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UNIQUE ERGODICITY, STABLE ERGODICITY AND THE MAUTNER PHENOMENON FOR DIFFEOMORPHISMS

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Introduction

Beginning with [GPS] the first two authors have been studying stable ergodicity of volume preserving partially hyperbolic diffeomorphisms on a compact manifold M. The most recent survey on the subject is [PS3]. A key issue is the way in which the strong stable and strong unstable manifolds foliate M. To prove ergodicity one assumes **essential accessibility**, namely that every Borel set $S \subset M$ which consists simultaneously of whole strong stable leaves and whole strong unstable leaves has measure zero or one. Such a set S is said to be **us-saturated**. As essential accessibility is a measure theory concept, it is difficult to verify and even more difficult to prove stable under perturbation. A stronger assumption is **full accessibility**[1] in which it is required that M and the empty set are the only us-saturated sets. In many cases full accessibility is stable under perturbation, and this leads to stable ergodicity.

We have conjectured that the stably ergodic diffeomorphisms are open and dense among C^2 volume preserving partially hyperbolic diffeomorphisms. Our plan of attack was to prove that an open and dense subset of partially hyperbolic diffeomorphisms are fully accessible. As already noted, this seems far easier than the similar assertion for essential accesssibility, so we focused on the full accessibility property. The following recent developments, however, caused us to reconsider our position and to shift our attention more in the direction of essential accessibility.

(a) Among affine diffeomorphisms of finite volume, compact homogeneous spaces those which are stably ergodic among left translations are precisely those with the essential accessibility property [St3]. In other words, affine stable ergodicity is equivalent to essential accessibility. The proof relies significantly on the structural properties of Lie groups.
(b) As was shown by Federico Rodgriguez Hertz, essential accessibility without full accessibility sometimes leads to (nonlinear) stable ergodicity, [RH].
(c) The Mautner phenomenon from representation theory leads to a proof of half of the affine stable ergodicity result mentioned in (a), while in [PS2] we establish a nonlinear version of the Mautner phenomenon, which we apply to nonlinear stable ergodicity.

Below, we give a proof of the Mautner phenomenon in the case it is used for (a), but instead of structural properties of Lie groups or their representation theory,

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[1] In previous papers, we referred to full accessibility as us-accessibility, and to a stronger condition as homotopy accessibility. The latter is always stable under perturbation and is often a consequence of the former.

we use Birkhoff's theorem as in the Hopf-Anosov argument for the ergodicity of Anosov systems. This proof makes us feel we have landed in the right place.

We would like to see a unified explanation of these Mautner phenomena, one that might generalize Rodriguez-Hertz's theorem to *all* essentially accessible affine diffeomorphisms. To this end we wondered what more of a potentially useful nature could be said about the strong stable and unstable manifold foliations. The third author has extended the results in his monograph [St1], and answered question 6.8 of [BPSW] for affine diffeomorphisms – namely, in the affine, essentially accessible case, the strong stable or strong unstable manifold foliations are uniquely ergodic. See Theorem 0.7 below.

The Mautner phenomenon

Roughly speaking the Mautner phenomenon refers to invariance of a function along trajectories of one flow implying invariance along certain transverse flows. Mautner first observed the phenomenon in an affine ergodicity proof – invariance of a function along the geodesic flow (for a compact surface of constant negative curvature) implies invariance along the horocycle flows. It has been generalized considerably by Auslander-Green, Dani and Moore for the ergodic theory of flows on homogeneous spaces. See [St1] for references, proofs and a discussion of the results.

A version the Mautner phenomenon applies precisely to prove the ergodicity of the essentially accessible affine diffeomorphisms of finite volume, compact homogeneous spaces. In [PS3] we sketched a proof of this result. Below, we do a better job. The proof is quite close in structure to the best proof we have for partially hyperbolic diffeomorphisms with the essential accessibility property, see [PS2] and [PS3].

Let G be a connected Lie group, and $B \subset G$ a closed subgroup such that G/B is compact and of finite volume, i.e., G/B admits finite G-invariant volume.[2] Let $f \in \mathrm{Aff}(G/B)$ be an **affine map** of G/B, i.e., $f = L_a \circ A$, where $L_a : G/B \to G/B$ is left transation by a fixed element $a \in G$ and $A : G/B \to G/B$ is a map induced by a fixed automorphism $\overline{A} \in \mathrm{Aut}(G)$ such that $\overline{A}(B) = B$. The covering map $\overline{f} = L_a \circ \overline{A} : G \to G$ induces an automorphism $d\overline{f}$ of the Lie algebra $\mathfrak{g}$. With respect to $d\overline{f}$, the Lie algebra $\mathfrak{g}$ splits into generalized eigenspaces $\mathfrak{g} = \mathfrak{g}^u \oplus \mathfrak{g}^c \oplus \mathfrak{g}^s$ such that the eigenvalues of $d\overline{f}$ are respectively outside, on, or inside the unit circle. The corresponding connected subgroups G^u, G^c, and G^s are the unstable, neutral, and stable **horospherical subgroups**. Their orbits form the (strong) unstable, center, and (strong) stable foliations for f on G/B. These facts are proved in [PSS].

Let $H \subset G$ be the subgroup generated by G^u and G^s. It is normal and called the **hyperbolic subgroup** for f. See [PS1]. Under the previous conditions, the version of the Mautner phenomenon that we prove is:

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[2] "Finite volume" includes the requirement that the measure on G/B be G-invariant. In particular,

(a) If Γ is a uniform discrete subgroup of $G = \mathrm{SL}(2, \mathbb{R})$ then the homogeneous space G/Γ is of finite volume, but
(b) if T is the subgroup of G consisting of upper triangular matrices then the homogeneous space $G/T \approx S^1$ is not of finite volume because there is no G-invariant measure on it.

Theorem 0.1. *Every f-invariant L^1 function $\phi : G/B \to \mathbb{R}$ is essentially constant on cosets $x\overline{HB}$.*

Since the values of an L^1 function are ambiguous on a zero set, the meaning of Theorem 0.1 is this: we assume that for almost every $y = xB \in G/B = M$, $\phi(f(y)) = \phi(y)$, and we conclude that there is a zero set $Z \subset M$ such that for each pair $y, y' \in G \setminus Z$ that lie in a common left coset of $\overline{HB}$, we have $\phi(y) = \phi(y')$.

Corollary 0.2. *If $\overline{HB} = G$ then f is ergodic.*

In the same way, we say that ϕ is **essentially constant along the cells of a partition** $\mathcal{P}$ of M if, excluding a zero set from M, $\phi(x) = \phi(x')$ whenever x, x' lie in a common cell of $\mathcal{P}$. To prove Theorem 0.1 we use the following four applications of Fubini's Theorem. The well educated reader may find them superfluous.

Lemma 0.3. *Suppose that X, Y are σ-finite measure spaces and $\phi_n : X \times Y \to \mathbb{R}$, $n = 1, 2, \dots,$ is a sequence of measurable functions that converges almost everywhere to a limit ϕ. If each ϕ_n is essentially constant with respect to x then the same is true for ϕ.*

Proof The hypothesis means that for each n, there is a zero set Z_n, and if we call its y-slice

$$Z_n(y) = \{x \in X : (x, y) \in Z_n\}$$

then the function $x \mapsto \phi_n(x, y)$ is constant on $X \setminus Z_n(y)$. By Fubini's Theorem the set of slices $X \times y$ for which $Z_n(y)$ has positive X-measure is a zero set Z_n^*. Let Z be the union of the zero sets Z_n^* and the zero set on which $\phi_n(x, y)$ fails to converge to $\phi(x, y)$. It is a zero set. On each slice not in Z, $\phi_n(x, y)$ is almost everywhere constant and converges pointwise to $\phi(x, y)$. Hence $\phi(x, y)$ is essentially constant with respect to x.

Proposition 0.4. *If $f : M \to M$ is a C^2 measure preserving, partially hyperbolic diffeomorphism and $\phi : M \to \mathbb{R}$ is a measurable f-invariant function then ϕ is essentially constant along the leaves of the strong stable and strong unstable foliations.*

Proof Suppose at first that ϕ is L^1. The Birkhoff Ergodic Theorem provides a continuous projection $\beta : L^1(M, \mathbb{R}) \to \mathrm{Inv}^1(f)$ where

$$\beta(\psi)(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(x)),$$

the limit exists almost everywhere, and $\mathrm{Inv}^1(f)$ is the space of L^1 invariant functions. Clearly $\beta(\phi) = \phi$. If ψ is continuous then it is straightforward to see that if $\beta(\psi)(x)$ exists at one point of a strong stable manifold $W^{ss}(p)$ then it exists at all points of $W^{ss}(p)$ and has the same value. Thus, $\beta(\psi)$ is essentially constant along the leaves of the strong stable foliation.

The space $C^0(M, \mathbb{R})$ is dense in $L^1(M, \mathbb{R})$, and so there is a sequence of continuos functions ψ_n that converges to ϕ in the L^1 sense. Since β is continuous, $\beta(\psi_n)$ converges to ϕ in the L^1 sense. The Riesz Lemma gives a subsequence such that

$$\lim_{k\to\infty} \beta(\psi_{n_k})(x) = \phi(x) \text{ almost everywhere.}$$

Applying Lemma 0.3 in a $\mathcal{W}^{ss}$-foliation box is permissable because the foliation is absolutely continuous. Hence ϕ is essentially constant along strong stable plaques.

Covering M with foliation boxes completes the proof that ϕ is essentially constant along strong stable leaves. Replacing f with f^{-1} gives the same assertion for the strong unstable foliation.

Finally, if ϕ is measurable but not L^1, we replace it by a cut-off version

$$\phi_L(x) \;=\; \begin{cases} \phi(x) & \text{if } |\phi(x)| \le L \\ 0 & \text{if } |\phi(x)| > L. \end{cases}$$

Clearly, ϕ_L is L^1 and f-invariant. Essential constancy of ϕ_L along the leaves of the strong stable and strong unstable foliations implies the same for ϕ

The next lemma generalizes the fact that almost everywhere invariance of a measurable function along orbits of a flow is implied by almost everywhere invariance for each time-t map.

Lemma 0.5. *Suppose that the smooth manifold M carries a smooth measure m, $\phi : M \to \mathbb{R}$ is a measurable function, G is a Lie group that acts smoothly on M, and the G-orbits foliate M. The following are equivalent:*

(a) *ϕ is essentially constant along the orbits of G.*
(b) *For each $g \in G$, there is a zero set $Z_g \subset M$ such that for all $x \in M \setminus Z_g$, $\phi(x) = \phi(gx)$.*
(c) *There is a single zero set Z such that for all $x \in M \setminus Z$, and for all $g \in G$, $\phi(x) = \phi(gx)$.*

Proof We write the action of G on M as $x \mapsto gx$, and show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

Assume (a). Then there is a zero set $Z \subset M$ such that if $x, x' \in M \setminus Z$ belong to a common G-orbit then $\phi(x) = \phi(x')$. Fix $g \in G$ and define

$$Z_g = Z \cup g^{-1}Z.$$

Since G acts smoothly, Z_g is a zero set. If $x \in M \setminus Z_g$ then $x, gx \in M \setminus Z$, and by (a), $\phi(x) = \phi(gx)$. This gives (b).

Assume (b). Uncountability of G precludes taking Z as the union of the Z_g, $g \in G$. We first look at the question in a foliation box $U \subset M$, which we coordinatize as $W \times Y$ where the local G-orbits are plaques $P = W \times y$. We claim that there is a zero set $Z(U) \subset U$ such that if $x, x' \in U \setminus Z(U)$ lie on a common plaque then $\phi(x) = \phi(x')$. Fix any $a < b$ and consider the sets

$$A = \{x \in U : \phi(x) \le a\} \text{ and } B = \{x \in U : \phi(x) \ge b\}.$$

Let $Z(a, b)$ be the union of those plaques such that both slices $A_y = \{w : (w, y) \in A\}$ and $B_y = \{w : (w, y) \in B\}$ have positive W-area. It is enough to show that $Z(a, b)$ is a zero set for then we can take

$$Z(U) = \bigcup Z(a, b)$$

where (a, b) ranges over all rationals with $a < b$.

Suppose the local assertion is false: there is a set $Y_0 \subset Y$ of positive Y-area such that for each $y \in Y_0$, both slices A_y and B_y have positive W-area. Let A_0 be the set of density points of A. It differs from A by a zero set. Fubini's Theorem implies that almost every plaque meets A in a W-zero set if and only if it meets A_0 in a W-zero set. The same is true for B. Thus, almost every plaque in $W \times Y_0$ contains density points of both A and B. Fix such a pair of density points p, q in a common plaque. Note that p, q need not lie in A, B, and their slices need not have positive W-area.

Since p, q lie in a common plaque we can fix a $g \in G$ with $gp = q$, where we write the action as $h_g(x) = gx$. Since G acts smoothly on M, $F : x \mapsto gx$ is a diffeomorphism that slides along plaques and sends p to q. The plaques meeting Z_g in sets of non-zero W-area form a zero set, and discarding it from U produces a subset U' of full measure such that for every plaque $P \subset U'$,

$$\phi(gx) = \phi(x) \text{ almost everywhere with respect to } W\text{-area on } P.$$

Thus, for each plaque $P \subset U'$,

$$F(A' \cap P) = A' \cap P \text{ modulo a } W\text{-zero set on } P,$$

where $A' = A \cap U'$. Since A and A' differ by a zero set, p is a density point of A'. A diffeomorphism preserves all density points, so $F(p) = q$ is a density point of A', and hence of A. This is obviously impossible – A and B cannot have a common density point.

Globalization presents no problem. For M can be covered by countably many open foliation boxes U, and discarding every G-orbit that meets one of the zero sets $Z(U)$ leaves a full measure set $M' \subset M$, and ϕ is essentially constant along every G-orbit in M'. This gives (c).

Assume (c). Then there is a zero set $Z \subset M$, such that for all $g \in G$ and all $x \in M \setminus Z$, we have $\phi(x) = \phi(gx)$. Now, if $x, x' \in M \setminus Z$ lie on the same G-orbit then $x' = gx$ for some $g \in G$, and hence $\phi(x) = \phi(x')$, which gives (a).

Lemma 0.6. *If $\phi : M \to \mathbb{R}$ is measurable and $h : G \to \mathrm{Homeo}(M)$ is a nice action then the stabilizer*

$$\mathrm{St}(\phi) = \{g \in G : \ \phi(x) = \phi(h_g(x)) \ a.e.\}$$

is a closed subgroup of G.

Proof Here, "nice" means that M is locally compact, metrizable, h is continuous, μ is a regular probability measure on M, and the Radon-Nikodym derivatives of h_g are locally uniformly bounded. In the case at hand, μ is a G-invariant measure on the homogeneous space $M = G/B$, and the action is left or right G-multiplication.

Suppose that $g, g' \in \mathrm{St}(\phi)$. Absolute continuity implies that $h_{g'}$ is a zero-set-preserving change of variables. Hence

$$\phi \circ h_g = \phi \ \ (a.e.) \Rightarrow \ \ \phi \circ h_g \circ h_{g'} = \phi \circ h_{g'} = \phi \ \ (a.e).$$

Since $h_{gg'} = h_g \circ h_{g'}$, we have $gg' \in \mathrm{St}(\phi)$. Similarly, absolute continuity of $h_{g^{-1}}$ gives

$$\phi \circ h_g = \phi \ \ (a.e.) \Rightarrow \ \ \phi \circ h_g \circ h_{g^{-1}} = \phi \circ h_{g^{-1}} \ \ (a.e),$$

and hence $g^{-1} \in \mathrm{St}(\phi)$, which completes the proof that the stabilizer is a subgroup of G.

To prove closedness, suppose that $g_n \to g$ and $g_n \in \mathrm{St}(\phi)$ for all n. Call $h_{g_n} = h_n$ and $h_g = h$. We must show that $\phi \circ h = \phi$ almost everywhere. Since the issue is local, it is enough to choose a compact neighborhood N of an arbitrary $x_0 \in X$ and show that $\phi \circ h = \phi$ almost everywhere on N. Continuity of the action and compactness of N imply that $h_n|_N \to h|_N$ uniformly. Thus there is a compact neighborhood W of $h(N)$ such that for all $n \geq$ some n_0, we have

$$h_n(N) \subset W.$$

Lusin's Theorem states that ϕ is uniformly continuous on a compact subset $K \subset W$ where we can make $\mu(W \setminus K)$ as small as we want. Uniform local boundedness of the Radon-Nikodym derivatives implies that we can thereby force $\mu(N \setminus h^{-1}K)$ and $\mu(N \setminus h_n^{-1}K)$ to be as small as we want.

Let $\epsilon > 0$ be given. Choose K as above so that for each $n \geq n_0$,

$$\mu(S_n) < \epsilon \text{ where } S_n = N \setminus (h^{-1}(K) \cup h_n^{-1}(K)).$$

Metrize W with d. There is a $\delta > 0$ such that if $y, y' \in K$ and $d(y, y') < \delta$ then $|\phi(y) - \phi(y')| < \epsilon$. There is also an $n_1 \geq n_0$ such that for each $n \geq n_1$ and each $x \in N$ we have

$$|h_n(x) - h(x)| < \delta.$$

Hence, for each $n \geq n_1$ and for all $x \in N \setminus S_n$ we have

$$|\phi \circ h_n(x) - \phi \circ h(x)| < \epsilon,$$

and consequently

$$\mu\{x \in N : |\phi \circ h_n(x) - \phi \circ h(x)| \geq \epsilon\} < \epsilon.$$

This means that $\phi \circ h_n|_N$ converges to $\phi \circ h|_N$ in measure. By Riesz's Lemma, there is a subsequence converging almost everywhere. Since $\phi \circ h_n = \phi$, we get $\phi \circ h|_N = \phi|_N$ almost everywhere, and hence $g \in \mathrm{St}(\phi)$ as claimed.

Proof[Proof of Theorem 0.1] We are given an L^1 function $\phi : G/B \to \mathbb{R}$ that is invariant under the affine diffeomorphism $f : G/B \to G/B$, and we claim that ϕ is essentially constant along left cosets of $\overline{HB}$. By Proposition 0.4, ϕ is essentially constant along the leaves of the strong stable and strong unstable foliations, which, as stated above, are the left orbits of Lie subgroups G^s and G^u. By Lemma 0.5 there is a zero set Z such that for all $g \in G^u \cup G^s$ and for all $y \in M \setminus Z$ we have $\phi(y) = \phi(gy)$. We claim that ϕ is also essentially constant along H-orbits, where H is the subgroup generated by G^u and G^s.

Because H is normal, its orbits foliate G/B. Take any $h \in H$. It is expressed as a product

$$h = g_1 \cdot g_2 \cdots g_n$$

where $g_1, \ldots, g_n$ are alternately in G^u and G^s. Now

$$Z_h = g_n^{-1} \cdots g_2^{-1}(Z)$$

is a zero set, and for all $y \in M \setminus Z_h$ we have $\phi(y) = \phi(hy)$. Lemma 0.5 implies that ϕ is essentially constant along left H-orbits. Since H is normal, left orbits are the same as right orbits and Lemma 0.5 gives a zero set Z^r such that for all $h \in H$ and for all $y \in M \setminus Z^r$ we have

$$\phi(y) = \phi(yh)$$

where $yh = xhB$ and $y = xB$. Clearly we also have

$$\phi(yhb) = \phi(xhbB) = \phi(xhB) = \phi(yh) = \phi(y),$$

so HB is contained in the stabilizer of ϕ. Lemma 0.6 implies that the stabilizer contains also the closure $\overline{HB}$. Thus, for each $g \in \overline{HB}$ we have a zero set Z_g such that for all $y \in M \setminus Z_g$, $\phi(y) = \phi(yg)$. Lemma 0.5 implies that ϕ is essentially constant along left cosets of $\overline{HB}$.

Unique ergodocity

Due to Dani (see [St1] §1), $f \in \mathrm{Aff}(G/B)$ is well known to be a K-automorphism of G/B iff $G = \overline{HB}$. We prove the following result.

Theorem 0.7. *Let $f : G/B \to G/B$ be an affine map on a compact space G/B of finite volume. Then the following conditions are equivalent:*
(1) *f is a K-automorphism of G/B, i.e. $G = \overline{HB}$,*
(2) *G^s is ergodic on G/B,*
(3) *G^s is minimal on G/B,*
(4) *G^s is strictly ergodic on G/B.*

We do not know how to prove Theorem 0.7 directly. Instead, we show how to derive it (at least, in many cases; the general case is considered somewhat differently) from the following classical result. Recall that f is said to be *semisimple* if $d\overline{f} : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable over $\mathbb{C}$.

Theorem 0.8. [B],[V],[EP] *Let $f : G/B \to G/B$ be a semisimple affine map on a compact space G/B of finite volume. Then the conditions (2), (3), and (4) from Theorem 1 are equivalent to the following condition:*
(1′) *f is weak mixing on G/B.*

Remark 1. Formally speaking, [EP] had to do with the case when f is a pure translation on G/B, i.e. $f = L_g$ for some $g \in G$. But the proof does work for affine maps as well. Also, in Theorem 0.8 it suffices to assume that $d\overline{f}$ is semisimple on $\mathfrak{g}^c$ only, i.e., is isometric on the neutral foliation of G/B (with respect to a metric on G/B induced by suitable right G-invariant metric on G).

Remark 2. If f is semisimple then (1) and (1′) are equivalent. In fact, let f be weak mixing on G/B. Then it is weak mixing on $G/\overline{HB}$. On the other hand, if f is semisimple then it is isometric on $G/\overline{HB}$. Hence $G/\overline{HB}$ is trivial, i.e., f is K-automorphism. In the general case (1) is of course stronger than (1′).

To derive Theorem 0.7 from Theorem 0.8 we make two observations.

Observation 1. With no loss of generality we can assume that B is a *uniform lattice* in G, i.e., a discrete subgroup whose quotient space G/B is compact. In fact, let $D \subset B$ be the maximal connected subgroup that is normal in G. Since $\overline{A}(B) = B$, it follows that $\overline{A}(D) = D$. So we can replace G/B by G'/B', where $G' = G/D$ and $B' = B/D$. Hence we can assume that D is trivial, i.e., B is a *quasi-lattice* in G. On the other hand, due to Witte it is known (see [St1], Section 9) that (1) and (2) both imply that the radical of G is nilpotent and B is Zariski dense in the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. This means, in particular, that any connected subgroup of G normalized by B, is normal in G. Hence, if B is a quasi-lattice in G then its identity component B_0 is trivial, i.e., B is in fact discrete.

Observation 2. It is clear that (4) ⇒ (3) ⇒ (2) for any (not necessarily compact) space G/B of finite volume. Furthermore, the G^s-action on $G/\overline{HB}$ is trivial because $G^s \subset H$ and H is normal in G. Hence (2) ⇒ (1).

So, it suffices to prove that (1) ⇒ (4). We start with the case when f is a pure transation, i.e., $f = L_g$. Since the radical of G is nilpotent, it follows that any element of G admits a Jordan decomposition, i.e., $g = s \times u$, where $s \in G$ is semisimple and $u \in G$ is unipotent. Since s and g have common horospherical subgroups, it follows that H is hyperbolically generated subgroup both for $f = L_g$

and L_s. Hence (1) implies that L_s is a K-automorphism as well. Then it suffices to apply Theorem 0.8 to L_s and conclude that (1) implies (4).

Now let $f = L_g \circ A$. One can try to reduce the general case to the previous one using a suspension construction as in [PSS]. The idea is to embed some power A^k into one-parameter subgroup $C \subset \mathrm{Aut}(G)$ and replace G/B by G'/B', where $G = C \cdot G$ and $B' = A^{k\mathbb{Z}} \cdot B$. Then G/B is invariant under pure translation $L_h : G'/B' \to G'/B'$ for some $h = h(g, A, k) \in G'$, and the action of L_h on G/B coincides with that of f^k. So, in a sense, in [PSS] we reduced the general case to the case of pure translation. The problem, however, is that L_h is only ergodic on G'/B' and not even weak mixing. (Clearly, the G^s-action on G'/B' is not ergodic either.)

So we need something else to apply Theorem 0.8 in the general case. One can try to find a semisimple affine map $f' : G/B \to G/B$ having the same horospherical subgroups as f as was done for pure translations. This is easy to do in the cases when G is either semisimple or nilpotent.

In fact, let G be semisimple. Then it is well known that the group of inner automorphisms is of finite index in $\mathrm{Aut}(G)$. Hence there exists $k \in \mathbb{N}$ and $a \in G$ such that $A^k(x) = axa^{-1},\ x \in G$. It follows that $f^k = L_h$ for some $h = h(g, A, k) \in G$. We fall into the pure translation case, so we can replace $f^k = L_h$ by its semisimple part $L_s,\ s \in G$.

Now consider the abelian case: $G = \mathbb{R}^n,\ B = \mathbb{Z}^n$ and assume for simplicity that $f = A \in \mathrm{SL}(n, \mathbb{Z})$. It is well known that for some $k \in \mathbb{N}$, $A^k = S \times U$, where S is semisimple, U is unipotent, and both S and U are in $\mathrm{SL}(n, \mathbb{Z})$. Hence in this case one can replace $f^k = A^k$ by its semisimple part S^k.

This can be generalized to the nilpotent case to prove that for some $k \in \mathbb{N}$, f^k admits Jordan decomposition inside $\mathrm{Aff}(G/B)$ into semisimple and unipotent parts. As for the general case, we know that (1) implies that G is a semidirect product $G = P \cdot N$, where P is a semisimple subgroup and N is the nilradical of G. Apparently, combining semisimple and nilpotent cases, one can decompose some power of f into commuting semisimple and unipotent affine maps on G/B to apply Theorem 0.8.

However, this seems rather technically involved and in the general case we prefer not to use Theorem 0.8 directly. First we demonstrate that (1) implies (2).

Lemma 0.9. *Let G/B be of finite volume and $f \in \mathrm{Aff}(G/B)$ be a K-automorphism. Then the G^s-action on G/B is ergodic.*

Proof. G^s is a unipotent subgroup of G and according to the Mautner phenomenon (see [St1], Section 2), the ergodic decomposition for the G^s-action on G/B is of the form $\{x\overline{MB},\ x \in G\}$, where $M \subset G$ is the smallest normal subgroup of G containing G^s. Since B is Zariski dense in the Ad-representation, it follows that $M' = (\overline{MB})_0$ is normal in G. Clearly, $\overline{A}(M) = M$. Since $\overline{A}(B) = B$, it follows that $\overline{A}(M') = M'$ and f covers an affine map $f' : G'/B' \to G'/B'$, where $G' = G/M',\ B' = M'B/M'$. Since G'/B' is of finite volume, it follows that f' keeps G'-invariant measure on G'/B' invariant. But the stable horospherical subgroup for f' in G' is trivial, so the same is valid for the unstable horospherical subgroup. It follows that $G^u \subset M'$ and then $H \subset M'$. But $G = \overline{HB}$ and hence $M' = G$, and thus the G^s-action on G/B is ergodic. □

It remains to prove that the G^s-action on the compact space G/B is minimal and strictly ergodic whenever it is ergodic. If G is semisimple, this can be deduced from Theorem 0.8 as above. Also, there is no problem if G is nilpotent. In fact, by Furstenberg's theorem (see [St1], §3) any ergodic homogeneous flow on a nilmanifold is minimal and strictly ergodic. In the general case this approach can be extended as follows.

Theorem 0.10. [St2] *Let $f : G/B \to G/B$ be an affine map on a compact space of finite volume. Then the G^s-action on each invariant homogeneous subspace $x\overline{HB} \subset G/B,\ x \in G$, is ergodic and minimal.*

Remark 3. The proof of this result develops methods used by Dani in his study of horospherical flows (see [St1], Section 13) and eventually involves Theorem 0.8. The formulation and the proof were given in [St2] for pure translations only. However, using the suspension construction as above, one easily reduces to the pure translation case (note that no assumptions on ergodic properties of the f-action are needed in Theorem 0.10).

It follows from Theorem 0.10 that (1) implies (2) and (3). On the other hand, from the fundamental results of Ratner on unipotent flows (see [St1], Chapter II]) it follows that (3) implies (4) and we are done. □

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