\infty}^{\infty} f dx = 0\} \neq L^2(\mathbb{R})$. Hence $A$ in not invertible.

The proposition above illustrates the importance of finding inverses of operators. The following simple statement gives a powerful criterion for existence of inverses.

Theorem B.1. Assume an operator $A : X \to X$ is invertible and an operator $B : X \to X$ is bounded with the norm satisfying the inequality
\[ \|B\| < \|A^{-1}\|^{-1} \]
Then the operator $A + B$ (defined on the domain of $A$) is invertible. Moreover, its inverse is given by the absolutely convergent series
\[ (A + B)^{-1} = \sum_{n=0}^{\infty} A^{-1}(-BA^{-1})^n, \tag{B.6} \]
called the Neumann series (for $A + B$).
Exercise 28. Prove this theorem. Hint: Show that the series (B.6) is absolutely convergent and gives the inverse to \( A + B \) and use that if \( T_n \) is a Cauchy sequence of operators from \( X \) to \( X \), then there is an operator \( T : X \to X \) s.t. \( T_n \to T \) (see Theorem B.2 below).

Exercise 29 (*). Consider the Schrödinger operator \(-\Delta + V(x) + \lambda\), where the potential \( V(x) \) is a real bounded below and above function. Show that for \( \lambda > 0 \), \( \|V\|_{\infty} < \lambda^{-1} \), this operator is invertible. Hint: Use the inequality \( \|(-\Delta + \lambda)^{-1}\|_{\infty} < \lambda^{-1} \), which can be shown by going to the Fourier transform and using the Plancherel theorem.

B.2 Special classes of linear operators

**Adjoints.** Consider an operator \( A \) on a Hilbert space \( X \). With it we associate its adjoint \( A^* \) defined by the relation \( \langle A^*u, v \rangle = \langle u, Av \rangle \), for all \( v \in D(A) \), and for all \( u \)'s such that \( \sup_{v \in D(A), \|v\|=1} |\langle u, Av \rangle| < \infty \) (those \( u \)'s form the domain of the operator \( A^*, D(A^*) \)).

If \( A \) is a bounded operator then \( \sup_{\|v\|=1} |\langle u, Av \rangle| \leq \|u\| \|A\| < \infty \) and it suffices to check only the relation \( \langle A^*u, v \rangle = \langle u, Av \rangle \).

Exercise B.3. Show that if operators \( A \) and \( B \) are bounded, then
(a) \( A^* \) is a bounded operator and \( \|A\| = \|A^*\| \) (Hint: use that \( \|A\| = \sup_{\|u\|,\|v\|=1} |\langle u, Av \rangle| \), see Exercise ??(2)).
(b) \( (A + B)^* = A^* + B^* \),
(c) \( (\alpha A)^* = \overline{\alpha} A^* \),
(d) \( (AB)^* = B^* A^* \),
(e) \( (A^{-1})^* = (A^*)^{-1} \).

An important class of operators on a Hilbert space is the class of self-adjoint operators. By definition, an operator \( A \) is called self-adjoint if and only if \( A^* = A \). By definition, every self-adjoint operator is symmetric, i.e. \( \langle Au, v \rangle = \langle u, Av \rangle \), for all \( u, v \in D(A) \). Notice that the converse is not true. If an operator \( A \) is symmetric then all we know is that \( D(A) \subseteq D(A^*) \) (show this!). However, a symmetric operator \( A \) obeying \( D(A) = D(A^*) \) is also self-adjoint. Thus every symmetric bounded operator is self-adjoint.

Consider the examples of the operators 1)–5) above. We have the following: \( M_f \) is symmetric if and only if \( f \) is a real function; \( \frac{\partial}{\partial x_j} \) is anti–symmetric, so the differentiation operator is not symmetric, but \( -i \frac{\partial}{\partial x_j} \) is symmetric; the identity operator is obviously symmetric; \( \Delta \) is symmetric and so is \( -\Delta + V(x) \) for \( V(x) \) real; the integral operator is symmetric if \( K(x, y) = \overline{K(y, x)} \) (cf with matrices!). In addition, if we know that \( K(x, y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \), then the operator \( K \) is self-adjoint. A point is that while the symmetry property is easy to verify, the self-adjointness property is hard. See [15, 16, 28] for a proof of self-adjointness of \( -i \frac{\partial}{\partial x_j}, \Delta \) and \( -\Delta + V(x) \).

We observe that for any operator \( A \) on a Hilbert space \( X \), we can write
\[
X = \text{null } A \oplus \overline{\text{Ran } A^*} \tag{B.7}
\]
Here, the null space is defined as \( \text{null } A := \{ u \in X : Au = 0 \} \).

Exercise 30. Show that for a bounded operator \( A \), null \( A \) is a closed set, and show (B.7).
**Projections.** A bounded operator $P$ on $X$ is called a projection operator (or simply a projection) if and only if it satisfies

$$P^2 = P.$$ 

This relation implies $\|P\| \leq \|P\|^2$, i.e. $\|P\| \geq 1$. We have

$$v \in \text{Ran } P \iff P v = v \quad \text{and} \quad v \in (\text{Ran } P)^\perp \iff P^* v = 0. \quad (B.8)$$

Indeed, if $v \in \text{Ran } P$, then there is a $u \in X$ s.t. $v = Pu$, so $P v = P^2 u = Pu = v$; the second statement is left as an

**Exercise 31.** Prove that (a) $P^* v = 0$ if and only if $v \perp \text{Ran } P$, (b) $\text{Ran } P$ is closed and (c) $P^*$ is also a projection.

**Examples.** The following are projection operators:

1) on a space of functions $u(x)$, $\chi_{x \in E} : u(x) \mapsto \chi_E(x)u(x)$, where $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$,

2) on a space of functions $u(x)$, $\chi_{x \in E} = F^{-1} \chi_{x \in E} F$ acting as $u(x) \mapsto (\chi_E(k)\hat{u}(k))^{-}(x)$,

3) on any Hilbert space, where $\varphi, \psi$ are two fixed elements s.t. $\langle \varphi, \psi \rangle = 1$: $u \mapsto \langle \varphi, u \rangle \psi$

4) as in 3), but where now $\{\psi_i\}$ is an orthonormal set (i.e. $\langle \psi_i, \psi_j \rangle = \delta_{i,j}$): $u \mapsto \sum_i \langle \psi_i, u \rangle \psi_i$.

A projection $P$ is called an orthogonal projection if and only if it is self-adjoint, i.e. if and only if $P = P^*$. Let $P$ be an orthogonal projection, then (B.8) implies that

$$v \perp \text{Ran } P \iff P v = 0, \text{ i.e., } \text{null } P = (\text{Ran } P)^\perp. \quad (B.9)$$

The projections in Examples 1), 2) and 4) above are orthogonal. The projection in Example 3) is orthogonal if and only if $\varphi = \psi$.

**Exercise 32.** Let $P$ be an orthogonal projection. Show that

(a) $\|P\| \leq 1$, and therefore $\|P\| = 1$ (Hint: Use (B.8)),

(b) $I - P$ is also an orthogonal projection and $\text{Ran } (I - P)^\perp \text{Ran } P$ and $\text{null } I - P = \text{Ran } P$,

(c) $X = \text{Ran } P \oplus \text{null } P$.

**Remark.** Orthogonal projections on $X$ are in one-to-one correspondence with closed subspaces of a Hilbert space $X$. This correspondence is obtained as follows. Let $V = \text{Ran } P$. Then $V$ is a closed subspace of $X$. To show that $V$ is closed, let $\{v_n\} \subset V$, and $v_n \to v \in X$, and show that $v \in V$. Since $P$ is a projection, we have $v_n = P v_n$, so $\|v - P v\| = \|v_n - P(v_n - v)\| \leq \|v - v_n\| + \|P\| \|v - v_n\| \to 0$, as $n \to \infty$. Therefore $v = P v$, so $v \in V$, and $V$ is closed.

Conversely, given a closed subspace $V$, define a projection operator $P$ by

$$P u = v, \text{ where } u = v + v^\perp \in V \oplus V^\perp. \quad (B.10)$$

**Exercise 33.** Show that $P$ defined in (B.10) is an orthogonal projection with $\text{Ran } P = V$. For any given $V$, show that there is only one orthogonal projection (the one given in (B.10)) such that $\text{Ran } P = V$.

(*Galerkin approximation $(A \to PAP)$*)
The space of bounded linear operators \( \mathcal{L}(X,Y) \) We assume that \( X \) and \( Y \) are normed vector spaces over \( \mathbb{C} \), and consider the set of all bounded linear operators from \( X \) into \( Y \), i.e. each such operator is defined on the entire space \( X \), and its range lies in \( Y \). This set of operators is denoted by \( \mathcal{L}(X,Y) \).

For \( A,B \in \mathcal{L}(X,Y) \), we define a new operator, called \( A+B \), by setting \( (A+B)u := Au + Bu \), for all \( u \in X \). Also, for \( \lambda \in \mathbb{C} \) and \( A \in \mathcal{L}(X,Y) \), we define a new operator \( \lambda A \) as \( (\lambda A)u := \lambda Au \), for all \( u \in X \). If in addition to these two operations on operators, we equip the set \( \mathcal{L}(X,Y) \) with the norm introduced in (B.3), then \( \mathcal{L}(X,Y) \) is a normed vector space.

**Exercise 34.** Show that \( \mathcal{L}(X,Y) \) is a vector space.

An important question is: when is \( \mathcal{L}(X,Y) \) a Banach space? The answer is given in the following theorem, which is not difficult to prove (see e.g. [10], Proposition 5.3):

**Theorem B.2.** If \( Y \) is a Banach space, then \( \mathcal{L}(X,Y) \) is a Banach space.

The dual space. In the special case when \( Y = \mathbb{C} \), the space \( \mathcal{L}(X,Y) \) is called the dual space of \( X \) (or simply the dual, or adjoint space or conjugate space of \( X \)), and it is denoted as \( X' \).

Hence the elements of \( X' := \mathcal{L}(X,\mathbb{C}) \) are linear maps from \( X \) to \( \mathbb{C} \), and they are called linear functionals. Remark also that since \( \mathbb{C} \) is complete, then the last theorem shows that \( X' \) is always a Banach space, whether \( X \) is complete or not.

The operator norm induces a norm on \( X' \): if \( l \in X' \), then

\[
\| l \| = \sup_{\|x\| = 1} |l(x)|.
\]

If \( X \) is a space of functions, then \( X' \) can be identified with either a space of functions or a space of distributions or a space of measures. Here are some examples of dual spaces:

1) \( (L^p)' = L^q \), where \( 1/p + 1/q = 1 \), if \( 1 \leq p < \infty \) (space of functions),
2) \( (L^\infty)' \) is a space of measures which is much larger than \( L^1 \),
3) \( (H_s)' = H_{-s} \) (space of distributions if \( s > 0 \)).

Note that \( (L^p)' \supset L^q \), for \( 1 \leq p < \infty \) follows from the Hölder inequality. In fact, given \( f \in L^q \), define \( l_f(u) := \int f u \). Since \( |l_f(u)| \leq \|f\|_q \|u\|_p \), we see that \( l_f \) is a bounded linear functional on \( L^p \). It can be shown that in fact any bounded linear functional on \( L^p \) can be represented by \( l_f \) for some \( f \in L^q \).

**B.3 Spectrum**

Consider an operator \( A \) acting on a Banach space \( X \) with a domain \( \mathcal{D}(A) \) (i.e., \( A : \mathcal{D}(A) \rightarrow X \)). The **spectrum**, \( \sigma(A) \), of an operator \( A \) is the set in \( \mathbb{C} \) defined by

\[
\sigma(A) := \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible} \}.
\]  \hspace{1cm} \text{(B.11)}

For notational convenience, the operator “multiplication by \( z \in \mathbb{C} \)” will be simply written as \( z I \). Clearly, eigenvalues of \( A \) belong to \( \sigma(A) \) (in fact, if \( \lambda \) is an eigenvalue, then \( Au_\lambda = \lambda u_\lambda \) for some nonzero \( u_\lambda \in X \) (\( u_\lambda \) is called an **eigenvector**), so \( (A - \lambda)u_\lambda = 0 \), and \( A - \lambda \) is not invertible). In general, the spectrum can also contain continuous pieces and it can take very peculiar forms.
Exercise 35. The spectrum of the multiplication operator introduced in example 1) above is \(\sigma(M_f) = \text{Ran} f\), the differentiation operator 2) has spectrum \(\sigma\left(\frac{\partial}{\partial x_j}\right) = i\mathbb{R}\), and the identity operator 3) has the spectrum consisting of one point \(\sigma(\mathbb{I}) = \{1\}\).

The study of the spectra of operators is called spectral analysis.

The complement of the spectrum is called the resolvent set \(\rho(A)\):

\[\rho(A) := \mathbb{C} \setminus \sigma(A).\]

The following result states important property of the spectra; its proof is based on a useful identity.

**Theorem B.3.** The set \(\sigma(A)\) is closed (and, consequently, the set \(\rho(A)\) is open).

**Proof.** We prove the equivalent statement that the set \(\rho(A)\) is open. Let \(z_0 \in \rho(A)\). Then \(A - z_0\) is invertible. Using \(A - z = A - z_0 + z_0 - z\), we write

\[A - z = (A - z_0)[\mathbb{I} + (z_0 - z)(A - z_0)^{-1}].\]

If \(|z_0 - z| < \|(A - z_0)^{-1}\|^{-1}\), then the operator in the square brackets on the right is invertible by the Neumann theorem (see Theorem B.1). Therefore the operator \(A - z\) is invertible for \(|z - z_0| < \|(A - z_0)^{-1}\|^{-1}\) as a product of two invertible operators. \(\square\)

For \(z \in \rho(A)\) the operator \(A - z\) has a bounded in \(X\) inverse. Denote this inverse as

\[R_A(z) := (A - z)^{-1}.\]

It is called the resolvent of \(A\) at \(z \in \rho(A)\). It plays an important role in analysis of operators. The proof of the theorem above shows that the resolvent is an analytic operator valued function in \(z \in \rho(A)\) in the sense that for any \(z_0 \in \rho(A)\) and for any \(z\) such that \(|z - z_0| < \|R_A(z_0)\|^{-1}\), the resolvent \(R_A(z)\) can be expanded in the series

\[R_A(z) = R_A(z_0) \sum_{n=0}^{\infty} ((z - z_0)R_A(z_0))^n \quad (B.12)\]

which is absolutely convergent is the sense that

\[\sum_{n=0}^{\infty} \|(z - z_0)R_A(z_0)\|^n = \sum_{n=0}^{\infty} |z - z_0|^n \|R_A(z_0)\|^n < \infty.\]

Indeed the series above is just the Neumann series for the inverse of the operator \(A - z = (A - z_0)[\mathbb{I} + (z_0 - z)(A - z_0)^{-1}]\) (see Theorem B.1).

The resolvent satisfies two equations:

\[R_A(z) - R_A(w) = (z - w)R_A(z)R_A(w) \quad (B.13)\]

and

\[R_A(z) - R_B(z) = R_A(z)(B - A)R_B(z), \quad (B.14)\]

called the first and second resolvent equations. The first equation follows from the second one with \(B = (z - w)\mathbb{I}\) and the second equation is equivalent to the identity \(\frac{1}{A} - \frac{1}{B} = \frac{1}{A}(B - A)\frac{1}{B}\) which can be easily verified.
Proposition B.2. If $T$ is a bounded operator then $\sigma(T) \subset \{ z \in \mathbb{C} \mid |z| \leq \|T\| \}$.

Proof. Expand formally by the Newmann series

$$\frac{1}{A - z} = \sum_{n=0}^{\infty} \left( \frac{z}{A} \right)^n.$$ 

This shows that $z \in \rho(A)$ (i.e., $(A - z)^{-1}$ is bounded) if and only if $\|A/z\| < 1$. Equivalently, $z \in \rho(A)$ if and only if $|z| > \|A\|$. Therefore, $z \in \sigma(A)$ if and only if $|z| \leq \|A\|$. \qed

B.4 Perron-Frobenius Theory

Consider a bounded operator $T$ on the Hilbert space $X = L^2(\Omega)$.

Definition 12. An operator $T$ is called positivity preserving/improving if and only if $u \geq 0$, $u \neq 0 \implies Tu \geq 0/Tu > 0$.

Note if $T$ is positivity preserving, then $T$ maps real functions into real functions.

Theorem B.4. Let $T$ be a bounded positive and positivity improving operator and let $\lambda$ be an eigenvalue of $T$ with an eigenvector $\varphi$. Then

a) $\lambda = \|T\| \Rightarrow \lambda$ is simple and $\varphi > 0$ (modulo a constant factor).

b) $\varphi > 0$ and $\|T\|$ is an eigenvalue of $T \Rightarrow \lambda$ is simple and $\lambda = \|T\|$.

Proof. a) Let $\lambda = \|T\|$, $T\varphi = \lambda \varphi$ and $\varphi$ be real. Then $|\varphi| \pm \varphi \geq 0$ and therefore $T(|\varphi| \pm \varphi) > 0$. The latter inequality implies that $|T\varphi| \leq T|\varphi|$ and therefore

$$\langle |\varphi|, T|\varphi| \rangle \geq \langle |\varphi|, T\varphi \rangle \geq \langle \varphi, T\varphi \rangle = \lambda |\varphi|^2.$$ 

Since $\lambda = \|T\| = \sup_{|\varphi| = 1} \langle \varphi, T\varphi \rangle$, we conclude using variational calculus (see e.g. [15] or [?]) that

$$T|\varphi| = \lambda |\varphi| \quad \text{(B.15)}$$

i.e., $|\varphi|$ is an eigenfunction of $T$ with the eigenvalue $\lambda$. Indeed, since $\lambda = \|T\| = \sup_{|\varphi| = 1} \langle \varphi, T\varphi \rangle$, $|\varphi|$ is the maximizer for this problem. Hence $|\varphi|$ satisfies the Euler-Lagrange equation $T|\varphi| = \mu |\varphi|$ for some $\mu$. This implies that $\mu ||\varphi||^2 = \langle |\varphi|, T|\varphi| \rangle = \lambda ||\varphi||^2$ and hence $\mu = \lambda$. Equation (B.15) and the positivity improving property of $T$ imply that $|\varphi| > 0$.

Now either $\varphi = \pm |\varphi|$ or $|\varphi| + \varphi$ and $|\varphi| - \varphi$ are nonzero. In the latter case they are eigenfunctions of $T$ corresponding to the eigenvalue $\lambda : T(|\varphi| \pm \varphi) = \lambda (|\varphi| \pm \varphi)$. By the positivity improving property of $T$ this implies that $|\varphi| \pm \varphi > 0$ which is impossible. Thus $\varphi = \pm |\varphi|$.

If $\psi_1$ and $\psi_2$ are two real eigenfunctions of $T$ with the eigenvalue $\lambda$ then so is $a\psi_1 + b\psi_2$ for any $a, b \in \mathbb{R}$. By the above, either $a\psi_1 + b\psi_2 > 0$ or $a\psi_1 + b\psi_2 < 0 \forall a, b \in \mathbb{R} \setminus \{0\}$, which is impossible. Thus $T$ has a single real eigenfunction associated with $\lambda$.

Let now $\psi$ be a complex eigenfunction of $T$ with the eigenvalue $\lambda$ and let $\psi = \psi_1 + i\psi_1$ where $\psi_1$ and $\psi_2$ are real. Then the equation $T\psi = \lambda \psi$ becomes

$$T\psi_1 + iT\psi_2 = \lambda \psi_1 + i\lambda \psi_2.$$
Since $T\psi_2$ and $T\psi_2$ and $\lambda$ are real (see above) we conclude that $T\psi_i = \lambda\psi_2$, $i = 1, 2$, and therefore by the above $\psi_2 = c\psi_1$ for some constant $c$. Hence $\psi = (1 + ic)\psi_1$ is positive and unique modulo a constant complex factor.

b) By a) and eigenfunction, $\psi$, corresponding to $\nu := ||T||$ can be chosen to be positive, $\psi > 0$. But then

$$\lambda \langle \psi, \varphi \rangle = \langle \psi, T\varphi \rangle = \langle T\psi, \varphi \rangle = \nu \langle \psi, \varphi \rangle$$

and therefore $\lambda = \nu$ and $\psi = c\varphi$. 

*Question: Can the condition that $||T||$ is an eigenvalue of $T$ (see b) be removed?*

Now we consider the Schrödinger operator $H = -\Delta + V(x)$ with a real, bounded potential $V(x)$. The above result allows us to obtain the following important

**Theorem B.5.** Let $H = -\Delta + V(x)$ have an eigenvalue $E_0$ with an eigenfunction $\varphi_0(x)$ and let $\inf \sigma(H)$ be an eigenvalue. Then

$$\varphi_0 > 0 \Rightarrow E_0 = \inf \{\lambda | \lambda \in \sigma(H)\} \text{ and } E_0 \text{ is non-degenerate}$$

and, conversely,

$$E_0 = \inf \{\lambda | \lambda \in \sigma(H)\} \Rightarrow E_0 \text{ is non-degenerate and } \varphi_0 > 0$$

(modulo multiplication by a constant factor).

**Proof.** To simplify the exposition we assume $V(x) \leq 0$ and let $W(x) = -V(x) \geq 0$. For $\mu > \sup W$ we have

$$(-\Delta - W + \mu)^{-1} = (-\Delta + \mu)^{-1} \sum_{n=0}^\infty [W(-\Delta + \mu)^{-1}]^n$$

(B.16)

where the series converges in norm as

$$||W(-\Delta + \mu)^{-1}|| \leq ||W|| ||(-\Delta + \mu)^{-1}|| \leq ||W|| L^\infty \mu^{-1} < 1$$

by our assumption that $\mu > \sup W = ||W|| L^\infty$. To be explicit we assume that $d = 3$. Then the operator $(-\Delta + \mu)^{-1}$ has the integral kernel

$$\frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|} > 0$$

while the operator $W(-\Delta + \mu)^{-1}$ has the integral kernel

$$W(x)\frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|} \geq 0.$$ 

Consequently, the operator

$$(-\Delta + \mu)^{-1} f(x) = \frac{1}{4\pi} \int \frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|} f(y) \, dy$$

is positivity improving ($f \geq 0, f \neq 0 \Rightarrow (-\Delta + \mu)^{-1} f > 0$) while the operator
\[
W(-\Delta + \mu)^{-1}f(x) = \frac{1}{4\pi} \int W(x) \frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|} f(y) \, dy
\]

is positivity preserving \((f \geq 0 \Rightarrow W(-\Delta + \mu)^{-1}f \geq 0)\). The latter fact implies that the operators \([W(-\Delta + \mu)^{-1}]^n, n \geq 1\), are positivity preserving (prove this!) and consequently the operator

\[
(-\Delta + \mu)^{-1} + \sum_{n=1}^{\infty} [W(-\Delta + \mu)^{-1}]^n
\]

is positivity improving (prove this!).

Thus we have shown that the operator \((H+\mu)^{-1}\) is positivity improving. Series (B.16) shows also that \((H+\mu)^{-1}\) is bounded. Since

\[
\langle u, (H + \mu)u \rangle \geq (-\sup W + \mu)\|u\|^2 > 0,
\]

we conclude that the operator \((H+\mu)^{-1}\) is positive (as an inverse of a positive operator). Finally, \(\|(H+\mu)^{-1}\| = \sup \sigma((H+\mu)^{-1}) = (\inf \sigma(H) + \mu)^{-1}\) is an eigenvalue by the condition of the theorem. Hence the previous theorem applies to it. Since \(H\varphi_0 = E_0\varphi_0 \iff (H + \mu)^{-1}\varphi_0 = (E_0 + \mu)^{-1}\varphi_0\), the theorem under verification follows. *This paragraph needs details!* \(\square\)

## C Semigroups

Let \(X\) be a Banach space and \(A\) be a linear (closed) operator on \(X\) with dense domain \(Y = \mathcal{D}(A)\). Our goal is to solve the initial value problem

\[
\frac{\partial u}{\partial t} = Au, \quad u|_{t=0} = u_0 \in Y
\]

for \(u \in C(\mathbb{R}, Y) \cap C^1(\mathbb{R}, X)\). We use the following terminology.

- The family, \(U(t), t \geq 0\), of operators on \(X\) is called a \textbf{(strongly continuous) semigroup} if and only if
  - (a) \(U(t)\) are bounded \(\forall t \geq 0\).
  - (b) \(U(0) = 1\) and \(U(t+s) = U(t)U(s)\).
  - (c) \(t \to U(t)\varphi\) is continuous \(\forall \varphi \in X\).

- The family \(U(t)\) is called a \textbf{contraction semigroup} if and only if \(U(t)\) is a semigroup and \(\|U(t)\| \leq 1\).

- The \textbf{(closed)} operator

\[
Au := \lim_{s \to 0} \frac{1}{s} (U(s) - 1)u \tag{C.1}
\]

on \(X\) with the domain

\[
\mathcal{D}(A) := \{ u \in X \mid \text{the limit on the r.h.s of (C.1) exists} \}
\]

is called the \textbf{generator} of \(U(t)\). If \(A\) is the generator of the semigroup \(U(t)\), then we write
\[ U(t) = e^{At}. \]

**Theorem C.1.** If \( A \) is the generator of \( U(t) \), then \( U(t)D(A) \subset D(A) \) and \( \forall u_0 \in D(A), u := U(t)u_0 \) solves the equation

\[
\frac{\partial u}{\partial t} = Au
\]

with the initial condition \( u|_{t=0} = u_0 \).

**Proof.** If \( u \in D(A) \), then

\[
AU(t)u = \lim_{s \to 0} \frac{1}{s} (U(s) - 1)U(t)u = U(t) \lim_{s \to 0} \frac{1}{s} (U(s) - 1)u \text{ exists.}
\]

Hence \( U(t)u \in D(A) \). Furthermore,

\[
AU(t)u = \lim_{s \to 0} \frac{1}{s} (U(t + s)u - U(t)u) \equiv \frac{\partial}{\partial t} U(t)u.
\]

\( \square \)

**Corollary C.2.** If an operator \( A \) is the generator of a semigroup \( U(t) \), then the initial value problem

\[
\frac{\partial u}{\partial t} = Au, \quad u|_{t=0} = u_0
\]

has a solution for any \( u_0 \in D(A) \) and this solution is given by the formula \( u = U(t)u_0 \).

Thus the main question here is: when does an operator \( A \) generate a semigroup? There are three general situations where we can show that an operator \( A \) generates a semigroup:

- \( A \) is a bounded operator;
- \( A \) is either self-adjoint (\( A^* = A \) and bounded above, \( A \leq C \) for some \( C < \infty \) (\( \langle u, Au \rangle \leq C \| u \|^2 \)) or anti-self-adjoint (\( A^* = -A \));
- \( A \) is a ‘constant coefficient pseudo-differential’ operator; more precisely, \( A = a(-i\nabla_x) \), where \( a \) is some decent function.

For a bounded operator \( A \), the exponential \( e^{At} \) can be defined by

\[
e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.
\]

(C.3)

The series on the r.h.s. converges absolutely since

\[
\left\| \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\| (tA)^n \|}{n!} \leq \sum_{n=0}^{\infty} \left[ \frac{t^n}{n!} \| A \|^n \right] = e^{t\| A \|} < \infty.
\]

**Exercise 36.** Show that equation (C.3) defines a semigroup and that this semigroup is generated by \( B \).
In the third case \( A \) and \( e^{At} \) are defined using the Fourier transform as \((\hat{A}u)(k) = a(k)\hat{u}(k)\) and \((e^{At}u)(k) = e^{a(k)t}\hat{u}(k)\), respectively (see Appendix A and [15] for more on the Fourier transform). In the second case, the result follows from the following theorem (see [15] for more details):

**Theorem C.3** (Hille-Yosida). Let \( A \) be a closed operator such that (a) \((0, \infty) \subset \rho(A)\) and (b) \(||(A - \lambda)^{-1}|| \leq 1/\lambda\) for any \( \lambda > 0 \). Then \( A \) generates a unique semigroup and this semigroup is contractive.

**Proof.** The idea is very simple: we approximate the operator \( A \) by bounded operators \( A_{\lambda} \) so that \( A_{\lambda}u \to Au \ \forall u \in \mathcal{D}(A) \) as \( \lambda \to \infty \); construct the semigroup, \( U_{\lambda}(t) \), for \( A_{\lambda} \) by the formula

\[
U_{\lambda}(t) = \sum_{n=0}^{\infty} \frac{1}{n!}(tA_{\lambda})^n
\]

(see equation (C.3)); define the semigroup, \( U(t) \), for \( A \) as the limit

\[
U(t)u = \lim_{\lambda \to \infty} U_{\lambda}(t)u
\]

for any \( u \in \mathcal{D}(A) \) and then, by continuity, extend \( U(t) \) to the entire space \( X \).

We define \( A_{\lambda} \) as

\[
A_{\lambda} := A\lambda(\lambda - A)^{-1}.
\]

(Note \( A - \lambda \) for \( \lambda > 0 \) is invertible by the condition that \((0, \infty) \subset \rho(A)\).) Then \( \forall u \in \mathcal{D}(A) \), by (b)

\[
\lambda(\lambda - A)^{-1}u - u = (\lambda - A)^{-1}Au \to 0,
\]

as \( \lambda \to \infty \), and

\[
||\lambda(\lambda - A)^{-1}|| \leq 1.
\]

Hence, by an \( \epsilon/3 \) argument,

\[
\lambda(\lambda - A)^{-1}u \to u \ \forall u \in X.
\]

This implies \( \forall u \in \mathcal{D}(A) \),

\[
A_{\lambda}u = \lambda(\lambda - A)^{-1}Au \to Au.
\]

Now we consider the semigroup \( U_{\lambda}(t) \) defined in (C.4). Due to (b) and the relation \( A_{\lambda} = \lambda^2(\lambda - A)^{-1} - \lambda \), we have

\[
||e^{A_{\lambda}t}|| \leq e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!}||t\lambda^2(\lambda - A)^{-1}||^n
\]

\[
\leq e^{-\lambda} \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^{2n} ||(\lambda - A)^{-1}||^n
\]

\[
\leq e^{-\lambda} \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^n.
\]
Hence

\[ ||e^{A\lambda t}|| \leq 1, \quad \text{(C.7)} \]

i.e., \( e^{A\lambda t} \) is the contractive semigroup.

Now we show that \( \{ e^{A\lambda t}, \lambda > 0 \} \) is a Cauchy family in the sense that

\[ ||(e^{A\lambda' t} - e^{A\lambda t})u|| \to 0 \quad \text{(C.8)} \]

as \( \lambda, \lambda' \to \infty \) for any \( u \in X \). To prove this we represent the operator acting on \( u \) inside the norm as an integral of a derivative

\[ e^{A\lambda' t} - e^{A\lambda t} = \int_0^t \frac{\partial}{\partial s} e^{A\lambda' s} e^{A\lambda(t-s)} ds. \]

Using the equation

\[ \frac{\partial}{\partial s} e^{A\lambda s} = A\lambda e^{A\lambda s} = e^{A\lambda s} A\lambda \]

gives

\[ e^{A\lambda' t} - e^{A\lambda t} = \int_0^t e^{A\lambda' s} e^{A\lambda(t-s)} (A\lambda' - A\lambda) ds. \]

The last equation together with (C.7) yields

\[ ||(e^{A\lambda' t} - e^{A\lambda t})u|| \leq \int_0^t ||e^{A\lambda' s} e^{A\lambda(t-s)} (A\lambda' - A\lambda) u|| ds \]
\[ \leq \int_0^t ||(A\lambda' - A\lambda) u|| ds = t ||(A\lambda' - A\lambda) u|| \]

and therefore (C.8) follows for any \( t \geq 0 \) first \( \forall u \in \mathcal{D}(A) \) and then by continuity \( \forall u \in X \).

Now equation (C.8) implies that the limit on the r.h.s. of (C.5) exists \( \forall t \in [0, \infty) \) \( \forall u \in X \) and satisfies

\[ ||U(t)|| \leq 1. \]

Equations (C.5) and (C.7) imply also that \( U(t) \) is the semigroup: \( U(t+s) = U(t)U(s) \) and \( U(0) = 1 \). It remains to prove that \( U(t) \) is strongly continuous and is generated by the operator \( A \). Using the relation \( U\lambda(s) - 1 = \int_0^s dt U\lambda(t) A\lambda \), we find for \( u \in \mathcal{D}(A) \)

\[ \lim_{s \to 0} ||(U(s) - 1)u|| = \lim_{s \to 0} \lim_{\lambda \to \infty} ||(U\lambda(s) - 1)u|| \]
\[ = \lim_{s \to 0} \lim_{\lambda \to \infty} \int_0^s dt ||U\lambda(t) A\lambda u|| \]
\[ \leq \lim_{s \to 0} \lim_{\lambda \to \infty} \int_0^s dt ||A\lambda u|| \]
\[ = \lim_{s \to 0} s ||A u|| = 0. \]

Hence \( U(s)u \to u \) as \( s \to 0 \) \( \forall u \in \mathcal{D}(A) \). Since \( U(t) \) is bounded uniformly in \( t \) we conclude that it is strongly continuous. Similarly, using the relation \( U\lambda(s) - 1 - A\lambda = \int_0^s dt (U\lambda(t) - 1) A\lambda \) and the strong continuity of \( U\lambda(t) \) shown above, we conclude that \( U(t) \) is generated by \( A \). \( \square \)
How do we check the conditions of the Hille-Yosida theorem? Usually this is a hard business. However, there are several cases where this can be easily done. Below, $X$ is a Hilbert Space.

A) $A = -A^*$ ($A$ is anti-self-adjoint). Then $\sigma(A) \subset i\mathbb{R}$ and $\|(A - \lambda)^{-1}\| \leq \lambda^{-1}$ for $\lambda > 0$.

B) $A = A^* \leq 0$ ($A$ is non-positive). Then $\sigma(A) \subset (-\infty, 0]$ and $\|(A - \lambda)^{-1}\| \leq \lambda^{-1}$ for $\lambda > 0$.

C) Perturbations of generators. For example, $A = A_0 + B$ where $A_0$ generates a semigroup and $\|B(A_0 - \lambda)^{-1}\| \leq \beta$ with $\beta < 1 \forall \lambda \geq \lambda_0$, for some $\lambda_0 > 0$. Indeed, in this case we have for $\lambda > \lambda_0$

$$A - \lambda = [1 + T_\lambda](A_0 - \lambda)$$

where $T_\lambda = B(A_0 - \lambda)^{-1}$. By the condition above $\|T_\lambda\| \leq \beta \forall \lambda \geq \lambda_0$. Since $\beta < 1$, $1 + T_\lambda$ is invertible and therefore so is the r.h.s. of (C.9) $\forall \lambda \geq \lambda_0$. Therefore $(\lambda_0, \infty) \subset \rho(A)$. Moreover, (C.9) implies that

$$(A - \lambda)^{-1} = (A_0 - \lambda)^{-1}(1 + T_\lambda)^{-1}$$

and therefore

$$\|(A - \lambda)^{-1}\| \leq \|(A_0 - \lambda)^{-1}\| \|(1 + T_\lambda)^{-1}\|.$$ 

Since $A_0$ generates a semigroup, we have that $\|(A_0 - \lambda)^{-1}\| \leq 1/\lambda$. Since $\|T_\lambda\| \leq \beta < 1$ we have that $\|(1 + T_\lambda)^{-1}\| \leq (1 - \beta)^{-1}$. Collecting the last three estimates we conclude that $\forall \lambda \geq \lambda_0$

$$\|(A - \lambda)^{-1}\| \leq (1 - \beta)^{-1}\lambda^{-1}.$$ 

This is not quite what we need (remember (b)). However, for the operator $A_\mu = A - \mu$ with $\mu = \min(\lambda_0, (1 - \beta)^{-1})$, we obtain

$$\|(A_\mu - \lambda)^{-1}\| = \|(A - (\mu + \lambda))^{-1}\| \leq \mu(\mu + \lambda)^{-1} < \lambda^{-1}.$$ 

Thus the operator $A_\mu$ generates a contraction semigroup $e^{A_\mu t}$, $\|e^{A_\mu t}\| \leq 1$. Now, $e^{At} := e^{A_\mu t}e^{\mu t}$ gives a semigroup for the operator $A$ and this semigroup satisfies the estimate

$$\|e^{At}\| \leq e^{\mu t}.$$

**Exercise 37.** Check the last statement.

**Examples.**

1) The Schrödinger equation. In this case $A = -iH$ where $H$ is a self-adjoint operator (e.g., a Schrödinger operator $H := -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$).

2) The heat equation. In this case $A = A^* \leq 0$ or more generally $A = -A_0 - A_1$ with $A_0 = A_0^* > 0$ and

$$\|A_0^{-1/2}A_1A_0^{-1/2}\| < 1.$$ 

For example $A = -\sum \partial_{x_i}a_{ij}(x)\partial_{x_j} + \sum b_i(x)\partial_{x_i} + c(x)$ with the matrix $(a_{ij}(x)) \geq \delta 1$, $(\sum |b_i(x)|^2)^{1/2} \leq \gamma$ and $c(x) \geq \gamma^2/\delta$.

3) The wave equation. In this case
\[
A = \begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix} \quad \text{with} \quad H = H^* \geq 0.
\] (C.10)

Indeed, if \( v \) satisfies the wave equation
\[
\frac{\partial^2 v}{\partial t^2} = -H v \quad \text{with} \quad H \geq 0,
\]
then the element \( u = (v, \partial v/\partial t) \) satisfies the equation
\[
\frac{\partial u}{\partial t} = Au
\]
with the operator \( A \) given in (C.10). On the Hilbert space \( X = \mathcal{D}(H) \oplus L^2 \) with the inner product
\[
\langle u, w \rangle_X = \langle u_1, Hw_1 \rangle + \langle u_2, w_2 \rangle
\]
where \( u = (u_1, u_2) \) and \( w = (w_1, w_2) \), the operator (C.10) is anti-self-adjoint, \( A = -A^* \), and therefore it generates a unique contraction semigroup.

**Example.** The acoustical wave equation.
\[
\frac{\partial^2 v}{\partial t^2} = c^2 \rho \nabla \cdot \frac{1}{\rho} \nabla v.
\]
The operator \( H := -c^2 \rho \nabla \cdot \frac{1}{\rho} \nabla \) is self-adjoint and, in fact, non-negative on the space \( L^2(\mathbb{R}^3, (c^2 \rho)^{-1} dx) \).

**Exercise 38.** Show that \( H \) is symmetric, i.e.,
\[
\langle Hu, v \rangle = \langle u, Hv \rangle \quad \forall u, v \in \mathcal{D}(H).
\]

4) **Maxwell equations.** In a vacuum, Maxwell’s equations for the electric and magnetic fields, \( E(x, t) \) and \( H(x, t) \), read
\[
\begin{align*}
\text{curl } E &= -\mu \frac{\partial H}{\partial t}, \\
\text{curl } H &= \epsilon \frac{\partial E}{\partial t} \\
\text{div}(\epsilon E) &= 0, \\
\text{div}(\mu H) &= 0
\end{align*}
\]
where \( \epsilon \) and \( \mu \) are dielectric constant and magnetic permeability, respectively. These equations can be written as
\[
\partial_t u = JAu,
\]
where \( u = (E, H), J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and
\[
A = \begin{pmatrix} \mu^{-1} \text{curl} & 0 \\ 0 & \epsilon^{-1} \text{curl} \end{pmatrix}
\]
on the Hilbert space (we use that \( \epsilon \) and \( \mu \) are independent of \( x \))
\[
\mathcal{H}_1 = \{(E, H) \mid E, H \in H_1(\mathbb{R}^3, \mathbb{R}^3), \text{div} E = 0, \text{div} H = 0\}. 
\]
One can show that the operator $A$ with the domain $\mathcal{H}_1 \subseteq L^2(\mathbb{R}^3, \mathbb{R}^3) \oplus L^2(\mathbb{R}^3, \mathbb{R}^3)$ is self-adjoint (show that it is symmetric) and $\sigma_{\text{ess}}(A) = [0, \infty)$. Hence the operator $JA$ generates a contraction semigroup according to criterion D) above.

\section{Gâteaux and Fréchet derivatives}

Our goal is to develop a differential calculus of maps between Banach spaces. Let $X$ and $Y$ be Banach spaces and $M$, an open subset of $X$. We consider a map $F : M \to Y$. The map $F$ is called Gâteaux differentiable at $u \in M$ if and only if there exists a bounded, linear map $dF(u) : X \to Y$, s.t. for any $\xi \in X$:

$$dF(u)\xi := \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} F(u + \lambda \xi). \quad (D.1)$$

The map $dF(u)$ is called the Gâteaux derivative or sometimes, the variational derivative.

The map $F$ is called continuously differentiable at $\pi \in X$ if and only if it is Gâteaux differentiable for $u$ in a neighborhood of $\pi$, and moreover, $u \mapsto dF(u)$ is a continuous map from $X$ to $\mathcal{L}(X,Y)$ at the point $\pi$; i.e. if $u_n \to \pi$ in $X$, then $dF(u_n) \to dF(\pi)$ in $\mathcal{L}(X,Y)$. The map $F$ is called continuously differentiable, or $C^1$ (written $F \in C^1$) if and only if it is continuously differentiable for every $u \in M$.

**Example D.1 (formal).**

1) If $F(u) = Lu$, where $L$ is a linear map, then $dF(u) = L$ (independently of $u$). Indeed, $dF(u)\xi = \frac{\partial}{\partial \lambda} (L(u + \lambda \xi)|_{\lambda=0} = \frac{\partial}{\partial \lambda} (Lu + \lambda L \xi)|_{\lambda=0} = L \xi$. Thus if $L$ is bounded, then $F$ is $C^1$.

2) If $F(u) = f \circ u$ (composition map), for a fixed $C^1$ function $f : \mathbb{R} \to \mathbb{R}$, and $u : \mathbb{R}^n \to \mathbb{R}$, then $dF(u)\xi$ is the multiplication operator by $f'(u)$. Indeed, $dF(u)\xi = \frac{\partial}{\partial \lambda} F(u + \lambda \xi)|_{\lambda=0} = \frac{\partial}{\partial \lambda} f(u(x) + \lambda \xi(x))|_{\lambda=0} = f'(u)\xi$. So if $f'(u)$ is a bounded function, say for some $u \in L^p(\mathbb{R}^n)$, then $F : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is differentiable at $u$.

**Exercise D.1.** Compute $dF(u)$ for $F : \mathbb{R}^n \to \mathbb{R}^m$, and for

$$F(u) = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

**Exercise 39.** Let $F(u) = f \circ u$, and let $\Omega$ be a domain in $\mathbb{R}^n$ with a smooth boundary. Show that if $f \in C^{k+1}(\mathbb{R})$, then $F : C^k(\overline{\Omega}) \to C^k(\overline{\Omega})$, and $F$ is $C^1$ with $dF(u)\xi = f'(u)\xi$.

As usual, the symbol $o(\|\xi\|)$ stands for a map $R : X \to Y$ s.t.

$$\frac{\|R\|}{\|\xi\|} \to 0, \quad \text{as } \|\xi\| \to 0.$$

The following properties of the Gâteaux derivative play an important role in applications.

**Theorem 13.** (a) (The fundamental theorem of calculus) If $F$ is Gâteaux differentiable at $u \in M$, then we have

$$F(u + \xi) - F(u) = \int_0^1 ds dF(u + s\xi)\xi.$$

• (b) (The chain rule) If $F$ and $G$ are Gâteaux differentiable at $u \in M$ and $F(u) \in F(M)$, then we have $d(G \circ F)(u) = dG(F(u))dF(u)$.

(c) Let $K$ be a convex subset of a Banach space $X$ (i.e. if $u, v \in K$, then $su + (1 - s)v \in K$, for all $s \in [0, 1]$). Show that if $F : K \to K$ satisfies $\|dF(\psi)\| \leq \alpha$, $\forall \psi \in K$, then $F$ is Lipschitz: $\|F(\psi) - F(\varphi)\| \leq \alpha\|\psi - \varphi\|$, $\forall \psi, \varphi \in K$.

(d) If $F$ is $C^1$ at $u \in X$, then as $\|\xi\| \to 0$

$$F(u + \xi) = F(u) + dF(u)\xi + o(\|\xi\|).$$

(D.2)

Proof. Define the function $g : [0, 1] \to Y$ by

$$g(t) = F(u + t\xi),$$

for $u, \xi \in X$ fixed. According to the definition of the Gâteaux derivative (D.1), we have

$$g'(t) = dF(u + t\xi)\xi.$$

By Fundamental Theorem of Calculus,

$$g(1) - g(0) = \int_0^1 g'(t)dt. \quad \text{(D.3)}$$

Since $g'(t) = dF(u + t\xi)\xi$, this gives the first statement.

**Exercise D.2.** Show the properties (c) and (d).

Using (D.3), we find $g(1) - g(0) = g'(0) + \int_0^1 (g'(t) - g'(0))dt$, which in turn gives

$$\|F(u + \xi) - F(u) - dF(u)\xi\|$$

$$= \|g(1) - g(0) - g'(0)\|$$

$$\leq \sup_{0 < t < 1} \|g'(t) - g'(0)\|$$

$$\leq \sup_{0 < t < 1} \|dF(u + t\xi) - dF(u)\| \|\xi\|$$

$$= o(\|\xi\|).$$

In the last step, we used the continuity of $dF(u)$.

**Discussion. The Fréchet derivative.** Though the Gâteaux derivative is straightforward to compute, for theoretical considerations, one needs often a stronger notion of derivative: the Fréchet derivative. Before we define the Fréchet derivative, let us remark that equation (D.1) is equivalent to

$$F(u + \lambda\xi) - F(u) = \lambda dF(u)\xi + o(\lambda),$$

(D.4)

were $o(\lambda)$ is a vector in $Y$ satisfying $\lim_{\lambda \to 0} \|o(\lambda)\|/\lambda = 0$. Notice that in general, $o(\lambda)$ depends on $\xi$.

The map $F$ is called Fréchet differentiable at $u \in X$ if and only if there exists a bounded linear map $dF(u) \in L(X, Y)$ s.t. (D.2) holds as $\|\xi\| \to 0$. The operator $dF(u)$ satisfying (D.2) is called the Fréchet derivative of $F$ at the point $u$.

From this definition and equation (D.4), it is clear that if $F$ is Fréchet differentiable at $u$, with Fréchet derivative $dF(u)$, then $F$ is Gâteaux differentiable at $u$ with Gâteaux derivative given by the same operator $dF(u)$. In the opposite direction, if $F \in C^1$, then the preceding theorem implies...
Theorem D.1. If $F$ is continuously Gâteaux differentiable at $u \in X$, with Gâteaux derivative $dF(u)$, then $F$ is Fréchet differentiable at $u$, and the Fréchet derivative is given by the same operator $dF(u)$.

For a detailed discussion of Fréchet and Gâteaux derivatives, we refer to [38].

In everything that follows, by the derivative $dF(u)$ we understand the Gâteaux derivative. We point out that in most of our applications, we deal with $C^1$ maps, in which case the Fréchet and Gâteaux derivatives coincide, according to the last theorem.

We conclude this section with some useful rigorous results about Gâteaux derivatives of composition operators $F(u) = f \circ u$, where $f$ is a fixed function and $u$ belongs to the space of differentiable functions. Such operators appear often in applications. The statements below are useful in this context. An important result in this direction is the following.

Proposition 14. Let $\Omega \subset \mathbb{R}^n$. Let $F(u) = f \circ u$ with $f \in C^1(\Omega)$ and obeying the estimates

$$|f^{(k)}(u)| \leq c|u|^{p+1-k} \quad \text{for } k = 0, 1, 2 \quad (D.5)$$

for some $p \geq 1$. Then $F : H^r(\Omega) \to L^2(\Omega)$ and is $C^1$, provided the indices $p$ and $r$ satisfy the relation $p < \frac{2r}{(n-2r)_+}$ and $r > 0$.

Proof. Let $u \in H^r(\Omega)$. Then by the Sobolev embedding theorem (see Section ??) $u \in L^{2(p+1)}(\Omega)$ since $\frac{n}{p} > \frac{n}{2} - r$. Hence

$$|f(u)| \leq c|u|^{p+1} \in L^2.$$ 

This shows that $F : H^r(\Omega) \to L^2(\Omega)$. To show that $F \in C^1$, we compute for $h > 0$

$$\frac{1}{h}(f(u + h\xi) - f(u)) = f'(u)\xi + R_h(\xi)$$

where $R_h(\xi) = \frac{1}{h} \int_0^h (f'(u + t\xi) - f'(u))\xi dt$. Now we estimate by the mean value theorem

$$||R_h(\xi)||_2 \leq \frac{1}{h} \int_0^h \|f''(u + \bar{\xi})\xi^2\|_2 t dt$$

for some $0 \leq \bar{\xi} \leq t$. Using the estimate $|f''(u)| \leq c|u|^{p-1}$ and the triangle and Hölder inequalities we derive furthermore

$$||R_h(\xi)||_2 \leq \frac{c}{h} \int_0^h \|u + \bar{\xi}|^{p-1}\xi^2\|_2 t dt$$

$$\leq h c \left(\|u\|^{p-1}\xi^2\|_2 + \|\xi\|^{p+1}\|_2\right)$$

$$\leq c h \left(\|u\|^{p-1}\|\xi\|^{2(p+1)} + \|\xi\|^{p+1}\|_{2(p+1)}\right).$$

Hence $||R_h(\xi)||_2 \to 0$ and therefore $\frac{1}{h}(f(u + h\xi) - f(u)) - f'(u)\xi \to 0$ as $h \to 0$. Thus $F$ is $C^1$. 

Corollary 15. Assume $f : \mathbb{C} \to \mathbb{C}$ satisfies estimates $(D.5)$ with $1 \leq p < \frac{2r}{(n-2r)_+}$, $r > 0$. Define $F(u) = -\Delta + f(u)$. Then $F : H^r(\Omega) \to L^2(\Omega)$ and is $C^1$.

Furthermore, we have
Theorem 16. Let $F(u) = f \circ u$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $k > n/2$. If $f \in C^{k+1}(\mathbb{R})$, then $F : H^k(\Omega) \to H^k(\Omega)$, and $F$ is $C^1$.

Exercise 40. Prove this theorem for $\frac{n}{2} < k \leq 2$ and $l = 0, k$.

For a complete proof of the Theorem, see [24], page 221.

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