WEIGHTED HODGE IDEALS OF REDUCED DIVISORS

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Abstract. We study the Hodge and weight filtrations on the localization along a hypersurface, using methods from birational geometry and the V-filtration induced by a local defining equation. These filtrations give rise to ideal sheaves called weighted Hodge ideals, which include the adjoint ideal and a multiplier ideal. We analyze their local and global properties, from which we deduce applications related to singularities of hypersurfaces of smooth varieties.

A. Introduction

In this paper, we continue the study of weighted Hodge ideals that started in [Ola22], where the focus was the 0-th weighted Hodge ideals, also called weighted multiplier ideals. We show that several results satisfied by the weighted multiplier ideals can be generalized under suitable conditions.

Let X be a smooth complex variety of dimension n. To an effective reduced divisor D on X one can associate a sequence of ideal sheaves $I_p(D) \subseteq \mathscr{O}_X$, called the Hodge ideals of D and studied in a series of papers [MP19a],[MP18],[MP19b], [MP20b], [MP20a]. They arise from the theory of mixed Hodge modules of M. Saito, which induces a Hodge filtration $F_\bullet \mathscr{O}_X(*D)$ by coherent $\mathscr{O}_X$-modules on $\mathscr{O}_X(*D)$, the sheaf of functions with poles along D, seen as a left $\mathscr{D}_X$-module. This $\mathscr{D}$-module underlies the mixed Hodge module $j_*\mathbb{Q}_U^H[n]$, where $j : U = X \setminus D \hookrightarrow X$. Saito showed that the Hodge filtration is contained in the pole order filtration, that is,

$$F_p\mathscr{O}_X(*D) \subseteq \mathscr{O}_X((p+1)D)$$

for all $p \geq 0$. Consequently, we can define the Hodge ideal $I_p(D)$ by

$$F_p\mathscr{O}_X(*D) = \mathscr{O}_X((p+1)D) \otimes I_p(D).$$

The $\mathscr{D}_X$-module $\mathscr{O}_X(*D)$ is also endowed with a weight filtration $W_\bullet \mathscr{O}_X(*D)$ by $\mathscr{D}_X$-submodules. The Hodge filtration of these submodules satisfies

$$F_pW_{n+l}\mathscr{O}_X(*D) \subseteq F_p\mathscr{O}_X(*D) \subseteq \mathscr{O}_X((p+1)D),$$

and similarly we can define the weighted Hodge ideals by

$$F_pW_{n+l}\mathscr{O}_X(*D) = \mathscr{O}_X((p+1)D) \otimes I_p^{W_l}(D).$$

The weighted Hodge ideals form a chain of inclusions

$$I_p^{W_0}(D) \subseteq I_p^{W_1}(D) \subseteq \cdots \subseteq I_p^{W_n}(D).$$

We can always understand the two extreme ideals in this chain. The first element in the list admits an easy description:

$$I_p^{W_0}(D) = \mathscr{O}_X(-(p+1)D).$$

On the other end, the last ideal in this chain is the usual p-th Hodge ideal, that is,

$$I_p(D) = I_p^{W_n}(D).$$

Unlike $I_p^{W_0}(D)$, for all the other degrees, the support of the scheme defined by $I_p^{W_l}(D)$ is contained in the singular locus of D.

Birational definition We give an alternative description of the weighted Hodge ideals in terms of a resolution of singularities. Let $f : Y \to X$ be a resolution of singularities of the pair (X, D) which is an isomorphism over $X \setminus D$, and let $E := (f^*D)_{\mathrm{red}}$. This description stems from the birational definition of Hodge ideals in [MP19a, §9], and uses right $\mathscr{D}$-modules. The $\mathscr{D}_Y$-module $\omega_Y(*E)$ admits a filtered resolution by $\mathscr{D}_Y$-modules given by

$$B^\bullet = 0 \to \mathscr{D}_Y \to \Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to \cdots \to \omega_Y(E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to 0.$$

Similarly, using the weight filtration on the sheaves of logarithmic p-forms (see (1.4)), we show that the complex

$$W_l B^\bullet = 0 \to \mathscr{D}_Y \to W_l\Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to \cdots \to W_l\omega_Y(E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to 0$$

is filtered quasi-isomorphic to the $\mathscr{D}_X$-module $W_{n+l}\omega(*E)$ (see Proposition 4.1).

The $\mathscr{D}_X$-module $\omega_X(*D)$ can be described using the filtered resolution of $\omega_Y(*E)$ described above. More precisely, we can define the complex $A^\bullet$ by

$$0 \to f^*\mathscr{D}_X \to \Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \to \cdots \to \omega_Y(E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \to 0$$

placed in degrees $-n, \ldots, 0$, and we have that,

$$R^0 f_* A^\bullet \cong \omega_X(*D)$$

(see [MP19a, §9]). To give the alternative description of the weighted Hodge ideals, we introduce the complex $C^\bullet_{l,p-n}$ defined as

$$0 \to f^*F_{p-n}\mathscr{D}_X \to W_l\Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} f^*F_{p-n+1}\mathscr{D}_X \to \cdots \to W_l\omega_Y(E) \otimes_{\mathscr{O}_Y} f^*F_p\mathscr{D}_X \to 0$$

and we show that the image of

$$R^0 f_* C^\bullet_{l,p-n} \to R^0 f_* A^\bullet = \omega_X(*D)$$

is precisely $F_{p-n}W_{n+l}\omega_X(*D) = I_p^{W_l}(D) \otimes \omega_X((p+1)D)$ (see Proposition 4.3).

Description of weighted Hodge ideals using the V-filtration. A very convenient local description of Hodge ideals was given in terms of the Kashiwara-Malgrange V-filtration of the graph embedding $i_+\mathscr{O}_X$ in [MP20b, Theorem A'] (see (5.1)), which works in the more general setting of Hodge ideals of $\mathbb{Q}$-divisors. In this case, we suppose that the reduced divisor $D \subseteq X$ can be defined by a regular function $f \in \mathscr{O}_X(X)$. Weighted Hodge ideals admit a similar description.

Theorem A. *Let X be a smooth complex variety and D a reduced divisor defined by a regular function $f \in \mathscr{O}_X(X)$. Then,*

$$I_p^{W_l}(D) = \left\{ \sum_{j=0}^{p} Q_j(1) f^{p-j} v_j \ : v = \sum_{j=0}^{p} v_j \partial_t^j \delta \in V^1 i_+ \mathscr{O}_X \text{ and } (t\partial_t)^l v \in V^{>1} i_+ \mathscr{O}_X \right\}.$$

The proof is based on two ideas. First, we can relate the Hodge filtration of $V^1 i_+ \mathscr{O}_X$ with that of $\mathscr{O}_X(*D)$ (see 5.2). Second, the weight filtration on the nearby cycles sheaf can be related to that of the local cohomology sheaf (Proposition 5.3). This is enough to understand all the weighted Hodge ideals in the case when D only has isolated weighted-homogeneous singularities (see Remark 5.7).

The description in Theorem A is useful to relate the weighted Hodge ideals with some invariants of the singularities, like the minimal exponent. Recall that to the variety $D \subseteq X$ we can associate the Bernstein-Sato polynomial $b_D(s)$. The polynomial $(s+1)$ divides $b_D(s)$, and we denote $\widetilde{b_D}(s) = b_D(s)/(s+1)$. The negative of the largest root of $\widetilde{b_D}(s)$ is called the *minimal exponent* of a D and is denoted $\widetilde{\alpha_D}$. This invariant encodes important properties of the singularities of D. For instance, it is a refined version of the log-canonical threshold, since $lct(X, D) = \min\{\widetilde{\alpha_D}, 1\}$. In particular, this implies that (X, D) is log-canonical if and only if $\widetilde{\alpha_D} \geq 1$. Moreover, it is a result of Saito that D has rational singularities if and only if $\widetilde{\alpha_D} > 1$.

The notions of log-canonicity and rationality can be described in terms of weighted Hodge ideals. Recall that 0-th weighted Hodge ideals, or weighted multiplier ideals, form a sequence of ideals interpolating between the adjoint ideal and a multiplier ideal. This is the case, as $I_0^{W_1}(D) = \mathrm{adj}(D)$ (see for instance [Ola22, Theorem A]) and $I_0(D) = \mathcal{J}((1-\varepsilon)D)$ for $0 < \epsilon \ll 1$ [BS05]. These two ideals identify if a singularity is respectively rational or log-canonical. We give an analogous description for the higher weighted Hodge ideals. The Hodge ideal $I_p(D)$ is trivial if and only if $\widetilde{\alpha_D} \geq p+1$, in which case we say that (X, D) is p-log-canonical. Also, the weighted Hodge ideal $I_p^{W_1}(D)$ is trivial if and only if $\widetilde{\alpha_D} > p+1$ (see Corollary 5.10), which some authors referred to as D being p-rational. The rest of the p-weighted Hodge ideals filter and measure the "distance" between (X, D) having p-log-canonical singularities and D being p-rational.

Isolated singularities. Recall that the weighted Hodge ideals satisfy

$$I_p^{W_{l-1}}(D) \subseteq I_p^{W_l}(D).$$

The difference between the two ideals can be described by the coherent sheaf $F_p \operatorname{gr}_{n+l}^W \mathscr{O}_X(*D)$ (see (6.1)). If D has isolated singularities, we give a description of the dimension of this sheaf at the singular points in terms of a resolution of singularities. For this, possibly after restricting to an open set, assume D has one isolated singularity $x \in D$. In this case, there exists a pure Hodge structure H_l for $l \geq 2$, such that the dimension of their Hodge pieces describes the desired dimension. More concretely,

$$\dim(F_p(\mathrm{gr}^W_{n+l}\,\mathscr{O}_X(*D))_x) = \sum_{r=0}^{p} \binom{n+p-r}{p-r} \dim(\mathrm{Gr}_F^{n-r} H_l) \tag{0.1}$$

(see §6 for more details). For this reason, to find the difference between two consecutive weighted Hodge ideals, it is enough to compute the dimensions of the spaces $\mathrm{Gr}_F^{n-p} H_l$.

Theorem B. *Let $g : \widetilde{D} \to D$ be a log-resolution of singularities that is an isomorphism outside of x. Let $G \subseteq \widetilde{D}$ be the exceptional divisor. Then*

$$\dim(\mathrm{Gr}_F^{n-p} H_l) = h^{p,n-l-p}(H^{n-2}(G))$$

if $l \geq 3$, and

$$\dim(\mathrm{Gr}_F^{n-p} H_2) = h^{p,n-p-2}(H^{n-2}(G)) - h^{n-p-1,p+1}(H^n(G)),$$

where $H^k(G) = H^k(G, \mathbb{C})$ and $h^{p,q}(H^k(G)) = \dim(H^{p,q}(\mathrm{Gr}^W_{p+q} H^k(G)))$.

When $p = 0$ the second summand in the description of $\dim(\mathrm{Gr}_F^{n-p} H_2)$ is 0 because the dimension of G is $n-2$, and therefore these dimensions are described as Hodge numbers of the middle cohomology of G. For $p \geq 1$ we cannot expect this term to be 0 in general, but this dimension admits a geometric interpretation (see Remark 6.8).

Vanishing results. Weighted Hodge ideals satisfy global results under suitable conditions. Let X be a smooth projective variety and D an ample divisor with at most isolated singularities. Under this assumptions, when $p = 0$ we have that

$$H^i(X, \omega_X(D) \otimes I_0^{W_l}(D)) = 0$$

for $i \geq 1$ and $l \geq 2$ [Ola22, Theorem E]. To generalize this result for all $p \geq 1$, we require the condition that $I_{p-1}^{W_l}(D) = \mathscr{O}_X$.

Theorem C. *Let X be a smooth projective variety of dimension n, and D an ample reduced effective divisor with at most isolated singularities. Suppose that $I_{p-1}^{W_1}(D)$ is trivial. Then*

(1) For $l \geq 2$ and $i \geq 2$,

$$H^i(X, \omega_X((p+1)D) \otimes I_p^{W_l}(D)) = 0.$$

(2) If $H^j(X, \Omega_X^{n-j}((p-j+1)D)) = 0$ for all $1 \leq j \leq p$, then

$$H^1(X, \omega_X((p+1)D) \otimes I_p^{W_l}(D)) = 0$$

for $l \geq 2$.

When $l = 1$ and $i = 1$ the vanishing does not hold in general. For an example see Remark 7.2. A Kodaira-type vanishing result is also satisfied for all $l \geq 1$, and the proof is based on a vanishing result by Saito [Sai90, Proposition 2.33] (see Proposition 8.1).

Applications. The global and local results we have discussed can be used to obtain results about the geometry of certain isolated singularities of hypersurfaces in $\mathbb{P}^n$. This is because the vanishing condition in Theorem C is satisfied when X is a toric variety.

Corollary D. *Let $D \subseteq \mathbb{P}^n$ be a hypersurface of degree d with at most isolated singularities. Let $Z_{l,p}$ be the scheme defined by $I_p^{W_l}(D)$. Then,*

$$H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \twoheadrightarrow H^0(\mathbb{P}^n, \mathscr{O}_{Z_{l,p}})$$

for $k \geq (p+1)d - n - 1$ if $l \geq 2$, and $k \geq (p+1)d - n$ if $l \geq 1$.

This result gives a bound on a certain type of isolated singularities we describe next. For simplicity, suppose D has at most one isolated singularity $x \in D$, and assume $\widetilde{\alpha_D} = p+1$. We describe first the case $p = 0$. This case corresponds to a log-canonical and not rational singularity. In this case, according to (0.1), the length of the scheme described by $I_0^{W_1}(D)$ is determined by $\mathrm{Gr}_F^0(H^{n-2}(G))$, using the notation of Theorem B. Ishii proved that in this case, $\dim(\mathrm{Gr}_F^0(H^{n-2}(G))) = 1$ [Ish85, Proposition 3.7]. This means that the ideal $I_0^{W_1}(D)$ is the maximal ideal of x in X, and that there exists exactly one degree $l \geq 2$ such that

$$\dim(\mathrm{Gr}_F^n(H_l)) = 1,$$

while the dimension for the other degrees is 0. A log-canonical singularity is of type $(0, n-l)$ in this case [Ish85, Definition 4.1].

Assume now that $\widetilde{\alpha_D} = p+1$ for an $p \in \mathbb{Z}_{\geq 0}$, and that D has at most one isolated singularity $x \in D$. In this case, we have an analogous picture. Namely, the ideal $I_p^{W_1}(D)$ is the maximal ideal of x in X (see Proposition 9.1), or equivalently, as $I_p(D) = \mathscr{O}_X$, the length of the scheme described by $I_p^{W_1}(D)$ is 1. This in particular means that

$$\mathrm{Gr}_F^{n-r} H_l = 0$$

for $l \geq 2$ and $0 \leq r \leq p-1$ by (0.1) and Theorem B. Moreover, by the same results, we know that there exists exactly one degree $l \geq 2$ such that

$$\dim(\mathrm{Gr}_F^{n-p} H_l) = 1,$$

while the dimension for all the other degrees is 0. Related invariants in similar conditions have been studied by Friedman and Laza in [FL22, Theorem 6.11 and Corollary 6.14].

In analogy to the case of log-canonical singularities, we call the singularity described above of type $(p, n-l-p)$ (see Definition 9.3). Weighted homogeneous singularities with $\widetilde{\alpha_f} = p+1$ are examples of singularities of type $(p, n-2-p)$ and the origin in $Z(x^2+y^2+z^2+u^2w^2+u^4+w^5) \subseteq \mathbb{A}^5$ gives an example of a singularity of type $(1, 5-3-1) = (1,1)$ (see Example 9.5). For a hypersurface of $\mathbb{P}^n$ with at most isolated singularities and $\widetilde{\alpha_D} = p+1$, we give a bound on the number of these singularities (see Corollary 9.6).

Restriction theorem. Finally, we study the behavior of weighted Hodge ideals of a pair (X, D) under the restriction of a hypersurface of X. Let $H \subseteq X$ be a smooth hypersurface, and D_H the restriction of D to H. If D_H is reduced, then we can also consider the pair (H, D_H) and their respective weighted Hodge ideals.

Theorem E. *Let X be a smooth variety and D an effective reduced divisor. Let $H \subseteq X$ be a smooth divisor such that $H \subsetneq \mathrm{Supp}(D)$ and $D_H = D\big|_H$ is reduced. Then, for every $p \geq 0$ and $l \geq 0$ we have*

$$I_p^{W_l}(D_H) \subseteq I_p^{W_l}(D) \cdot \mathscr{O}_H.$$

Moreover, if H is general, then we have an equality.

This is the analogue of the Restriction Theorem for Hodge ideals [MP18, Theorem A], and for multiplier ideals [Laz04, Theorem 9.5.1].

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B. Preliminaries

1. Mixed Hodge modules. In this section, we recall some facts about mixed Hodge modules and set up the notation we use throughout this paper.

Let X be a smooth variety of dimension n. Mixed Hodge modules introduced by Saito in [Sai88] are the main object used throughout this article. For a graded-polarizable mixed Hodge module M, we denote the underlying left regular holonomic $\mathscr{D}_X$-module by $\mathcal{M}$. In some contexts, it is more useful to use right $\mathscr{D}_X$-modules. Recall that if $\mathcal{M}$ is a left $\mathscr{D}_X$-module, the corresponding right $\mathscr{D}_X$-module is $\mathcal{M} \otimes_{\mathscr{O}_X} \omega_X$, where ω_X is the canonical sheaf. We mostly use left $\mathscr{D}$-modules, and in case we are using right $\mathscr{D}$-modules instead, we will say it explicitly.

A mixed Hodge module M is endowed with a weight filtration, which we denote by $W_\bullet M$, and

$$\mathrm{gr}_l^W M := W_l M / W_{l-1} M$$

is the quotient, which is a polarizable Hodge module of weight l. We denote by $F_\bullet \mathcal{M}$ the Hodge filtration. The de Rham complex is defined as:

$$\mathrm{DR}(\mathcal{M}) = \big[\mathcal{M} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathcal{M} \to \cdots \to \omega_X \otimes_{\mathscr{O}_X} \mathcal{M}\big][n],$$

and the Hodge filtration of $\mathcal{M}$ induces a filtration on this complex:

$$F_p \,\mathrm{DR}(\mathcal{M}) = \big[F_p\mathcal{M} \to \Omega_X^1 \otimes_{\mathscr{O}_X} F_{p+1}\mathcal{M} \to \cdots \to \omega_X \otimes_{\mathscr{O}_X} F_{p+n}\mathcal{M}\big][n].$$

The p-th subquotient of this filtration is the complex

$$\mathrm{gr}_p^F \mathrm{DR}(\mathcal{M}) = \big[\,\mathrm{gr}_p^F \mathcal{M} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathrm{gr}_{p+1}^F \mathcal{M} \to \cdots \to \omega_X \otimes_{\mathscr{O}_X} \mathrm{gr}_{p+n}^F \mathcal{M}\big][n].$$

Let D be a reduced effective divisor. The mixed Hodge module we mostly study in this paper is $j_*\mathbb{Q}_U^H[n]$, where $j : U = X \smallsetminus D \hookrightarrow X$, whose underlying $\mathscr{D}_X$-module is the sheaf of functions with poles along D denoted by $\mathscr{O}_X(*D)$. To study $\mathscr{O}_X(*D)$, it is sometimes convenient to use a resolution of singularities, and the properties of pushforwards. Fix a log-resolution of singularities of (X, D), that is, a proper birational morphism $f : Y \to X$ such that Y is smooth, it is an isomorphism over U, and $(f^*D)_{red} = E$ is a divisor with simple normal crossings. In this setup, we have that

$$f_+\mathscr{O}_Y(*E) \cong H^0 f_+\mathscr{O}_Y(*E) \cong \mathscr{O}_X(*D) \tag{1.1}$$

(see for example [MP19a, Lemma 2.2]). Since E is a simple normal crossings divisor, the weight filtration of the $\mathscr{D}_Y$-module $\mathscr{O}_Y(*E)$ can be described in terms of the intersections of its irreducible components. The lowest degree of the weight filtration is $n = \dim Y$, that is:

$$W_{n-1}\mathscr{O}_Y(*E) = 0.$$

The lowest piece corresponds to the canonical Hodge module of Y:

$$W_n\mathscr{O}_Y(*E) \cong \mathscr{O}_Y.$$

To describe the rest of the subquotients, we introduce the following very useful notation. Let

$$E = \bigcup_{i\in I} E_i.$$

The variety

$$E(l) = \bigsqcup_{\substack{J\subseteq I\\ |J|=l}} E_J,$$

with $E_J = \bigcap_{j\in J} E_j$, is a smooth and possibly disconnected variety. We denote $i_l : E(l) \to Y$ the map such that on each component is the inclusion. We have that

$$\operatorname{gr}^W_{n+l} \mathscr{O}_Y(*E) \cong i_{l+}\mathscr{O}_{E(l)} \tag{1.2}$$

with a Tate twist (see [KS21, Prop 9.2]).

In order to describe the weight filtration of a pushforward of a projective morphism, a useful tool is to use the spectral sequence associated to the weight filtration:

$$E_1^{p,q} = H^{p+q} f_+(\operatorname{gr}^W_{-p} \mathscr{O}_Y(*E)) \Rightarrow H^{p+q} f_+\mathscr{O}_Y(*E), \tag{1.3}$$

which degenerates at E_2, and there is an isomorphism:

$$E_2^{p,q} \cong \operatorname{gr}^W_q H^{p+q} f_+\mathscr{O}_Y(*E)$$

[Sai90, Proposition 2.15].

Finally, recall that the sheaf of p-forms with logarithmic poles along E denoted by $\Omega^p_Y(\log E)$ are endowed with a weight filtration. This increasing filtration consists of subsheaves

$$W_l\Omega^p_Y(\log E) \subseteq \Omega^p_Y(\log E) \tag{1.4}$$

such that if $z_1, \ldots, z_n$ are local coordinates on an open set V, and E is given by the equation

$$z_1 \cdots z_r = 0,$$

then in V, $W_l\Omega^p(\log E)$ is a $\mathscr{O}_V$ module generated by elements of the form

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_s}}{z_{i_s}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_{p-s}}$$

with $i_l \leq r$ and $s \leq k$ (see [CEZGL14, 3.4.1.2] for more details). For $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_{p-s}\}$ we use the notation

$$\frac{dz_I}{z_I} \wedge dz_J = \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_s}}{z_{i_s}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_{p-s}}.$$

C. Characterizations

2. Definition. In this section, we introduce weighted Hodge ideals using the theory of mixed Hodge modules.

A fundamental result by Saito about the Hodge filtration on $\mathscr{O}_X(*D)$ states that

$$F_p\mathscr{O}_X(*D) \subseteq \mathscr{O}_X((p+1)D)$$

(see [Sai93, Proposition 0.9]). The definition of Hodge ideals follows from this result. These ideals are denoted by $I_p(D)$, and are defined using the formula

$$F_p\mathscr{O}_X(*D) = I_p(D) \otimes \mathscr{O}_X((p+1)D)$$

(see [MP19a, Definition 9.4]). In this article, we study weighted Hodge ideals which are defined similarly using the weight filtration with which $\mathscr{O}_X(*D)$ is endowed. The Hodge filtration of the sub-$\mathscr{D}_X$ modules $W_{n+l}\mathscr{O}_X(*D)$ satisfies

$$F_pW_{n+l}\mathscr{O}_X(*D) \subseteq F_p\mathscr{O}_X(*D) \subseteq \mathscr{O}_X((p+1)D)$$

for all $p \geq 0$.

Definition 2.1 (Weighted Hodge ideals)**.** Let X be a smooth complex variety and D a reduced divisor. For $l \geq 0$ and $p \geq 0$, we define the ideal sheaf $I_p^{W_l}(D)$ on X by the formula

$$F_pW_{n+l}\mathscr{O}_X(*D) = I_p^{W_l}(D) \otimes \mathscr{O}_X((p+1)D).$$

We call $I_p^{W_l}(D)$ the *l-th weighted p-th Hodge ideal* of D.

There is in fact a chain of inclusions

$$I_p^{W_1}(D) \subseteq I_p^{W_2}(D) \subseteq \cdots \subseteq I_p^{W_{n-1}}(D) \subseteq I_p^{W_n}(D) \tag{2.2}$$

for all $p \geq 0$. Indeed, the weight filtration of $\mathscr{O}_X(*D)$ is an increasing filtration, hence

$$F_pW_{n+l}\mathscr{O}_X(*D) \subseteq F_pW_{n+l+1}\mathscr{O}_X(*D),$$

or equivalently

$$\mathscr{O}_X((p+1)D) \otimes I_p^{W_l}(D) \subseteq \mathscr{O}_X((p+1)D) \otimes I_p^{W_{l+1}}(D).$$

3. Simple normal crossings divisor. Weighted Hodge ideals can be described completely when the reduced divisor D has simple normal crossings. In this case, the Hodge filtration of $\mathscr{O}_X(*D)$ is fully understood, and from this information we can deduce the Hodge filtration of $W_{n+l}\mathscr{O}_X(*D)$.

Let D be a simple normal crossings divisor. In this case, the Hodge filtration of $\mathscr{O}_X(*D)$ admits a simple description:

$$F_p\mathscr{O}_X(*D) = F_p\mathscr{D}_X \cdot \mathscr{O}_X(D) \tag{3.1}$$

if $p \geq 0$, and 0 otherwise. Using this, one obtains a local description of the Hodge ideals. Let $x_1, \ldots, x_n$ be coordinates around $z \in X$, such that D is defined by $(x_1 \cdots x_r = 0)$. For every $p \geq 0$, the ideal $I_p(D)$ is generated around z by

$$\{x_1^{a_1} \cdots x_r^{a_r} : \ 0 \leq a_i \leq p, \ \sum a_i = p(r-1)\} \tag{3.2}$$

[MP19a, Proposition 8.2]. Weighted Hodge ideals of D admit a similar local description.

Proposition 3.3. *Let $x_1, \ldots, x_n$ be coordinates around $z \in X$, such that D is defined by $(x_1 \cdots x_r = 0)$. Then, for every $p \geq 0$ and $l \leq r$, $I_p^{W_l}(D)$ is generated around z by*

$$\{x_{j_1}^{a_1} \cdots x_{j_l}^{a_l} x_{I\setminus J}^{p+1} : \ J = \{j_1, \ldots, j_l\} \subseteq I, \ 0 \leq a_i \leq p \text{ and } \sum a_i = p(l-1)\},$$

where $I = \{1, \ldots, r\}$. For $l \geq r$, $I_p^{W_l}(D) = I_p(D)$ around z.

Proof. The Hodge filtration of $W_{n+l}\mathscr{O}_X(*D)$ also admits a simple description:

$$F_pW_{n+l}\mathscr{O}_X(*D) = F_p\mathscr{D}_X \cdot F_0W_{n+l}\mathscr{O}_X(*D). \tag{3.4}$$

Indeed, this follows from the fact that $\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D) \cong i_{l+}\mathscr{O}_{E(l)}$ with a Tate Twist, so that the analogous statement of (3.4) is true for $i_{l+}\mathscr{O}_{E(l)}$ (see e.g. [Sai09, Remark 1.1 iii].)

For the rest of the proof we use right $\mathscr{D}$-modules. By [Ola22, Proposition 4.1],

$$F_0W_{n+l}\omega_X(*D) = W_l\omega_X(D).$$

Around z, $W_l\omega_X(D)$ is generated by

$$\left\{\frac{\omega}{x_J}\right\}_{J\subseteq I, \ |J|=l}$$

where ω is the standard generator of ω_X. It is clear that $W_l\omega_X(D) \cdot F_p\mathscr{D}_X$ is generated by

$$\left\{\frac{\omega}{x_{j_1}^{1+b_1} \cdots x_{j_l}^{1+b_l}} : \ \sum b_i = p, \ J \subseteq I, \text{ and } \ |J| = l\right\}.$$

The result follows from the equation $\frac{\omega}{x_{j_1}^{1+b_1} \cdots x_{j_l}^{1+b_l}} = \frac{\omega}{x_I^{p+1}}(x_{j_1}^{p-b_1} \cdots x_{j_l}^{p-b_l} x_{I\setminus J}^{p+1})$. The last statement follows from the fact that, if $l > r$, around z, $W_l\omega_X(D) = \omega_X(D)$. $\square$

4. Birational definition. Let X be a smooth variety and D a reduced divisor. Consider a log-resolution $f : Y \to X$ of the pair (X, D), which is an isomorphism over $X \setminus D$, and denote $E = (f^*D)_{\mathrm{red}}$. A birational definition is given for Hodge ideals in [MP19a, §9]. In this section, we give a similar equivalent definition for weighted Hodge ideals. For the rest of this section we use right $\mathscr{D}$-modules, as it is more convenient for the construction. Recall that the right $\mathscr{D}_X$-module corresponding to $\mathscr{O}_X(*D)$ is $\omega_X(*D)$, and

$$F_{p-n}\omega_X(*D) = I_p(D) \otimes \omega_X((p+1)D).$$

Consider the following complex which we denote by $A^\bullet$:

$$0 \to f^*\mathscr{D}_X \to \Omega^1_Y(\log E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \to \cdots \to \omega_Y(E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \to 0$$

placed in degrees $-n, \ldots, 0$. The results in [MP19a, §3] say that the complex $A^\bullet$ represents the object $\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}$ in the derived category of filtered right $f^{-1}\mathscr{D}_X$-modules. Moreover, $R^0 f_* A^\bullet \cong \omega_X(*D)$.

For $p \geq 0$ define the subcomplex $C^\bullet_{p-n} = F_{p-n}A^\bullet$ of $A^\bullet$ by

$$0 \to f^*F_{p-n}\mathscr{D}_X \to \Omega^1_Y(\log E) \otimes_{\mathscr{O}_Y} f^*F_{p-n+1}\mathscr{D}_X \to \cdots \to \omega_Y(E) \otimes_{\mathscr{O}_Y} f^*F_p\mathscr{D}_X \to 0.$$

The pushforward of this complex admits the following interpretation:

$$R^0 f_* C^\bullet_{p-n} = F_{p-n}\omega_X(*D) = I_p(D) \otimes \omega_X((p+1)D)$$

by [MP19a, Remark 9.3, Corollary 12.1].

We prove similar results in order to obtain a birational definition. Consider the complex $B^\bullet$:

$$0 \to \mathscr{D}_Y \to \Omega^1_Y(\log E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to \cdots \to \omega_Y(E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to 0$$

in degrees $-n, \ldots, 0$, where the map

$$\Omega^p_Y(\log E) \otimes \mathscr{D}_Y \xrightarrow{d'} \Omega^{p+1}_Y(\log E) \otimes \mathscr{D}_Y$$

is given by $\omega \otimes P \to d\omega \otimes P + \sum (dz_i \wedge \omega) \otimes \partial_i P$. The complex $B^\bullet$ is filtered quasi-isomorphic to the object $\omega_Y(*E)$ in degree 0 [MP19a, Proposition 3.1].

Proposition 4.1. *The complex*

$$W_l B^\bullet = 0 \to \mathscr{D}_Y \to W_l\Omega^1_Y(\log E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to \cdots \to W_l\omega_Y(E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to 0$$

in degrees $-n, \ldots, 0$ *is quasi-isomorphic to* $W_{n+l}\omega_Y(*E)$.

Proof. We see first that the complex $W_l B^\bullet$ is exact in degrees $-n, \ldots, -1$. Fix a degree $-p$. We need to see that

$$W_l\Omega^{n-p-1}_Y(\log E) \otimes \mathscr{D}_Y \to W_l\Omega^{n-p}_Y(\log E) \otimes \mathscr{D}_Y \xrightarrow{b} W_l\Omega^{n-p+1}_Y(\log E) \otimes \mathscr{D}_Y$$

is exact. Let $x \in X$ be a point and $\{z_1, \ldots, z_n\}$ be a set of coordinates in an open neighborhood around the point. We localize at x, take the completion, and identify the completion of $\mathscr{O}_{X,x}$ with $\mathbb{C}[\![z_1, \ldots, z_n]\!]$. Let $\eta \in \ker \hat{b}$. By exactness of $B^\bullet$, there exists ω in the completion of $\Omega^{n-p-1}_Y(\log E) \otimes \mathscr{D}_Y$ such that $d'\omega = \eta$ (we keep calling d' the differentials of this complex). We can write $\omega = \sum g_{I,J,\alpha} \frac{dz_I}{z_I} \wedge dz_J \otimes \partial^\alpha$, with $g_{I,J,\alpha} \in \mathbb{C}[\![z_1, \ldots, z_n]\!]$, since every element $P \in \mathscr{D}_Y$ can be written as $P = \sum g_\alpha \partial^\alpha$. Moreover, expanding each $g_{I,J,\alpha}$, we can write

$$\omega = \sum C^\beta_{I,J,\alpha} z^\beta \frac{dz_I}{z_I} \wedge dz_J \otimes \partial^\alpha$$

so that no z_i that appears in z_I divides $C^\beta_{I,J,\alpha} z^\beta$. From this description, it follows that for each summand $C^\beta_{I,J,\alpha} z^\beta \frac{dz_I}{z_I} \wedge dz_J \otimes \partial^\alpha$, $|I|$ determines the weight where the form $C^\beta_{I,J,\alpha} z^\beta \frac{dz_I}{z_I} \wedge dz_J$ lies.

Next, we write, $\omega = \omega_{\leq l} + \omega_{>l}$, where the first term consists of the summands with $|I| \leq l$, and the latter of the terms with $|I| > l$. Using the description of d', we see that $d'\omega_{\leq l}$ is in the completion of $W_l \Omega^{n-p}_Y(\log E) \otimes \mathscr{D}_Y$, and each summand of $d'\omega_{>l}$ is not. Indeed,

$$\begin{aligned}
&d'(C^\beta_{I,J,\alpha} z^\beta \frac{dz_I}{z_I} \wedge dz_J \otimes \partial^\alpha) \\
=& \sum_k C^\beta_{I,J,\alpha} \beta_k z^{\beta - e_k} dz_k \wedge \frac{dz_I}{z_I} \wedge dz_J \otimes \partial^\alpha + \sum_k dz_k \wedge (C^\beta_{I,J,\alpha} z^\beta \frac{dz_I}{z_I} \wedge dz_J) \otimes \partial_k \partial^\alpha \\
=& \sum_k ((-1)^{|I|} C^\beta_{I,J,\alpha} \beta_k) \ z^{\beta - e_k} \ \frac{dz_I}{z_I} \wedge (dz_k \wedge dz_J) \otimes \partial^\alpha \\
&+ \sum_k ((-1)^{|I|} C^\beta_{I,J,\alpha}) \ z^\beta \ \frac{dz_I}{z_I} \wedge (dz_k \wedge dz_J) \otimes \partial_k \partial^\alpha.
\end{aligned}$$

Since $\eta \in \ker \hat{b}$, $d'\omega_{>l} = 0$, and $d'\omega_{\leq l} = \eta$, with $\omega_{\leq l}$ in the completion of $W_l \Omega^{n-p-1}_Y(\log E) \otimes \mathscr{D}_Y$.

Consider now the map,

$$W_l \omega_Y(E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \to W_{n+l} \omega_Y(*E)$$

given by $\frac{\omega}{f} \otimes P \to \frac{\omega}{f} \cdot P$. Fixing a degree of the Hodge filtration, and using the description of the Hodge filtration of $W_{n+l}\omega_Y(*E)$ (see for example Proposition 3.3), we see that this map is surjective. That the kernel is the image of $W_l \Omega^{n-1}_Y(\log E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$ follows from [MP19a, Proposition 3.1] and an argument similar to the one above.

□

Consider next the complex

$$W_l A^\bullet = 0 \to f^*\mathscr{D}_X \to W_l \Omega^1_Y(\log E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \to \cdots \to W_l \omega_Y(E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \to 0.$$

We have that $W_l A^\bullet = W_l B^\bullet \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}$, where $\mathscr{D}_{Y \to X} = \mathscr{O}_Y \otimes_{f^{-1}\mathscr{O}_X} f^{-1}\mathscr{D}_X$ is the transfer module. Note that when we see it as an $\mathscr{O}_Y$ module, we simply write $f^*\mathscr{D}_X$.

Lemma 4.2. *The complex $W_l A^\bullet$ represents $W_{n+l}\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}$ in the derived category of filtered right $f^{-1}\mathscr{D}_X$-modules.*

Proof. It is enough to show that the elements $W_l B^k$ are acyclic with respect to $-\otimes_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}$. For any k consider the following spectral sequence:

$$E_2^{p,q} = \mathcal{T}or_p^{\mathscr{D}_Y}(\mathcal{T}or_q^{\mathscr{O}_Y}(W_l\Omega_Y^k(\log E), \mathscr{D}_Y), \mathscr{D}_{Y\to X}) \Rightarrow \mathcal{T}or_{p+q}^{\mathscr{O}_Y}(W_l\Omega_Y^k(\log E), f^*\mathscr{D}_X)$$

[Wei94, Theorem 5.6.6]. As $\mathscr{D}_Y$ is a locally free $\mathscr{O}_Y$-module, then $E_2^{p,q} = 0$ for $q \neq 0$. Therefore,

$$\mathcal{T}or_p^{\mathscr{D}_Y}(W_l\Omega_Y^k(\log E) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y, \mathscr{D}_{Y\to X}) \cong \mathcal{T}or_p^{\mathscr{O}_Y}(W_l\Omega_Y^k(\log E), f^*\mathscr{D}_X) = 0$$

for $p \neq 0$, where the last equality follows from the fact that $f^*\mathscr{D}_X$ is locally free. □

The map

$$R^0 f_*(W_{n+l}\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}) \xrightarrow{\varphi} R^0 f_*(\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$$

is precisely the morphism

$$H^0 f_+(W_{n+l}\omega_Y(*E)) \to H^0 f_+(\omega_Y(*E)) = \omega_X(*D),$$

whose image is $W_{n+l}\omega_X(*D)$. Moreover, the complex $C_{p-n}^\bullet$ described above corresponds to the $F_{p-n}(\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$ using the identification

$$\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X} \cong B^\bullet \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}.$$

By strictness, there is an injective map

$$R^0 f_* C_{p-n}^\bullet \hookrightarrow R^0 f_* A^\bullet \cong \omega_X(*D)$$

whose image is $F_{p-n}\omega_X(*D) = I_p(D) \otimes \omega_X((p+1)D)$ (see [MP19a, Sections 4, 9, and 12]).

Similarly, we define $C_{l,p-n}^\bullet$ by

$$0 \to f^*F_{p-n}\mathscr{D}_X \to W_l\Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} f^*F_{p-n+1}\mathscr{D}_X \to \cdots \to W_l\omega_Y(E) \otimes_{\mathscr{O}_Y} f^*F_p\mathscr{D}_X \to 0$$

which corresponds to $F_{p-n}(W_{n+l}\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X})$ under the identification

$$W_{n+l}\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_Y} \mathscr{D}_{Y\to X} \cong W_l B^\bullet \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y\to X}$$

given by Lemma 4.2.

Proposition 4.3. *Using the notation above,*

$$I_p^{W_l}(D) \otimes \omega_X((p+1)D) = \mathrm{im}[R^0 f_* C_{l,p-n}^\bullet \hookrightarrow R^0 f_* W_l A^\bullet \to R^0 f_* A^\bullet \cong \omega_X(*D)].$$

Proof. By strictness, we have an injective map

$$R^0 f_* C_{l,p-n}^\bullet \hookrightarrow R^0 f_* W_l A^\bullet$$

whose image is $F_{p-n}H^0 f_+ W_{n+l}\omega_X(*D)$ (see for instance [MP19a, §4]). Taking the composition

$$R^0 f_* C_{l,p-n}^\bullet \hookrightarrow R^0 f_* W_l A^\bullet \to R^0 f_* A^\bullet \cong \omega_X(*D),$$

and using strictness in the middle morphism (since it underlies a morphism of mixed Hodge modules), the image corresponds to $F_{p-n}W_{n+l}\omega_X(*D) = I_p^{W_l}(D) \otimes \omega_X((p+1)D)$. □

The description in Proposition 4.3 for $I_0^{W_l}(D)$ coincides with the description in [Ola22, Proposition 3], since $f_*W_l\omega_Y(E) \to f_*\omega_Y(E)$ is an inclusion. The complex $C^\bullet_{l,1-n}$ also has a simple description. Recall that by definition

$$C^\bullet_{l,1-n} = [W_l\Omega_Y^{n-1}(\log E) \to W_l\omega_Y(E) \otimes f^*F_1\mathscr{D}_X]$$

in degrees -1 and 0. Moreover, the map

$$\Omega_Y^{n-1}(\log E) \to \omega_Y(E) \otimes f^*F_1\mathscr{D}_X$$

is injective [MP19a, Lemma 3.4]. Using the fact that $W_l\Omega_Y^{n-1}(\log E) \hookrightarrow \Omega_Y^{n-1}(\log E)$ and $W_l\omega_Y(E) \otimes f^*F_1\mathscr{D}_X \hookrightarrow \omega_Y(E) \otimes f^*F_1\mathscr{D}_X$ are injective (since $F_1\mathscr{D}_X$ is a locally free $\mathscr{O}_X$-module), we obtain that the differential in $C^\bullet_{l,1-n}$ is also an inclusion. Let $\mathcal{F}_{l,1}$ be the cokernel. This means that

$$I_1^{W_l}(D) \otimes \omega_X(2D) = \mathrm{im}[f_*\mathcal{F}_{l,1} \to \omega_X(D)].$$

This map can be interpreted by using the complex $C^\bullet_{1-n}$. Indeed, let $\mathcal{F}_1$ be the cokernel of the differential in $C^\bullet_{1-n}$. We have an induced map $\mathcal{F}_{l,1} \to \mathcal{F}_1$. Since $f_*\mathcal{F}_1 = I_1(D) \otimes \omega_X(2D)$,

$$I_1^{W_l}(D) \otimes \omega_X(2D) = \mathrm{im}[f_*\mathcal{F}_{l,1} \to f_*\mathcal{F}_1].$$

Note that since weighted Hodge ideals were defined in terms of the Hodge and weight filtrations of $\mathscr{O}_X(*D)$, the constructions presented in this section are independent of the resolution of singularities.

5. Weighted Hodge ideals and V-filtration. Let X be a smooth variety and D be an effective reduced divisor defined by the global equation $f \in \mathscr{O}_X(X)$. The Hodge ideals $I_p(D)$ can be described using the V-filtration of $i_+\mathscr{O}_X$, where i is the graph embedding defined by f. Namely,

$$I_p(D) = \left\{ \sum_{j=0}^{p} Q_j(1) f^{p-j} v_j \ : \ \sum_{j=0}^{p} v_j \partial_t^j \delta \in V^1 i_+\mathscr{O}_X \right\}, \tag{5.1}$$

where $Q_j(x) = \prod_{i=0}^{j-1}(x+i)$, [MP20b, Theorem A']. An equivalent description is obtained using the following map:

$$\tau : V^1 i_+\mathscr{O}_X \to \mathscr{O}_X(*D)$$

given by

$$\tau\left(\sum_{i=0}^{p} v_i \partial_t^i \delta\right) = \sum_{i=0}^{p} Q_i(1) \frac{v_i}{f^{i+1}}.$$

The map τ[1] is a surjective morphism of $\mathscr{D}_X$-modules, and

$$I_p(D) \otimes \mathscr{O}_X((p+1)D) = F_p\mathscr{O}_X(*D) = \tau(F_{p+1}V^1 i_+\mathscr{O}_X), \tag{5.2}$$

[1]The map τ corresponds to τ_1 in the notation of [MP20a]. See §1 for the discussion about the reduced case.

see [MP20a, Proposition 5.4 and Lemma 5.1]. Moreover, the map τ induces a map

$$\bar{\tau} : \operatorname{gr}_V^1 i_+\mathscr{O}_X \to \mathscr{O}_X(*D)/\mathscr{O}_X.$$

Indeed, it is enough to see that $\tau(V^{>1}i_+\mathscr{O}_X) \subseteq \mathscr{O}_X$. This follows from the fact that $V^{>1}i_+\mathscr{O}_X = V^{1+\alpha}i_+\mathscr{O}_X = t \cdot V^\alpha i_+\mathscr{O}_X$, with $\alpha > 0$, and that if $j > 0$, $tu\partial_t^j\delta = fu\partial_t^j\delta - ju\partial_t^{j-1}\delta$, and $tu\delta = fu\delta$. For $v = \sum_{j=0}^p v_j\partial_t^j\delta \in V^{>1}i_+\mathscr{O}_X$, there exists $u = \sum_{j=0}^p u_j\partial_t^j\delta \in V^\alpha i_+\mathscr{O}_X$ such that, $tu = v$. Hence,

$$\tau(v) = \tau(fu_0\delta + \sum_{j=1}^p (fu_j\partial_t^j\delta - ju_j\partial_t^{j-1}\delta)) = u_0,$$

as

$$Q_j(1)\frac{fu}{f^{j+1}} - jQ_{j-1}(1)\frac{u}{f^j} = 0$$

because $Q_j(1) = jQ_{j-1}(1)$.

The $\mathscr{D}_X$-module $\operatorname{gr}_V^1 i_+\mathscr{O}_X$ underlies the mixed Hodge module $\psi_{f,1}\mathscr{O}_X$ and its weight filtration can be described in terms of the nilpotent operator $t\partial_t$. In order to complete the description in Theorem 5.6, we first need to show that the map $\bar{\tau}$ also preserves the weight filtration.

Proposition 5.3. *The map $\bar{\tau}$ sends the weight and Hodge pieces to the same image as the map $\tau_{\mathscr{D}_X}$ that underlies a morphism of mixed Hodge modules*

$$\tau^H : \psi_{f,1}\mathscr{O}_X(-1) \to \mathcal{H}_D^1(\mathscr{O}_X).$$

Proof. The map $\bar{\tau}$ is surjective and using its description, we observe that its kernel is the image of the map $\partial_t t - 1$ on $\operatorname{gr}_V^1 i_+\mathscr{O}_X$. The same is true for the map $\tau_{\mathscr{D}_X}$. Indeed, the map $\partial_t t - 1$ underlies the composition $Var \circ can$ on $\psi_{f,1}\mathscr{O}_X$. As $can : \psi_{f,1}\mathscr{O}_X \to \phi_{f,1}\mathscr{O}_X$ is surjective because $i_+\mathscr{O}_X$ has strict support (see for instance [Sch14, §11]), the cokernel of $Var \circ can$ coincides with the cokernel of

$$Var : \phi_{f,1}\mathscr{O}_X \to \psi_{f,1}\mathscr{O}_X(-1).$$

The cokernel of Var is isomorphic to $i_{D*}\mathcal{H}^1 i_D^! \mathscr{O}_X$, where $i_D : D \to X$ is the inclusion [Sai90, Corollary 2.24]. Moreover, $i_{D*}\mathcal{H}^1 i_D^! \mathscr{O}_X$ is isomorphic to $\mathcal{H}_D^1(\mathscr{O}_X)$ [Sai09, §2.2]. This means that $\bar{\tau}$ and $\tau_{\mathscr{D}_X}$ could only differ by a $\mathscr{D}_X$-automorphism of $\mathcal{H}_D^1(\mathscr{O}_X)$ and the result is a consequence of Lemma 5.4. □

Lemma 5.4. *A $\mathscr{D}_X$-automorphism of $\mathcal{H}_D^1(\mathscr{O}_X)$ preserves the Hodge and weight filtration.*

Proof. We can restrict to an open affine subset. Let $X = \operatorname{Spec} R$ where D is defined by $f \in R$, and φ an $\mathscr{D}_R$-automorphism of R_f/R. Let $m \geq 2$, then $\varphi[\frac{1}{f^m}] = [\frac{g_m}{f^m}]$ for some $g_m \in R$, since $f^m\varphi[\frac{1}{f^m}] = 0$. Using that φ is $\mathscr{D}_R$-linear, we see that for every $T \in Der_{\mathbb{C}}(R)$, $T(g_m) \in (f^{m-1})$. This implies that around each smooth point of $P \in D$, using a regular system of parameters, we have an h such that $h(P) \neq 0$, and $g_m - g_m(P) \in f^m \cdot R_h$. Restricting the automorphism to the open set defined by h, we see that φ_h acts by multiplying by a constant. This means that this constant doesn't depend on m, and after restricting to double intersections, we see that this constant doesn't depend on the point. Let λ be the constant. Since $\varphi - \lambda \cdot Id$ is 0 on all the smooth points, $\varphi = \lambda \cdot Id$ everywhere. In particular, φ preserves the Hodge and the weight filtration. □

We are very grateful to Mircea Mustaţă for suggesting the argument of Lemma 5.4.

Remark 5.5. A simpler proof of the Lemma was suggested by a referee. Using the Riemann-Hilbert correspondence, and Verdier duality, it is enough to verify the conclusion of the Lemma on the perverse sheaf $\mathbb{Q}_D[n-1]$. We leave the original proof to have an argument using only $\mathscr{D}$-modules.

A consequence of the result above, is that we can write a description of the weighted Hodge ideals in a similar way to (5.1). Let $W_l V^1 i_+ \mathscr{O}_X$ be the submodule of $V^1 i_+ \mathscr{O}_X$ which maps to $W_{n+l-2} \operatorname{gr}_V^1 i_+ \mathscr{O}_X$ via the canonical projection.

Proposition 5.6. *Using the notation above,*

$$I_p^{W_l}(D) = \left\{ \sum_{j=0}^{p} Q_j(1) f^{p-j} v_j \ : \ \sum_{j=0}^{p} v_j \partial_t^j \delta \in W_l V^1 i_+ \mathscr{O}_X \right\}.$$

Proof. It follows from Proposition 5.3 that $\tau(F_{p+1} W_l V^1 i_+ \mathscr{O}_X) = F_p W_{n+l} \mathscr{O}_X(*D) = I_p^{W_l}(D) \otimes \mathscr{O}_X((p+1)D)$. □

The result above can be simplified even more using the description of the weight filtration of $\psi_{f,1}\mathscr{O}_X$, and that is the statement of Theorem A.

Proof of Theorem A. First, we note that if $v \in V^1 i_+ \mathscr{O}_X$, then $\tau(t\partial_t v) = 0$. Indeed, let $v = \sum_{j=0}^{p} v_j \partial_t^j \delta \in V^1 i_+ \mathscr{O}_X$, then

$$t\partial_t v = \sum_{j=0}^{p} (f v_j \partial_t^{j+1} \delta - (j+1) v_j \partial_t^j \delta),$$

and

$$\tau(t\partial_t v) = \sum_{j=0}^{p} \left(Q_{j+1}(1) \frac{f v_j}{f^{j+2}} - (j+1) Q_j(1) \frac{v_j}{f^{j+1}} \right) = 0.$$

The weight filtration of $\psi_{f,1}\mathscr{O}_X$ admits the following description for $k \geq 0$:

$$W_{n-1+k} \psi_{f,1} \mathscr{O}_X = \sum_{m \geq 0} (t\partial_t)^m (\ker (t\partial_t)^{2m+k+1})$$

(see [Sai94, 2.7] and for the monodromy filtration see e.g. [SZ85, Remark 2.3]). The only piece that is not an image of $(t\partial_t)$ is $\ker (t\partial_t)^{k+1}$. That means that the subset $\ker (t\partial_t)^l \subseteq W_l V^1 i_+ \mathscr{O}_X$ has the same image as $W_l V^1 i_+ \mathscr{O}_X$ via τ. □

Remark 5.7. Let (X, D) be a pair such that D has at most isolated weighted homogeneous singularities. Theorem A gives a complete description of the weighted Hodge ideals using the description of the V-filtration in [Sai09]. Using the notation above, in this case, $(t\partial_t)^2 u \in$

$V^{>1}i_+\mathscr{O}_X$ for all $u \in V^1 i_+\mathscr{O}_X$. For this reason, $I_p^{W_2}(D) = I_p(D)$, for all $p \geq 0$. An argument without the use of the V-filtration in the case of $p = 0$ is described in [Ola22, §10].

A direct application of Theorem A is that we can recover the following result proved in [MP19a, Theorem C]. The proof we give differs from the one in [MP19a] and is also much shorter.

Corollary 5.8. *Let X be a smooth variety and D an effective reduced divisor. Then*

$$I_p(D) \subseteq \mathrm{adj}(D)$$

for all $p \geq 1$.

Proof. Recall that $\mathrm{adj}(D) = I_0^{W_1}(D)$ [Ola22, Theorem A]. Moreover, as $I_p(D) \subseteq I_1(D)$ [MP19a, Proposition 13.1], it is enough to prove that $I_1(D) \subseteq I_0^{W_1}(D)$.

Let $u \in I_1(D)$. By (5.1), $u = u_0 f + u_1$, where f is defining equation of D, and $u_0\delta + u_1\partial_t\delta \in V^1 i_+\mathscr{O}_X$. We also have that

$$V^1 i_+\mathscr{O}_X \ni (f-t)(u_0\delta + u_1\partial_t\delta) = u_1\delta,$$

and

$$\partial_t(u_1\delta) = u_1\partial_t\delta + u_0\delta - u_0\delta \in V^{>0}i_+\mathscr{O}_X.$$

Finally, as $\delta \in V^{>0}i_+\mathscr{O}_X$, then $u_0 f\delta = t(u_0\delta) \in V^{>1}i_+\mathscr{O}_X \subseteq W_1V^1 i_+\mathscr{O}_X$. This means that $u_0 f\delta + u_1\delta \in W_1V^1 i_+\mathscr{O}_X$, hence $u_0 f + u_1 \in I_0^{W_1}(D)$. □

There is a relation between the minimal exponent of f and the weighted Hodge ideals. Recall that if we denote $b_f(s)$ the Bernstein-Sato polynomial, and $\widetilde{b}_f(s)$ the reduced one, we call $\widetilde{\alpha_f}$ the negative of the largest root of $\widetilde{b}_f(s)$. Saito proved in [Sai16] that $I_p(D) = \mathscr{O}_X$ if and only if $\widetilde{\alpha_f} \geq p+1$ (c.f. [MP20b, Corollary 6.1]). Moreover, this result also holds in the case of $\mathbb{Q}$-divisors, and it can be stated in the following form.

Lemma 5.9 ([MP20a, Lemma 1.2])**.** *For an integer p and $\alpha \in (0,1]$,*

$$\partial_t^p\delta \in V^\alpha i_+\mathscr{O}_X \Leftrightarrow \widetilde{\alpha_f} \geq p + \alpha.$$

Using these ideas, we obtain the following result for the 1st weighted Hodge ideals.

Corollary 5.10. *Using the notation above,*

$$I_p^{W_1}(D) = \mathscr{O}_X \textit{ if and only if } \widetilde{\alpha_f} > p+1.$$

Proof. Suppose first that $\widetilde{\alpha_f} > p+1$. Then, by Lemma 5.9, $\partial_t^p\delta \in V^1 i_+\mathscr{O}_X$. Moreover, there exists $\alpha \in (0,1]$ such that $\widetilde{\alpha_f} \geq p+1+\alpha$. Again, by Lemma 5.9, $\partial_t^{p+1}\delta \in W_1V^1 i_+\mathscr{O}_X$, and therefore, $I_p^{W_1}(D) = \mathscr{O}_X$.

Suppose now that $I_p^{W_1}(D) = \mathscr{O}_X$. Then, $I_p(D) = \mathscr{O}_X$, and in particular $\delta, \partial_t\delta, \ldots, \partial_t^p\delta \in V^1 i_+\mathscr{O}_X$. Moreover, there exists $v = \sum_{j=0}^p v_j\partial_t^j\delta \in W_1V^1 i_+\mathscr{O}_X$ such that $\sum_{j=0}^p Q_j(1)f^{p-j}v_j = 1$. It is enough to show that $\partial_t^p\delta \in W_1V^1 i_+\mathscr{O}_X$. Indeed, by Proposition 5.6 and the injectivity of $t : \mathrm{gr}_V^0 i_+\mathscr{O}_X \to \mathrm{gr}_V^1 i_+\mathscr{O}_X$ (see e.g. [Sch14, §11]), this means that $\partial_t^{p+1}\delta \in V^\alpha i_+\mathscr{O}_X$

with $\alpha \in (0,1]$, and therefore, $\widetilde{\alpha_f} \geq p+1+\alpha > p+1$. We argue by induction. Suppose $p=0$. Then $v = v_0\delta$ and by the second condition, $v_0 = 1$. Hence, $\delta \in W_1V^1 i_+\mathscr{O}_X$. By the induction hypothesis, we assume now that $\partial_t^k\delta \in W_1V^1 i_+\mathscr{O}_X$ for $k=0,\ldots,p-1$. It follows from the description of v that

$$Q_p(1)v_p = 1 - f(\sum_{j=0}^{p-1} Q_j(1) f^{p-1-j} v_j),$$

and then

$$v = \partial_t^p\delta - f(\sum_{j=0}^{p-1} Q_j(1) f^{p-1-j} v_j)\partial_t^p\delta + \sum_{j=0}^{p-1} v_j\partial_t^j\delta.$$

The result follows if we show that $f\partial_t^p\delta \in W_1V^1 i_+\mathscr{O}_X$, and this is a consequence of $f\partial_t^p\delta = t\partial_t(\partial_t^{p-1}\delta) + p\partial_t^{p-1}\delta \in W_1V^1 i_+\mathscr{O}_X$.

□

Remark 5.11. In general, we cannot obtain more information about the other p-weighted Hodge ideals. In [Ola22, §13], the case of isolated log-canonical singularities, that are not rational, is discussed. This case corresponds to $\widetilde{\alpha_f} = 1$. By the discussion above, it is clear that $I_0(D) = \mathscr{O}_X$ and that $I_0^{W_1}(D)$ is not trivial. For $l = 2,\ldots,n-1$, there are examples of f where the weighted multiplier ideals $I_0^{W_l}(D)$ are trivial, and other examples where they are non-trivial [Ish85, Theorem 5.2].

D. Local study

6. Measuring the difference between weighted Hodge ideals. There is a short exact sequence that arises from the definition of the weight filtration on $\mathscr{O}_X(*D)$:

$$0 \to W_{n+l-1}\mathscr{O}_X(*D) \to W_{n+l}\mathscr{O}_X(*D) \to \mathrm{gr}^W_{n+l}\mathscr{O}_X(*D) \to 0.$$

Applying F_p, we obtain the short exact sequence

(6.1)

$$0 \to I_p^{W_{n+l-1}}(D) \otimes \mathscr{O}_X((p+1)D) \to I_p^{W_{n+l}}(D) \otimes \mathscr{O}_X((p+1)D) \to F_p\,\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D) \to 0.$$

When D has at most isolated singularities and $l \geq 2$, $\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D)$ is supported on the singular points. To simplify the notation, we use the following definition.

Definition 6.2. Suppose D has at most one isolated singularity $x \in D$, and let $i_x : \{x\} \hookrightarrow X$. For $l \geq 2$, we denote by H_l the complex pure Hodge structure of weight $n+l$ such that

$$\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D) \cong (i_x)_+H_l.$$

In order to describe the dimension of $F_p(i_x)_+H_l$, it is enough to describe the dimension of $\mathrm{Gr}_F^{n-k} H_l$ for $0 \leq k \leq p$. This is a consequence of the local description of the Hodge filtration of $(i_x)_+H_l$. Let $x_1,\ldots,x_n$ be a set of coordinates around the point $x \in X$. We have the following description of the pushforward of H_l as a $\mathscr{D}$-module:

$$(i_x)_+H_l = (i_x)_*H_l \otimes_{\mathbb{C}} \mathbb{C}[\partial_1,\cdots,\partial_n], \tag{6.3}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and

$$F_p(i_x)_+H_l = \bigoplus_{\nu\in\mathbb{Z}^n_{\geq 0}} (i_x)_* F_{p-|\nu|-n} H_l \otimes \partial^\nu, \tag{6.4}$$

where $\partial^\nu = \partial_1^{\nu_1}\cdots\partial_n^{\nu_n}$, $|\nu| = \nu_1 + \ldots + \nu_n$, and $F_k H_l = F^{-k}H_l$. Since the lowest degree of the Hodge filtration of $\mathscr{O}_X(*D)$ is 0, and $\mathrm{DR}((i_x)_+H_l) \cong (i_x)_*H_l$, that is, the push-forward of the pure Hodge structure H_l is a skyscraper sheaf, then the highest degree of the Hodge filtration of H_l is n, in other words, $F^{n+1}H_l = 0$. Using this, we obtain, for instance, that

$$F_0(i_x)_+H_l = (i_x)_*F^nH_l\otimes 1 = (i_x)_*\operatorname{Gr}^n_F H_l \otimes 1,$$

and

$$F_1(i_x)_+H_l = (i_x)_*F^{n-1}H_l\otimes 1 \oplus \bigoplus_i (i_x)_*F^nH_l\otimes\partial_i.$$

Since $F_p(i_x)_+H_l$ is a skyscraper sheaf, we denote by $\dim(F_p(i_x)_+H_l)$ the dimension of the complex vector space J_p that satisfies $F_p(i_x)_+H_l = (i_x)_*J_p$. From the discussion above, we obtain that

$$\dim(F_0(i_x)_+H_l) = \dim(\operatorname{Gr}^n_F H_l),$$

$$\dim(F_1(i_x)_+H_l) = \dim(F^{n-1}H_l) + n\dim(\operatorname{Gr}^n_F H_l) = \dim(\operatorname{Gr}^{n-1}_F H_l) + (n+1)\dim(\operatorname{Gr}^n_F H_l),$$

and in general

(6.5)

$$\begin{aligned}\dim(F_p(i_x)_+H_l) &= \sum_{k=0}^{p}\binom{n-1+k}{k}\dim(F^{n-p+k}H_l) = \sum_{r=0}^{p}\dim(\operatorname{Gr}^{n-r}_F H_l)\sum_{k=0}^{p-r}\binom{n-1+k}{k}\\ &= \sum_{r=0}^{p}\binom{n+p-r}{p-r}\dim(\operatorname{Gr}^{n-r}_F H_l).\end{aligned}$$

The dimension of $\operatorname{Gr}^{n-k}_F H_l$ is described in Theorem B.

Proof of Theorem B. We can and will assume that X is a projective variety. Indeed, there is an open set around x which has a smooth projective compactification $\bar{X}$. Let $\bar{D}$ be the closure of D in $\bar{X}$. Consider a log-resolution of $(\bar{X}\smallsetminus x, \bar{D}\smallsetminus x)$ given by a sequence of blow ups with centers over the singular locus of $\bar{D}\smallsetminus x$. By blowing up the same sequence of centers over $\bar{X}$, we obtain a map $X_1 \to \bar{X}$. Let D_1 be the strict transform of $\bar{D}$. By construction, the map is an isomorphism over (X, D), and D_1 has only one isolated singularity corresponding to $x \in D$. We replace (X, D) with (X_1, D_1).

First, we prove that these dimensions do not depend on the log-resolution of singularities that is an isomorphism outside of $\{x\}$. Since for a pair of resolution of singularities one can find a third one that dominates the two of them, it is enough to show that the dimensions are equal if we have two resolutions of singularities $g_1 : D_1 \to D$ and $g_2 : D_2 \to D$ such that there is a morphism $h : D_1 \to D_2$ such that $g_1 = g_2 \circ h$. Let $G_i \subseteq D_i$ be the exceptional divisor of g_i. Consider the exact sequence of mixed Hodge structures

$$\cdots \to H^{k-1}(G_1) \to H^k(D_2) \to H^k(D_1) \oplus H^k(G_2) \to H^k(G_1) \to \cdots$$

(see [PS08, Proof of Theorem 6.15]). For $l \geq 3$, applying $H^{p,n-l-p}$, we obtain that

$$H^{p,n-l-p}(H^{n-2}(G_2)) \cong H^{p,n-l-p}(H^{n-2}(G_1)).$$

For $l = 2$, applying $H^{p,n-p-2}$ and $H^{n-p-1,p+1}$, and noting that $h^{p,n-p-2}(D_i) = h^{n-p-1,p+1}(D_i)$, we obtain that

$$h^{p,n-p-2}(H^{n-2}(G_1)) - h^{n-p-1,p+1}(H^n(G_1)) = h^{p,n-p-2}(H^{n-2}(G_2)) - h^{n-p-1,p+1}(H^n(G_2)).$$

Let $f : Y \to X$ be a log-resolution that is an isomorphism outside of x, and $E := f^{-1}(D)_{red}$. This resolution defines a log-resolution of singularities $g : \widetilde{D} \to D$ by restriction, that is an isomorphism outside of x. We use the spectral sequence (1.3) for the constant map from X to a point. In this case, it says

$$E_1^{-n-l,q} = \mathbb{H}^{q-n-l}(X, \mathrm{DR}(\mathrm{gr}^W_{n+l} \mathscr{O}_X(*D))) \Rightarrow H^{q-l}(U, \mathbb{C}), \tag{6.6}$$

noting that $\mathrm{DR}(\mathscr{O}_X(*D)) \cong \mathbf{R}j_*\mathbb{C}_U[n]$, where $j : U = X \smallsetminus D \hookrightarrow X$. We also have the isomorphism

$$E_2^{-n-l,q} \cong \mathrm{Gr}^W_q H^{q-l}(U).$$

Consider the maps

$$E_1^{-n-l-1,n+l} \to E_1^{-n-l,n+l} \to E_1^{-n-l+1,n+l},$$

corresponding to

$$\mathbb{H}^{-1}(X, \mathrm{DR}(\mathrm{gr}^W_{n+l+1} \mathscr{O}_X(*D))) \to \mathbb{H}^0(X, \mathrm{DR}(\mathrm{gr}^W_{n+l} \mathscr{O}_X(*D))) \to \mathbb{H}^1(X, \mathrm{DR}(\mathrm{gr}^W_{n+l-1} \mathscr{O}_X(*D))).$$

Moreover, the degeneration of the Hodge-to-de-Rham spectral sequence says that

$$\mathrm{gr}^F_{-n+p} \mathbb{H}^i(X, \mathrm{DR}(\mathrm{gr}^W_{n+l} \mathscr{O}_X(*D))) \cong \mathbb{H}^i(X, \mathrm{gr}^F_{-n+p} \mathrm{DR}(\mathrm{gr}^W_{n+l} \mathscr{O}_X(*D))) \tag{6.7}$$

(see for example [MP19a, Example 4.2]).

Consider first the case $l \geq 3$. Noting that $\mathbb{H}^i(X, \mathrm{DR}((i_x)_+ H_l)) = 0$ if $i \neq 0$ for $l \geq 2$, we obtain that

$$E_2^{-n-l,n+l} \cong H_l.$$

Applying gr^F_{-n+p}, using (6.7), and the E_2-degeneration of the spectral sequence, we obtain that

$$\mathrm{gr}^F_{-n+p} E_2^{-n-l,n+l} \cong \mathrm{gr}^F_{-n+p} H_l = \mathrm{Gr}_F^{n-p} H_l \cong H^{n-p,l+p}(H^n(U)) \cong H^{p,n-l-p}(H^n_c(U))^*,$$

where the last isomoprhism follows from Poincaré duality (see [PS08, Theorem 6.23]). Using the long exact sequence of the pair (X, D), we obtain that

$$H^{p,n-l-p}(H^n_c(U)) \cong H^{p,n-l-p}(H^{n-1}(D)),$$

as $H^{n-1}(X)$ and $H^n(X)$ have pure Hodge structures. Finally, as g has $\{x\}$ as discriminant, we have a long exact sequence,

$$H^{n-2}(\widetilde{D}) \to H^{n-2}(G) \to H^{n-1}(D) \to H^{n-1}(\widetilde{D}).$$

As this is a sequence of mixed Hodge structures, we obtain

$$H^{p,n-l-p}(H^{n-1}(D)) \cong H^{p,n-l-p}(H^{n-2}(G)).$$

Consider now $l = 2$. In this case, the maps

$$E_1^{-n-3,n+2} \to E_1^{-n-2,n+2} \to E_1^{-n-1,n+2} \to E_1^{-n,n+2}$$

correspond to

$$0 \to H_2 \xrightarrow{\widetilde{\beta}} H^n(D)(-1) \xrightarrow{\widetilde{\gamma}} H^{n+2}(X).$$

Indeed, the first two terms follow from the explanation above. The third term follows from the fact that $\mathrm{DR}(\mathrm{gr}^W_{n+1} \mathscr{O}_X(*D)) \cong IC_D(-1)$, a Tate twist of the intersection complex of D [Sai09, §2.2]. Furthermore, $IH^n(D) \cong H^n(D)$ [GM80, §6.1]. The last term in the complex, follow as $\mathrm{DR}(\mathrm{gr}^W_n \mathscr{O}_X(*D)) \cong \mathbb{C}_X[n]$. From the short exact sequence

$$\ker\beta \to \mathrm{Gr}_F^{n-p} H_2 \to \mathrm{im}\,\beta,$$

where $\beta = \mathrm{gr}^F_{-n+p}\widetilde{\beta}$ and $\gamma = \mathrm{gr}^F_{-n+p}\widetilde{\gamma}$, we obtain that

$$\begin{aligned}\dim(\mathrm{Gr}_F^{n-p} H_2) &= \dim\ker\beta + \dim\mathrm{im}\,\beta \\ &= h^{p,n-p-2}(H^n_c(U)) + h^{n-p-1,p+1}(H^n(D)) \\ &\quad - h^{p,n-p-2}(X) + h^{p,n-p-2}(H^{n-2}_c(U)) - h^{p,n-p-2}(H^{n-1}_c(U)).\end{aligned}$$

Indeed, this follows from the descriptions of $E_2^{n-2+s,n+2}$ for $s = 0, 1, 2$ and Poincaré duality. More precisely, we have three short exact sequences

$$0 \to H^{p,n-p-2}(H^n_c(U))^* \to \mathrm{Gr}_F^{n-p} H_2 \to \mathrm{im}\,\beta \to 0,$$

$$0 \to \ker\gamma \to \mathrm{Gr}_F^{n-p} H^{n+1}(D) \to \mathrm{im}\,\gamma \to 0,$$

$$0 \to \mathrm{im}\,\gamma \to H^{p,n-p-2}(X)^* \to H^{p,n-p-2}(H^{n-2}_c(U))^* \to 0,$$

and also that $\mathrm{Gr}_F^{n-p} E_2^{n-1,n+2} \cong H^{p,n-p-2}(H^{n-1}_c(U))^*$. Using the long exact sequence associated to the pair (X, D) to relate these three sequences, we obtain

$$\dim(\mathrm{Gr}_F^{n-p} H_2) = h^{n-p-1,p+1}(H^n(D)) - h^{p,n-p-2}(H^{n-2}(D)) + h^{p,n-p-2}(H^{n-1}(D)).$$

Finally, using that the map g has $\{x\}$ as discriminant, we obtain that

$$\dim(\mathrm{Gr}_F^{n-p} H_2) = h^{p,n-p-2}(H^{n-2}(G)) - h^{n-p-1,p+1}(H^n(G)).$$

□

Remark 6.8. In general, the term $h^{n-p-1,p+1}(H^n(G))$ might not be 0. Consider for instance $n = 4$ and $p = 1$. In this case, $h^{2,2}(H^4(G)) = k$ where k is the number of irreducible components of G. Using similar computations as above, we also see that

$$h^{p,n-p-2}(H^{n-2}(G)) - h^{n-p-1,p+1}(H^n(G)) = h^{n-p-1,p+1}(H^n(D)) - h^{p,n-p-2}(H^{n-2}(D)),$$

that is, the failure of Poincaré duality. Still, in the case $p = 0$, the term $h^{n-p-1,p+1}(H^n(G))$ is always 0, as G is $(n-2)$-dimensional (see [Ola22, Theorem B]).

E. Vanishing Theorems

7. Ample divisors. Let X be a smooth projective variety of dimension n, and D an ample divisor. Let $U = X \smallsetminus D$. As U is smooth and affine, $H^{i+n}(U) = 0$ for $i > 0$ (see for instance [Laz04, Theorem 3.1.1]). In this setting we have the following result.

Lemma 7.1. *There is a short exact sequence*

$$\begin{aligned} 0 \to & H^i(X, \mathrm{DR}(W_{n+l}\mathscr{O}_X(*D)) \to H^i(X, \mathrm{DR}(\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D))) \to \\ \to & H^{i+1}(X, \mathrm{DR}(W_{n+l-1}\mathscr{O}_X(*D))) \to 0. \end{aligned}$$

Proof. In [Ola22, Proof of Proposition 12.1], using the spectral sequences

$$E_1^{-n-l,q} = H^{q-n-l}(X, \mathrm{DR}(\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D))) \Rightarrow H^{q-l}(U, \mathbb{C})$$

and

$$E_1'^{-n-l,q} = H^{q-n-l}(X, \mathrm{DR}(\mathrm{gr}^W_{n+l} W_{n+k}\mathscr{O}_X(*D))) \Rightarrow H^{q-n-l}(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D)))$$

and noting that

$$E_2^{-n-l,q} \cong \mathrm{Gr}^W_q H^{q-l}(U, \mathbb{C})$$
$$E_2'^{-n-l,q} \cong \mathrm{gr}^W_q H^{q-n-l}(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D)))$$

we obtained:

(a) For $s \geq 1$,
$$\mathrm{gr}^W_{n+k+i-s} H^i(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D))) \cong \mathrm{Gr}^W_{n+k+i-s} H^{i+n}(U, \mathbb{C}).$$

(b) For $s \geq 1$,
$$\mathrm{gr}^W_{n+k+i+s} H^i(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D))) = 0.$$

(c) Let
$$\alpha_{k+1} : H^{i-1}(X, \mathrm{DR}(\mathrm{gr}^W_{n+k+1}\mathscr{O}_X(*D))) \to H^i(X, \mathrm{DR}(\mathrm{gr}^W_{n+k}\mathscr{O}_X(*D)))$$
corresponding to the map $E_1^{-n-k-1,i+n+k} \to E_1^{-n-k,i+n+k}$. Then we have the following short exact sequence:
$$0 \to \mathrm{im}\,\alpha_{k+1} \to \mathrm{gr}^W_{i+n+k} H^i(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D))) \to \mathrm{Gr}^W_{i+n+k} H^{i+n}(U, \mathbb{C}) \to 0.$$

If $i \geq 1$, then
$$\mathrm{im}\,\alpha_{k+1} \cong \mathrm{gr}^W_{i+n+k} H^i(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D))) \cong H^i(X, \mathrm{DR}(W_{n+k}\mathscr{O}_X(*D))).$$

Consider now the complex

$$E_1^{-n-l-1,n+l+i} \overset{\alpha_{l+1}}{\to} E_1^{-n-l,n+l+i} \overset{\alpha_l}{\to} E_1^{-n-l+1,n+l+i}.$$

As $E_2^{-n-l,n+l+i} = 0$, using the analysis above, we obtain a short exact sequence

$$0 \to \mathrm{im}\,\alpha_{l+1} \to H^i(X, \mathrm{DR}(\mathrm{gr}^W_{n+l}\mathscr{O}_X(*D))) \to \mathrm{im}\,\alpha_l \to 0,$$

and the result follows. □

When $p = 0$, the result above is enough to obtain that

$$H^i(X, \omega_X(D) \otimes I_0^{W_l}(D)) = 0$$

for $l \geq 2$ and $i \geq 1$. Indeed, as 0 is the lowest degree of the Hodge filtration on $\mathscr{O}_X(*D)$, we have

$$H^i(X, \omega_X(D) \otimes I_0^{W_l}(D)) \cong \operatorname{gr}^F_{-n} H^i(X, \mathrm{DR}(W_{n+l}\mathscr{O}_X(*D))).$$

This is no longer the case when we consider $\operatorname{gr}^F_{-n+p}$ for $p \geq 1$ instead. Nonetheless, following the idea in [MP19a, Proof of Theorem F], we give conditions in Theorem C to obtain an analogue vanishing theorem.

Proof of Theorem C. Since $I_{p-1}^{W_l}(D) = \mathscr{O}_X$, we have the following short exact sequence

$$0 \to \omega_X(pD) \to \omega_X((p+1)D) \otimes I_p^{W_l}(D) \to \omega_X \otimes \operatorname{gr}^F_p(W_{n+l}\mathscr{O}_X(*D)) \to 0.$$

Using the long exact sequence of cohomologies and Kodaira-vanishing, we note that it is enough to prove that

$$H^i(X, \omega_X \otimes \operatorname{gr}^F_p(W_{n+l}\mathscr{O}_X(*D))) = 0.$$

Consider now the complex

$$C^\bullet := \operatorname{gr}^F_{-n+p} \mathrm{DR}(W_{n+l}\mathscr{O}_X(*D)).$$

The complex $C^\bullet$ can be identified with the complex

$$\left[\Omega_X^{n-p} \otimes \mathscr{O}_X(D) \to \Omega_X^{n-p+1} \otimes \mathscr{O}_D(2D) \to \cdots \to \Omega_X^{n-1} \otimes \mathscr{O}_D(pD) \to \omega_X \otimes \operatorname{gr}^F_p(W_{n+l}\mathscr{O}_X(*D))\right]$$

concentrated in degrees $-p$ to 0, since $F_0W_{n+l}\mathscr{O}_X(*D) = \mathscr{O}_X(D)$ and $\operatorname{gr}^F_k W_{n+l}\mathscr{O}_X(*D) \cong \mathscr{O}_D((k+1)D)$ for $k \leq p-1$ (see §1 for the definition of $\operatorname{gr}^F_p \mathrm{DR}$).

Suppose now that D has at most isolated singularities. By Lemma 7.1, we obtain that

$$\mathbb{H}^i(X, \mathrm{DR}(W_{n+l}\mathscr{O}_X(*D))) = 0$$

for $i \geq 1$ and $l \geq 2$. In particular, this means that

$$\mathbb{H}^i(X, C^\bullet) = 0$$

for the same indices, by the Hodge-to-de-Rham degeneration. Next, we use the exact sequence

$$E_1^{p,q} = H^q(X, C^p) \Rightarrow \mathbb{H}^{p+q}(X, C^\bullet).$$

Note that

$$E_1^{0,q} = H^q(X, \omega_X \otimes \operatorname{gr}^F_p(W_{n+l}\mathscr{O}_X(*D))).$$

Since

$$E_1^{-1,q} = H^q(X, \Omega_X^{n-1} \otimes \mathscr{O}_D(pD)),$$

then $E_1^{-1,q} = 0$ if $q \geq 2$ by Nakano vanishing. Moreover, $E_1^{-1,1} = 0$ by our hypothesis.

We continue with a similar analysis in the higher pages of the spectral sequence. More precisely, we show that the hypothesis implies that $E_r^{-r,q+r-1} = 0$ for all $r \geq 2$. Note that this is enough to complete the proof. Indeed, if this is the case, we obtain that

$$E_\infty^{0,q} = H^q(X, \omega_X \otimes \mathrm{gr}_p^F(W_{n+l}\mathscr{O}_X(*D))) = 0$$

for $q \geq 1$, where the last equality follows from the established equality with $C^\bullet$.

To complete the proof, note that

$$E_1^{-r,q+r-1} = 0$$

for $r \geq p$. Indeed, this is clear for the strict inequality by the degrees on which $C^\bullet$ is concentrated, and

$$E_1^{-p,q+p-1} = H^{q+p-1}(X, \Omega_X^{n-p} \otimes \mathscr{O}_X(D)).$$

If $q \geq 2$, then this spaces vanishes by Nakano vanishing, and if $q = 1$, it vanishes by our assumption. Finally, for $r \leq p-1$, we have

$$E_1^{-r,q+r-1} = H^{q+r-1}(X, \Omega_X^{n-r} \otimes \mathscr{O}_D((p+1-r)D)).$$

This space fits the a long exact sequence

$$\to H^{q+r-1}(X, \Omega_X^{n-r}((p+1-r)D)) \to E_1^{-r,q+r-1} \to H^{q+r}(X, \Omega_X^{n-r}((p-r)D)).$$

If $q \geq 2$, then the two other terms vanish by Nakano vanishing, and if $q = 1$, they vanish by the assumption. □

Remark 7.2. This result does not hold in general for $l = 1$ (see [Ola22, Remark 9]).

8. Kodaira-type vanishing. Using a similar idea to the one in the proof of Theorem C, we obtain a vanishing theorem for weighted Hodge ideals. This is the analogue result to [MP19a, Theorem F].

Proposition 8.1. *Let X be a smooth projective variety of dimension n, and D a reduced effective divisor. Let L be a line bundle such that $L(kD)$ is ample for $0 \leq k \leq p$, and assume $I_{p-1}^{W_1}(D)$ is trivial. Then*

(1) For $l \geq 1$ and $i \geq 2$,

$$H^i(X, \omega_X((p+1)D) \otimes L \otimes I_p^{W_l}(D)) = 0.$$

(2) If $H^j(X, \Omega_X^{n-j} \otimes L((p-j+1)D)) = 0$ for all $1 \leq j \leq p$, then

$$H^1(X, \omega_X((p+1)D) \otimes L \otimes I_p^{W_l}(D)) = 0$$

for $l \geq 1$.

Proof. Since $I_{p-1}^{W_l}(D) = \mathscr{O}_X$, we have the following short exact sequence

$$0 \to \omega_X \otimes L(pD) \to \omega_X \otimes L((p+1)D) \otimes I_p^{W_l}(D) \to \omega_X \otimes L \otimes \mathrm{gr}_p^F(W_{n+l}\mathscr{O}_X(*D)) \to 0.$$

By Kodaira-vanishing, it is enough to prove

$$H^i(X, \omega_X \otimes L \otimes \mathrm{gr}_p^F(W_{n+l}\mathscr{O}_X(*D))) = 0.$$

We have that

$$\mathbb{H}^i(X, L \otimes \mathrm{gr}^F_{-n+p} \mathrm{DR}(W_{n+l}\mathscr{O}_X(*D)) = 0$$

for $i \geq 1$ and $l \geq 1$ as a consequence of a vanishing result by Saito [Sai90, Proposition 2.33]. To complete the proof, we use the same spectral sequence as in the proof of Theorem C. □

9. Applications. In this section, we combine the local study and the vanishing results. To obtain applications, we use the vanishing theorems of the previous sections. A class varieties where the vanishing condition in Theorem C and Proposition 8.1 is satisfied, is toric varieties. In this case, the Bott-Danilov-Steenbrink vanishing theorem says that if A is an ample line bundle on the toric variety X, then

$$H^i(X, \Omega_j \otimes A) = 0$$

for $j \geq 0$ and $i \geq 1$ (see e.g. [Mus02, Theorem 2.4]). For the applications, we discuss the case of $X = \mathbb{P}^n$. We start with the proof of Corollary D.

Proof of Corollary D. Consider the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n}(k) \otimes I_p^{W_l}(D) \to \mathscr{O}_{\mathbb{P}^n}(k) \to \mathscr{O}_{Z_{p,l}}(k) \to 0.$$

The result follows from passing to cohomology and applying Theorem C. □

9.1. *Isolated p-log-canonical singularities.* Suppose the pair (X, D) is p-log-canonical, and has at most isolated singularities. If $p = 0$, the pair is log-canonical and in this case, $I_0^{W_1}(D)$ is the maximal ideal at each isolated singularity that is not rational, by a result of Ishii (see [Ola22, §5.3]). For simplicity, let $x \in D$ be the only singularity and $i : \{x\} \hookrightarrow X$ the inclusion, and suppose that it is log-canonical singularity and not rational. The result above means that if we denote

$$i_* H_l = \mathrm{DR}(\mathrm{gr}^W_{n+l} \mathscr{O}_X(*D))$$

for $l \geq 2$, there exists exactly one degree l such that $\dim(\mathrm{gr}^F_{-n} H_l) = 1$, and the rest are 0. In this case, using [Ola22, Theorem B], we say that the singularity is of type $(0, n-l)$ [Ish85, Definition 4.1]. There is a similar picture for the cases $p \geq 1$ we describe next.

Non-rational log-canonical singularities correspond to the case where the minimal exponent at the singularity is 1. We then consider singularities with minimal exponent $p+1$, in which case $I_p(D) = \mathscr{O}_X$ and $I_p^{W_1}(D)$ is non-trivial by Corollary 5.10. These singularities generalize the example of non-rational log-canonical singularities in the following sense.

Proposition 9.1. *Suppose D has at most one isolated singularity $x \in D$, and $\widetilde{\alpha_D} = p+1$. Then,*

$$I_p^{W_1}(D) = \mathfrak{m}_x,$$

the maximal ideal of x in X.

Proof. Suppose that D is defined by $f \in \mathscr{O}_X$. Recall from the proof of Corollary 5.10, that as $\widetilde{\alpha_f} = p+1$, then $\delta, \partial_t\delta, \ldots, \partial_t^p\delta \in V^1 B_f$. Moreover, we also know that $\delta, \partial_t\delta, \ldots, \partial_t^{p-1}\delta \in$

$W_1V^1B_f$. It is then enough to show that $g\partial_t^p\delta \in W_1V^1B_f$ if and only if $g \in \mathfrak{m}_x$. As D has an isolated singularity, we have that

$$\operatorname{gr}_p^F \operatorname{gr}_V^\alpha B_f \text{ is annihilated by } \mathfrak{m}_x$$

for $\alpha < 1$ [DS12, 4.11.1].

We also know that $\partial_t^p\delta \in V^1B_f \smallsetminus W_1V^1B_f$, and this means that $\partial_t^{p+1}\delta \in V^0B_f \smallsetminus V^{>0}B_f$. In particular, the class of $\partial_t^{p+1}\delta$ in $\operatorname{Gr}_p^F \operatorname{gr}_V^0 B_f$ is not zero. Using the result above, for any $g \in \mathfrak{m}_x$, the class of $g\partial_t^{p+1}\delta$ in $\operatorname{Gr}_p^F \operatorname{gr}_V^0 B_f$ is zero. This means that $g\partial_t^{p+1}\delta \in V^{>0}B_f$, and equivalently, $g\partial_t^p\delta \in W_1V^1B_f$. Using the description of Theorem A, we obtain that $g \in I_p^{W_1}(D)$ for any $g \in \mathfrak{m}_x$, and we know that the ideal is not trivial, hence we have an equality. □

In other words, if D has one isolated singularity $x \in D$, and $\widetilde{\alpha_D} = p+1$, then

$$\sum_{l\geq 2} \dim(\operatorname{Gr}_F^{n-p} H_l) = 1$$

by Theorem B, that is, there is exactly one $l \geq 2$ such that

$$\dim(\operatorname{Gr}_F^{n-p} H_l) = 1,$$

and the rest are 0. Moreover, by the same result, $\sum_{l\geq 2} \dim(\operatorname{Gr}_F^{n-r} H_l) = 1$, for $0 \leq r \leq p-1$.

Remark 9.2. Friedman and Laza have studied related invariants of singularities in similar conditions in [FL22, Theorem 6.11 and Corollary 6.14].

Definition 9.3. Let $x \in D$ be an isolated singularity such that $\widetilde{\alpha_{D_x}} = p+1$, that is an isolated p-log-canonical that is not p-rational. Let l be the degree such that $\dim(\operatorname{Gr}_F^{n-p} H_l) = 1$. Then, we say that the singularity is of type $(p, n-l-p)$.

Remark 9.4.
i) Definition 9.3 is analogous to the definition of isolated log-canonical singularities of type $(0, s)$ [Ish85, Definition 4.1], when $x \in D$ is an isolated singularity and D is a hypersurface of a smooth variety.
ii) Ishii defined these singularities more generally for normal isolated 1-Gorenstein log-canonical singularities. It is an open question how to generalize this definition for non-hypersurface singularities.
iii) The possible types are $(p,p), (p,p+1), \dots (p, n-2-p)$. This is a consequence of the fact that the nilpotency order of the vanishing cohomology is bounded by Briançon-Skoda exponent [Sch80, Main Theorem]. This nilpotency order gives a bound for the nilpotency order of $(\partial_t t)$ on $\operatorname{gr}_V^0 B_f$, which in turn gives a bound for the order of $(t\partial_t)$ on $\operatorname{gr}_V^1 B_f$. The Briançon-Skoda exponent is bounded by $n-2p-1$ (see for instance [JKSY22a]), which means that $n-l-p \geq p$.

Example 9.5. Suppose that $f \in \mathbb{C}[x_1, \dots, x_n]$ is a polynomial with an isolated singularity at the origin, and a non-degenerate Newton boundary. Let $\Gamma_+(f) = \Gamma$ the Newton polyhedron of f, $\Gamma(f)$ the union of the compact faces of $\Gamma_+(f)$, and $\mathcal{F}$ the set of compact facets.

For each $F \in \mathcal{F}$, there is a unique vector $B_F \in (\mathbb{Q}_{\geq 0})^n$ such that $\langle A, B_F \rangle = 1$ for all $A \in F$. For every monomial x^ν, we define

$$\tilde{\rho}_F(x^\nu) = \langle \nu + \mathbf{1}, B_F \rangle,$$

where $\mathbf{1} = (1, \dots, 1)$, and for any $g \in \mathscr{O}$, $g = \sum g_A x^A$,

$$\tilde{\rho}_F(g) = \min\{\tilde{\rho}_F(x^A) \ : \ g_A \neq 0\}.$$

Finally, we define

$$\tilde{\rho}(g) = \min\{\tilde{\rho}_F(g) \ : \ F \in \mathcal{F}\}.$$

In this case, the minimal exponent is $\tilde{\rho}(1)$.

Suppose $\widetilde{\alpha}_f = p + 1$, which implies that $\partial_t^p \delta \in V^1 i_+ \mathscr{O}_X$. Using the description of the microlocal V-filtration (see [Sai94, Proposition 3.2]), we see that if

$$r = \#\{F \in \mathcal{F} \ : \ \tilde{\rho}_F(1) = \tilde{\rho}(1)\},$$

then $(t\partial_t)^{r+1} \partial_t^p \delta \in V^{>1} i_+ \mathscr{O}_X$, or equivalently,

$$1 \in I_p^{W_{r+1}}(D).$$

In general, $r + 1$ is not the degree with $\mathrm{Gr}_F^{n-p} H_l \neq 0$.

i) Weighted homogeneous singularities with $\widetilde{\alpha}_f = p+1$ are examples of singularities of type $(p, n-2-p)$ (see Remark 5.7). Isolated singularities with non-degenerate Newton boundary give examples for different degrees of l. For instance, $f = x^2 + y^2 + z^2 + u^2 w^2 + u^4 + w^5 \in \mathbb{C}^5$ satisfies that $\widetilde{\alpha_f} = 2$, and $r = 2$, using the notation above. We can also verify that $(t\partial_t)^2 \partial_t \delta \notin V^{>1} i_+ \mathscr{O}_X$, since $w^5 \partial_t^3 \delta \in V^0 \setminus V^{>0}$. Indeed, this follows from the fact that $w^5 \notin J(f)$, where $J(f)$ is the Jacobian ideal, and [JKSY22b, Proposition 1.3]. Therefore, this singularity is of type $(1, 1)$.

ii) Let Δ_0 be the compact face that contains $\frac{1}{p+1}\mathbf{1}$ in its relative interior, and let $s = \dim \Delta_0$. Assume also that the Newton polyhedron is simplicial. The number r defined above satisfies that $s = n - r$. Let l be the degree such that $\mathrm{Gr}_F^{n-p} H_l \neq 0$. Then $l \leq r + 1 = n - s + 1$, if $s > 0$, and $l \leq n$ is $s = 0$.

iii) If $p = 0$, the previous inequalities are equalities (without the simplicial assumption) by a result of Watanabe that says that the singularities are log-canonical of type $(0, s-1)$ if $s > 0$, and $(0, 0)$ if $s = 0$, which is equivalent to the equalities [Wat87, Corollary 3.14].

Using Proposition 9.1 and the vanishing results, we obtain a bound on the number of these singularities in a hypersurface of $\mathbb{P}^n$.

Corollary 9.6. *Let D be a reduced hypersurface of $\mathbb{P}^n$ of degree d with at most isolated singularities. Assume that the pair $(\mathbb{P}^n, D)$ is strictly p-log-canonical, that is, $\widetilde{\alpha_D} = p + 1$. Let Z be the union of the strictly p-log-canonical singular points of D and Z_2 the union of* *those of type* $(p,p),\dots,(p,n-3-p)$*. Then,*

$$\#Z_2 \leq \binom{(p+1)d-1}{n},$$

and

$$\#Z \leq \binom{(p+1)d}{n}.$$

Proof. By Proposition 9.1, the scheme Z is defined by the ideal $I_p^{W_l}(D)$. Therefore, the result follows from Corollary D.

□

F. Restriction Theorem

Let $(\mathcal{M},F)$ be a filtered right $\mathscr{D}$-module underlying a mixed Hodge module M on X. Let $H \subseteq X$ be a smooth hypersurface and $i : H \hookrightarrow X$ the inclusion. In this section, we change the notation of the V-filtration by $V_k = V^{-k}$, which is the notation used in [MP18]. There exists a canonical morphism

$$\operatorname{gr}_0^V \mathcal{M} \xrightarrow{\sigma} \operatorname{gr}_{-1}^V \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(H) \tag{9.7}$$

satisfying

$$\mathcal{H}^0 i^! \mathcal{M} \cong \ker(\sigma) \quad \text{and} \quad \mathcal{H}^1 i^! \mathcal{M} \cong \operatorname{coker}(\sigma)$$

with the filtrations induced by the filtrations on $\mathcal{M}$ (see [MP18, §2]). Moreover, on an open set $U \subseteq X$ where H is given by a local equation t, this map corresponds to

$$\operatorname{Var} = \cdot t : \operatorname{gr}_0^V \mathcal{M} \to \operatorname{gr}_{-1}^V \mathcal{M}$$

between the vanishing and nearby cycles along H.

In the proof of [MP18, Theorem A] the authors defined for all k a morphism

$$F_k \mathcal{H}^1 i^! \mathcal{M} \to F_k \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H(H). \tag{9.8}$$

First, we define a morphism

$$\eta : F_k \operatorname{gr}_{-1}^V \mathcal{M} = \frac{F_k V_{-1} \mathcal{M}}{F_k V_{<-1} \mathcal{M}} \to F_k \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H$$

such that for $u \in F_k V_{-1} \mathcal{M}$, $\eta(u)$ is the class of u in $F_k \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H$. This map is well defined, as on an open set U where H is defined by an equation t, the V-filtration satisfies

$$(F_k V_\alpha \mathcal{M}) \cdot t = F_k V_{\alpha-1} \mathcal{M} \quad \text{for } \alpha < 0,$$

and $F_k \mathcal{M} \cdot t$ maps to 0 in $F_k \mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H$. The map η induces a map on $F_k \mathcal{H}^1 i^! \mathcal{M}$. Indeed, since locally σ is right multiplication by t, the image of σ is mapped to 0 by $\eta \otimes \mathscr{O}_X(H)$.

Proof of Theorem E. Let $\mathcal{M} = W_{n+l}\omega_X(*D)$. For every k we have the canonical morphism (9.8):

$$F_k\mathcal{H}^1 i^! \mathcal{M} \to F_k\mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H(H).$$

Note that the sheaf

$$F_{k-n}\mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H(H) = I_k^{W_l}(D) \otimes \omega_X((k+1)D) \otimes \mathscr{O}_H(H) \cong I_k^{W_l}(D) \otimes \omega_H((k+1)D_H).$$

Consider the short exact sequence

$$0 \to \mathcal{M} \to \omega_X(*D) \to \mathcal{C} \to 0.$$

Applying the functor $i^!$ and taking cohomology we obtain an exact sequence

$$0 \to \mathcal{H}^0 i^1 \mathcal{C} \to \mathcal{H}^1 i^! \mathcal{M} \to \mathcal{H}^1 i^! \omega_X(*D) \to \mathcal{H}^1 i^! \mathcal{C} \to 0,$$

as $\mathcal{H}^0 i^! \omega_X(*D) = 0$. As $\mathrm{gr}_i^W \mathcal{C} = 0$ for $i < n+l+1$,

$$\mathrm{gr}_i^W \mathcal{H}^0 i^! \mathcal{C} = 0 \quad \text{for } i < n+l+1,$$

and

$$\mathrm{gr}_i^W \mathcal{H}^1 i^! \mathcal{C} = 0 \quad \text{for } i < n+l+2$$

by [Sai90, Proposition 2.26]. Therefore, we obtain a short exact sequence

$$0 \to W_{n+l+1}\mathcal{H}^0 i^1 \mathcal{C} \to W_{n+l+1}\mathcal{H}^1 i^! \mathcal{M} \to W_{n+l+1}\mathcal{H}^1 i^! \omega_X(*D) \to 0.$$

Note that as

$$\mathrm{Ext}^1(W_{n+l+1}\mathcal{H}^1 i^! \omega_X(*D), W_{n+l+1}\mathcal{H}^0 i^1 \mathcal{C}) = 0$$

(see [Sch14, §23]), there is a split map

$$W_{n+l+1}\mathcal{H}^1 i^! \omega_X(*D) \to W_{n+l+1}\mathcal{H}^1 i^! \mathcal{M}. \tag{9.9}$$

The source of this maps admits the following interpretation:

$$W_{n+l+1}\mathcal{H}^1 i^! \omega_X(*D) \cong W_{n-1+l}\omega_H(*D_H).$$

Indeed,

$$\mathcal{H}^1 i^! \omega_X(*D) \cong \omega_H(*D_H)(-1)$$

[MP18, Proof of Theorem A].

Taking the corresponding piece of the Hodge filtration in (9.9) and composing it with (9.8), we obtain a morphism

$$F_k W_{n+l+1}\mathcal{H}^1 i^! \omega_X(*D) \to F_k\mathcal{M} \otimes_{\mathscr{O}_X} \mathscr{O}_H(H).$$

Using the morphism above and switching k to $k-n$, we obtain a map

$$F_{k-n+1}W_{n-1+l}\omega_H(*D_H) = I_k^{W_l}(D_H) \otimes \omega_H((k+1)D_H) \to I_k^{W_l}(D) \otimes \omega_H((k+1)D_H),$$

and hence

$$I_k^{W_l}(D_H) \to I_k^{W_l}(D) \otimes \mathscr{O}_H.$$

Composing this map with $I_k^{W_l}(D) \otimes \mathscr{O}_H \to I_k^{W_l}(D) \cdot \mathscr{O}_H$, we obtain a morphism

$$I_k^{W_l}(D_H) \to I_k^{W_l}(D) \cdot \mathscr{O}_H. \tag{9.10}$$

By construction, this map is compatible with restriction to open sets. Let $V = H \setminus D_H$ be the complement. When restricted to V, this map is the identity on $\mathscr{O}_V$, and therefore it is an inclusion.

For the last statement, we note that a general H is in particular non-characteristic with respect to $\omega_X(*D)$. By the description of the V-filtration in this case [Sai88, Lemma 3.5.6], the map σ is the zero map, and therefore (9.8) is a surjection. Moreover, in this case

$$\mathcal{H}^1 i^! \mathcal{M} = \mathcal{H}^1 i^! W_{n+l}\omega_X(*D) \cong W_{n+l+1}\mathcal{H}^1 i^! \omega_X(*D)$$

where the first equality is the definition of $\mathcal{M}$ and the isomorphism is a result of Saito [Sai90, Lemma 2.25]. Hence, in this case (9.10) is an isomorphism. □

Remark 9.11. A similar result can be obtained when H is an intersection of several general hyperplane sections. For more details, see [Ola22, Remark 12].

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