SINGULARITIES OF SECANT VARIETIES FROM A HODGE THEORETIC PERSPECTIVE

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Abstract. We study the singularities of secant varieties of smooth projective varieties using methods from birational geometry when the embedding line bundle is sufficiently positive. We give a necessary and sufficient condition for these to have p-Du Bois singularities. In addition, we show that the singularities of these varieties are never higher rational, by giving a classification of the cases when they are pre-1-rational. From these results, we deduce several consequences, including a Kodaira-Akizuki-Nakano type vanishing result for the reflexive differential forms of the secant varieties.

A. Introduction

Secant varieties have been vastly studied in the literature. In particular, there has been a great deal of interest in understanding their defining equations and syzygies as well as their singularities ([ENP20, Rai12, SV09, SV11, Ver01, Ver08, Ver09] and the references therein). The research on these varieties is also partly motivated by topics in algebraic statistics and algebraic complexity theory ([SS06, LW09]). A few years ago, under some positivity conditions on the embedding line bundle, [Ull16] had shown the normality of these varieties, thereby completing some results of Vermeire. More recently, the singularities of the secant varieties were shown to be Du Bois, which is an important class of singularities (see [KS11] for a survey), in [CS18], and also a characterization for when these are rational was given.

Very recently, the notions of Du Bois and rational singularities have been extended substantially in a series of papers [MOPW23, JKSY22, FL22a, SVV23]. For this reason, it is natural to ask: *to which of these newly defined singularity classes do the secant varieties belong?* To set up the context, we start with a smooth projective variety X of dimension n and a very ample line bundle L on X. We denote by $\Sigma = \Sigma(X, L)$ the secant variety of X with respect to L. Under a mild assumption on the positivity of L, which requires L to be 3-very ample (see Definition 2.1), $\dim \Sigma = 2n + 1$ and the singular locus of Σ is $X \subset \Sigma$ (see Proposition 2.2).

In [Ull16] and [CS18], the singularities of the secant varieties were studied when X is embedded by a sufficiently positive adjoint linear series. As a natural extension, throughout this article, we will mostly be concerned with the situation described in Set-up 2.28, where we assume that one of the following holds:

- $n = 1$ and $\deg(L) \geq 2g + 3$ where g is the genus of X; or

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- $n \geq 2$ and $L = L_{l,d} := lK_X + dA + B$ where A and B are very ample and nef line bundles respectively, with a certain positivity assumption described in (2.29).

The positivity assumption we indicated is purely numerical in terms of (l, d) depending on a parameter s. Under these assumptions, L is 3-very ample (see Remark 2.30). Roughly speaking, when $n \geq 2$, the larger s for which (l, d) satisfies the condition (2.29) implies the better singularities of Σ. For example, the tuple $(1, d)$ with $d \geq 2n + 2$ satisfies (2.29) with $s = 0$; and the main results of Ullery and Chou–Song can be stated as follows:

Theorem 0.1 ([Ull16, CS18], Remark 2.31)**.** *Assume one of the following holds:*

- *$n = 1$ and $\deg(L) \geq 2g + 3$ where g is the genus of X; or*
- *$n \geq 2$ and $L = L_{1,d} := K_X + dA + B$ where A and B are very ample and nef line bundles respectively, with $d \geq 2n + 2$.*

Then Σ is normal and has Du Bois singularities. In particular, the same conclusions hold in the situation of Set-up 2.28 if (l, d) satisfies (2.29) with $s = 0$ when $n \geq 2$.

The present work is devoted to understanding the singularities of Σ, where we additionally assume that (l, d) satisfies (2.29) with $s = p$ when $n \geq 2$ for some $1 \leq p \leq n$. For the convenience of the reader, here we remark that for $n \geq 2$, the line bundle

$$L = 2K_X + dA + B$$

satisfies (2.29) with $s = 1$, when $d \geq 3n + 4$ (if $n \geq 3$, then $d = 3n + 3$ also satisfies the condition). Moreover, for $n \geq 2$ and $1 \leq p \leq n$, the line bundle

$$L = \begin{cases} \left[\binom{n-1}{p-1} + 1\right] K_X + \left[\left(\binom{n-1}{p-1} + 1\right)(n+2) + 2p\left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)\right] A + B & \text{if } p - 1 \leq \lfloor \binom{n-1}{2} \rfloor; \\ \left[\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 1\right] K_X + \left[\left(\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 1\right)(n+2) + 2p\left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)\right] A + B & \text{otherwise} \end{cases}$$

satisfies the positivity assumptions described in (2.29) with $s = p$.

Higher Du Bois singularities. We now recall that associated to any complex variety Z there is the Du Bois complex $\underline{\Omega}^\bullet_Z$, introduced in [DB81], which is an object in the derived category of filtered complexes on Z. The associated graded objects

$$\underline{\Omega}^k_Z := \mathrm{Gr}^k_F \underline{\Omega}^\bullet_Z[k]$$

are objects in the derived category of coherent sheaves. A variety Z is said to have Du Bois singularities if the canonical morphism $\mathcal{O}_Z \to \underline{\Omega}^0_Z$ is a quasi-isomorphism. This notion has been generalized by requiring the canonical morphisms $\Omega^k_Z \to \underline{\Omega}^k_Z$ to be quasi-isomorphisms for $0 \leq k \leq p$ as well. This condition is well-behaved for varieties whose singularities are local complete intersections (see [MP22a, MP22b]) but usually fails otherwise, as is the case of secant varieties as they are in general not even Cohen-Macaulay ([CS18, Theorem 1.3]).

For this reason, a new definition of varieties having p-Du Bois singularities was introduced in [SVV23], which generalizes the previously described notion for varieties with local complete intersection singularities. Our first result is about a vanishing condition on the cohomology of the Du Bois complexes. Following [SVV23], we say that a variety has *pre-p-Du Bois singularities* if these complexes are concentrated in degree zero for $0 \leq k \leq p$ (see

Definition 1.1). When $n \geq 2$ and (l, d) satisfies (2.29) for sufficiently large s, Σ satisfies this for every non-negative integers p. Also, Σ has pre-p-Du Bois singularities for all $p \geq 0$ when X is a curve embedded by the complete linear series of a line bundle of degree $\geq 2g+3$:

Theorem A. *Let $p \in \mathbb{N}$. In the situation of Set-up 2.28, assume (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \geq 2$. Then the natural maps*

$$\mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \underline{\Omega}^k_\Sigma$$

are quasi-isomorphisms for all $0 \leq k \leq p$; in particular, the singularities of Σ are pre-p-Du Bois.

In fact, we prove a more general statement that does not require the Set-up 2.28 conditions. Briefly stated, it is enough to verify for $i, j \geq 1$

$$H^i(\Omega^q_{F_x} \otimes b_x^*(jL)(-2jE_x)) = 0 \text{ for all } 0 \leq q \leq p \text{ and for all } x \in X \tag{0.2}$$

where $b_x : F_x := \mathrm{Bl}_x X \to X$ is blow-up at x and E_x is the exceptional divisor, to obtain the same conclusion as in Theorem A (see Proposition 3.3). Other results presented below are also valid if (0.2) is satisfied with some extra mild conditions.

We show in Theorem 3.1 that in the situation of Set-up 2.28, if (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \geq 2$, then (0.2) is satisfied. Other examples where (0.2) is satisfied, include when $b_x^*L(-2E_x)$ is ample and F_x satisfies Bott vanishing for all $x \in X$. In the case of curves embedded by line bundles of smaller degrees, we refer to Example 4.14.

In addition to requiring Z to have pre-p-Du Bois singularities, following [SVV23], a variety is said to have p-Du Bois singularities if two extra conditions are satisfied. The first is a codimension condition on the singular locus, and the second is a condition in degree zero for $\underline{\Omega}^k_Z$ when $0 \leq k \leq p$ (see Definition 1.2). Secant varieties satisfy the first condition up to a certain range; and in relation to the last condition, we show:

Theorem B. *Let p be a positive integer. Suppose we are in the situation of Set-up 2.28 and assume (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \geq 2$. Then the natural maps*

$$\delta_k : \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \Omega^{[k]}_\Sigma$$

are isomorphisms for $1 \leq k \leq p$ if and only if $H^k(\mathcal{O}_X) = 0$ for $1 \leq k \leq p$.

As before, we remark here that if Σ is normal, then its singularities are Du Bois and the conclusion of Theorem B holds if (0.2) is satisfied.

We now introduce an useful invariant

$$\nu(X) := \max\left\{i \mid 1 \leq i \leq n-1 \text{ such that } H^j(\mathcal{O}_X) = 0 \text{ for all } 1 \leq j \leq i\right\},$$

where we set the convention $\nu(X) = 0$ if the set above is empty. The importance of this invariant was observed in [CS18]; it follows from *loc. cit.* that if $n \geq 2$ and (l, d) satisfies (2.29) for some $s \geq 0$, or if X is a curve embedded by $|L|$ with $\deg(L) \geq 2g+3$, then Σ is Cohen-Macaulay if and only if $\nu(X) = n-1$ (notice that this always holds when $n = 1$), and it has rational singularities if and only if $\nu(X) = n-1$ and $H^n(\mathcal{O}_X) = 0$. The following is an immediate consequence of Theorem A and Theorem B.

Corollary C. *Let $p \in \mathbb{N}$ with $p \leq \lfloor \frac{n}{2} \rfloor$. Suppose we are in the situation of Set-up 2.28 and assume (l, d) satisfies (2.29) with $s = p$ when $n \geq 2$. Then Σ has p-Du Bois singularities if and only if $p \leq \nu(X)$.*

When L is 3-very ample, the codimension of the singular locus of Σ is $n+1$ whence it follows from the definition of p-Du Bois singularity and Proposition 2.2, that in this case Σ has p-Du Bois singularity for some $p > \lfloor \frac{n}{2} \rfloor$ if and only if $\Sigma = \mathbb{P}(H^0(L))$.

We further study the morphisms $\delta_k : \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \Omega^{[k]}_\Sigma$ and in Theorem 5.1, we also discuss the case for higher degrees not considered in the definition of p-Du Bois singularities. Recall that a variety Z is said to have weakly rational singularities if the Grauert-Riemenschneider sheaf ω^{GR}_Z, which is by definition the push-forward of the canonical bundle from a resolution of singularities of Z, is reflexive. In the situation of Set-up 2.28, [CS18, Theorem 1.4] shows that the natural map $\omega^{\mathrm{GR}}_\Sigma \hookrightarrow \omega_\Sigma := \mathcal{H}^{-2n-1}(\omega^\bullet_\Sigma)$ is an isomorphism if and only if $H^n(\mathcal{O}_X) = 0$. Notice that under our set-up, this map is an isomorphism if and only if the singularities of Σ are weakly rational as Σ is normal and ω_Σ is reflexive. When $n \geq 2$ and (l, d) satisfies (2.29) with $s = n$, *loc. cit.* combined with Theorem 5.1 (1) establishes the equivalences:

$$\Sigma \text{ has weakly rational singularities} \iff H^n(\mathcal{O}_X) = 0 \iff \delta_{2n+1} \text{ is an isomorphism}$$
$$\iff \delta_{2n} \text{ is an isomorphism.}$$

Higher rational singularities. We discuss next the extension of the notion of rational singularities. Recall that a variety Z is said to have rational singularities if for a resolution of singularities $f : \widetilde{Z} \to Z$, the canonical morphism $\mathcal{O}_Z \to \mathbf{R}f_*\mathcal{O}_{\widetilde{Z}}$ is a quasi-isomorphism. Intrinsically, this is equivalent to requiring that the morphism $\mathcal{O}_Z \to \mathbf{D}_Z(\underline{\Omega}^{\dim Z}_Z)$ is a quasi-isomorphism, where $\mathbf{D}_Z(-)$ is the Grothendieck dual. Analogous to the definition of pre-p-Du Bois singularities, a variety Z is said to have *pre-p-rational singularities* if the complexes $\mathbf{D}_Z(\underline{\Omega}^{\dim Z - k}_Z)$ are concentrated in degree zero for any $0 \leq k \leq p$ (see Definition 1.4). When $p < \mathrm{codim}_Z(Z_{\mathrm{sing}})$ and $f : \tilde{Z} \to Z$ is a strong log resolution with $E := f^{-1}(Z_{\mathrm{sing}})_{\mathrm{red}}$ a simple normal crossing divisor (see Definition 1.5), this condition is equivalent to requiring that the complexes $\mathbf{R}f_*\Omega^k_{\widetilde{Z}}(\log E)$ are concentrated in degree zero for $0 \leq k \leq p$. It was shown in [SVV23] that for a normal variety, pre-p-rational singularities are pre-p-Du Bois. However, it turns out that the singularities of secant varieties are almost never pre-1-rational:

Theorem D. *Suppose we are in the situation of Set-up 2.28, and assume (l, d) satisfies (2.29) with $s = 1$ when $n \geq 2$. Then Σ has pre-1-rational singularities if and only if $X \subset \mathbb{P}(H^0(L))$ is a rational normal curve of degree ≥ 3.*

The above result is a consequence of a more general fact that when $n \geq 2$, if (l, d) satisfies (2.29) with $s = 1$, then $\mathcal{H}^1(\mathbf{D}_\Sigma(\underline{\Omega}^{\dim \Sigma - 1}_\Sigma)) \neq 0$, and similarly if $n = 1$, $\deg(L) \geq 2g + 3$ and $g > 0$.

Following [FL22a, SVV23], a variety is said to have p-rational singularities if Z is normal, has pre-p-rational singularities, and additionally satisfies a codimension condition of the singular locus (see Definition 1.6). By Proposition 2.2, this last condition is never satisfied when $X \subset \mathbb{P}(H^0(L))$ is a rational normal curve of degree ≥ 4.

It is worth mentioning that, unlike the local complete intersection case, where p-Du Bois singularities are $(p-1)$-rational ([CDM22, MP22b, FL22b]), our study produces secant varieties whose singularities are p-Du Bois for some $p \geq 2$ but not even pre-1-rational. This feature is also shared by the singularities of non-simplicial affine toric varieties, see [SVV23, Proposition E]. We also obtain secant varieties whose singularities are p-Du Bois for large p but not rational, a feature that is shared by the singularities of certain affine cones over smooth projective varieties, see [SVV23, Proposition F].

Some examples. We provide concrete examples to highlight the scope of our results:

Example 0.3 (Curves)**.** Let $C \subset \mathbb{P}^N$ be a smooth curve of genus g, embedded by the complete linear series of a line bundle L with $\deg(L) \geq 2g+3$. Then:

- Σ is normal, Cohen-Macaulay, and has Du Bois singularities. It has weakly rational singularities $\iff$ it has rational singularities $\iff$ $g = 0$.
- The singularities of Σ are pre-p-Du Bois for all $p \geq 0$. They are pre-1-rational if and only if $g = 0$.
- The singularities are not p-Du Bois for any $p \geq 1$ unless $\Sigma = \mathbb{P}^N$ (one can show that this happens if and only if $C \subset \mathbb{P}^N$ is a twisted cubic, i.e., $(C, L) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$).

Example 0.4 (Higher dimensions)**.** Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension $n \geq 2$, embedded by the complete linear series of

$$L := \left[\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 1\right] K_X + \left[\left(\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + 1\right)(n+2) + 2n\left(\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1\right)\right] A + B$$

where A and B are very ample and nef line bundles respectively. Then Σ is normal, and has Du Bois singularities. Moreover, the singularities of Σ has the following properties:

- They are pre-p-Du Bois for all $p \geq 0$. However, they are never pre-1-rational.
- If X is rationally connected, then the singularities are rational and $\lfloor \frac{n}{2} \rfloor$-Du Bois.
- If X is Calabi-Yau in the strong sense (i.e., $H^i(\mathcal{O}_X) = 0$ for all $1 \leq i \leq n-1$) then they are $\lfloor \frac{n}{2} \rfloor$-Du Bois, but not rational. In this case, Σ is Cohen-Macaulay, but not weakly rational.
- If X is hyper-Kähler (recall that they live in even dimensions), then they are 1-Du Bois, but not 2-Du Bois. In this case, Σ is Cohen-Macaulay if and only if X is a K3 surface. However, the singularities of Σ are not weakly rational.

One could also obtain many other examples with various features.

Consequences. As the associated graded objects $\underline{\Omega}_Z^k$ of the Du Bois complex of a variety Z are generalizations of the sheaf of k-forms in the smooth case, we can use Theorem A and Theorem B to prove a Kodaira-Akizuki-Nakano type vanishing theorem:

Corollary E (Analogue of Kodaira-Akizuki-Nakano vanishing theorem)**.** *Let p be a positive integer. Suppose we are in the situation of Set-up 2.28, and assume (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \geq 2$. let $\mathcal{L}$ be an ample line bundle on Σ. If $H^k(\mathcal{O}_X) = 0$ for $1 \leq k \leq p$, then*

$$H^q(\Omega_\Sigma^{[p]} \otimes \mathcal{L}) = 0 \ \ \textit{when } p + q > \dim \Sigma = 2n + 1.$$

We also apply our results to obtain consequences for h-differentials, an introduction to which can be found in [HJ14]. It has been proven in [KS21] that if Z is a variety with rational singularities, then $\Omega_h^p|_Z \cong \Omega_Z^{[p]}$ for all p. Here we obtain the isomorphisms $\Omega_h^p|_\Sigma \cong \Omega_\Sigma^{[p]}$ up to a certain range of p, even when the singularities of Σ are not rational:

Corollary F (Description of the h-differentials)**.** *Let p be a positive integer. Suppose we are in the situation of Set-up 2.28, and assume (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \geq 2$. If $H^k(\mathcal{O}_X) = 0$ for $1 \leq k \leq p$, then then there is a natural isomorphism $\Omega_h^p|_\Sigma \cong \Omega_\Sigma^{[p]}$.*

The previous two results are consequences of more general Corollary 5.16 and Corollary 5.17. Finally, pre-1-rational singularities of rational normal curves of degree ≥ 3 give consequences for the Hodge-Du Bois numbers of their secant varieties:

Corollary G (Symmetry of Hodge-Du Bois numbers)**.** *Let $X \subset \mathbb{P}^{c+1}$ be a rational normal curve of degree ≥ 3. Then*

$$\underline{h}^{p,q}(\Sigma) = \underline{h}^{q,p}(\Sigma) = \underline{h}^{3-p,3-q} \tag{0.5}$$

for all $0 \leq p, q \leq 3$, where $\underline{h}^{p,q}(\Sigma) := \dim \mathbb{H}^q(\Sigma, \underline{\Omega}_\Sigma^p)$.

As a concluding note, we remark that there is a third measure of singularities that is natural to consider, which is the local cohomological dimension $\operatorname{lcd}(\mathbb{P}^N, \Sigma)$ of Σ inside $\mathbb{P}^N := \mathbb{P}(H^0(L))$, and to study the filtrations on the local cohomology sheaves along the direction of [MP22a]. This will be the topic of a future study.

The structure of this article can be summarized as follows: Sect. B is divided in two parts, §1 and §2. In §1, we provide a brief review on higher Du Bois and higher rational singularities. Next, in §2, we recall the basics on secant varieties and describe our set-up. The proofs of Theorem A and Theorem B are given in Sect. C. The proof of Theorem D will appear in Sect. D.

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B. Preliminaries

We work over the field $\mathbb{C}$ of complex numbers. By a *variety*, we mean an integral separated scheme of finite type over $\mathbb{C}$. For a variety Z, we denote by Z_{sing} its singular locus.

1. Hodge theory. In this section we recall the basics on higher Du Bois and rational singularities. In pursuit of a general theory of these singularities beyond the local complete intersection case, [SVV23] extracted their key features that we will describe.

1.1. *Higher Du Bois singularities.* The first property of this type of singularities is a vanishing condition:

Definition 1.1. A variety Z is said to have *pre-p-Du Bois singularities* for $p \in \mathbb{N}$ if

$$\mathcal{H}^i(\underline{\Omega}_Z^k) = 0 \text{ for all } i \geq 1, 0 \leq k \leq p.$$

Equivalently, the complexes $\underline{\Omega}_Z^k$ are concentrated in degree zero for k in the given range.

Recall that by definition, a variety Z has Du Bois singularities if the morphism $\mathcal{O}_Z \to \underline{\Omega}_Z^0$ is a quasi-isomorphism. Also recall that Z is called *seminormal* if $\mathcal{H}^0(\underline{\Omega}_Z^0) \cong \mathcal{O}_Z$. In particular, Z has Du Bois singularities if and only if it is seminormal and its singularities are pre-0-Du Bois. The picture generalizes through the following

Definition 1.2. A variety Z is said to have *p-Du Bois singularities* for $p \in \mathbb{N}$ if it is seminormal, and the following conditions are satisfied:

(1) $\operatorname{codim}_Z(Z_{\text{sing}}) \geq 2p + 1$,
(2) Z has pre-p-Du Bois singularities,
(3) $\mathcal{H}^0(\underline{\Omega}_Z^k)$ is reflexive for all $0 \leq k \leq p$.

A related condition on a variety Z is the requirement that the morphisms

$$\Omega_Z^k \to \underline{\Omega}_Z^k \text{ are quasi-isomorphisms for } 0 \leq k \leq p. \tag{1.3}$$

The above condition is equivalent to the conditions stated in Definition 1.2 when Z is a local complete intersection, but the requirement (1.3) is generally more restrictive. If Z satisfies (1.3), then its singularities are called *strict-p-Du Bois* in [SVV23].

1.2. *Higher rational singularities.* We now proceed towards the definition of higher rational singularities. Let us first introduce the notation for the (shifted) Grothendieck duality functor: given a variety Z, we set

$$\mathbf{D}_Z(-) := \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(-, \omega_Z^\bullet)[-\dim Z].$$

While defining higher rational singularities, one is concerned with the complex $\mathbf{D}_Z(\underline{\Omega}_Z^{\dim Z - k})$:

Definition 1.4. A variety Z is said to have *pre-p-rational singularities* for $p \in \mathbb{N}$ if

$$\mathcal{H}^i(\mathbf{D}_Z(\underline{\Omega}_Z^{\dim Z - k})) = 0 \text{ for all } i \geq 1, 0 \leq k \leq p.$$

Equivalently, $\mathbf{D}_Z(\underline{\Omega}_Z^{\dim Z - k})$ is concentrated in degree zero for k in the given range.

Recall that Z is said to have rational singularities if the map $\mathcal{O}_Z \to \mathbf{D}_Z(\underline{\Omega}_Z^{\dim Z})$ is a quasi-isomorphism; this is equivalent to the requirement that Z is normal and $R^i f_* \mathcal{O}_{\tilde{Z}} = 0$ for a log resolution $f : \tilde{Z} \to Z$ with reduced exceptional divisor E simple normal crossings. In particular, Z has rational singularities if and only if it is normal and its singularities are pre-0-rational. In the upcoming sections, we use the following kind of resolutions of singularities.

Definition 1.5. Let Z be a variety. By a *strong log resolution* of Z, we mean a proper morphism $\mu : \tilde{Z} \to Z$ that is an isomorphism over $Z_{\mathrm{sm}} := Z \setminus Z_{\mathrm{sing}}$ with $\tilde{Z}$ smooth, and $\mu^{-1}(Z_{\mathrm{sing}})_{\mathrm{red}}$ is a divisor with simple normal crossings.

The picture for rational singularities generalizes through the following:

Definition 1.6. A variety Z is said to have *p-rational singularities* for $p \in \mathbb{N}$ if it is normal, and

(1) $\operatorname{codim}_Z(Z_{\mathrm{sing}}) > 2p + 1$,
(2) $R^i f_* \Omega^k_{\tilde{Z}}(\log E) = 0$ for all $i > 0$ and $0 \leq k \leq p$ and for any strong log resolution $f : \tilde{Z} \to Z$.

The second condition above is equivalent to the requirement that the singularities of Z are pre-p-rational. When Z is a local complete intersection, the conditions of Definition 1.6 are equivalent to requiring the morphisms

$$\Omega^k_Z \to \mathbf{D}_Z(\underline{\Omega}^{\dim Z - k}_Z) \text{ are quasi-isomorphisms for } 0 \leq k \leq p. \tag{1.7}$$

However, the above condition is more restrictive, and if Z satisfies this, then its singularities are called *strict-p-rational* in [SVV23]. We refer the interested readers to *loc. cit.* where various relationships between these singularities are discussed.

2. Preliminaries on secant varieties. In this section, we describe the geometry of secant varieties and provide several computational tools that will be used throughout the rest of the article.

2.1. *Strong log resolution of secant varieties.* Let X be a smooth projective variety of dimension n. Let L be a very ample line bundle on X inducing the embedding $X \hookrightarrow \mathbb{P}(H^0(L))$. Further, let $X^{[2]}$ be the Hilbert scheme of two points on X, which is a smooth projective variety. Consider the universal family $\Phi \subset X \times X^{[2]}$ that comes with two natural projections $q : \Phi \to X$ and $\theta : \Phi \to X^{[2]}$. It is known that $\Phi \cong \mathrm{Bl}_\Delta(X \times X)$ and let $b_\Delta : \Phi \cong \mathrm{Bl}_\Delta(X \times X) \to X \times X$ be the blow-up morphism where $\Delta \subset X \times X$ is the diagonal. We have the following commutative diagram:

$$\begin{tikzcd}
 & \Phi \arrow[r, "b_\Delta"] \arrow[dl, "\theta"'] \arrow[d, "q"] & X \times X \arrow[dl, "p_1"] \\
X^{[2]} & X &
\end{tikzcd}$$

We set $\mathcal{E}_L := \theta_* q^* L$ and notice that this bundle is globally generated as L is very ample, i.e., the evaluation map

$$H^0(\mathcal{E}_L) \otimes \mathcal{O}_{X^{[2]}} \to \mathcal{E}_L$$

is surjective. Observe that $H^0(\mathcal{E}_L) \cong H^0(L)$ whence the above surjection induces a map $f : \mathbb{P}(\mathcal{E}_L) \to \mathbb{P}(H^0(L))$ which surjects onto the secant variety $\Sigma := \Sigma(X, L)$ which, by definition, is the Zariski closure of the union of 2-secant lines of $X \hookrightarrow \mathbb{P}(H^0(L))$. This way, we obtain the surjective map $t : \mathbb{P}(\mathcal{E}_L) \to \Sigma$. In what follows, we set $\pi : \mathbb{P}(\mathcal{E}_L) \to X^{[2]}$ to be the structure morphism. Recall the following

Definition 2.1. A bundle L on a smooth projective variety X is called *k–very ample* if for any 0–dimensional subscheme ξ of length $k+1$, the evaluation map of global sections $H^0(L) \to H^0(L \otimes \mathcal{O}_\xi)$ surjects.

The singular locus of secant varieties is very simple under the mild condition that L is 3-very ample. Moreover, in this case the map t described above is a resolution of singularities. Although these facts are well-known to experts, we include the proofs for the convenience of the readers.

Proposition 2.2. *Assume L is 3-very ample. Then the following statements hold:*

(1) $t|_{\mathbb{P}(\mathcal{E}_L)\backslash t^{-1}(X)} : \mathbb{P}(\mathcal{E}_L)\backslash t^{-1}(X) \to \Sigma\backslash X$ is an isomorphism. In particular $\Sigma_{sing} \subseteq X$.

(2) If in addition $\Sigma \neq \mathbb{P}(H^0(L))$, then $\Sigma_{sing} = X$.

We proceed to the proof of the above result. In what follows, we use that given a zero-dimensional subspace $\xi \subset X$ with ideal sheaf $\mathcal{I}_\xi$, the linear subspace $\langle\xi\rangle$ spanned by ξ is isomorphic to $\mathbb{P}(H^0(L)/H^0(L \otimes \mathcal{I}_\xi))$. Observe that if $\xi_1 \subset X$ and $\xi_2 \subset X$ are distinct zero-dimensional subschemes of length 2, then $\langle\xi_1\rangle \neq \langle\xi_2\rangle$ when L is 3-very ample.

Lemma 2.3. *Assume L is* 3*-very ample. Let $\xi_1 \subset X$ and $\xi_2 \subset X$ be zero-dimensional subschemes of length 2 with Supp$(\xi_1) \cap$ Supp$(\xi_2) = \phi$. Then $\langle\xi_1\rangle \cap \langle\xi_2\rangle = \phi$.*

Proof. Suppose to the contrary, we have $z \in (l_1 = \langle\xi_1\rangle) \cap (l_2 = \langle\xi_2\rangle)$. Observe that $\langle\xi\rangle \cong \mathbb{P}^2$ where $\xi := \xi_1 \cup \xi_2$. On the other hand, $H^0(L) \to H^0(L \otimes \mathcal{O}_\xi)$ is surjective since L is 3-very ample, whence $\langle\xi\rangle \cong \mathbb{P}(H^0(L)/H^0(L \otimes \mathcal{I}_\xi)) \cong \mathbb{P}^3$, a contradiction. □

Lemma 2.4. *Assume L is* 3*-very ample, and let $x \in \Sigma$. If there is a unique zero-dimensional subscheme $\xi \subset X$ of length 2 such that $x \in \langle\xi\rangle$, then there exists an open set $x \in V \subset \Sigma$ such that $t|_{t^{-1}(V)} : t^{-1}(V) \to V$ is an isomorphism; in particular $x \in \Sigma$ is smooth.*

Proof. Put $l = \langle\xi\rangle$ and consider the Cartesian square:

$$\begin{tikzcd} t^{-1}(x) \arrow[r, hook] \arrow[d] & \pi^{-1}(\xi) \arrow[d, "\cong"] \\ \{x\} \arrow[r, hook] & l \end{tikzcd}$$

Since $t|_{\pi^{-1}(\xi)} : \pi^{-1}([\xi]) \to l$ is an isomorphism by the construction of the resolution t, the scheme-theoretic preimage $t^{-1}(x)$ is a reduced point. Thus, there exists an open set $x \in U \subset \Sigma$ such that for all $y \in U$, we have $\dim(t^{-1}(y)) = 0$. Consequently, $t|_U : t^{-1}(U) \to U$, is a finite morphism. Using base change of affine morphism, and upper semi-continuity of ranks of coherent sheaves, we conclude that there exists an open subset $x \in V \subseteq U$ such that $t|_V : t^{-1}(V) \to V$ is an isomorphism. The last assertion follows since $t^{-1}(V)$ is smooth. □

Proof of Proposition 2.2 (1). Thanks to Lemma 2.4, it is enough to show that for any $x \in \Sigma\backslash X$, there exists a unique zero-dimensional subscheme $\xi \subset X$ of length 2 such that $x \subset \langle\xi\rangle$. Suppose to the contrary, $x \in \Sigma\backslash X$ with $x \in (l_1 = \langle\xi_1\rangle) \cap (l_2 = \langle\xi_2\rangle)$, where $\xi_i \subset X$ are distinct zero-dimensional subschemes of length 2. Since $\langle\xi_1\rangle \neq \langle\xi_2\rangle$ and since two lines intersect at most at one point, we conclude that $\mathrm{Supp}(\xi_1) \cap \mathrm{Supp}(\xi_2) = \phi$ (see the left schematic diagram in Figure 2.1).

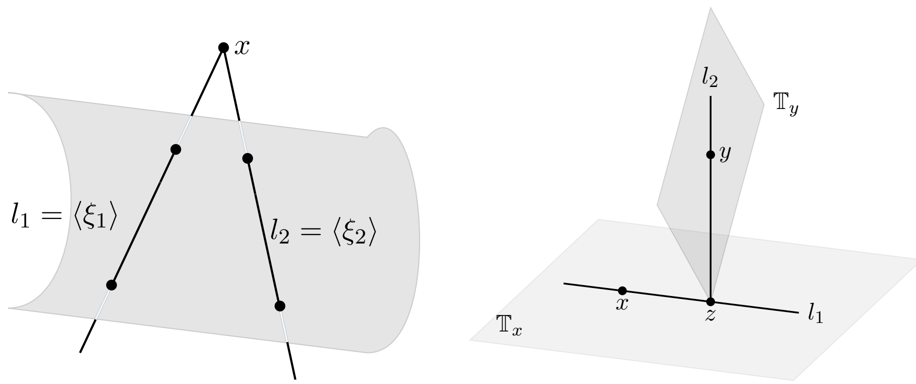

FIGURE 2.1. Left: $x \in (l_1 = \langle\xi_1\rangle) \cap (l_2 = \langle\xi_2\rangle)$. Right: $z \in \mathbb{T}_x \cap \mathbb{T}_y$ where $z \notin \{x, y\}$

But this contradicts Lemma 2.3. □

We now proceed to prove Proposition 2.2 (2).

Lemma 2.5. *Assume L is 3-very ample. Let $x, y \in X$ be two distinct points with the embedded tangent spaces $\mathbb{T}_x \cap \mathbb{T}_y \neq \phi$. Then either $\mathbb{T}_x \cap \mathbb{T}_y = \{x\}$ or $\mathbb{T}_x \cap \mathbb{T}_y = \{y\}$.*

Proof. For the sake of contradiction, assume $z \in \mathbb{T}_x \cap \mathbb{T}_y$ where $z \notin \{x, y\}$. Consider the lines $l_1 = \overline{xz}$, $l_2 = \overline{yz}$ (see the right schematic diagram in Figure 2.1). Notice that l_1 (resp. l_2) intersects X at x (resp. y) with multiplicity ≥ 2. Consequently, we get zero-dimensional subschemes $\xi_1 \subset X$ and $\xi_2 \subset X$ of length 2, with $\mathrm{Supp}(\xi_1) = \{x\}$ and $\mathrm{Supp}(\xi_2) = \{y\}$ with $z \in \langle\xi_1\rangle \cap \langle\xi_2\rangle$. This contradicts Lemma 2.3. □

Lemma 2.6. *Assume L is 3-very ample. Let $x \in X$. Then for general $y \in X$, $\mathbb{T}_x \cap \mathbb{T}_y = \phi$.*

Proof. Since $X \subset \mathbb{P}(H^0(L))$ is non-degenerate (in particular, $X \not\subseteq \mathbb{T}_x$), for general $y \in X$, we have $\mathbb{T}_x \cap \mathbb{T}_y \neq \{y\}$. For $y_1, y_2 \in X$ distinct points, both distinct from x, assume $\mathbb{T}_x \cap \mathbb{T}_{y_1} = \mathbb{T}_x \cap \mathbb{T}_{y_2} = \{x\}$. Then, as in the proof of Lemma 2.5, working with $\overline{xy_1}$ and $\overline{xy_2}$, we obtain length 2 subschemes $\xi_1, \xi_2 \subset X$ with $\mathrm{Supp}(\xi_1) \cap \mathrm{Supp}(\xi_2) = \phi$ and $x \in \langle\xi_1\rangle \cap \langle\xi_2\rangle$, which contradicts Lemma 2.3. The conclusion follows from Lemma 2.5. □

Proof of Proposition 2.2 (2). In view of (1), we suppose to the contrary that $\Sigma_{\mathrm{sing}} \subsetneq X$, then fix $x \in X \backslash \Sigma_{\mathrm{sing}}$. For general $y \in X$, it follows from Lemma 2.6 that $\dim(\mathrm{span}(\mathbb{T}_x, \mathbb{T}_y)) = 2n+1$. Applying Terracini's lemma (cf. [Laz04a, Lemma 3.4.28]), we deduce that $\mathrm{span}(\mathbb{T}_x, \mathbb{T}_y) = \mathbb{T}_x\Sigma$. On the other hand, by the generality of y, Σ is smooth at y, so another application of Terracini's lemma yields that $\mathrm{span}(\mathbb{T}_x, \mathbb{T}_y) = \mathbb{T}_y\Sigma$. Therefore $X \subseteq \mathbb{T}_x\Sigma$. Recall that $X \subset \mathbb{P}(H^0(L))$ is non-degenerate, so $\mathbb{T}_x\Sigma = \mathbb{P}(H^0(L))$. Since x is a smooth point of Σ, we obtain $\Sigma = \mathbb{P}(H^0(L))$, a contradiction. □

From here until the end of §2.2, we tacitly assume that L is 3-very ample. In this case, $\Sigma_{\mathrm{sing}} \subseteq X$ and $\mathbb{P}(\mathcal{E}_L) \subset X^{[2]} \times \mathbb{P}(H^0(L))$ together with the second projection provides a natural resolution of singularities $t : \mathbb{P}(\mathcal{E}_L) \to \Sigma(X,L)$ by Proposition 2.2. It follows from [Ver01, Lemma 3.8] that scheme-theoretically we can identify the exceptional divisor $t^{-1}(X) \cong \Phi$ and the restriction of t on it coincides with the surjection $q : \Phi \to X$.

As an immediate consequence of the above discussion and Proposition 2.2, we obtain

Corollary 2.7. *Assume L is 3-very ample and $\Sigma \neq \mathbb{P}(H^0(L))$. Then, the morphism t is a strong log resolution of Σ.*

Strictly speaking, we don't use that t is a *strong* log resolution when $\Sigma \neq \mathbb{P}(H^0(L))$ in the proof of Theorem D. This is because, to check whether Σ has pre-1-rational singularities through its birational description, we only need a resolution which is an isomorphism outside a locus of codimension at least two. The morphism t satisfies this when L is 3-very ample as in this case $\mathrm{codim}_{\Sigma}(X) = n+1 \geq 2$. See Remark 6.1 for more details.

In summary, for any $x \in X$ we have the following diagram with Cartesian squares where the vertical arrows are surjections

$$\begin{tikzcd} F_x \arrow[r, hook] \arrow[d] & \Phi \arrow[r, hook] \arrow[d, "q"] & \mathbb{P}(\mathcal{E}_L) \arrow[d, "t"] \arrow[dr, "f"] & \\ \{x\} \arrow[r, hook] & X \arrow[r, hook] & \Sigma \arrow[r, hook] & \mathbb{P}(H^0(L)) \end{tikzcd} \tag{2.8}$$

and $F_x \cong \mathrm{Bl}_x X$ is the blow-up of X at x. We set $b_x : F_x \cong \mathrm{Bl}_x X \to X$ to be the blow-up morphism. In the sequel, we will often use the fact that the map $q : \Phi \to X$ is smooth by [CS18, Lemma 2.1] without any further reference.

2.2. *Useful isomorphisms and exact sequences.* We start by recalling some basic facts about the log resolution of Σ described in §2.1 that are used crucially in the proofs of our main results.

First of all, by [Ull16, Proof of Lemma 2.3], we have the isomorphisms

$$\mathcal{N}^*_{F_x/\Phi} \cong \mathcal{O}_{F_x}^{\oplus n}, \text{ and } \mathcal{N}^*_{\Phi/\mathbb{P}(\mathcal{E}_L)}|_{F_x} \cong b_x^* L(-2E_x), \tag{2.9}$$

where E_x is the exceptional divisor of b_x. Moreover, by [Ull16, Proof of Lemma 2.3], the normal bundle sequence of $F_x \subset \Phi \subset \mathbb{P}(\mathcal{E}_L)$ is split. Consequently, by using (2.9) one obtains the following isomorphism:

$$\mathcal{N}^*_{F_x/\mathbb{P}(\mathcal{E}_L)} \cong \mathcal{O}_{F_x}^{\oplus n} \oplus b_x^* L(-2E_x). \tag{2.10}$$

Denoting the ideal sheaf of $F_x \subset \mathbb{P}(\mathcal{E}_L)$ by $\mathcal{I}_{F_x}$, and using (2.10), we also obtain the isomorphisms

$$\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x} \cong \mathrm{Sym}^j \mathcal{N}^*_{F_x/\mathbb{P}(\mathcal{E}_L)} \cong \bigoplus_{m=0}^{j} \bigoplus^{\binom{n+j-m-1}{n-1}} b_x^*(mL)(-2mE_x). \tag{2.11}$$

Next, observe that we have the following commutative diagram with exact rows and columns for $p \geq 1$:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \arrow[r] \arrow[d, equal] & \Omega^p_{\mathbb{P}(\mathcal{E}_L)} \arrow[r] \arrow[d] & \Omega^p_{\Phi} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \arrow[r] & \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \arrow[r] \arrow[d] & \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{\Phi} \arrow[r] \arrow[d] & 0 \\
& & \Omega^{p-1}_{\Phi} \arrow[r, equal] \arrow[d] & \mathcal{F} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd} \tag{2.12}$$

Since Ω^{p-1}_{Φ} is locally free, for any $x \in X$, restricting the resulting exact sequence appearing in the right vertical column on F_x, we obtain the following short exact sequence for any $p \geq 1$:

$$0 \to \Omega^p_{\Phi}|_{F_x} \to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \to \Omega^{p-1}_{\Phi}|_{F_x} \to 0. \tag{2.13}$$

The above exact sequence will be essential for us in the sequel.

We now prove a proposition crucially needed in the proof of Theorem B:

Proposition 2.14. *There is an isomorphism*

$$H^k(\Phi, \mathcal{O}_{\Phi}) \cong \bigoplus_{j=0}^{k} H^{k-j}(X, H^j(X, \mathcal{O}_X) \otimes \mathcal{O}_X). \tag{2.15}$$

In particular,

$$h^{0,k}(\Phi) = h^{0,k}(X)h^{0,0}(X) + h^{0,k-1}(X)h^{0,1}(X) + \cdots + h^{0,0}(X)h^{0,k}(X). \tag{2.16}$$

Proof. We note first that since the map q is smooth, we have an isomorphism

$$\mathbf{R}q_*\mathcal{O}_{\Phi} \cong \bigoplus R^j q_*\mathcal{O}_{\Phi}[-j] \tag{2.17}$$

in $\mathbf{D}^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves on X. Indeed, this comes from the fact that

$$\mathbf{R}q_*\mathbb{C}_{\Phi} \cong \bigoplus R^j q_*\mathbb{C}_{\Phi}[-j]$$

[Del68, Theorem 2.11], and taking $\mathrm{gr}^0_{\mathrm{F}}$, where F is the Hodge filtration. Moreover,

$$R^j q_*\mathcal{O}_{\Phi} \cong H^j(X, \mathcal{O}_X) \otimes \mathcal{O}_X \tag{2.18}$$

by [CS18, Lemma 2.2]. Taking hypercohomology $\mathbb{H}^k$ of (2.17), this says that

$$H^k(\Phi, \mathcal{O}_\Phi) \cong \bigoplus H^{k-j}(X, H^j(X, \mathcal{O}_X) \otimes \mathcal{O}_X)$$

which is (2.15). Lastly, (2.16) follows immediately from this. □

We remark that there is a more elementary proof of (2.16) using the fact that Φ is the blow-up of $X \times X$ along the diagonal. Lastly, we compute the direct and higher direct images of Ω^1_Φ that will be required in n the proof of Theorem D:

Lemma 2.19. *The following statements hold:*

(1) If $n = 1$, then we have the following isomorphisms for all j:

$$R^j q_* \Omega^1_\Phi \cong H^j(\mathcal{O}_X) \otimes \Omega^1_X \oplus H^j(\Omega^1_X) \otimes \mathcal{O}_X.$$

(2) Assume $n \geq 2$. Then:

(i) $R^j q_* \Omega^1_\Phi \cong H^j(\mathcal{O}_X) \otimes \Omega^1_X \oplus H^j(\Omega^1_X) \otimes \mathcal{O}_X$ for all $j \neq 1$;

(ii) We have an exact sequence

$$0 \to H^1(\mathcal{O}_X) \otimes \Omega^1_X \oplus H^1(\Omega^1_X) \otimes \mathcal{O}_X \to R^1 q_* \Omega^1_\Phi \to \mathcal{O}_X \to 0.$$

Proof. Let us denote by p_1 and p_2 the two projections from $X \times X$ to its factors. Using the diagram

$$\begin{tikzcd} X \times X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "q_2"] \\ X \arrow[r, "q_1"] & \bullet \end{tikzcd}$$

and flat base change, we deduce that $R^j p_{1*} p_2^* \Omega^1_X \cong q_1^* R^j q_{2*} \Omega^1_X \cong H^j(\Omega^1_X) \otimes \mathcal{O}_X$. Combining this with projection formula and [CS18, Lemma 2.2], we obtain

$$R^j p_{1*} \Omega^1_{X \times X} \cong R^j p_{1*}(p_1^* \Omega^1_X \oplus p_2^* \Omega^1_X) \cong H^j(\mathcal{O}_X) \otimes \Omega^1_X \oplus H^j(\Omega^1_X) \otimes \mathcal{O}_X \ \text{ for all } j. \tag{2.20}$$

We recall that $q = p_1 \circ b_\Delta$ and note that we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}^{n-1} \cong E_x \arrow[r, hook] \arrow[d] & E_\Delta \arrow[r, hook, "j'_\Delta"] \arrow[d, "q_\Delta"] & \Phi \cong \mathrm{Bl}_\Delta(X \times X) \arrow[d, "b_\Delta"] \arrow[dd, bend left, "q"] \\ \{(x,x)\} \arrow[r, hook] & \Delta \arrow[r, hook, "j_\Delta"] \arrow[dr, "p_0"', "\cong"] & X \times X \arrow[d, "p_1"] \\ & & X \end{tikzcd}$$

When $n = 1$, b_Δ is an isomorphism, whence $R^j q_* \Omega^1_\Phi \cong R^j p_{1*} \Omega^1_{X \times X}$ and the conclusion follows from (2.20). This proves (1).

Now assume $n \geq 2$. Recall that $R^j b_{\Delta *} \mathcal{O}_\Phi = 0$ for all $j \geq 1$, and $b_{\Delta *} \mathcal{O}_\Phi \cong \mathcal{O}_{X \times X}$. Using Leray spectral sequence and projection formula, we obtain the following isomorphisms for all j:

$$R^j q_* b_\Delta^* \Omega^1_{X \times X} \cong R^j p_{1*} \Omega^1_{X \times X} \ \text{ and } \ R^j q_* j'_{\Delta *} \Omega^1_{E_\Delta/\Delta} \cong p_{0*} R^j q_{\Delta *} \Omega^1_{E_\Delta/\Delta}. \tag{2.21}$$

Notice that $q_\Delta : E_\Delta \cong \mathbb{P}(\mathcal{N}_\Delta^*) \to \Delta$ is the structure morphism of the projective bundle, where $\mathcal{N}_\Delta$ is the normal bundle of $\Delta \hookrightarrow X \times X$. Consequently, passing to the long exact sequence corresponding to $q_{\Delta *}$ of the exact sequence

$$0 \to \Omega^1_{E_\Delta/\Delta} \to q_\Delta^* \mathcal{N}_\Delta^*(-1) \to \mathcal{O}_{E_\Delta} \to 0,$$

we obtain

$$R^j q_{\Delta *} \Omega^1_{E_\Delta/\Delta} \cong \begin{cases} \mathcal{O}_\Delta & \text{if } j = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

Also, passing to the long exact sequence corresponding to q_* of the following short exact sequence

$$0 \to b_\Delta^* \Omega^1_{X\times X} \to \Omega^1_\Phi \to j'_{\Delta *} \Omega^1_{E_\Delta/\Delta} \to 0,$$

and using (2.21), (2.22) we obtain the isomorphisms

$$\begin{gathered} q_* \Omega^1_\Phi \cong p_{1*} \Omega^1_{X\times X}, \\ R^j q_* \Omega^1_\Phi \cong R^j p_{1*} \Omega^1_{X\times X} \text{ for all } j \geq 3. \end{gathered} \tag{2.23}$$

Moreover, we also obtain the following exact sequence

$$0 \to R^1 p_{1*} \Omega^1_{X\times X} \to R^1 q_* \Omega^1_\Phi \to \mathcal{O}_X \to R^2 p_{1*} \Omega^1_{X\times X} \to R^2 q_* \Omega^1_\Phi \to 0. \tag{2.24}$$

Now, the map

$$\mathcal{O}_X \to R^2 p_{1*} \Omega^1_{X\times X}$$

is injective if it is non-zero. We claim that it is the zero map. Indeed, for otherwise $R^1 p_{1*} \Omega^1_{X\times X} \cong R^1 q_* \Omega^1_\Phi$. The Leray spectral sequence

$$E_2^{i,j} := H^i(R^j q_* \Omega^1_\Phi) \implies H^{i+j}(\Omega^1_\Phi),$$

being a first quadrant spectralsequence, induces the exact sequence

$$0 \to E_2^{1,0} \to H^1(\Omega^1_\Phi) \to E_2^{0,1} \to E_2^{2,0}.$$

We conclude that

$$h^0(R^1 p_{1*} \Omega^1_{X\times X}) = h^0(R^1 q_* \Omega^1_\Phi) \geq h^1(\Omega^1_\Phi) - h^1(q_* \Omega^1_\Phi). \tag{2.25}$$

By (2.20), we compute

$$h^0(R^1 p_{1*} \Omega^1_{X\times X}) = (h^{1,0}(X))^2 + h^{1,1}(X). \tag{2.26}$$

On the other hand, $h^1(\Omega^1_\Phi) - h^1(q_* \Omega^1_\Phi) = h^1(\Omega^1_{X\times X}) + 1 - h^1(q_* \Omega^1_\Phi)$ as $\Phi \cong \mathrm{Bl}_\Delta(X \times X)$, whence using (2.20) once again, we obtain

$$h^1(\Omega^1_\Phi) - h^1(q_* \Omega^1_\Phi) = (h^{1,0}(X))^2 + h^{1,1}(X) + 1. \tag{2.27}$$

But (2.26) and (2.27) contradicts (2.25). Thus, (2.24) breaks off into the desired short exact sequence and the isomorphism. This, combined with (2.23) and (2.20) proves (2).

The final assertion is an immediate consequence of (1) and (2) using the fact that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = 0$ for all $i \geq 1$ if $\mathcal{F}$ is locally free. The proof is now complete. $\square$

2.3. *Description of the Set-up.* We now formally introduce

Set-up 2.28. *Let X be a smooth projective variety of dimension n. Let L be a line bundle on X that satisfies:*

- *If $n = 1$, we assume $deg(L) \geq 2g + 3$.*
- *If $n \geq 2$, then $L = L_{l,d} := lK_X + dA + B$ where A and B are very ample and nef line bundles respectively. We additionally assume for a given $s \in \mathbb{N}$ (which will be specified in the statement of our results) that $(l, d) \in \mathbb{N} \times \mathbb{N}$ satisfies the following conditions*[1]*:*

$$l \geq \max_{-1 \leq i \leq s-1} \left\{ \binom{n-1}{i} + 1 \right\},$$
$$d \geq \max \left\{ \begin{array}{l} l(n+2) + 2, l(n+1) + 2 + s, (n+1)(l+1), \\ \max\limits_{-1 \leq i \leq s-1} \left\{ (n+2) \left(l - \binom{n-1}{i} - 1 \right) + 2(i+1) \left(\binom{n}{i+1} + 1 \right) \right\} \end{array} \right\} \tag{2.29}$$

Let $\Sigma := \Sigma(X, L)$ be the secant variety of $X \hookrightarrow \mathbb{P}(H^0(L))$.

The next two remarks will be used repeatedly, often without any further reference.

Remark 2.30. We note, once and for all, that in the situation of Set-up 2.28, L is 3-very ample. This is evident for $n = 1$. To see this for $n \geq 2$, use the well-known fact that $K_X + (n + 2)A + B'$ is very ample for any nef line bundle B' as A is very ample. Write

$$L_{l,d} = [K_X + (n + 2)A + B] + [(l - 1)K_X + (d - n - 2)A].$$

Recall that a line bundle is 1-very ample is equivalent to it being very ample, and by [HTT05, Theorem 1.1], a tensor product of a and b-very ample line bundles is $(a + b)$-very ample when $a, b \geq 0$. Thus, to prove the assertion, we simply note that $d - n - 2 \geq (l - 1)(n + 2) + 2$ as $d \geq l(n + 2) + 2$ by assumption.

Remark 2.31. We note that in the situation of Set-up 2.28, Σ is normal and has Du Bois singularities (in particular, it is seminormal). For $n = 1$, this follows from [Ull16, Corollary A] and [CS18, Theorem 1.2]. To see this for $n \geq 2$, use the well-known fact that $K_X + (n + 1)A + B$ is nef as A and B are very ample and nef line bundles respectively. write

$$L_{l,d} = K_X + (2n + 2)A + [(l - 1)K_X + (d - 2n - 2)A + B].$$

Using [Ull16, Corollary B] and [CS18, Theorem 1.2], it is enough to show that $d - 2n - 2 \geq (l - 1)(n + 1)$ which holds as $d \geq (l + 1)(n + 1)$ by assumption. In other words, the above discussion verifies that Set-up 2.28 satisfies [CS18, Assumption 1.1].

C. Du Bois complex of secant varieties

Let X be a smooth projective variety of dimension n and L a 3-very ample line bundle. We discuss the Du Bois complex of the secant variety $\Sigma = \Sigma(X, L)$. In particular, we prove Theorem A and Theorem B.

[1]Here we introduce the convention that $\binom{a}{b} = 0$ if $b < 0$ or if $b > a$ or if $a = 0$.

3. Secant varieties with pre-p-Du Bois singularities. In this section, we give sufficient conditions for the secant variety $\Sigma = \Sigma(X, L)$ to have pre-p-Du Bois singularities. Recall that this is a condition on the complex $\underline{\Omega}^p_\Sigma$. The first result that we are after is a local vanishing statement:

Theorem 3.1. *Let $p \in \mathbb{N}$. Suppose we are in the situation of Set-up 2.28 and assume (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \geq 2$. Then $R^i t_* \Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) = 0$ for all $i \geq 1$ and $0 \leq k \leq p$.*

Remark 3.2. This condition is sufficient for $\mathcal{H}^j(\underline{\Omega}^p_\Sigma) = 0$ for $j \neq 0$, as is explained in the proof of Theorem A (see (3.21)).

We need some preparations to prove the above result.

Proposition 3.3. *Let $i, p \geq 1$ be integers and let L be a 3-very ample line bundle. Suppose that for all $x \in X$, $j \geq 1$, and $0 \leq q \leq p$,*

$$H^i(\Omega^q_{F_x} \otimes b_x^*(jL)(-2jE_x)) = 0. \tag{3.4}$$

Then $R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) = 0$.

Proof. We prove the assertion using the following claims.

Claim 3.5. *If $H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0$ for all $x \in X$ and $m \geq 1$, then $R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) = 0$.*

Proof. Using the formal function theorem for $x \in \Sigma$, we obtain the isomorphism

$$\left(R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi)\right)^{\widehat{}}_x \cong \varprojlim H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^j_{F_x})).$$

Since $(R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi))_y = 0$ for $y \in U_X$, it is enough to check that for $x \in X$,

$$H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^j_{F_x})) = 0 \text{ for } j \geq 1. \tag{3.6}$$

Passing to the cohomology of the following exact sequence

$$\begin{aligned} 0 \to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x}) &\to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^{j+1}_{F_x}) \\ &\to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^j_{F_x}) \to 0, \end{aligned}$$

we conclude that it is enough to verify that

$$H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x})) = 0 \text{ for all } j \geq 0.$$

Using (2.11) and (2.9), we conclude that

$$\begin{aligned} &H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \otimes (\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x})) \\ \cong &\bigoplus_{q=0}^{j} \bigoplus^{\binom{n+j-q-1}{n-1}} H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \otimes b_x^*((q+1)L)(-(2q+2)E_x)) \end{aligned}$$

and the conclusion follows. $\square$

Claim 3.7. *Let $m \geq 0$. If $H^i(\Omega^q_\Phi|_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0$ for $q = p-1, p$ and for all $x \in X$, then*

$$H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0.$$

Proof. Follows by twisting (2.13) by $b_x^*(mL)(-2mE_x)$, and passing to cohomology. □

Claim 3.8. *Assume $m, q \geq 0$. If*

$$H^i(\Omega^{q'}_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0 \ \text{for all } 0 \leq q' \leq q \ \text{and for all } x \in X, \tag{3.9}$$

then $H^i(\Omega^q_\Phi|_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0$ for all $x \in X$.

Proof. To show this, we use the following short exact sequence

$$0 \to \mathcal{N}^*_{F_x/\Phi} \to \Omega^1_\Phi|_{F_x} \to \Omega^1_{F_x} \to 0. \tag{3.10}$$

Since all these sheaves are locally free, there exists a filtration

$$\Omega^q_\Phi|_{F_x} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^q \supseteq F^{q+1} = 0 \ \text{with } F^l/F^{l+1} \cong \bigwedge^l \mathcal{N}^*_{F_x/\Phi} \otimes \Omega^{q-l}_{F_x}. \tag{3.11}$$

We prove by induction that

$$H^i(F^l \otimes b_x^*(mL)(-2mE_x)) = 0 \ \text{for } l = 0, \ldots, q.$$

For the base case, we use the short exact sequence

$$0 \to F^{q+1} = 0 \to F^q \to \bigwedge^q \mathcal{N}^*_{F_x/\Phi} \to 0.$$

We twist the sequence by $b_x^*(mL)(-2mE_x)$ and take cohomology, and then the vanishing follows from the hypothesis (3.9) by (2.9). Assume next that we know the result for l. We use the short exact sequence

$$0 \to F^l \to F^{l-1} \to \bigwedge^{l-1} \mathcal{N}^*_{F_x/\Phi} \otimes \Omega^{q-l+1}_{F_x} \to 0.$$

Twisting by $b_x^*(mL)(-2mE_x)$ and taking cohomology, the result follows from the induction hypothesis, and (3.9) by (2.9). □

The assertion of the proposition follows by combining the above three claims. □

We show next that under the positivity condition in Theorem 3.1 when $n \geq 2$, the conditions of Proposition 3.3 are satisfied. Recall that A is a very ample and B is a nef line bundle on X, and moreover for a given $l, d \in \mathbb{N}$, we have

$$L_{l,d} := lK_X + dA + B.$$

Also recall that a vector bundle $\mathcal{E}$ on X is called *k–jet ample* if for every choice of t distinct points $x_1, \cdots, x_t \in X$ and for every tuple $(k_1, \cdots, k_t)$ of positive integers with $\sum k_i = k+1$, the evaluation map

$$H^0(\mathcal{E}) \to H^0\left(\mathcal{E} \otimes \left(\mathcal{O}_X/(\mathcal{I}^{k_1}_{x_1} \otimes \cdots \otimes \mathcal{I}^{k_t}_{x_t})\right)\right) = \bigoplus_{i=1}^t H^0\left(\mathcal{E} \otimes \left(\mathcal{O}_X/\mathcal{I}^{k_i}_{x_i}\right)\right)$$

surjects where $\mathcal{I}_{x_i}$ is the ideal sheaf of $x_i \in X$.

Lemma 3.12. *Let $n \geq 2$, $0 \leq p \leq n$ and $j \geq \max\left\{\frac{p}{2}, 1\right\}$. If $l \geq 0$ and $d \geq l(n+1)+2+p$ then $\Omega_X^p(jL_{l,d})$ is $(2j-p)$-jet ample.*

Proof. We write $\Omega_X^p(jL_{l,d}) = \Omega_X^p(2pA)(jlK_X + (jd-2p)A + jB)$. It is well-known that $\Omega_X(2A)$ is globally generated (i.e. 0-jet ample), whence so is $\Omega_X^p(2pA)$. Elementary consideration of Castelnuovo-Mumford regularity and Kodaira vanishing theorem guarantees that $K_X + (n+1)A$ and $K_X + (n+1)A + jB$ are also globally generated. Moreover, since A is a very ample line bundle, it is 1-jet ample (see [BDRS99, page 3]). Observe that we have

$$jd - 2p - jl(n+1) \geq 2j - p$$

by our assumption on d. The conclusion follows from [BDRS99, Proposition 2.3]. ☐

Lemma 3.13. *Let $n \geq 2$, $0 \leq p \leq n$. Further assume*

$$l \geq \binom{n-1}{p-1} + 1 \text{ and } d \geq \max\left\{l(n+2), (n+2)\left(l - \binom{n-1}{p-1} - 1\right) + 2p\left(\binom{n}{p} + 1\right)\right\}.$$

Then $H^i(\Omega_X^p(jL_{l,d})) = 0$ for all $i, j \geq 1$.

Proof. It is easy to see using a splitting principle that $\det(\Omega_X^p) = \binom{n-1}{p-1}K_X$. Consequently, we obtain

$$\Omega_X^p(jL_{l,d}) = K_X \otimes \Omega_X^p(2pA) \otimes \det(\Omega_X^p(2pA)) \otimes Q$$

where

$$Q = \left(jl - \binom{n-1}{p-1} - 1\right)K_X + \left(jd - 2p - \binom{n}{p}2p\right)A + jB.$$

Notice that

$$jd - 2p - \binom{n}{p}2p \geq (n+2)\left(jl - \binom{n-1}{p-1} - 1\right)$$

by assumption. Since $K_X + (n+2)A$ is very ample, we see that Q is ample. Since $\Omega_X^p(2pA)$ is nef, the assertion follows from Griffiths' vanishing theorem ([Laz04b, Variant 7.3.2]). ☐

Proposition 3.14. *Let $n \geq 2$, $0 \leq p \leq n$ and assume (l, d) satisfies (2.29) with $s = p$. Then*

$$H^i(\Omega_{F_x}^{p'} \otimes b_x^*(jL_{l,d})(-2jE_x)) = 0$$

for all $i, j \geq 1$, $0 \leq p' \leq p$ and for all $x \in X$.

Proof. Recall from Remark 2.31 that we can write $L_{l,d} = K_X + (2n+2)A + B'$ where $B' = (l-1)K_X + (d-2n-2)A + B$ is nef. Also recall from [CS18, Proof of Proposition 3.2] that

$$b_x^*(jL_{l,d})(-2jE_x) \cong K_{F_x} + (n+1)P + Q$$

where $P = b_x^*(2A)(-E_x)$ and $Q = (j-1)(K_{F_x} + (n+1)P + b_x^*B') + b_x^*B'$. It is well-known that $K_{F_x} + (n+1)P$ is very ample. Since b_x^*A is nef and big, we conclude that

$$b_x^*(jL_{l,d})(-2jE_x) \text{ and } b_x^*(jL_{l,d})(-2E_x) - K_{F_x} \text{ are both ample } \forall j \geq 1,\ \forall x \in X. \tag{3.15}$$

Thus, the conclusion follows for $p' = 0$ by Kodaira vanishing theorem. Henceforth we assume $p' \geq 1$ (whence $p \geq 1$). Since b_x is the blow-up of X at x, we observe that

$\Omega^1_{F_x}(\log E_x)(-E_x) \cong b_x^*\Omega^1_X$ and we deduce the following exact sequence for any $x \in X$ and for any $1 \le p' \le p \le n$, $j \ge 1$
(3.16)
$$0 \to b_x^*(\Omega_X^{p'}(jL_{l,d}))(-(2j-p'+1)E_x) \to \Omega^{p'}_{F_x}\otimes b_x^*(jL_{l,d})(-2jE_x) \to \Omega^{p'}_{E_x}\otimes b_x^*(jL_{l,d})(-2jE_x) \to 0.$$

Notice that (3.15) also yields

$$H^i(\Omega^{p'}_{E_x}(b_x^*(jL_{l,d})(-2jE_x))) = 0 \text{ for all } i,j \ge 1 \text{ and for all } x \in X \tag{3.17}$$

by Bott vanishing theorem. Now assume $2j - p' \ge 0$, and we use the exact sequence

$$0 \to \Omega^{p'}_X(jL_{l,d}) \otimes \mathcal{I}_x^{2j-p'+1} \to \Omega^{p'}_X(jL_{l,d}) \to \Omega^{p'}_X(jL_{l,d}) \otimes (\mathcal{O}_X/\mathcal{I}_x^{2j-p'+1}) \to 0. \tag{3.18}$$

Using Lemma 3.12, we observe that the map $H^0(\Omega^{p'}_X(jL_{l,d})) \to H^0(\Omega^{p'}_X(jL_{l,d})\otimes(\mathcal{O}_X/\mathcal{I}_x^{2j-p'+1}))$ surjects. Passing to the cohomology of (3.18), and using Lemma 3.13, we get that

$$H^i(\Omega^{p'}_X(jL_{l,d}) \otimes \mathcal{I}_x^{2j-p'+1}) = 0 \text{ for } i \ge 1 \text{ and for all } x \in X \text{ if } 2j - p' \ge 0. \tag{3.19}$$

It is well-known (see for example [BEL91, Proof of Lemma 1.4]) that for $0 \le s \le n-1$

$$R^i b_{x*}\mathcal{O}_{F_x}(sE_x) = \begin{cases} \mathcal{O}_{F_x} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

whence $H^i(b_x^*(\Omega^{p'}_X(jL_{l,d}))(-(2j-p'+1)E_x) = 0$ by Lemma 3.13 for $i \ge 1$ if $2j - p' < 0$. Thus, using (3.19) we get

$$H^i(b_x^*(\Omega^{p'}_X(jL_{l,d}))(-(2j-p'+1)E_x) = 0 \text{ for all } i,j \ge 1 \text{ and for all } x \in X. \tag{3.20}$$

The assertion follows from (3.17), (3.20) and the cohomology sequence of (3.16). □

Proof of Theorem 3.1. Note that the statement follows for $p = 0$ by [CS18, Proposition 3.2]. Thus we assume $p \ge 1$. Now, if $n = 1$ and $\deg(L) \ge 2g+3$, then $H^i(\Omega^{p'}_{F_x}\otimes b_x^*(jL)(-2jE_x)) = 0$ for all $i, j \ge 1$, $x \in X$ and $p' \ge 0$. This is because in this case $\deg(b_x^*(L)(-2E_x)) \ge 2g+1$. Consequently, the assertion follows by Proposition 3.3. Finally, we consider the case when $n \ge 2$ and $p \ge 1$. But in this case the conclusion is a consequence of Proposition 3.3 thanks to Proposition 3.14. □

Now we are ready to provide the

Proof of Theorem A. Thanks to [CS18, Theorem 1.2], we assume $k \ge 1$. By [Ste85, Proposition 3.3] (see also [MOPW23, §2.1]), we have an exact triangle:

$$\mathbf{R}t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) \to \underline{\Omega}^k_\Sigma \to \underline{\Omega}^k_X \xrightarrow{+1} \tag{3.21}$$

Since $\mathcal{H}^i(\underline{\Omega}^k_X) = 0$ for $i \neq 0$ as X is smooth, passing to the cohomology of the above, we obtain $\mathcal{H}^i(\underline{\Omega}^k_\Sigma) = 0$ for $i \neq 0$ as $R^it_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) = 0$ for all $i \ge 1$ and $0 \le k \le p$ by Theorem 3.1. □

4. Reflexivity condition on $\mathcal{H}^0(\underline{\Omega}^p_\Sigma)$. We now aim to describe the sheaf $\mathcal{H}^0(\underline{\Omega}^p_\Sigma)$. Recall that for a seminormal variety with pre-p-Du Bois singularities, satisfying the codimension condition on the singular locus, the condition missing for it to have p-Du Bois singularities is that on degree zero, the associated graded complex of the Du Bois complex is reflexive. For this, we discuss a reflexivity condition of the push-forward of the sheaf of p-forms discussed in [KS21].

We work with the standing assumption that L is 3-very ample. We introduce the following notation: $U_X := \Sigma\backslash X$ and we have the inclusion $j_{U_X} : U_X \to \Sigma$.

Remark 4.1. We have the isomorphism ${j_{U_X}}_* j_{U_X}^* \Omega^{[p]}_\Sigma \cong {j_{U_X}}_* \Omega^p_{U_X}$ for all $p \geq 0$ where $\Omega^{[p]}_\Sigma := (\Omega^p_\Sigma)^{**}$. Thus, when Σ is normal, we have the isomorphism

$$ {j_{U_X}}_* \Omega^p_{U_X} \cong \Omega^{[p]}_\Sigma. \tag{4.2} $$

On the other hand, we have the natural inclusion

$$ \phi_p : t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)} \hookrightarrow {j_{U_X}}_* \Omega^p_{U_X}. $$

Thus, when Σ is normal, composing ϕ_p with the isomorphism (4.2), we obtain the maps

$$ \varphi_p : t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)} \to \Omega^{[p]}_\Sigma. $$

Proposition 4.3. *The natural inclusion $\phi_p : t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)} \hookrightarrow {j_{U_X}}_* \Omega^p_{U_X}$ is an isomorphism for $0 \leq p \leq n-1$. In particular, if Σ is normal, then $\varphi_p : t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)} \to \Omega^{[p]}_\Sigma$ is an isomorphism for $0 \leq p \leq n-1$.*

Proof. We recall from [KS21, (2.3.5)] that for all $l \geq 0$, Saito's formalism leads to a decomposition

$$ \mathbf{R} t_* \Omega^l_{\mathbb{P}(\mathcal{E}_L)} \cong K_l \oplus R_l $$

where $K_l, R_l \in \mathbf{D}^b(\mathrm{Coh}(\Sigma))$. Among other properties, K_l and R_l enjoy the following (see *loc. cit.* (2.3.6), (2.3.7)):

$$ \mathrm{Supp}(R_l) \subseteq \Sigma_{\mathrm{sing}} \subseteq X, \tag{4.4} $$

$$ \mathcal{H}^k(K_l) = 0 \text{ for } k \geq 2n - l + 2 \tag{4.5} $$

where (4.4) follows from Proposition 2.2. Since $t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}$ is torsion-free, (4.4) implies that $t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)} \cong \mathcal{H}^0(K_p)$. By [KS21, (2.3.8)] we also have the isomorphism

$$ \mathbf{R}\mathcal{H}om_{\mathcal{O}_\Sigma}(K_p, \omega^\bullet_\Sigma) \cong K_{2n+1-p}[2n+1]. $$

Thus, by [KS21, (2.3.9)], it is enough to show that

$$ \dim \Big(X \cap \mathrm{Supp} \Big(\mathcal{H}^k(K_{2n+1-p}) \Big) \Big) \leq 2n - 1 - k \;\text{ for all } k \in \mathbb{Z}. \tag{4.6} $$

Observe that $\dim \left(X \cap \mathrm{Supp} \left(\mathcal{H}^k(K_{2n+1-p}) \right) \right) \leq n$ whence (4.6) holds for $k \leq n-1$. On the other hand, since the dimension of the fibers of t is $\leq n$, we have $R^k t_* \Omega^{2n+1-p}_{\mathbb{P}(\mathcal{E}_L)} = 0$ for

$k \geq n+1$, whence (4.6) holds if $k \geq n+1$. Finally, since $n \geq p+1$ by assumption, we have $\mathcal{H}^n\left(K_{2n+1-p}\right) = 0$ by (4.5), and the assertion follows. $\square$

We need one more result in order to prove Theorem B. From the short exact sequence

$$0 \to q^*\Omega^1_X \to \Omega^1_\Phi \to \Omega^1_{\Phi/X} \to 0, \tag{4.7}$$

we have an induced morphism

$$\gamma_p : \Omega^p_X \to q_*\Omega^p_\Phi$$

by taking the wedge product of the first map and then pushing forward.

Proposition 4.8. *The maps γ_k for $k = 1, \ldots, p$ are isomorphisms if and only if $h^0(X, \Omega^k_X) = 0$ for $k = 1, \ldots, p$.*

Proof. We prove first that given the cohomological conditions, γ_p is an isomorphism. For this we use (4.7) again, and the fact that all these sheaves are locally free, to obtain a filtration

$$\Omega^k_\Phi = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^k \supseteq F^{k+1} = 0$$

with quotients $F^l/F^{l+1} \cong q^*\Omega^l_X \otimes \Omega^{k-l}_{\Phi/X}$. We prove by induction that $q_*F^l \cong \Omega^k_X$. The base case is

$$0 \to F^{k+1} = 0 \to F^k \to q^*\Omega^k_X \to 0,$$

and the claim is clear for F^k by Projection Formula and [CS18, Lemma 2.2]. Suppose next that $q_*F^l \cong \Omega^k_X$. Consider the short exact sequence

$$0 \to F^l \to F^{l-1} \to q^*\Omega^{l-1}_X \otimes \Omega^{k-l+1}_{\Phi/X} \to 0.$$

We pushforward the short exact sequence and by the Projection Formula, we have

$$0 \to \Omega^k_X \to q_*F^{l-1} \to \Omega^{l-1}_X \otimes q_*\Omega^{k-l+1}_{\Phi/X}.$$

Since $h^0(\Omega^{k-l+1}_{F_x}) = h^0(\Omega^{k-l+1}_X) = 0$ because F_x is birational to X for all $x \in X$, by Grauert's Theorem ([Har77, III, Corollary 12.9]), $q_*\Omega^{k-l+1}_{\Phi/X} = 0$ and then, $q_*F^{l-1} \cong \Omega^k_X$.

Suppose next that the maps γ_k are isomorphisms for $k = 1, \ldots, p$. We argue by induction. The base case is $p = 1$, in which case the assumption says that $\Omega^1_X \cong q_*\Omega^1_\Phi$, and therefore, by taking H^0 we obtain $h^{1,0}(X) = h^{1,0}(\Phi)$. By Proposition 2.14 we have

$$h^{1,0}(\Phi) = h^{0,1}(\Phi) = h^{0,1}(X) + h^{0,1}(X).$$

Therefore, $h^{0,1}(X) = h^{0,1}(\Phi) = 0$. This means that $h^0(X, \Omega^1_X) = 0$. Suppose now that the result is known for $p = r$, and that γ_k are isomorphisms for $k = 1, \ldots, r+1$. By the induction hypothesis, $h^0(X, \Omega^k_X) = 0$ for $k = 1, \ldots, r$. Moreover, since $\Omega^{r+1}_X \cong q_*\Omega^{r+1}_\Phi$, taking H^0 we obtain that $h^{r+1,0}(X) = h^{r+1,0}(\Phi)$. By Proposition 2.14 and the induction hypothesis, we have

$$h^{0,r+1}(\Phi) = h^{0,r+1}(X) + h^{0,r+1}(X).$$

Therefore, $h^{0,r+1}(X) = h^{0,r+1}(\Phi) = 0$. This means that $h^0(X, \Omega^{r+1}_X) = 0$. $\square$

Let us describe the maps δ_p in detail. The functoriality of Du Bois complexes induces a canonical map $\underline{\Omega}^p_\Sigma \to \mathbf{R}t_*\underline{\Omega}^p_{\mathbb{P}(\mathcal{E}_L)} = \mathbf{R}t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}$, which yields $\beta_p : \mathcal{H}^0(\underline{\Omega}^p_\Sigma) \to t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}$. In particular, when Σ is normal we obtain for all $p \geq 0$, the natural map

$$\delta_p := \varphi_p \circ \beta_p : \mathcal{H}^0(\underline{\Omega}^p_\Sigma) \to \Omega^{[p]}_\Sigma. \tag{4.9}$$

We record a fact that we will use without any further reference.

Remark 4.10. Assume Σ is normal. The map δ_k is an isomorphism if and only if β_k and φ_k are both isomorphisms. Indeed, this follows immediately from the injectivity of φ_k.

We are now ready to provide the

Proof of Theorem B. Recall that Σ is normal and has Du Bois singularities. In particular, we have the isomorphisms

$$\Omega^{[2n+1]}_\Sigma \cong j_{U_X*}\omega_{U_X} \cong \omega_\Sigma := \mathcal{H}^{-(2n+1)}\omega^\bullet_\Sigma. \tag{4.11}$$

Also recall that by Theorem 3.1

$$R^1t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) = 0 \text{ for all } 0 \leq k \leq p. \tag{4.12}$$

We work with the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) \arrow[r] \arrow[d, equal] & \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \arrow[r] \arrow[d, "\beta_k"] & \Omega^k_X \arrow[r] \arrow[d, "\gamma_k"] & 0 \\ 0 \arrow[r] & t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) \arrow[r] & t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)} \arrow[r] & q_*\Omega^k_\Phi \arrow[r] & 0 \end{tikzcd} \tag{4.13}$$

where the top sequence is obtained by passing to the cohomology of (3.21), the bottom row is obtained by taking the direct images of the sequence

$$0 \to \Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) \to \Omega^k_{\mathbb{P}(\mathcal{E}_L)} \to \Omega^k_\Phi \to 0.$$

Both rows are exact on the right because of (4.12).

First assume $H^0(\Omega^k_X) = 0$ for $1 \leq k \leq p$. Then, by Proposition Proposition 4.8, γ_k is an isomorphism for all $1 \leq k \leq p$ whence β_k's are isomorphisms in the same range. If $p \leq n-1$, then the conclusion follows since $\varphi_k : t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)} \to \Omega^{[k]}_\Sigma$ are isomorphism by Proposition 4.3. If $p \geq n$, then $H^n(\mathcal{O}_X) = 0$ by assumption, whence $t_*\omega_{\mathbb{P}}(\mathcal{E}_L) \cong \omega_\Sigma$ by [CS18, Theorem 5.8]. Thus, the conclusion in this case follows by [KS21, Theorem 1.4]. Conversely, assume $\delta_k : \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \Omega^{[k]}_\Sigma$ are isomorphisms for $1 \leq k \leq p$, whence β_k's are isomorphisms. Now the conclusion follows again from Proposition 4.8. This completes the proof. □

Proof of Corollary C. Under our assumptions, Σ has pre-p-Du Bois singularities by Theorem A and satisfies the codimension condition as $p \leq \lfloor \frac{n}{2} \rfloor$. Also recall that Σ is normal and has Du Bois singularities, and in particular is seminormal. We may assume that $n \geq 2$, $p \geq 1$. Note that $\mathcal{H}^0(\underline{\Omega}^k_\Sigma)$ is reflexive for $1 \leq k \leq p$ if and only if δ_k's are isomorphisms for $1 \leq k \leq p$. The conclusion now follows from Theorem B. □

Example 4.14 (Curves embedded by line bundles of smaller degree)**.** Suppose X is a curve and L is a non-special (i.e., $H^1(L)=0$) 3-very ample line bundle on X. In this case:

(i) Our proof shows that the singularities of Σ are pre-p-Du Bois for all $p \geq 0$.

(ii) [CS18, Proof of Theorem 3.4] shows that the singularities of Σ are Du Bois if Σ is normal.

(iii) The main result of [Ull16] asserts that Σ is normal if $L(-2x)$ is projectively normal for $x \in X$. In particular, via [GL86, Theorem 1] (see [Ull16, Proof of Corollary B]), if one of the following holds:

  (1) $\deg(L)=2g+1$ and $\mathrm{Cliff}(X) \geq 2$ (equivalently, X is not hyperelliptic, trigonal, or plane quintic); or

  (2) $\deg(L)=2g+1$ and $\mathrm{Cliff}(X) \geq 1$ (equivalently X is not hyperelliptic),

then Σ is normal and has Du Bois singularities.

On a complementary direction, although the secant varieties of canonical curves with $\mathrm{Cliff}(X) \geq 3$ is normal by [Ull16, Corollary B], it was shown in [CS18] that in this case the singularities of Σ are not Du Bois.

5. Further results. We can actually say more about the associated graded pieces of the Du Bois complex that are usually not considered in the definition of higher Du Bois singularities. In particular, we aim to prove the following:

Theorem 5.1. *Suppose we are in the situation of Set-up 2.28, and let $\delta_k : \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \Omega^{[k]}_\Sigma$ be the natural maps. Then the following statements hold.*

(1) Assume (l,d) satisfies (2.29) with $s=n$ when $n \geq 2$. Then the following are equivalent:

  (i) $H^n(\mathcal{O}_X)=0$,

  (ii) δ_{2n+1} is an isomorphism,

  (iii) δ_{2n} is an isomorphism.

(2) Assume $n \geq 3$ and (l,d) satisfies (2.29) with $s=n$. Let $n+2 \leq p \leq 2n-1$. If $H^k(\mathcal{O}_X)=0$ for all $k \geq p-n-1$, then δ_p is an isomorphism.

In order to prove part (2) of the above theorem, we need another local vanishing result:

Proposition 5.2. *Let $i,p \geq 1$ be integers and let L be a 3-very ample line bundle. Suppose that for all $x \in X$, $j \geq 0$, and $0 \leq q \leq p$,*

$$H^i(\Omega^q_{F_x} \otimes b_x^*(jL)(-2jE_x))=0. \tag{5.3}$$

Then $R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)=0$.

Proof. As we did in the proof of Proposition 3.3, we first prove the following

Claim 5.4. *If $H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \otimes b_x^*(mL)(-2mE_x))$ for all $x \in X$ and for all $m \geq 0$ then $R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)=0$.*

Proof. Clearly $(R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi))_y = 0$ if $y \in \Sigma$ and $y \notin X$. For $x \in X$, we use the formal function theorem

$$\left(R^i t_* \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)\right)^{\widehat{}}_x \cong \varprojlim H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^j_{F_x})).$$

Thus, in order to prove the assertion, it is enough to show that

$$H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^j_{F_x})) = 0 \text{ for all } j \geq 1. \tag{5.5}$$

Passing to the cohomology of the exact sequence

$$\begin{aligned} 0 \to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x}) \to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^{j+1}_{F_x}) \\ \to \Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}/\mathcal{I}^{j}_{F_x}) \to 0, \end{aligned} \tag{5.6}$$

we conclude that to prove (5.5), it is enough to show that

$$H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x})) = 0 \text{ for all } j \geq 0. \tag{5.7}$$

Observe that by (2.11), we have

$$H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \otimes (\mathcal{I}^j_{F_x}/\mathcal{I}^{j+1}_{F_x})) \cong \bigoplus_{m=0}^{j} \bigoplus^{\binom{n+j-m-1}{n-1}} H^i(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \otimes b_x^*(mL)(-2mE_x)) \tag{5.8}$$

and the conclusion follows. ☐

We now invoke Claim 3.7 and Claim 3.8, which immediately completes the proof. ☐

The previous result can be used to obtain one more reflexivity statement:

Proposition 5.9. *Let $n \geq 3$ and $n+2 \leq p \leq 2n-1$. Assume the following conditions:*

(i) $H^n(\Omega^q_{F_x} \otimes b_x^*(jL)(-jE_x)) = 0$ for all $j \geq 1$, $0 \leq q \leq 2n+1-p$,

(ii) $H^0(\Omega^{n-q}_X) = 0$ for all $0 \leq q \leq 2n+1-p$.

Then the two natural maps

$$t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \hookrightarrow t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)} \hookrightarrow {j_{U_X}}_*\Omega^p_{U_X}$$

are isomorphisms. In particular, if (i) and (ii) hold and if Σ is normal, then the map

$$t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \to \Omega^{[p]}_\Sigma$$

is an isomorphism.

Proof. We first note the following isomorphisms

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{O}_\Sigma}\left(\mathbf{R}t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi), \omega^\bullet_\Sigma\right) &\cong \mathbf{R}t_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_{\mathbb{P}(\mathcal{E}_L)}}\left(\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi), \omega_{\mathbb{P}(\mathcal{E}_L)}[2n+1]\right) \\ &\cong \mathbf{R}t_*\Omega^{2n+1-p}_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)[2n+1], \end{aligned}$$

where the first one is obtained via duality. Observe that it is enough to show that the natural map

$$t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)(-\Phi) \hookrightarrow {j_{U_X}}_*\Omega^p_{U_X}$$

is an isomorphism. Thus, by [KS21, Proposition 6.4], it is enough to show that

$$\dim\left(X \cap \operatorname{Supp}\left(R^k\Omega^{2n+1-p}_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)\right)\right) \leq 2n-1-k \text{ for all } k \in \mathbb{Z}. \tag{5.10}$$

As in the proof of Proposition 4.3, (5.10) holds if $k \leq n-1$ or if $k \geq n+1$. Thus it is enough to show that

$$R^n\Omega^{2n+1-p}_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi) = 0. \tag{5.11}$$

Notice that

$$H^n(\Omega^q_{F_x} \otimes b_x^*(jL)(-jE_x)) = 0 \text{ for all } j \geq 0, 0 \leq q \leq 2n+1-p. \tag{5.12}$$

Indeed (5.12) follows from hypothesis (i) for $j \geq 1$. For $j = 0$, the vanishing follows from hypothesis (ii) as we have $h^n(\Omega^q_{F_x}) = h^0(\Omega^{n-q}_{F_x}) = h^0(\Omega^{n-q}_X)$. Now (5.11) follows from Proposition 5.2. □

Proof of Theorem 5.1. Recall that under our assumption, Σ is normal and has Du Bois singularities. We also recall that δ_k is an isomorphism if and only if β_k and φ_k are isomorphisms.

We first prove (1) and set $p = 2n+1$. It follows from the middle column of (4.13) that

$$\mathcal{H}^0(\underline{\Omega}^{2n+1}_\Sigma) \cong t_*\omega_{\mathbb{P}(\mathcal{E}_L)}.$$

Thus, using (4.11), we see that the equivalence of (i) and (ii) is a consequence of [CS18, Theorem 5.8]. We now show the equivalence of (i) and (iii). If (i) holds, then as before, we see that the inclusion $t_*\omega_{\mathbb{P}(\mathcal{E}_L)} \hookrightarrow j_{U_X*}\omega_{U_X} \cong \omega_\Sigma$ is an isomorphism by [CS18, Theorem 5.8]. Thus, by [KS21, Theorem 1.5], we see that the two morphisms

$$t_*\Omega^{2n}_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) \hookrightarrow t_*\Omega^{2n}_{\mathbb{P}(\mathcal{E}_L)} \hookrightarrow j_{U_X*}\Omega^{2n}_{U_X}$$

are both isomorphisms. Consequently, it follows from (4.13) that β_{2n} and φ_{2n} are both isomorphisms. Conversely, assume (iii) holds. Then (4.13) implies $H^0(\omega_\Phi) = 0$ whence by Proposition 2.14 we get $H^0(\omega_X) = 0$ which is (i).

Now we prove (2). Since $p \geq n+2$ by assumption, it follows from the first row of (4.13) that

$$t_*\Omega^p_{\mathbb{P}(\mathcal{E}_L)}(\log\Phi)(-\Phi) \cong \mathcal{H}^0(\underline{\Omega}^p_\Sigma).$$

The conclusion now follows from (4.13), Proposition 5.9 and Proposition 3.14. □

Remark 5.13. If the natural map $\delta_k : \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \Omega^{[k]}_\Sigma$ is an isomorphisms for some $k \geq 1$, then

$$\min\left\{h^0(\Omega^{k-i}_X), h^0(\Omega^i_X)\right\} = 0 \text{ for all } 0 \leq i \leq k.$$

Indeed, since δ_k is an isomorphism, β_k is also an isomorphism. Consequently, from (4.13) we see that γ_k is an isomorphism, whence $h^0(\Omega^k_X) = h^0(\Omega^k_\Phi)$. The conclusion now follows from Proposition 2.14.

Corollary 5.14. *Let $p \in \mathbb{N}$. Suppose we are in the situation of Set-up 2.28. If one of the following holds:*

(i) $n = 1$ (whence $deg(L) \geq 2g+3$ by our assumption), $p \geq 0$, and $H^k(\mathcal{O}_X) = 0$ for $1 \leq k \leq p$;

(ii) $n \geq 2$, $p \geq 0$, (l,d) satisfies (2.29), with $s = \min\{p,n\}$, and $H^k(\mathcal{O}_X) = 0$ for $1 \leq k \leq p$;

(iii) $n \geq 2$, (l, d) *satisfies* (2.29) *with* $s = n$, $p \in \{2n, 2n+1\}$, *and* $H^n(\mathcal{O}_X) = 0$;

(iv) $n \geq 3$, (l, d) *satisfies* (2.29) *with* $s = n$, $n + 2 \leq p \leq 2n - 1$, *and* $H^k(\mathcal{O}_X) = 0$ *for all* $k \geq p - n - 1$;

then there is a natural quasi-isomorphism $\underline{\Omega}^p_\Sigma \cong \Omega^{[p]}_\Sigma$.

Remark 5.15. Notice that the conditions (i) and (ii) are vacuous when $p = 0$.

Proof. This is an immediate consequence of Theorem A, Theorem B and Theorem 5.1. □

Corollary E and Corollary F are special cases of the following more general results:

Corollary 5.16. *Let* $p \in \mathbb{N}$. *Suppose we are in the situation of Set-up 2.28, and let* $\mathcal{L}$ *be an ample line bundle on* Σ. *If one of the conditions (i), (ii), (iii) or (iv) of Corollary 5.14 holds, then*

$$H^q(\Omega^{[p]}_\Sigma \otimes \mathcal{L}) = 0 \quad \text{when } p + q > \dim \Sigma = 2n + 1.$$

Proof. This is an immediate consequence of Corollary 5.14 and [GNAPGP88, Theorem V.5.1]. □

Corollary 5.17 (Description of the h-differentials)**.** *Let* $p \in \mathbb{N}$. *Suppose we are in the situation of Set-up 2.28. If one of the conditions (i), (ii), (iii) or (iv) of Corollary 5.14 holds, then there is a natural isomorphism* $\Omega^p_h|_\Sigma \cong \Omega^{[p]}_\Sigma$.

Proof. This is an immediate consequence of Corollary 5.14 and [HJ14, Theorem 7.12]. □

D. Higher rational singularities of secant varieties

As before, it is our standing assumption that X is a smooth projective variety and L is a 3-very ample line bundle on X. Here we study the dual $\mathbf{D}_\Sigma(\underline{\Omega}^p_\Sigma)$. In particular, we prove Theorem D.

6. Cohomology of the dual of $\underline{\Omega}^p_\Sigma$.

Recall that for a variety Z, we set

$$\mathbf{D}_Z(-) := \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(-, \omega^\bullet_Z)[-\dim Z].$$

If $Z \subset W$ is of codimension c, we have $\mathbf{D}_W(-) \cong \mathbf{D}_Z(-)[-c]$ for complexes supported on Z.

Remark 6.1. Since L is 3-very ample, we have (3.21), dualizing which (with obvious modifications of wedge powers) we obtain the exact triangle:

$$\mathbf{D}_\Sigma(\underline{\Omega}^{2n+1-k}_X) \to \mathbf{D}_\Sigma(\underline{\Omega}^{2n+1-k}_\Sigma) \to \mathbf{R}t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \xrightarrow{+1} . \tag{6.2}$$

Thus, we have the isomorphisms $\mathbf{D}_\Sigma(\underline{\Omega}^{2n+1-k}_\Sigma) \cong \mathbf{R}t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)$ for $0 \leq k \leq n$; in particular

$$\mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}^{2n+1-k}_\Sigma)) \cong R^i t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \ \text{ for all } \ 0 \leq k \leq n, i \geq 0.$$

We first have the following results for the cohomology of the dual complex:

Lemma 6.3. *The following assertions hold for all* $0 \leq k \leq 2n + 1$:

(1) $\mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}_\Sigma^{2n+1-k})) \cong R^i t_* \Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)$ *for all* $0 \le i \le n-1$.

(2) *If* $i \ge n+2$, *then* $\mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}_\Sigma^{2n+1-k})) = 0$.

Proof. We start by observing the following consequence of the smoothness of X:

$$\mathbf{D}_\Sigma(\underline{\Omega}_X^{2n+1-k}) \cong \mathbf{D}_X(\underline{\Omega}_X^{2n+1-k})[-n-1] \cong \begin{cases} \Omega_X^{k-n-1}[-n-1] & \text{if } k \ge n+1; \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Thus

$$\mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}_X^{2n+1-k})) \cong \begin{cases} \Omega_X^{k-n-1} & \text{if } i = n+1 \text{ and } k \ge n+1; \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Consequently, we obtain (1) from (6.2). Further, $R^i t_* \Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) = 0$ for $i \ge n+1$, whence by (6.2) we conclude that for all $0 \le k \le 2n-1$, we have

$$\mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}_X^{2n+1-k})) \cong \mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}_\Sigma^{2n+1-k})) \text{ for all } i \ge n+2.$$

The conclusions now follow from (6.5). □

We end this section by showing that we can say more about $\mathcal{H}^0(\tau_k)$, a fact that will not be used in the sequel. Recall that we have the natural map

$$\tau_k : \underline{\Omega}_\Sigma^k \to \mathbf{D}_\Sigma(\underline{\Omega}_\Sigma^{2n+1-k}).$$

The induced map $\mathcal{H}^0(\tau_k)$ via the isomorphism of Lemma 6.3 (1) can be identified with

$$\phi_k' \circ \beta_k : \mathcal{H}^0(\underline{\Omega}_\Sigma^k) \to t_* \Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)$$

where $\phi_k : t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)} \hookrightarrow j_{U_X *}\Omega^k_{U_X}$ is the composition of the two inclusions ϕ_k', ϕ_k'' described in the following diagram:

$$\begin{tikzcd} t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)} \arrow[r, hook, "\phi_k'"] \arrow[rr, bend right, "\phi_k"'] & t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \arrow[r, hook, "\phi_k''"] & j_{U_X *}\Omega^k_{U_X} \end{tikzcd}$$

In particular, when Σ is normal, the maps $\delta_k : \mathcal{H}^0(\underline{\Omega}_\Sigma^k) \to \Omega_\Sigma^{[k]}$ in (4.9) is the composition of $\mathcal{H}^0(\tau_k)$ and an inclusion.

Remark 6.6. We observe that

(a) $\mathcal{H}^0(\tau_k)$ is an isomorphism if and only if β_k and ϕ_k' are isomorphisms.

(b) ϕ_k is an isomorphism if and only if ϕ_k' and ϕ_k'' are isomorphisms.

Also recall that if Σ is normal, then δ_k is an isomorphism if and only if β_k and ϕ_k are isomorphisms. In particular, if Σ is normal and δ_k is an isomorphism, then $\mathcal{H}^0(\tau_k)$ is also an isomorphism.

We have the following consequence of Theorem B and Theorem 5.1:

Proposition 6.7. *Suppose we are in the situation of Set-up 2.28. Let*

$$\mathcal{H}^0(\tau_k) : \mathcal{H}^0(\underline{\Omega}^k_\Sigma) \to \mathcal{H}^0(\mathbf{D}_\Sigma(\underline{\Omega}^{2n+1-k}_\Sigma))$$

be the natural maps. Then the following hold:

(1) *Let $0 \le p \le 2n+1$, and assume (l, d) satisfies (2.29) with $s = \min\{p, n\}$ when $n \ge 2$. Then $\mathcal{H}^0(\tau_k)$'s are isomorphisms for all $0 \le k \le p$ if and only if $H^k(\mathcal{O}_X) = 0$ for all $1 \le k \le p$*[2]*.*
(2) *Assume (l, d) satisfies (2.29) with $s = n$ when $n \ge 2$. If $H^n(\mathcal{O}_X) = 0$ then $\mathcal{H}^0(\tau_{2n+1})$ and $\mathcal{H}^0(\tau_{2n})$ are isomorphisms.*
(3) *Assume $n \ge 2$, (l, d) satisfies (2.29) with $s = n$, and let $n + 2 \le p \le 2n - 1$. If $H^{n-k}(\mathcal{O}_X) = 0$ for all $0 \le k \le 2n + 1 - p$, then $\mathcal{H}^0(\tau_p)$ is an isomorphism.*

Proof. We first recall that Σ is normal. Notice that (2) and (3) follow by Theorem 5.1 (1), (2) and Remark 6.6.

We now prove (1). Recall that we have the diagram (4.13) with exact rows for all $0 \le k \le p$. Since $q_*\mathcal{O}_\Phi \cong \mathcal{O}_X$ by (2.18), it follows that β_0 is an isomorphism. Also, ϕ_0 is an isomorphism as Σ is normal whence ϕ'_0 is an isomorphism by Remark 6.6 (b). Thus $\mathcal{H}^0(\tau_0)$ is an isomorphism by Remark 6.6 (a).

We now assume $k \ge 1$, hence $p \ge 1$. If $\mathcal{H}^0(\tau_k)$ is an isomorphism for all $1 \le k \le p$, then β_k's whence γ_k's are isomorphisms in the same range. Thus $H^0(\Omega^k_X) = 0$ for all $1 \le k \le p$ by Proposition 4.8. The converse follows from Theorem B and Remark 6.6. □

7. Secant varieties with pre-1-rational singularities.

We now prove Theorem D.

Proof of Theorem D. Observe that by Remark 6.1

$$\mathcal{H}^i(\mathbf{D}_\Sigma(\underline{\Omega}^{2n+1-k}_\Sigma)) \cong R^i t_*\Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \text{ for all } 0 \le k \le 1, i \ge 0. \tag{7.1}$$

Assume (1) holds. Then by (7.1), we have $R^1 t_*\Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) = 0$. Using the restriction sequence and Theorem 3.1, we obtain that

$$R^1 t_*\Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \cong R^1 q_*\Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_\Phi = 0.$$

This, by the exact sequence

$$0 \to \Omega^1_\Phi \to \Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_\Phi \to \mathcal{O}_\Phi \to 0$$

implies that the resulting map $q_*\mathcal{O}_\Phi \to R^1 q_*\Omega^1_\Phi$ is surjective. But

$$\operatorname{rank}(q_*\mathcal{O}_\Phi) = 1 \text{ and } \operatorname{rank}(R^1 q_*\Omega^1_\Phi) = \begin{cases} nh^{1,0}(X) + h^{1,1}(X) + 1 & \text{if } n \ge 2; \\ nh^{1,0}(X) + h^{1,1}(X) & \text{if } n = 1 \end{cases}$$

where the second equality follows from Lemma 2.19. Thus, we conclude that $n = 1$ and (X, L) is a rational normal curve of degree ≥ 3.

[2] Notice that this condition is vacuous when $p = 0$.

Now we prove the converse, and assume (X, L) is a rational normal curve of degree ≥ 3. We claim that

$$R^i t_* \Omega^k_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi) \text{ for all } 0 \leq k \leq 1, i \geq 1. \tag{7.2}$$

Notice that (7.2) holds for $k = 0$ by [CS18, Corollary 1.5]. Since (7.2) holds for $i \geq 2$, it is enough to show (7.2) for $i = k = 1$. We first show that

$$H^1(\Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x}) = 0 \text{ for all } x \in X. \tag{7.3}$$

Notice that in this case, $F_x \cong \mathbb{P}^1$, $\Phi \cong \mathbb{P}^1 \times \mathbb{P}^1$ and one sees easily that $h^1(\Omega^1_\Phi|_{F_x}) = h^1(\Omega^1_{F_x}) = 1$. Passing to the cohomology of the exact sequence

$$0 \to \Omega^1_\Phi|_{F_x} \to \Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \to \mathcal{O}_{F_x} \to 0, \tag{7.4}$$

and by (3.10), we obtain the composite map

$$H^0(\mathcal{O}_{F_x}) \to H^1(\Omega^1_\Phi|_{F_x}) \to H^1(\Omega^1_{F_x}) \tag{7.5}$$

Now we observe that the composition of the maps (7.5) sends $1 \in H^0(\mathcal{O}_{F_x})$ to $\mathrm{cl}(\Phi|_{F_x}) \in H^1(\Omega^1_{F_x})$ whence it is injective as $\mathrm{cl}(\Phi|_{F_x})$ is not cohomologically trivial. Consequently, the connecting map $H^0(\mathcal{O}_{F_x}) \to H^1(\Omega^1_\Phi|_{F_x})$ obtained from (7.4) is injective. Thus, (7.3) follows by passing to the cohomology of (7.4) as $H^1(\mathcal{O}_{F_x}) = 0$. Recall that by Claim 5.4, in order to show (7.2) it is enough to prove that

$$H^1(\Omega^1_{\mathbb{P}(\mathcal{E}_L)}(\log \Phi)|_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0 \text{ for all } m \geq 0, x \in X.$$

This holds for $m = 0$ by (7.3), and for $m \geq 1$ this is a consequence of

$$H^i(\Omega^q_\Phi|_{F_x} \otimes b_x^*(mL)(-2mE_x)) = 0 \text{ for all } m \geq 1, x \in X, q = 0, 1$$

via Claim 3.7, which is easy to see as the degree of the curve is ≥ 3 (or use Claim 3.8). □

Proof of Corollary G. Since $\dim(\Sigma) = 3$, it is enough to show (0.5) when $0 \leq p \leq 1$. This is an immediate consequence of Theorem D and [SVV23, Corollary 4.1]. □

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