Chapter 4

Representations of Groups

4.1 Definitions and Elementary Properties

Let G be a group and K a commutative ring.
A (linear) **representation** of G consists of a K-module V and an action $G \times V \mapsto V$ satisfying

$$g \cdot (av + bw) = ag \cdot v + bg \cdot w \quad \forall g \in G,\ a, b \in K,\ v, w \in W.$$

Equivalently, a rep. is a group homomorphism $G \mapsto \mathrm{Aut}_K(V)$.

Another formulation: Define a ring $K[G]$, called the **group ring**, as follows. As an abelian group,

$$K[G] = \{\text{free } K\text{-module with basis } G\}.$$

Multiplication is determined by $g \cdot h = gh$ (the left defines multiplication in $K[G]$; the right is multiplication in G). Then a rep. of G on V is a ring homomorphism $K[G] \mapsto \mathrm{End}_K(V)$. This makes V a left $K[G]$-module.

Note that as rings,

$$K[G \times H] = K[G] \otimes_{\mathbb{Z}} K[H].$$

$K[G]$ is commutative $\iff G$ is abelian.

Let G be finite. For a conjugacy class C, let

$$N_C := \sum_{x \in C} x \in K[G].$$

Definition 4.1.1. *Let R be a ring. The **center** of R is*

$$\mathrm{Z}(R) = \{a \in R \mid ax = xa\ \forall x \in R\}.$$

Proposition 4.1.2. *If G is finite then* $\mathrm{Z}(K[G])$ *is the free K-module*

$$KN_{C_1} \oplus \cdots \oplus KN_{C_k},$$

where $C_1, \ldots, C_k$ are the conjugacy classes of G.

Proof. For $g \in G$, and $C = C_j$,

$$g^{-1}N_C g = \sum_{x \in C} g^{-1}xg = \sum_{y \in g^{-1}Cg} y = N_C,$$

since $g^{-1}Cg = C$. Thus,

$$\bigoplus_{j=1}^{k} KN_j \subset \mathrm{Z}(K[G]).$$

Conversely, let

$$x = \sum_{g \in G} a_g g \in \mathrm{Z}(K[G]).$$

Then for all $h \in G$,

$$\begin{aligned}\sum_{g \in G} a_g g &= x \\ &= h^{-1}xh \\ &= \sum_{g \in G} a_g h^{-1}gh \\ &= \sum_{t \in G} a_{hth^{-1}} t.\end{aligned}$$

$\therefore a_g = a_{hgh^{-1}}\ \forall h, g$. ie. All elements of a given conjugacy class have the same coefficient in x. Thus,

$$x = \sum a_j N_{C_j}$$

where $a_j = a_g$ for any $g \in C_j$. So $x \in \bigoplus_{j=1}^{k} KN_{C_j}$. □

4.1.1 New Representations from Old

1. Direct sum of reps.
   Given reps.

$$G \times V \mapsto V \qquad G \times W \mapsto W,$$

form rep. of G on $V \oplus W$ by

$$g \cdot (v, w) = (g \cdot v, g \cdot w).$$

eg. If K is a field, $n = \dim V, m = \dim W$, then for $g \in G$, $\rho(g) \in GL_n(K), \tau(g) \in GL_m(K)$. The direct sum action is given by

$$\begin{pmatrix} \rho(g) & 0 \\ 0 & \tau(g) \end{pmatrix}.$$

Note: Sometimes write kV for $\overbrace{V \oplus \cdots \oplus V}^{k \text{ times}}$.

2. Tensor product of reps.
   Given reps.

$$G \times V \mapsto V \qquad G \times W \mapsto W,$$

form rep of G on $V \otimes_K W$ determined by

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$

This is the tensor product of V and W in the Hopf alg. sense. ie. The action is

$$K[G] \otimes V \otimes W \overset{\psi}{\longmapsto} K[G] \otimes K[G] \otimes V \otimes W \mapsto K[G] \otimes V \otimes K[G] \otimes W \overset{\mu_V \otimes \mu_W}{\longmapsto} V \otimes W,$$

where $\psi(g) = g \otimes g$ is induced by the diagonal map $G \mapsto G \times G$.

Let R be a ring. Recall that an R-module V is simple if it has no proper R-submodules except 0. In this context, such modules will often be called **irreducible**.

Definition 4.1.3. *An R-module $V \neq 0$ is called **indecomposable** if $\nexists$ R-modules $V_1 \neq 0, V_2 \neq 0$ s.t. $V \cong V_1 \oplus V_2$.*

When $R = K[G]$, we talk of "indecomposable reps." and "irreducible" (or "simple") reps.

Clearly, irreducible $\Rightarrow$ indecomposable. The reverse is not true. eg. Suppose K is a field. If the action of each elt. g of G has the form

$$\begin{pmatrix} P(g) & Q(g) \\ 0 & R(g) \end{pmatrix},$$

(where $P(g)$ is $n \times n$, $Q(g)$ is $n \times m$, $R(g)$ is $m \times m$), then $\exists$ an n-dim. subrepresentation $g \mapsto P(g)$, so not irreducible. But it might still be indecomposable if $Q(g) \neq 0$. In particular, take $G = \mathbb{Z}$, $n = m = 1$, let $P(k) = R(k) = 1$ and $Q(k) = k$ for all $k \in \mathbb{Z}$.

Goal: Let G be finite, K a field.

1. Show that there is (up to iso.) a finite list $V_1, \ldots, V_k$ of indecomposable $K[G]$-modules and find them.

2. Given a rep. V of G, show that the decomposition

$$V \cong V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_k^{n_k}$$

   into irreducible is unique, and give a method of determining the mult. n_k of each V_k.

3. In particular, find the decomposition

$$K[G] \cong V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_k^{n_k}.$$

Question. Given G, to what extent does this answer change with K? Does it depend on more that just char K?

4.2 Semisimple Rings

Note: Unless otherwise noted, module means left module.

Definition 4.2.1. *An R-module V is called* ***semisimple*** *if it is a direct sum of simple modules. R is called a* ***semisimple ring*** *if R is semisimple as a (left) R-module.*

Proposition 4.2.2. *If V is a semisimple module and $U \subset V$ then $V \cong U \oplus W$ for some W.*

Proof. Consider

$$S = \{\text{submodules } U' \subset V \text{ s.t. } U \cap U' = 0\}.$$

By Zorn's lemma, let $W \subset V$ be maximal s.t. $U \cap W = 0$. If $U \oplus W \subsetneq V$, choose $v \notin U \oplus W$. Write $v = v_1 + \cdots + v_n$, where $v_i \in V_i$ and $V_i \subset V$ is simple. Then $v_j \notin U \oplus W$ for some j. So

$$V_j \cap (U \oplus W) \subsetneq V_j$$

and since V_j is simple,

$$V_j \cap (U \oplus W) = 0.$$

But then $U \cap (W \oplus V_j) = 0$, so W is not maximal. Thus, $V = U \oplus W$. □

Definition 4.2.3. *V is* ***completely splittable*** *if $U \subset V \Rightarrow V = U \oplus W$ for some W.*

So V is semisimple $\Rightarrow$ V is completely splittable. Recalling Proposition 2.4.6, we see that V is completely splittable $\iff$ whenever $U \subset V$, if $i : U \hookrightarrow V$ is the inclusion then $\exists \sigma : V \mapsto U$ a homo. s.t. $\sigma i = 1_U$; σ is called a splitting of i.

Example 4.2.4. *Let K be a field, $R = M_{n\times n}(K)$,*

$$V = \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix}.$$

Claim. *V is a simple R-module.*

Proof of claim. Suppose $0 \subsetneq W \subset V$. Let

$$0 \neq x = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{pmatrix} \in W,$$

and suppose $x_j \neq 0$. Then W contains

$$\begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix} x = \begin{pmatrix} x_j & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}.$$

By dividing by x_j, W contains

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}.$$

Thus, W contains

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 1 & & \vdots \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix},$$

which is a basis of V. Hence $W = V$. □

Now,

$$R = \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & * & 0 & \cdots & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \cdots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

So R is semisimple.

Proposition 4.2.5. $\mathrm{Z}(M_{n\times n}(K)) = KI$.

Proof. Exercise. □

Let R be a ring, $x, y \in R$. The **commutator** of x and y is

$$[x, y] := xy - yx \in R.$$

The **commutator subspace** is

$$[R, R] = \{[x, y] \mid x, y \in R\}.$$

Note: $[R, R]$ is not an R-submodule of R, in general.

Example 4.2.6. *Let $R = M_{n\times n}(K)$. Then*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{pmatrix}.$$

Similarly, denoting by e^{ij} the matrix with 1 in the $(i, j)^{th}$ position and 0 elsewhere,

$$e^{ij} \in [R, R] \quad \forall i \neq j.$$

Also,

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ 0 & & & \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ 0 & & & \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ \vdots & & & -1 & & \vdots \\ & & & & 0 & \\ & & & & & \ddots \\ 0 & & & \cdots & & 0 \end{pmatrix} \in [R, R].$$

These matrices generate $\{M \mid \mathrm{Tr}M = 0\}$ *as a vector space. That is,*

$$[R, R] = \ker \mathrm{Tr} : R \mapsto K.$$

In particular, $\dim[R, R] = n^2 - 1$ *and* $\dim(R/[R, R]) = 1$.

Lemma 4.2.7. *If V is completely splittable and $U \subset V$ then U is completely splittable.*

Proof. Let $T \subset U$. Then

$$\begin{tikzcd} T \arrow[r, hook, "i"] \arrow[drr, "1_T"'] & U \arrow[r, hook, "j"] & V \arrow[d, "\exists\sigma"] \\ & & T \end{tikzcd}$$

Since V is completely splittable, $\exists\sigma$ s.t. $\sigma \circ ji = 1_T$, as in the diagram. So $\sigma \circ j$ is a splitting of i. □

Theorem 4.2.8. *If V is completely splittable then every submodule of V is semisimple.*

Corollary 4.2.9. *V is semisimple $\iff$ V is completely splittable.*

Corollary 4.2.10. *If V is semisimple then every submodule of V is semisimple.*

Proof of Theorem. Let $U \subset V$. Consider sets $\{S_i\}_{i \in I}$ of simple U-submodules which are "linearly independent", ie.

$$\langle S_i \rangle_{i \in I} = \bigoplus_{i \in I} S_i.$$

By Zorn's lemma, there is a maximal such set, $\{S_i\}_{i \in I}$. Let

$$S = \bigoplus_{i \in I} S_i.$$

By the lemma, S is completely splittable, so $\exists T$ s.t. $U = S \oplus T$. If $T \neq 0$, pick $0 \neq x \in T$. By Zorn's lemma,

$$\{T' \subset T \mid x \notin T'\}$$

has a maximal element, T_0. By the lemma, let T_1 be s.t.

$$T = T_0 \oplus T_1.$$

If T_1 is not simple then let $T_1 = A \oplus B$. x can't be in both $T_0 \oplus A$ and $T_0 \oplus B$ since their intersection is T_0. This contradicts the maximality of T_0, so T_1 is simple.

But T_1 can be added to $\{S_i\}_{i \in I}$ to get a still-linearly independent set of simple submodules. This contradicts the maximality of $\{S_i\}_{i \in I}$.

So $T = 0$ and $U = S = \bigoplus_{i \in I} S_i$ is semisimple. □

Corollary 4.2.11. *Let*

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of R-modules. If V is semisimple then U, W are semisimple.

Proof. $V \cong U \oplus W$, so $U \subset V$ and V has a submodule isomorphic to W. So by the theorem, U and W are semisimple. □

Theorem 4.2.12 (Maschke)**.** *If K is a field, G a finite group s.t.* $\operatorname{char} K \nmid |G|$ *then* $K[G]$ *is semisimple.*

Proof. Write $V = K[G]$, as a module over itself. Suppose U is a $K[G]$-submodule, and show that there exists a K-module splitting $p : V \mapsto U$.

As vector spaces, $\exists U_0$ s.t. $V \cong U \oplus U_0$ (U_0 is not necessarily a $K[G]$-module). This yields a linear map $p_0 : V \mapsto U$ (p_0 is not necessarily a $K[G]$-homomorphism).

Define $p : V \mapsto U$ by

$$p(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gv).$$

Then for $g' \in G$,

$$\begin{aligned} p(g'v) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gg'v) \\ &= \frac{1}{|G|} \sum_{f \in G} g' f^{-1} p_0(fv) \\ &= g' \frac{1}{|G|} \sum_{f \in G} f^{-1} p_0(fv) \\ &= g' p(v). \end{aligned}$$

So p is a $K[G]$-homomorphism.

Also, if $u \in U$ then

$$\begin{aligned} p(u) &= \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gu) \\ &= \frac{1}{|G|} \sum_{g \in G} g^{-1}(gu) \\ &= \frac{1}{|G|} \sum_{g \in G} u \\ &= u. \end{aligned}$$

$\therefore$ p is a splitting. □

Note: If K is not a field then the same proof works, provided $|G|$ is invertible in R and $\exists$ an R-module splitting $p_0 : V \mapsto U$.

Let R be a semisimple ring,

$$R \cong \bigoplus_{i \in I} V_i$$

where each V_i is simple.

Proposition 4.2.13. *Ever simple R-module appears (up to isomorphism) as V_i for some i.*

Proof. Let W be a simple R-module and $0 \neq w \in W$. Then

$$\begin{array}{ccc} R & \overset{\phi}{\longmapsto} & W \\ 1 & \longmapsto & w \end{array}$$

is not zero, so it is onto (since W is simple).

So W is a summand of R. Furthermore,

$$0 \neq \phi \in \hom_R(R, W) \cong \bigoplus_{i \in I} \hom_R(V_i, W),$$

so $\hom_R(V_i, W) \neq 0$ for some i. But any non-zero homo. between simple R-modules is an isomorphism, so W is isomorphic to some V_i. □

Proposition 4.2.14. *Let R be semisimple, $I \subset R$ a left ideal. Then $\exists$ an idempotent $e \in R$ s.t. $I = Re$.*

Note: If V is a simple R-module then $Rv = V$, for any $v \in V$. Moreover, since R is semisimple, $R \mapsto\!\!\!\!\rightarrow Rv$ splits, so V is isomorphic to a left ideal of R.

Proof. Since R is semisimple, $\exists J$ s.t. $R = I \oplus J$. Write $1 = e + f$ where $e \in I$, $f \in J$.

$e \in I \Rightarrow Re \subset I$. Conversely, given $x \in I$, $x = xe + xf$. $xf = x - xe \in I$ and since $f \in J$, $xf \in J$. Thus

$$xf \in I \cap J = 0 \Rightarrow \ x = xe.$$

$\therefore I = Re$.

Now, $x = xe \ \forall x \in I$, and in particular, $e = e^2$. □

4.3 Artinian Rings

Recall that an R-module V is Noetherian if for any chain

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots$$

of submodules, $\exists N$ s.t. $V_n = V_N \ \forall n \geq N$. Likewise:

Definition 4.3.1. *An R-module V is **Artinian** if for any chain*

$$V_0 \supset V_1 \supset \cdots \supset V_n \supset \cdots$$

*of submodules, $\exists N$ s.t. $V_n = V_N \ \forall n \geq N$. R is an **Artinian ring** if R is Artinian as a (left) R-module.*

Example 4.3.2. *$\mathbb{Z}$ is Noetherian (in fact, it is a PID) but not Artinian, since we have:*

$$2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset \cdots \supset 2^n\mathbb{Z} \supset \cdots .$$

When G is finite and K is a field, $K[G]$ is both Noetherian and Artinian (by counting dimensions, can't have a strictly increasing chain longer than $|G| + 1$).

Proposition 4.3.3. *Let*

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of R-modules. Then V is Noetherian (respectively Artinian) $\iff U, W$ are Noetherian (resp. Artinian).

Corollary 4.3.4. *If*

$$V = \bigoplus_{i=1}^{n} V_i$$

then V is Noetherian (resp. Artinian) $\iff V_i$ is Noetherian (resp. Artinian) $\forall i$.

Proposition 4.3.5. *If*

$$V = \bigoplus_{i \in I} V_i$$

with $V_i \neq 0$ and V is finitely generated then $|I| < \infty$.

Proof. Each generator has only finitely many non-zero components. □

Corollary 4.3.6. *If V is finitely generated and semisimple then V is both Noetherian and Artinian.*

Proof. By the hypothesis,

$$V = \bigoplus_{i=1}^{n} V_i$$

where each V_i is simple. So for each i, the only chain is $0 \subset V_i$. Thus V_i is both Noetherian and Artinian. □

Corollary 4.3.7. *If R is semisimple then R is both Noetherian and Artinian.*

Proof. As an R-module, R is generated by the single element 1. □

Proposition 4.3.8. *Let G be finite, K Noetherian (resp. Artinian). Then $K[G]$ is Noetherian (resp. Artinian).*

Proof. If K is Noetherian (or Artinian) then, as a K-module, so is $K^{|G|}$, which is isomorphic, as a K-module, to $K[G]$. But every $K[G]$-submodule of $K[G]$ is a K-submodule, so if $K[G]$ is Noetherian (or Artinian) as a K-module then it has the same property as a $K[G]$-module. □

Lemma 4.3.9 (Schur)**.** *Let V be a simple R-module. Then:*

1. $\mathrm{End}_R(V)$ *forms a division ring.*
2. *If R is a finite dimensional algebra (eg. $R = K[G]$ with G finite) over an algebraically closed field K then* $\mathrm{End}_R(V) \cong K$.

Proof.

1. If $f : V \mapsto V$ is nonzero then $\mathrm{Im} f = V$ so V is onto. Also, since $f \neq 0$, $\ker f \neq V$, so $\ker f = 0$. Hence f is an isomorphism, so it has an inverse. ie. $\mathrm{End}_R(V)$ is a division ring.
2. Let $f \in \mathrm{End}_R(V)$, and show $f = \lambda I$ for some $\lambda \in K$. For any $0 \neq x \in V$, Rx forms a finite dimensional subspace of V (its dimension is $\leq \dim R$). Since V is simple, $Rx = V$, so V is finite dimensional.

   So $\exists$ an eigenvector $0 \neq v \in V$ for f, so that $fv = \lambda v$. Since V is simple, $Rv = V$. Hence $\forall w \in V$, $w = rv$ so

   $$f(w) = rf(v) = \lambda rv = \lambda w.$$

   ie. $f = \lambda I$.

□

4.4 Wedderburn's Theorem

Let R be semisimple. Then

$$R \cong n_1 V_1 \oplus n_2 V_2 \oplus \cdots \oplus n_k V_k$$

where $V_1, \ldots, V_k$ is the list of simple R-modules (one from each isomorphism class). (This is a finite decomposition by Proposition 4.3.5)

For a ring A,

$$\begin{aligned} A &\overset{\phi}{\longmapsto} \mathrm{End}_A(A) \\ A &\longmapsto \phi_a \\ \phi_a(b) &= ba \end{aligned}$$

is a bijection, since every endomorphism f is equal to $\phi_{f(1)}$. Then

$$\phi_a \phi_b(1) = \phi_a(b) = ba = \phi_{ba}(1).$$

$\therefore \phi$ is a ring isomorphism

$$\phi : A^{\mathrm{opp}} \mapsto \mathrm{End}_A(A),$$

where A^{opp} is the ring with the same group structure as A but $a(\cdot_{\mathrm{opp}})b = ba$.

Set $D_j = \mathrm{End}_R(V_j)$, a division ring. Then

$$\begin{aligned} R &\cong (\mathrm{End}_R(R))^{\mathrm{opp}} \\ &\cong \prod_{j=1}^{k} \left(\mathrm{End}_R(n_j V_j)\right)^{\mathrm{opp}} \\ &\cong \prod_{j=1}^{k} \left(M_{n_j \times n_j}(\mathrm{End}_R(V_j)\right)^{\mathrm{opp}} \\ &\cong \prod_{j=1}^{k} \left(M_{n_j \times n_j}(D_j)\right)^{\mathrm{opp}} \\ &\cong \prod_{j=1}^{k} M_{n_j \times n_j}(D_j^{\mathrm{opp}}) \end{aligned}$$

where on the last line, the isomorphism is given by the transpose map.

Under this isomorphism, $V_j \subset n_j V_j \subset R$ corresponds to

$$\hom_R(n_j V_j, V_j) \cong \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} \subset M_{n_j \times n_j}(D_j^{\text{opp}}).$$

In particular, suppose R is an algebra over a field K and $D_j \cong K$ (eg. if K is algebraically closed). Then:

1. $\dim V_j = n_j$ for each j.
2. $\dim R = \sum_{j=1}^{k} n_j^2$.

Example 4.4.1.

1. $R = \mathbb{C}(S_2)$
   By Maschke's Theorem, $\operatorname{char} K = 0 \Rightarrow K[G]$ *is semisimple. Here,*

   $$2 = 1^2 + 1^2$$

   and there are no other possibilities, so R has 2 indecomposable reps., each on a 1-dimensional space.

   They are: Let $\dim V = 1$ *with basis v. $S_2 = \{e, T\}$, with $T^2 = e$. The trivial rep. is:*

   $$e \cdot v = v,$$
   $$T \cdot v = v.$$

   The sign rep. is:

   $$e \cdot v = v,$$
   $$T \cdot v = -v.$$

2. $R = \mathbb{C}(S_3)$
   Either

   $$6 = 1^2 + 1^2 + \cdots + 1^2 \qquad \textit{or} \qquad 6 = 1^2 + 1^2 + 2^2.$$

   Easy to see that the trivial rep. and the sign rep. ($\sigma \cdot v = (-1)^{\operatorname{sgn}\sigma} v$, sgn *is the homomorphism $\epsilon : S_n \mapsto \{1, -1\}$ used to define A_n in section 1.6.2) are the only possible reps. of R on a 1-dim. V. Hence R has 3 indecomposable reps.: trivial rep., sign rep., a 2-dim. rep.*

The 2*-dim. rep. is:*

$$(1\ 2) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix},$$
$$(1\ 3) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$(1\ 2\ 3) = (1\ 2)(1\ 3) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

3. $R = \mathbb{C}(S_4)$

$$24 = 1^2 + 1^2 + (a)^2 + \cdots + (\ )^2, \quad a \geq 2.$$

Looking at congruence $\mod 4$*, need* 3^2 *(*$2^2, 4^2$ *are divisible by* 4*, but* $24 - 1^2 - 1^2$ *is not). Hence, the only possibility is*

$$24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2.$$

ie. two 1*-dim. reps., one* 2*-dim. rep., two* 3*-dim. reps.*

Theorem 4.4.2. *Let* G *be a finite group,* K *an algebraically closed field of characteristic* 0*. Then the number of isomorphic simple* $K[G]$*-modules is equal to the number of conjugacy classes of* G*.*

Proof. As seen earlier,

$$\mathrm{Z}(K[G]) = \text{Free } K\text{-module on } \left\{ \sum_{g \in C} g \mid C \text{ a conj. class} \right\}.$$

So the number of conjugacy classes is equal to $\dim \mathrm{Z}(K[G])$. Also,

$$K[G] = \prod_{j=1}^{k} M_{n_j \times n_j}(K).$$

Now, $\mathrm{Z}(M_{n\times n}(K)) = KI$, which has dimension 1. Thus,

$$\dim \mathrm{Z}(K[G]) = k = \#\text{ nonisomorphic simple } K[G]\text{-modules}.$$

□

4.5 Changing the Ground Ring

Example 4.5.1. *Let $G = C_3 = \{e, t, t^2\}$, $K = \mathbb{R}$. Then using $V = \mathbb{R}^2$,*

$$\begin{aligned} \rho : K[G] &\mapsto M_2(\mathbb{R}) \\ t &\mapsto \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}. \end{aligned}$$

(ρ is rotation by $\frac{2\pi}{3}$.) Then ρ is indecomposable. But if we use $K = \mathbb{C}$, and $\tilde{\rho} : \mathbb{C}[G] \mapsto M_2(\mathbb{C})$ induced by the same representation, then $\tilde{\rho}$ is decomposable since over $\mathbb{C}$, we can change basis and diagonalize:

$$\tilde{\rho}(t) = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix},$$

in an appropriate basis.

Given $f : R \mapsto S$ a ring homomorphism, f induces a functor

$$\begin{aligned} \{\text{(left) } R\text{-mods.}\} &\mapsto \{\text{(left) } S\text{-mods.}\} \\ V &\mapsto V_S := S \otimes_R V. \end{aligned}$$

The map f makes S a two-sided R-module (and in particular, a right module), so $S \otimes_R V$ makes sense. $S \otimes_R V$ is an S-module via the action

$$s'(s \otimes v) = (s's) \otimes v.$$

If

$$0 \to U \to V \to W \to 0$$

is a short exact sequence of left R-modules and M is a right R-module then

$$M \otimes_R U \to M \otimes_R V \to M \otimes_R W \to 0$$

is exact, although the first map may not be injective. However, if M is a free R-module, $M \cong R^n$ then $M \otimes_R N \cong N^n$, and so

$$M \otimes_R U \hookrightarrow M \otimes_R V$$

in this case.

In particular, if $f : R \hookrightarrow S$ makes S into a free R-module then when

$$0 \to U \to V \to W \to 0$$

is exact, so is

$$0 \to U_S \to V_S \to W_S \to 0,$$

ie. $(V/U)_S \cong V_S/U_S$.

In particular, if $K \subset M$ is a field extension then M is a free K-module.

$f : K \mapsto M$ induces $K[G] \mapsto M[G]$, and thus

$$K[G]\text{-mods.} \mapsto M[G]\text{-mods.}$$
$$V \mapsto V_M$$

ie. $(mg)(m' \otimes v) = (mm' \otimes gv)$, defines $M[G]$-action on V_M.
Note: If $v = kv'$ then $m' \otimes v = m' f(k) \otimes v'$, but then

$$mg(m' \otimes v) = mm' \otimes gv = mm' \otimes kgv' = mm' f(k) \otimes gv' = mg(m' f(k) \otimes v'),$$

so the action is well-defined. Also,

$$g(m(m' \otimes v)) = g(mm' \otimes v) = mm' \otimes gv = m(m' \otimes gv) = mg(m' \otimes v),$$

so the action of g is M-linear.

If $K \subset M$ is a field extension and $n = \dim V < \infty$,

$$\rho : K[G] \mapsto \mathrm{End}_K(V) \cong M_{n\times n}(K)$$

then $\dim V_M = n$ and for the induced map

$$\tilde{\rho} : M[G] \mapsto \mathrm{End}_M(V_M),$$

the matrix $\tilde{\rho}(g)$ for the action of g is just $\rho(g)$, regarded as a matrix in M (whose entries happen to lie in K).

As we have seen, V simple $\not\Rightarrow$ V_M is simple.

4.6 Composition Series

Let V be an R-module.

Definition 4.6.1. *A **composition series** for V consists of a chain of submodules*

$$0 = V_n \subset V_{n-1} \subset \cdots \subset V_1 \subset V_0 = V$$

s.t. V_{j-1}/V_j is simple $\forall j = 1, \ldots, n$.

The composition series

$$0 = V_n \subset \cdots \subset V_0 = V$$

and

$$0 = W_m \subset \cdots \subset W_0 = V$$

are called equivalent if $n = m$ and $\exists \sigma \in S_n$ s.t.

$$V_{j-1}/V_j \cong W_{\sigma(j)-1}/W_{\sigma(j)} \quad \forall j.$$

ie. the list of "composition factors" (including multiplicities) is the same, although the order may be different.

Proposition 4.6.2. *V has a composition series $\iff$ V is both Artinian and Noetherian. In this case, any series can be refined to a composition series.*

Proof.

$\Leftarrow$: Suppose V is Artinian and Noetherian. Let $V_0 = V$. Since V is Noetherian, V contains a maximal (proper) submodule, V_1 (by Theorem 2.6.2). Continuing, so long as $V_j \neq 0$, get

$$V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_j \supsetneq \cdots$$

s.t. V_{j+1} is maximal in V_j, ie. V_j/V_{j+1} is simple. Since V is Artinian, the chain must terminate.

$\Rightarrow$: Suppose

$$0 = V_n \subset \cdots \subset V_0 = V$$

is a composition series. Then we have the exact sequence

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

Since V_1 is simple, V_1 is Artinian and Noetherian. V/V_1 has a composition series of length $n-1$, so by induction, V/V_1 is Artinian and Noetherian. Thus, V is Artinian and Noetherian.

Finally, given any series

$$0 = V_n \subset \cdots \subset V_0 = V,$$

each V_{i-1}/V_i is Noetherian and Artinian, so it has a composition series. Using each of these series, we may refine the given series to a composition series. □

Theorem 4.6.3. *Any two comp. series for V are equivalent.*

Proof. Let

$$0 = V_n \subset \cdots \subset V_0 = V$$

and

$$0 = W_m \subset \cdots \subset W_0 = V$$

be comp. series. For $1 \leq i \leq n$ and $1 \leq j \leq m$, set

$$V_{ij} := V_i + (V_{i-1} \cap W_j) \quad \text{and} \quad W_{ji} := W_j + (W_{j-1} \cap W_j).$$

Claim.

$$\frac{V_{i,j-1}}{V_{ij}} \cong \frac{V_{i-1} \cap W_{j-1}}{(V_i \cap W_{j-1}) + (V_{i-1} \cap W_j)} \cong \frac{W_{j,i-1}}{W_{ji}}.$$

Proof of claim. Consider

$$\phi : V_{i-1} \cap W_{j-1} \hookrightarrow V_i + (V_{i-1} \cap W_j - 1) \mapsto\!\!\!\!\rightarrow \frac{V_i + (V_{i-1} \cap W_{j-i})}{V_i + (V_{i-1}} \cap W_j) = \frac{V_{i,j-i}}{V_{ij}}.$$

$V_i \subset V_{ij}$, so every element of $V_{i,j-1}$ is congruent modulo V_{ij} to one in $V_{i-1} \cap W_{j-1}$. ie. ϕ is surjective.

Clearly, $V_{i-1} \cap W_j \subset V_{ij}$, so

$$V_{i-1 \cap W_j} \subset \ker \phi.$$

Also, $V_j \cap W_{j-1} \subset V_i \subset V_{ij}$, so

$$V_i \cap W_{j-1} \subset \ker \phi.$$

Hence,

$$(V_{i-1} \cap W_j) + (V_i \cap W_{j-1}) \subset \ker \phi.$$

Conversely, suppose $x \in V_{i-1} \cap W_{j-1}$ lies in

$$\ker \phi = (V_{i-1} \cap W_{j-1}) \cap (V_i + (V_{i-1} \cap W_j)).$$

Write $x = y + z$ where $y \in V_i$ and $z \in V_{i-1} \cap W_j$. Since $x \in W_{j-1}$ and $z \in W_j \subset W_{j-1}$, it follows that $y \in W_{j-1}$. So

$$x = y + z$$

exhibits x as an elt. of $(V_i \cap W_{j-1}) + (V_{i-1} \cap W_j)$. □

Notice that since V_{i-1}/V_i and W_{j-1}/W_j are simple,

$$\frac{V_{i-1} \cap W_{j-1}}{(V_i \cap W_{j-1}) + (V_{i-1} \cap W_j)}$$

is either 0 or simple.

So we have

$$V = V_0 = V_{10} \supset V_{11} \supset \cdots \supset V_{1m} = V_1 = V_{20} \supset \cdots \supset \cdots \supset V_{n-1} = V_{n0} \supset \cdots \supset V_{nm} = 0. \qquad (*)$$

and similarly,

$$V = W_0 = W_{10} \supset W_{11} \supset \cdots \supset W_{1n} = W_1 = W_{20} \supset \cdots \supset \cdots \supset W_{m-1} = W_{m0} \supset \cdots \supset W_{mn} = 0. \qquad (**)$$

Notice that both chains have the same length, and by the claim, there is a bijection between the quotient modules, each of which is either simple or 0. So by shortening the chains by deleting entries which equal their predecessors, all the 0-quotient modules are deleted, and what is left are composition series. The number of 0-quotients deleted is the same (they are paired), so the resulting comp. series have the same length and same quotients, ie. they are equivalent.

But (*) reduces to

$$V_n \subset \cdots \subset V_0$$

and (**) reduces to

$$W_m \subset \cdots \subset W_0,$$

since they are respectively refinements of these series, and you can't refine a comp. series any further. So these two comp. series are equivalent. □

4.7 Characters

Let

$$\rho : K[G] \mapsto \mathrm{End}_K(V)$$

be a rep. of K on a free K-module V. Define

$$\chi_\rho : K[G] \mapsto K,$$

the **character** of ρ by $\chi_\rho = \mathrm{Tr}(\rho(x))$.

Since $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$, χ_ρ is determined by its values on the basis G for $K[G]$, so sometimes write $\chi_\rho : G \mapsto K$.

Recall that Tr is preserved under change of basis, since

$$\mathrm{Tr}(A^{-1}BA) = \mathrm{Tr}(AA^{-1}B) = \mathrm{Tr}(B).$$

So, if $h = x^{-1}gx$ then

$$\rho(h) = \rho(x)^{-1}\rho(g)\rho(x)$$

and thus, $\chi_\rho(h) = \chi_\rho(g)$.

Proposition 4.7.1. *Let*

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of $K[G]$-modules, each of which is free as a K-module. Then

$$\chi_V = \chi_U + \chi_W.$$

Proof. Since U is a $K[G]$-submodule, for all $g \in U$, the matrix for $\rho(g)$ has the form

$$\rho_V(g) = \begin{pmatrix} \rho_U(g) & * \\ 0 & \rho_W(g) \end{pmatrix}.$$

□

Proposition 4.7.2. $\chi_{V\otimes W} = \chi_V \chi_W$.

Proof. Let $\{e_i\}, \{f_j\}$ be bases for V, W respectively. Then $\{e_i \otimes f_j\}$ is a basis for $V \otimes W$, and

$$(A \otimes B)(e_i \otimes f_j) = a_{ii}b_{jj}(e_i \otimes f_j) + \text{ other terms}.$$

So,

$$\mathrm{Tr}(A \otimes B) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ii}b_{jj} = (\mathrm{Tr}A)(\mathrm{Tr}B).$$

□

Proposition 4.7.3. *Viewing $K[G]$ as a left $K[G]$-module,*

$$\chi_{K[G]}(g) = \begin{cases} |G|, & g = e, \\ 0, & g \neq e. \end{cases}$$

Proof. In the basis $\{g\}_{g \in G}$ for $K[G]$, the action of any elt. of G is given by a permutation matrix. So, by the definition of the trace,

$$\begin{aligned} \chi_{K[G]}(g) &= |\{x \in G \mid gx = x\}| \\ &= \begin{cases} |G|, & g = e, \\ 0, & g \neq e. \end{cases} \end{aligned}$$

□

Corollary 4.7.4. *Suppose*

$$K[G] \cong V_1 \oplus \cdots \oplus V_r$$

and K is a field s.t. $\operatorname{char} K \nmid |G|$. *Thus* $\chi_{K[G]} = \sum_{i=1}^{r} \chi_i$ *where* $\chi_i = \chi_{V_i}$.

Let $y = \sum_{g \in G} c_g g \in K[G]$. Then for any g,

$$c_g = \frac{1}{|G|} \sum_{i=1}^{r} \chi_i(yg^{-1}).$$

Proof. Pick $g \in G$.

$$y = \sum_{h \in G} c_h h = c_g g + \sum_{h \neq g} c_h h.$$

$\therefore yg^{-1} = c_g e + \sum_{h \neq g} c_h h g^{-1}$. Applying $\chi_{K[G]} = \sum_{i=1}^{r} \chi_i$,

$$\begin{aligned} \sum_{i=1}^{r} \chi_i(yg^{-1}) &= \chi_{K[G]}(yg^{-1}) \\ &= c_g \chi_{K[G]}(e) + \sum_{h \neq g} c_h \chi_{K[G]}(hg^{-1}) \\ &= |G| c_g + 0. \end{aligned}$$

□

Set $\mathrm{CF}_K(G) := \{f : G \mapsto K \mid f(y^{-1}xy) = f(x)\ \forall x, y \in G\}$. $\mathrm{CF}_K(G)$ is a ring using addition and multiplication of functions. It is called the ring of **class functions**.

$\mathrm{R}_K(G)$ is the abelian group generated by iso. classes of f.d. reps. of G, with the relation

$$[V] = [V'] + [V'']$$

for every short exact sequence

$$0 \to V' \to V \to V'' \to 0.$$

Define multiplication on $\mathrm{R}_K(G)$ by

$$[V][W] = [V \otimes W].$$

Then the preceeding implies that

$$\begin{aligned} \theta : \mathrm{R}_K(G) &\mapsto \mathrm{CF}_K(G) \\ [\rho] &\mapsto \chi_\rho \end{aligned}$$

is a ring homomorphism.

Set $\mathrm{Ch}_K(G) := \mathrm{Im}\theta$, the "ring of generalized K-characters of G", or simply the "character ring of G over K".

Lemma 4.7.5. *Let V, W be $K[G]$-modules and let $f \in \hom_K(V, W)$. Define $\tilde{f} : V \mapsto W$ by*

$$\tilde{f}(v) = \sum_{g \in G} g^{-1} f(gv).$$

Then $\tilde{f} \in \hom_{K[G]}(V, W)$.

If $V = W$ then $\mathrm{Tr}\,\tilde{f} = |G|\mathrm{Tr}(f)$.

Proof. For $x \in G$,

$$\tilde{f}(xv) = \sum_{g \in G} g^{-1} f(gxv) = \sum_{h \in G} xh^{-1} f(hv) = x\tilde{f}(v).$$

Now suppose $V = W$. Then,

$$\tilde{f} = \sum_{g \in G} M_g^{-1} f M_g$$

where M_g represents the action of g on V. Hence,

$$\begin{aligned} \mathrm{Tr}(\tilde{f}) &= \sum_{g \in G} \mathrm{Tr}(M_g^{-1} f M_g) \\ &= \sum_{g \in G} \mathrm{Tr}(f) \\ &= |G|\mathrm{Tr}(f). \end{aligned}$$

□

Let K be a field.

Lemma 4.7.6. *Let $\alpha : G \mapsto \mathrm{Aut}_K(V), \beta : G \mapsto \mathrm{Aut}_K(W)$ be non-isomorphic simple reps. Pick bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ for V and W. Let $[\alpha_{ij}(g)]$ and $[\beta_{ij}(g)]$ denote matrices for $\alpha(g), \beta(g)$ in these bases. Then for any i, j, k, t, $1 \leq i, j \leq n,\ 1 \leq k, t \leq m$,*

$$\sum_{g \in G} \beta_{ij}(g^{-1})\alpha_{kt}(g) = 0.$$

Proof. Let $f : V \mapsto W$ be the linear transformation which in chosen bases for V and W is given by the matrix E which is 1 in the $(j,k)^{\text{th}}$ position and 0 elsewhere. By the previous lemma, $\tilde{f} \in \hom_{K[G]}(V, W) = 0$ (since V, W are non-isomorphic and simple). The $(i,t)^{\text{th}}$ position of the matrix for $\tilde{f}$ is

$$\begin{aligned} 0 &= \sum_{g \in G} \sum_{r,s} \beta_{ir}(g^{-1}) E_{is} \alpha_{st}(g) \\ &= \sum_{g \in G} \beta_{ij}(g^{-1})\alpha_{kt}(g), \end{aligned}$$

since $E_{rs} = 0$ except when $r = j, s = k$. □

Corollary 4.7.7. *Let V, W be non-isomorphic simple $K[G]$-modules. Then*

$$\sum_{g \in G} \chi_V(g)\chi_W(g^{-1}) = 0.$$

Proof. Let $\alpha(g), \beta(g)$ be the matrices for the reps. Then

$$\sum_{g \in G} \chi_V(g)\chi_W(g^{-1}) = \sum_g \sum_t \sum_i \alpha_{tt}(g)\beta_{ii}(g^{-1}) = 0.$$

□

Theorem 4.7.8. *Let $\alpha : G \mapsto \mathrm{Aut}_K(V)$ be a simple G-rep. If K is algebraically closed and* $\mathrm{char}\, K = 0$ *then* $\dim Z \mid |G|$ *and*

$$\sum_{g \in G} \alpha_{ij}(g^{-1})\alpha_{kt}(g) = \delta_{jk}\delta_{it}\frac{|G|}{\dim V}.$$

Proof. Let $f : V \mapsto V$ be the linear transformation which in a chosen basis for V is given by the matrix E which is 1 in the $(j,k)^{\text{th}}$ position and 0 elsewhere. So $\tilde{f} \in \hom_{\mathbb{Z}[G]}(V, V)$. Since V is simple, $\hom_{K[G]}(V, V) = K$, and thus, $\hom_{\mathbb{Z}[G]}(V, V) = \mathbb{Z}$. That is, $\tilde{f} = cI$ for some $c \in \mathbb{Z}$.

As above, the $(i,t)^{\text{th}}$ entry of the matrix for $\tilde{f}$ is

$$\sum_{g\in G} \alpha_{ij}(g^{-1})\alpha_{ij}(g^{-1})\alpha_{kt}(g).$$

Now, $\mathrm{Tr}(\tilde{f}) = \mathrm{Tr}(cI) = c\dim V$. On the other hand, by the earlier lemma, $\mathrm{Tr}(\tilde{f}) = |G|\mathrm{Tr}(E)$. Thus,

$$\begin{aligned} c\dim V &= |G|\mathrm{Tr}(E), \\ c &= \frac{|G|\mathrm{Tr}(E)}{\dim V}. \end{aligned}$$

If $j \neq k$ then $\mathrm{Tr}(E) = 0$, so $c = 0$. Also, if $i \neq t$ then the $(i,t)^{\text{th}}$ entry of $\tilde{f}$ is 0, regardless of c. Hence,

$$\sum_{g\in G} \alpha_{ij}(g^{-1})\alpha_{kt}(g) = 0 \quad \text{unless } i = t \text{ and } j = k.$$

When $i = t$ and $j = k$, $\mathrm{Tr}(E) = 1$ so

$$\frac{|G|}{\dim V} = c = \sum_{g\in G} \alpha_{ij}(g^{-1})\alpha_{kt}(g),$$

and in particular, $\dim V \mid |G|$. □

Corollary 4.7.9. *Let V be a simple $K[G]$-module where K is algebraically closed and* $\mathrm{char}\, K = 0$. *Then*

$$\sum_{g\in G} \chi_V(g)\chi_V(g^{-1}) = |G|.$$

Proof. Let $\alpha(g)$ be the matrix for V. Set $s := \dim V$. Then

$$\begin{aligned} \sum_{g\in G} \chi_V(g)\chi_V(g^{-1}) &= \sum_{g\in G}\sum_{t=1}^{s}\sum_{i=1}^{s} \alpha_{tt}(g)\alpha_{ii}(g^{-1}) \\ &= \sum_{t=1}^{s}\sum_{i=1}^{s}\sum_{g\in G} \alpha_{tt}(g)\alpha_{ii}(g^{-1}) \\ &= \sum_{t=1}^{s}\sum_{i=1}^{s} \delta_{it}\frac{|G|}{s} \\ &= s\frac{|G|}{s} \\ &= |G|. \end{aligned}$$

□

If $\operatorname{char} K = 0$, can define an inner product on $\mathrm{Ch}_K(G)$ via

$$\langle \chi_V, \chi_W \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g^{-1}).$$

If K is algebraically closed, have just shown that $\{\chi_V \mid V \text{ simple}\}$ forms an orthonormal set in $\mathrm{Ch}_K(G)$. Since $K[G]$ is semisimple, every rep. is a sum of simple ones, so this is in fact a basis.

In particular:

Corollary 4.7.10. *If* $\operatorname{char} K = 0$ *then*

$$\mathrm{R}_K(G) \cong \mathrm{Ch}_K(G).$$

If K is algebraically closed, then

$$\mathrm{Ch}_K(G) = \mathrm{CF}_K(G).$$

Proof. We have just shown that $\{\chi_V \mid V \text{ simple}\}$ is an orthonormal set in $\mathrm{Ch}_K(G) \subset \mathrm{CF}_K(G)$ (the inner product extends in the obvious way to $\mathrm{CF}_K(G)$). Thus, this set is linearly independent, so $\theta : \mathrm{R}_K(G) \mapsto \mathrm{Ch}_K(G)$ is injective. By construction, $\mathrm{R}_K(G) \mapsto \mathrm{Ch}_K(G)$ is onto, so $\mathrm{R}_K(G) \cong \mathrm{Ch}_K(G)$.

Let $C_1, \ldots, C_r$ be the set of conj. classes of G. $\mathrm{CF}_K(G)$ has a basis $\{f_j : G \mapsto K\}$ where

$$f_j(g) = \begin{cases} 1 & g \in C_j, \\ 0 & g \notin C_j. \end{cases}$$

Hence, the dimension of $\mathrm{CF}_K(G)$ is the number of conj. classes of G, which, we have seen, is the number of simple $K[G]$-modules, ie. the dimension of $\mathrm{R}_K(G)$. □

4.8 Change of Group - Induction and Restriction

Let $H \leq G$, so $K[H] \subset K[G]$. Let N be a rep. of G, $G \times N \mapsto N$. Restricting to H produces an action $H \times N \mapsto N$. Denote the resulting rep. of H by N_H.

Conversely, let M be a rep. of G. Define the **induced** representation of G, denoted M_G, via

$$M^G := K[G] \otimes_{K[H]} M.$$

ie. M^G is generated as a K-module by

$$\{g \otimes m \mid g \in G, m \in M\}$$

where $gh \otimes m \sim g \otimes hm$. The G-action on M^G is defined by

$$g'(g \otimes m) = g'g \otimes m.$$

Let $g_1, \ldots, g_r$ be a set of representatives for the left cosets $\{gH\}$. Then $\{g_j \otimes m\}$ generates M^G. In fact, if K is a field and $m_1, \ldots, m_k$ is a basis for M then

$$\{g_j \otimes m_i \mid 1 \leq j \leq r, 1 \leq i \leq k\}$$

forms a basis for M^G. In particular,

$$\dim M^G = \frac{|G|}{|H|} \dim M,$$

whereas $\dim N_H = \dim N$.

This is a special case of a ground-ring change. A ring homo. $f : R \mapsto S$ induces

$$\begin{aligned} \{S\text{-modules}\} &\overset{P}{\longmapsto} \{R\text{-modules}\} \\ N &\longmapsto N, \end{aligned}$$

where N (on the right) is regarded as an R-module via the action through f. f also induces

$$\begin{aligned} \{R\text{-modules}\} &\overset{Q}{\longmapsto} \{S\text{-modules}\} \\ M &\longmapsto M \otimes_R M. \end{aligned}$$

Q and P are adjoint functors, ie.

$$\hom_S(QM, N) = \hom_R(M, PN) \quad \forall R\text{-mods. } M, S\text{-mods. } N.$$

To see this, given $\alpha : QM = S \otimes_R M \mapsto N$, define $\beta : M \mapsto PN = N$ by

$$\beta(m) = \alpha(1 \otimes m).$$

Then

$$\beta(rm) = \alpha(1 \otimes rm) = \alpha(r \otimes m) = r\alpha(1 \otimes m) = r\beta(m),$$

$\therefore \beta$ is an R-mod. homo.

Conversely, given $\beta : M \mapsto PN$, define $\alpha : QM \mapsto N$ by

$$\alpha(s \otimes m) = s\beta(m).$$

Then

$$\alpha(s'(s \otimes m)) = \alpha(ss' \otimes m) = ss'\beta(m) = s'\alpha(s \otimes m).$$

$\therefore \alpha$ is an S-mod. homo.

In our special case,

$$\hom_{K[G]}(M^G, N) = \hom_{K[H]}(M, N_H).$$

This is called Frobenius Reciprocity.

Also, if $A \leq B \leq C$ then

$$M^C \cong (M^B)^C$$

and

$$N_A \cong (N_B)_A.$$

Let $(\chi_N)_H := \chi_{N_H}$ and $(\chi_M)^G := \chi_{M^G}$ denote the characters of restricted and induced representations. $(\chi_N)_H$ is the composite function

$$H \hookrightarrow G \overset{\chi_N}{\longmapsto} K,$$

ie. $(\chi_N)_H = \chi_N|_H$. To describe $(\chi_M)^G$, let $g_1, \ldots, g_r$ be a set of left coset representatives for GH and let $m_1, \ldots, m_m$ be a basis for V, so that $\{g_i \otimes m_j\}$ is a basis for M^G.

For $g \in G$,

$$g \cdot (g_i \otimes m_j) = gg_i \otimes m_j = g_{i_g} \otimes hm_j,$$

where $gg_i = g_{i_g}h$, $h \in H$, $i_g = 1, \ldots, r$. Letting $\alpha_G(g)$ be the matrix representing the action of g on M^G and $\alpha_H(h)$ be the matrix for the action of h on M, the contribution to $\mathrm{Tr}\alpha_G(g)$ is:

$$\text{coeff. of } (g_i \otimes m_j) \text{ in } g(g_i \otimes m_j) = \begin{cases} 0 & i_g \neq i \\ (\alpha_H(h))_{jj} & i_g = g. \end{cases}$$

Note that if $i_g = i$ then $h = g_i^{-1} g g_i$. So

$$\begin{aligned}
\chi_M^G(g) &= \sum_{i,j} \begin{cases} 0 & g_i^{-1} g g_i \notin H \\ \alpha_H(g_i^{-1} g g_i)_{jj} & g_i^{-1} g g_i \in H \end{cases} \\
&= \sum_i \begin{cases} 0 & g_i^{-1} g g_i \notin H \\ \chi_M(g_i^{-1} g g_i) & g_i^{-1} g g_i \in H \end{cases} \\
&= \sum_i \chi_M(g_i^{-1} g g_i), \quad \text{using the convention } \chi_M(x) = 0 \text{ if } x \notin H \\
&= \frac{1}{|H|} \sum_{x \in G} \chi_M(x^{-1} g x).
\end{aligned}$$

4.9 Examples

4.9.1 $G = S_3$

We have $|G| = 6$.

Conj. class	# conjugates
$\lambda = (3), (1\ 2\ 3)$	2
$\lambda = (2, 1), (1\ 2)$	3
$\lambda = (1, 1, 1), e$	1

We have 3 conjugacy classes, so 3 indecomposable reps. So our dimensions are determined:

$$6 = 1^2 + 1^2 + 2^2.$$

The reps. are:

1. V_1 = trivial rep., $\dim V_1 = 1, \chi_1 = (\overset{(3)}{1}, \overset{(2,1)}{1}, \overset{(3)}{1})$.

2. V_2 = sign rep., $\dim V_2 = 1, \chi_2 = (1, -1, 1)$.

3. V_3 = natural rep., $\dim V_3 = 2$. By orthogonality of characters, $\chi_2 = (2, 0, -1)$. This representation is given on the space
$$\langle x_1, x_2, x_3\rangle / \langle x_1 + x_2 + x_3\rangle$$
by
$$\sigma(\overline{x_i}) = \overline{x_{\sigma(i)}}.$$

Altogether, our character table is

$$\chi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -1 \end{pmatrix}.$$

4.9.2 $G = D_8 = \langle a, b \mid a^4 = b^2 = e, bab^{-1} = a^{-1}\rangle$

$|G| = 8$. We view $G \subset S_4$ via $a = (1\ 2\ 3\ 4), b = (1\ 2)$.

Conj. class	# conjugates
$a = (1\ 3\ 2\ 4)$	2
$a^2 = (1\ 2)(3\ 4)$	1
$b = (1\ 2)$	2
$ab = (1\ 3)(2\ 4)$	2
e	1

The dimensions of the irred. reps. are determined:

$$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2.$$

The 1-dim. reps. are given by $\hom(D_8, S^1)$ where

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

We have

$$\hom(D_8, S^1) = \hom((D_8)_{ab}, S^1) = \hom(C_2 \times C_2, S^1) = \hom(C_2, S^1) \times \hom(C_2, S^1).$$

Dimension	Rep.	
1	V_1	trivial,
1	V_2	$a \cdot v = -v, b \cdot v = v$
1	V_3	$a \cdot v = v, b \cdot v = -v$
1	V_4	$a \cdot v = -v, b \cdot v = -v$
2	V_5	Find character by $\chi_{K[G]}(\sigma) = \begin{cases} \lvert G\rvert, & \sigma = e \\ 0, & \sigma \neq e. \end{cases}$

Our character table (columns indexed by e, a, a^2, b, ab) is:

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & -2 & 0 & 0 \end{pmatrix}.$$

Set $IC_{\langle a\rangle} := (\mathrm{Triv}C_{\langle a\rangle})^{D_8}$, that is, the 2-dimensional representation of D_8 obtained by induction from the trivial representation of the cyclic subgroup generated by a. If v is a basis for the 1-dimensional vector space V for the trivial 1-dimensional vector representation of $C_{\langle a\rangle}$, then a basis for $V^{D_8} = K[D_8] \otimes_{K[\langle a\rangle]} V$ is given $\langle 1 \otimes v, b \otimes v\rangle$, since 1 and b are a set of coset representatives. Since left multiplication by e, a, or a^2 preserve the cosets, in $IC_{\langle a\rangle}$ they are mapped to the identity matrix, while left multiplication by b or ab switches the cosets. Thus the traces are 2 and 0 respectively so $\chi_{IC_{\langle a\rangle}} = (2\ 2\ 2\ 0\ 0)$. Comparing this with the character table gives $IC_{\langle a\rangle} = V_1 + V_3$.

Let $\alpha^{3,1}$ denote the natural 3-dimensional representation of S_4 on $\langle x_1, x_2, x_3, x_4\rangle / \langle x_1 + x_2 + x_3 + x_4\rangle$. $\alpha^{3,1}|_{D_8}$ splits as $W \oplus \alpha^{3,1}|_{D_8}/W$, where $W = \langle w\rangle$ where $w = x_1 + x_2$.. Since $a \cdot w = x_3 + x_4 = -w$ and $b \cdot w = x_2 + x_1 = w$, $W \cong V_2$. and, it is easy to see (using characters or otherwise) that $W \oplus \alpha^{3,1}|_{D_8}/W \cong V_5$, so $\alpha^{3,1}|_{D_8} = V_2 + V_5$.

4.9.3 $G = \mathbb{H}_8$

$\mathbb{H}_8$ is the group of Quaternions. It consists of 8 elements,

$$\pm i, \pm j, \pm k, \pm 1$$

such that $(-1)^2 = 1$, $-1 \in Z(\mathbb{H}_8)$ and

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k, \ jk = i, \ ki = j.$$

Conj. class	# conjugates
$i \sim -i$	2
$j \sim -j$	2
$k \sim -k$	2
-1	1
1	1

The dimensions of the irred. reps. are determined:

$$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2.$$

$\mathbb{H}_8/\langle -1 \rangle \cong C_2 \times C_2$ is abelian, so $(\mathbb{H}_8)_{ab} = C_2 \times C_2$, and thus,

$$\hom(\mathbb{H}_8, S^1) = \hom(C_2, S^1) \times \hom(C_2, S^1).$$

Dimension	Rep.	
1	V_1	trivial
1	V_2	$i \cdot v = -v$, $j \cdot v = v$
1	V_3	$i \cdot v = v$, $j \cdot v = -v$
1	V_4	$i \cdot v = -v$, $j \cdot v = -v$
2	V_5	Find character by $\chi_{K[G]}(\sigma) = \begin{cases} \lvert G \rvert, & \sigma = e \\ 0, & \sigma \neq e. \end{cases}$

Our character table (columns indexed by $1, -1, i, j, k$) is:

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}.$$

The "natural" representation of $\mathbb{H}_8$ on $W = \langle x_1, x_i, x_j, x_k \rangle$ is given by $i \cdot x_1 = x_i$, $i \cdot x_i = -x_1$, $i \cdot x_j = x_k$, $i \cdot x_k = -x_j$, etc. By inspection $\chi_W = (4 \ -4 \ 0 \ 0 \ 0)$, which, from the character table is recognized as $2V_5$. The subspace $\langle x_1 + x_i, x_j + x_k \rangle \subset W$ is closed under the action of $\mathbb{H}_8$ and provides a natural description of V_5.

4.9.4 $G = C_7 \rtimes C_3 = \langle a, b \mid a^7 = e, b^3 = e, bab^{-1} = a^2 \rangle$

Conj. class	# conjugates
a	3
$a^3 \sim a^{-1}$	3
b	7
b^2	7
1	1

$G_{ab} = C_{\langle b \rangle} = C_3$, and thus,

$$\hom(G, S^1) = \hom(C_3, S^1).$$

yielding three 1-dimensional representatives.

The dimensions of the irred. reps. are determined:

$$21 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2.$$

Let $\omega = e^{2\pi i/3}$.

Dimension	Rep.	
1	V_1	trivial
1	V_2	$a \cdot v = v, b \cdot v = \omega v$
1	V_3	$i \cdot v = v, b \cdot v = \omega^2 v$
3	V_4	
3	V_5	

Our character table (columns indexed by $1, a, a^3, b, b^2$) looks like:

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \omega & \omega^2 \\ 1 & 1 & 1 & \omega^2 & \omega \\ 3 & x & y & s & t \\ 3 & x' & y' & s' & t' \end{pmatrix}.$$

for some x, y, s, t, x', y', s', t'.

Using $\chi_{K[G]}(\sigma) = \begin{cases} |G|, & \sigma = e \\ 0, & \sigma \neq e, \end{cases}$ we find that $x' = -(x+1)$, $y' = -(y+1)$, $s' = -s$, $t' = -t$.

Orthogonality of χ_1 and χ_4 gives $7(s+t) = -3 - 3x - 3y$ while orthogonality of the pairs χ_2, χ_4 and χ_3, χ_4 give $7(\omega s + \omega^2 t) = -3 - 2x - 3y$ and $7(\omega^2 s + \omega t) = -3 - 2x - 3y$ respectively. Thus $7(s+t) = 7(\omega s + \omega^2 t) = 7(\omega^2 s + \omega t)$, from which we deduce that $s = t = 0$ and so $(x + y + 1) = -\frac{7}{3}(s+t) = 0$. The inner product of χ_4 with itself gives $|G| = 21 = 9 + 3xy + 3xy$, which combined with $x + y + 1 = 0$ gives $x^2 + x + 2 = 0$, which determines x. Notice that the solution of $x^2 + x + 2 = 0$ satisfies $x = \zeta + \zeta^2 + \zeta^4$, where $\zeta = e^{2\pi i/7}$ and $1 - x = \zeta^3 + \zeta^5 + \zeta^6$,

Thus our character table is Our character table (columns indexed by $1, a, a^3, b, b^2$) looks like:

$$\chi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \omega & \omega^2 \\ 1 & 1 & 1 & \omega^2 & \omega \\ 3 & x & y & 0 & 0 \\ 3 & y & x & 0 & 0 \end{pmatrix}.$$

where $x = \zeta + \zeta^2 + \zeta^4$ and $y = \zeta^3 + \zeta^5 + \zeta^6$.

The representation V_4 is given explicitly by $a \mapsto \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ while V_5 is given by $a \mapsto \begin{pmatrix} \zeta^3 & 0 & 0 \\ 0 & \zeta^6 & 0 \\ 0 & 0 & \zeta^5 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

4.10 Symmetric Polynomials

For a free K-module V, let

$$T^K(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

called the **tensor algebra** on V. Multiplication is defined on $T^K(V)$ by

$$(x_1 \otimes \cdots \otimes x_k)(x_{k+1} \otimes \cdots \otimes x_\ell) = x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_\ell.$$

S_n acts on $V^{\otimes n}$ by permuting factors (called the **position action**), ie.

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Let

$$S(V) := T(V)/\sim$$

where $(x_1 \otimes \cdots \otimes x_n) \sim \sigma \cdot (x_1 \otimes \cdots \otimes x_n)$. This is called the **polynomial (symmetric) algebra** on V.

Example 4.10.1. *If $x_1, \ldots, x_m$ form a basis for V then*

$$S(V) \overset{\cong}{\longmapsto} K[x_1, \ldots, x_m]$$
$$x_{i_1} \otimes \cdots \otimes x_{i_n} \mapsto x_{i_1} \cdots x_{i_n}.$$

Likewise, the **exterior algebra** on V is

$$\Lambda(V) := T(V)/\sim$$

where

$$x_1 \otimes \cdots \otimes x_n \sim (-1)^\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

If $x_1, \ldots, x_m$ is a basis for V then S_m acts on V (on the right) by

$$x_j \cdot \sigma = x_{\sigma^{-1}(j)}.$$

$\therefore$ Get induced action of S_m on $T(V), S(V)$, and $\Lambda(V)$. This is called the **internal action**. Let

$$\Sigma(V) = \mathrm{Fix}^{S_m}(S(V)) = \{a \in S(V) \mid a = a \cdot \sigma \ \forall \sigma \in S_m\}.$$

When K is a field, the isomorphism $S(V) \cong K[x_1, \ldots, x_m]$ takes $\Sigma(V)$ to the ring of symmetric polynomials over K, as defined in Section 3.9. Recall the definition in that section of the elementary symmetric polynomials $s_1, \ldots, s_m$:

$$s_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

By identifying $S(V)$ with $K[x_1, \ldots, x_m]$, we have $s_j \in \Sigma(V)\ \forall j$.

Theorem 4.10.2. $\Sigma(V) \cong K[s_1, \ldots, s_m]$.

If rank$V = m$, write $\Sigma^K[m] = \Sigma^K(V)$.

$$\begin{aligned} K[x_1, \ldots, x_{m+1}] &\mapsto K[x_1, \ldots, x_m] \\ x_j &\mapsto x_j \quad j \leq m \\ x_{m+1} &\mapsto 0 \end{aligned}$$

induces the map

$$\begin{aligned} &\rho_{m+1} : \Sigma[m+1] \mapsto \Sigma[m] \\ &s_k(x_1, \ldots, x_{m+1}) \mapsto s_k(x_1, \ldots, x_m). \end{aligned}$$

Set $\Sigma := \varprojlim_m \Sigma[m]$, the inverse limit of graded rings. That is,

$$\Sigma = \{(a_m \in \Sigma[m])_{m=1}^{\infty} \mid \rho_{m+1}(a_{m+1}) = a_m \ \forall m\}.$$

Σ is a graded ring; the elements of Σ_n are sequences $(f[m])_{m=1}^{\infty}$, where $f[m]$ is a degree n symmetric poly. in m variables, and

$$f[m](x_1, \ldots, x_m) = f[m+1](x_1, \ldots, x_m, 0).$$

$\therefore$ $f[m]$ determines $f[k]$ for all $k \leq m$. However, since each $f[m]$ is of degree n, $f[n]$ determines $f[m]$ $\forall m$. ie. Given

$$f[n] = \rho(s_1(x_1, \ldots, x_n), \ldots, s_n(x_1, \ldots, x_n)),$$

we then have, for any $m \geq n$,

$$f[m] = \rho(s_1(x_1, \ldots, x_m), \ldots, s_n(x_1, \ldots, x_m)).$$

Equivalently, $f[m]$ is obtained from $f[n]$ by "symmetrizing over the m variables".

So, we may identify the sequence $(f[m])$ with the single element $f[n]$. ie. Σ_n has a basis consisting of the symmetric polynomials of degree n in n variables. (Alternatively, Σ_n has a basis consisting of the symmetric polynomials of degree n in m variables, for any $m \geq n$.) So

$$\Sigma \cong K[s_1, s_2, \ldots, s_k, \ldots].$$

Definition 4.10.3. *A **partition** of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of non-negative integers s.t.*

$$n = \lambda_1 + \cdots + \lambda_r.$$

$\lambda \vdash n$ means that λ is a partition of n.

Pick $n \geq 0$, let K be a field and let V be the free module with basis $x_1, \ldots, x_r$. For an unordered partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n, set V^λ to be the $K[S_n]$-submodule of $V^{\otimes n}$ (position action) generated by

$$x_1^{\otimes \lambda_1} \otimes \cdots \otimes x_r^{\otimes \lambda_r}.$$

That is, V^λ is the subspace of $V^{\otimes n}$ with basis

$$\{x_{i_1} \otimes \cdots \otimes x_{i_n} \mid \{i_1, \ldots, i_n\} \text{ contains } \lambda_j \text{ copies of } j\}.$$

Given $A \subset V^{\otimes n}$ a subspace, the **characteristic polynomial** of A is

$$\mathrm{Ch}(A) := \sum_{\lambda \vdash n} d_\lambda x^\lambda$$

where $d_\lambda = \dim(A \cap V^\lambda)$ and $x^\lambda = x_1^{\lambda_1} \ldots x_r^{\lambda_r}$.

It is clear from the definition that

$$\mathrm{Ch}(A \oplus B) = \mathrm{Ch}(A) + \mathrm{Ch}(B)$$
$$\mathrm{Ch}(A \otimes B) = \mathrm{Ch}(A)\mathrm{Ch}(B).$$

If A is closed under the internal action of S_r on V then $\mathrm{Ch}(A)$ is symmetric.

Let P be a projective $K[S_n]$-module, so that $P = K[S_n]e$ for some idempotent $e \in K[S_n]$. For any right $K[S_n]$-module N,

$$N \cong Ne \oplus N(1-e)$$

as vector spaces. Applying this in particular to $V^{\otimes n}$ with the position action,

$$V^{\otimes n} = V^{\otimes n}e \oplus V^{\otimes n}(1-e).$$

Set $P(V) := V^{\otimes n}e$. Then

$$K\text{-vector spaces} \mapsto K\text{-vector spaces}$$
$$V \mapsto P(V)$$

is a functor.

Example 4.10.4. *Suppose $p \nmid n!$. Then letting P be the trivial 1-dimensional rep. of S_n, P is an indecomposable proj. module with idempotent*

$$e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma.$$

We have:

$$\begin{aligned} P(V) &= \operatorname{span}\left\{ \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid v_1, \ldots, v_n \in V \right\} \subset V^{\otimes n} \\ &\cong S(V). \end{aligned}$$

Let $V = \langle x_1, \ldots, x_m \rangle$, ie. V is the vector space with basis $x_1, \ldots, x_m$. Then $\mathrm{Ch}(P(V))$ is a symmetric polynomial in $x_1, \ldots, x_m$ of degree n. In fact, if we let

$$P[m] = \mathrm{Ch}(P(\langle x_1, \ldots, x_m \rangle))$$

then

$$\begin{aligned} \langle x_1, \ldots, x_{m+1} \rangle &\mapsto \langle x_1, \ldots, x_m \rangle \\ x_j &\mapsto x_j \quad j \leq m \\ x_{m+1} &\mapsto 0 \end{aligned}$$

induces by functoriality a map

$$P(\langle x_1, \ldots, x_{m+1} \rangle)) \mapsto P(\langle x_1, \ldots, x_m \rangle))$$

so by applying $\mathrm{Ch}(\cdot)$, we get

$$\begin{aligned} P[m+1] &\mapsto P[m] \\ x_j &\mapsto x_j \quad j \leq m \\ x_{m+1} &\mapsto 0. \end{aligned}$$

ie. $(P[m])$ forms an elt. of $\Sigma^{\mathbb{Z}}$ (symmetric polys. with coeffs. in $\mathbb{Z}$). We write $\mathrm{Ch}(P)$ for this elt. of $\Sigma^{\mathbb{Z}}$. It is determined by the degree n symmetric polynomial $\mathrm{Ch}(P(V))$ in n vars. obtained from

$$V = \langle x_1, \ldots, x_n \rangle.$$

For an arbitrary $K[S_n]$-module P, we can write

$$P = \sum n_j P_j$$

where each P_j is an indecomposable proj. module and $n_j \geq 0$. Set $\mathrm{Ch}(P) := \sum n_j \mathrm{Ch}(P_j)$.

More generally, elements of $K_0(K[S_n])$ are sums

$$\sum n_j P_j$$

with $n_j \in \mathbb{Z}$. So by extending the definition to this case via

$$\mathrm{Ch}(\sum n_j P_j) = \sum n_j \mathrm{Ch}(P_j)$$

yields a homomorphism

$$\mathrm{Ch} : K_0(K[S_n]) \mapsto \Sigma^{\mathbb{Z}}[n].$$

We shall show, for char $K = 0$, that $\mathrm{Ch}(P)$ determines P.

For $n \geq 0$, set

$$\begin{aligned} R_n &:= \text{Underlying group of the representation ring } R(S_n) \\ &= K_0(K[S_n]) \\ &= \mathrm{span}_{\mathbb{Z}}\{\text{simple } K[S_n]\text{-modules}\} \\ &\cong \mathrm{span}_{\mathbb{Z}}\{\text{simple characters of } S_n\} \end{aligned}$$

and set $R = \bigoplus_n R_n$. The map $\mathrm{Ch} : R_n \mapsto \Sigma_n^{\mathbb{Z}}$ for each n yields $\mathrm{Ch} : R_n \mapsto \Sigma^{\mathbb{Z}}$.

Define a ring structure on R as follows: Let M be a $K[S_m]$-module, in R_m, and N a $K[S_n]$-module, in R_n. Set

$$M \cdot N = (M \otimes N)^{S_{m+n}} \in R_{m+n}.$$

ie. $M \otimes N$ is a $(S_m \times S_n)$-module in an obvious way, and $S_m \times S_n \subset S_{m+n}$; $M \cdot N$ is the induced S_{m+n}-module.

Theorem 4.10.5. $\mathrm{Ch} : R \mapsto \Sigma$ *is a ring isomorphism.*

Proof. We know that $\mathrm{Ch}(M \oplus n) = \mathrm{Ch}(M) + \mathrm{Ch}(N)$. We must show that $\mathrm{Ch}(M \cdot N) = \mathrm{Ch}(M)\mathrm{Ch}(N)$. It suffices to consider the case where M is a simple $K[S_m]$-module and N is a simple $K[S_n]$-module. Write $M = K[S_m] \cdot e$, $N = K[S_n] \cdot f$. Then $e \otimes f \in K[S_m] \otimes K[S_n] = K[S_m \times S_n]$, and

$$M \otimes N = K[S_m \times S_n] \cdot (e \otimes f).$$

Now $K[S_m \times S_n] \subset K[S_{m+n}]$ and so

$$M \cdot N = K[S_{m+n}] \cdot (e \otimes f).$$

For any V,

$$\begin{aligned} \mathrm{Ch}(M \cdot N(V)) &= \mathrm{Ch}(V^{\otimes(m+n)} \cdot (e \otimes f)) \\ &= \mathrm{Ch}(V^{\otimes m} \cdot e \otimes V^{\otimes n} \cdot f) \\ &= \mathrm{Ch}(V^{\otimes m} \cdot e)\mathrm{Ch}(V^{\otimes n} \cdot f) \\ &= \mathrm{Ch}(M(V))\mathrm{Ch}(N(V)) \end{aligned}$$

$\therefore \mathrm{Ch}(MN) = \mathrm{Ch}(M)\mathrm{Ch}(N)$. Thus, Ch is a ring homomorphism.

Since $\Sigma = \mathbb{Z}[s_1, s_2, \dots]$, to show Ch is onto, is suffices to show that $s_n \in \mathrm{Im}(\mathrm{Ch})\ \forall n$. Let P be the one-dimensional sign rep. of S_n. ie. $P = \langle w \rangle$ with $\sigma \cdot w = (-1)^{\mathrm{sgn}\sigma} w$. Then $P = K[S_n] \cdot e$ with

$$e = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}\sigma} \sigma,$$

an idempotent. For any vector space V,

$$P(V) = (V^{\otimes n}) \cdot e = \Lambda(V).$$

$\therefore \mathrm{Ch}(P(V)) = \mathrm{Ch}(\Lambda(V)) = s_n$. Thus, Ch is onto.

Claim. For each n, R_n and Σ_n are free abelian groups whose rank equals the number of partitions of n (into positive integers).

Proof of claim. The rank of R_n is equal to the number of non-isomorphic simple $K[S_n]$-reps, which is equal to the number of conjugacy classes in S_n. Each conjugacy class is determined by its cycle type, which is a partition of n (by Corollary 1.6.3). Moreover, it is obvious that every partition of n is the cycle type of some element in S_n. Thus, the rank of R_n is equal to the number of partitions of n.

For Σ_n, this follows from the fact that $\Sigma = \mathbb{Z}[s_1, s_2, \dots]$ and the degree of s_k is k. ie. A basis for Σ_n consists of monomials in $\{s_k\}$ of total degree n, and since $\deg s_k = k$, each such monomial corresponds to a partition of n via

$$(\lambda_1, \dots, \lambda_r) \leftrightarrow s_{\lambda_1} \dots s_{\lambda_r}.$$

□

Since Ch is one-to-one, this claim shows that Ch is also onto, whence an isomorphism. □

4.10.1 Other Bases for Σ_n

There are 6 bases for $\Sigma_n^{\mathbb{Q}}$ in "common" use, of which 5 form bases in $\Sigma_n^{\mathbb{Z}}$. All bases are indexed by partitions λ of n.

1. Elementary Symmetric Functions

$$s_\lambda = s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_r}.$$

eg.

$$s_{(2)} = x_1 x_2, \qquad s_{(1,2)} = (x_1 + x_2)^2.$$

2. Monomial Basis

$$m_\lambda = \text{symmetrization of } x_1^{\lambda_1} x_2^{\lambda_2} \dots x_r^{\lambda_r}.$$

eg.

$$m_{(2)} = x_1^2 + x_2^2, \qquad m_{(1,1)} = x_1 x_2.$$

3. Homogeneous Functions

Let

$$h_k = \sum_k \text{monomials of degree } k.$$

Then

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}.$$

eg.

$$h_{(2)} = x_1^2 + x_1 x_2 + x_2^2, \qquad h_{(1,1)} = (x_1 + x_2)^2.$$

4. Power Functions

Let

$$\psi_k = x_1^k + x_2^k + \cdots + x_n^k.$$

Then

$$\psi_\lambda = \psi_{\lambda_1} \psi_{\lambda_2} \cdots \psi_{\lambda_r}.$$

eg.

$$\psi_{(2)} = x_1^2 + x_2^2 \qquad \psi_{(1,1)} = (x_1 + x_2)^2.$$

5. Schur Functions

For $\mu = (\mu_1, \dots, \mu_n)$, with $\mu_j \geq 0 \ \forall j$, let

$$\begin{aligned} V_\mu &:= \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}\sigma} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n} \\ &= \det \begin{pmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{pmatrix}. \end{aligned}$$

In particular,

$$V_{(n-1,n-2,\dots,1,0)} = \prod_{i<j} (x_i - x_j),$$

called the Vandermonde determinant. For the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n (with $\lambda_j = 0$ allowed),

$$F_\lambda := \frac{V_{\lambda+(n-1,\ldots,1,0)}}{V_{(n-1,\ldots,1,0)}}.$$

eg.

$$\begin{aligned}
F_{(2)} = F_{(2,0)} &= \frac{\begin{vmatrix} x_1^3 & 1 \\ x_2^3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} \\
&= \frac{x_1^3 - x_2^3}{x_1 - x_2} \\
&= x_1^2 + x_1 x_2 + x_2^2, \\
F_{(1,1)} &= \frac{\begin{vmatrix} x_1^2 & x_1 \\ x_2^2 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} \\
&= \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} \\
&= x_1 x_2.
\end{aligned}$$

Note:

(a) $x_i = x_j \Rightarrow V_\mu = 0$. Thus, $V_{\lambda+(n-1,\ldots,1,0)}$ is divisible by $V_{(n-1,\ldots,1,0)}$, and so F_λ is a polynomial.

(b) Interchanging x_i, x_j multiplies both numerator and denominator by -1, so F_λ is symmetric.

6. Forgotten Basis
Let $m_\lambda = p(s_1, \ldots, s_k)$ be the expansion for m_λ in the elem. symmetric polys. Then

$$f_\lambda = p(h_1, \ldots, h_k).$$

eg. For $\lambda = (2)$,

$$\begin{aligned}
m_(2) &= x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = s_1^2 - 2s_2, \\
\therefore\ f_{(2)} &= h_1^2 - 2h_2 = (x_1 + x_2)^2 - 2(x_1^2 + x_1x_2 + x_2^2) = -x_1^2 - x_2^2.
\end{aligned}$$

For $\lambda = (1,1)$,

$$m_{(1,1)} = x_1 x_2 = s_2$$
$$\therefore \quad f_{(1,1)} = h_2 = x_1^2 + x_1 x_2 + x_2^2.$$

We know that 1 forms a basis for $\Sigma_n^{\mathbb{Q}}$ and it is trivial to see that 2 does. We have to prove that the others do.
Note: $\{\psi_\lambda\}$ does not form a basis for $\Sigma_n^{\mathbb{Z}}$. eg. $s_2 = \frac{1}{2}(\psi_{(1,1)} - \psi_{(2)})$ in $\sigma_n^{\mathbb{Q}}$, so

$$s_2 \notin \mathbb{Z}[\psi_1, \psi_2, \psi_3, \dots].$$

Generating Functions for s_n, h_n, ψ_n

The first three of our bases are defined as monomials in some other symmetric functions. Set

$$S(t) := \sum_{n=0}^{\infty} s_n t^n$$
$$H(t) := \sum_{n=0}^{\infty} h_n t^n$$
$$\Psi(t) := \sum_{n=0}^{\infty} \psi_n t^n.$$

By expanding and examining the coefficient of t^n, we see that

$$
\begin{aligned}
S(t) &= \prod_{j=1}^{\infty}(1 + x_j t), \\
H(t) &= \prod_{j=1}^{\infty}(1 + x_j t + x_j^2 t^2 + x_j^3 t^3 + \cdots) = \prod_{j=1}^{\infty} \frac{1}{1 - x_j t}, \\
\Psi(t) &= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} x_j^n t^{n-1} \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} x_j^n t^{n-1} \\
&= \sum_{j=1}^{\infty} \frac{x_j}{1 - x_j t} \\
&= \sum_{j=1}^{\infty} -\frac{d}{dt} \log(1 - x_j t) \\
&= \frac{d}{dt} \log\left(\prod_{j=1}^{\infty} \frac{1}{1 - x_j t}\right) \\
&= \frac{d}{dt} \log(H(t)) \\
&= \frac{H'(t)}{H(t)}.
\end{aligned}
$$

Thus,

$$S(t)H(-t) = 1 \tag{1}$$

$$\Psi(t) = \frac{H'(t)}{H(t)} \tag{2}$$

$$\Psi(-t) = \frac{H'(-t)}{H(-t)} = \frac{S'(t)}{S(t)} \tag{3}$$

(1) implies that

$$s_0 h_0 = 1$$

$$\sum_{j=0}^{n} (-1)^j s_j h_{n-j} = 0 \quad n > 0 \tag{1'}$$

Define $\omega : \Lambda^{\mathbb{Z}} = \mathbb{Z}[s_1, s_2, \dots] \mapsto \Lambda^{\mathbb{Z}}$ by $\omega(s_j) = h_j$. Since (1') is symmetrical in h, s, we get that ω is an isomorphism. In particular, $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$, and so the homogeneous functions form a basis.

Applying ω to (1') gives

$$\begin{aligned} 0 &= \sum_{j=0}^{n} (-1)^j h_j \omega(h_{n-j}) \\ &= \sum_{j=0}^{n} (-1)^{n-j} h_{n-j} \omega(h_j) \\ &= (-1)^n \sum_{j=0}^{n} (-1)^j \omega(h_j) h_{n-j} \quad \forall n > 0. \end{aligned}$$

Comparing with (1'), we see that $\omega(h_n) = s_n$, ie. $\omega^2 = 1$ (ω is an involution).

By (2),

$$\sum_{n=1}^{\infty} n h_n t^{n-1} = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \psi_j h_{n-j} t^{n-1}$$

$$\sum_{j=1}^{n} \psi_j h_{n-j} = n h_n \quad \forall n. \tag{2'}$$

Similarly,

$$\sum_{j=1}^{n} (-1)^{j-1} \psi_j s_{n-j} = n s_j \quad \forall n. \tag{3'}$$

Using (3'), each s_n can inductively be written as a polynomial in $\mathbb{Q}[\psi_1, \dots, \psi_n]$, so the power functions form a basis for $\Lambda^{\mathbb{Q}}$.

Since ω interchanges h, s, comparing (2') and (3') gives

$$\omega(\psi_n) = (-1)^{n-1} \psi_n.$$

To see that the Schur Functions form a basis for $\Lambda_n^{\mathbb{Z}}$, set $V_n := V_{(n-1,\dots,0)}$. Let A_k be the set of skew symmetric polynomials of degree k in n variables. Then we have an isomorphism

$$\begin{aligned} \Lambda_n &\mapsto A_{n+\binom{n}{2}} \\ f &\mapsto f V_n \end{aligned}$$

Since $\{F_\lambda V_n\}$ is the "monomial" basis for $A_{n+\binom{2}{2}}$ (ie. the basis obtained by skew symmetrizing each monomial), $\{F_\lambda\}$ forms a basis for Λ_n.