Chapter 4

Representations of Groups

4.1 Definitions and Elementary Properties

Let $G$ be a group and $K$ a commutative ring.
A (linear) representation of $G$ consists of a $K$-module $V$ and an action $G \times V \mapsto V$ satisfying

$$g \cdot (av + bw) = ag \cdot v + bg \cdot w \quad \forall g \in G, \; a, b \in K, \; v, w \in W.$$  

Equivalently, a rep. is a group homomorphism $G \mapsto \text{Aut}_K(V)$.

Another formulation: Define a ring $K[G]$, called the group ring, as follows. As an abelian group,

$$K[G] = \{\text{free } K\text{-module with basis } G\}.$$  

Multiplication is determined by $g \cdot h = gh$ (the left defines multiplication in $K[G]$; the right is multiplication in $G$). Then a rep. of $G$ on $V$ is a ring homomorphism $K[G] \mapsto \text{End}_K(V)$. This makes $V$ a left $K[G]$-module.

Note that as rings,

$$K[G \times H] = K[G] \otimes_Z K[H].$$  

$K[G]$ is commutative $\iff G$ is abelian.

Let $G$ be finite. For a conjugacy class $C$, let

$$N_C := \sum_{x \in C} x \in K[G].$$  

Definition 4.1.1. Let $R$ be a ring. The center of $R$ is

$$Z(R) = \{a \in R \mid ax = xa \ \forall x \in R\}.$$  

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Proposition 4.1.2. If $G$ is finite then $Z(K[G])$ is the free $K$-module

$$KN_{C_1} \oplus \cdots \oplus KN_{C_k},$$

where $C_1, \ldots, C_k$ are the conjugacy classes of $G$.

Proof. For $g \in G$, and $C = C_j$,

$$g^{-1}N_C g = \sum_{x \in C} g^{-1}xg = \sum_{y \in g^{-1}Cg} y = N_C,$$

since $g^{-1}Cg = C$. Thus,

$$\bigoplus_{j=1}^k KN_j \subset Z(K[G]).$$

Conversely, let

$$x = \sum_{g \in G} a_gg \in Z(K[G]).$$

Then for all $h \in G$,

$$\sum_{g \in G} a_gg = x = h^{-1}xh = \sum_{g \in G} a_g h^{-1}gh = \sum_{t \in G} a_{hth^{-1}}t.$$

\[\therefore a_g = a_{hgh^{-1}} \forall h, g\text{. ie. All elements of a given conjugacy class have the same coefficient in } x.\text{ Thus,}\]

$$x = \sum a_jN_{C_j}$$

where $a_j = a_g$ for any $g \in C_j$. So $x \in \bigoplus_{j=1}^k KN_{C_j}$. \hfill \square

4.1.1 New Representations from Old

1. Direct sum of reps.

Given reps.

$$G \times V \mapsto V \quad G \times W \mapsto W,$$
form rep. of $G$ on $V \oplus W$ by
\[ g \cdot (v, w) = (g \cdot v, g \cdot w). \]

eg. If $K$ is a field, $n = \dim V, m = \dim W$, then for $g \in G$, $\rho(g) \in GL_n(K), \tau(g) \in GL_m(K)$. The direct sum action is given by
\[
\begin{pmatrix}
\rho(g) & 0 \\
0 & \tau(g)
\end{pmatrix}^\times k.
\]

Note: Sometimes write $kV$ for $V \oplus \cdots \oplus V$.

2. Tensor product of reps.

Given reps.
\[
G \times V \mapsto V \quad G \times W \mapsto W,
\]
form rep of $G$ on $V \otimes_k W$ determined by
\[ g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w). \]

This is the tensor product of $V$ and $W$ in the Hopf alg. sense. ie. The action is
\[
K[G] \otimes V \otimes W \mapsto K[G] \otimes K[G] \otimes V \otimes W \mapsto K[G] \otimes V \otimes K[G] \otimes W \mapsto V \otimes W,
\]
where $\psi(g) = g \otimes g$ is induced by the diagonal map $G \mapsto G \times G$.

Let $R$ be a ring. Recall that an $R$-module $V$ is simple if it has no proper $R$-submodules except 0. In this context, such modules will often be called **irreducible**.

**Definition 4.1.3.** An $R$-module $V \neq 0$ is called **indecomposable** if $\nexists$ $R$-modules $V_1 \neq 0, V_2 \neq 0$ s.t. $V \cong V_1 \oplus V_2$.

When $R = K[G]$, we talk of “indecomposable reps.” and “irreducible” (or “simple”) reps.

Clearly, irreducible $\Rightarrow$ indecomposable. The reverse is not true. eg. Suppose $K$ is a field. If the action of each elt. $g$ of $G$ has the form
\[
\begin{pmatrix}
P(g) & Q(g) \\
0 & R(g)
\end{pmatrix},
\]
(where $P(g)$ is $n \times n$, $Q(g)$ is $n \times m$, $R(g)$ is $m \times m$), then $\exists$ an $n$-dim. subrepresentation $g \mapsto P(g)$, so not irreducible. But it might still be indecomposable if $Q(g) \neq 0$. In particular, take $G = \mathbb{Z}, n = m = 1$, let $P(k) = R(k) = 1$ and $Q(k) = k$ for all $k \in \mathbb{Z}$.

**Goal:** Let $G$ be finite, $K$ a field.
1. Show that there is (up to iso.) a finite list $V_1, \ldots, V_k$ of indecomposable $K[G]$-modules and find them.

2. Given a rep. $V$ of $G$, show that the decomposition

$$V \cong V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_k^{n_k}$$

into irreducible is unique, and give a method of determining the mult. $n_k$ of each $V_k$.

3. In particular, find the decomposition

$$K[G] \cong V_1^{m_1} \oplus V_2^{m_2} \oplus \cdots \oplus V_k^{m_k}.$$

*Question.* Given $G$, to what extent does this answer change with $K$? Does it depend on more that just char $K$?
4.2 Semisimple Rings

Note: Unless otherwise noted, module means left module.

**Definition 4.2.1.** An R-module V is called **semisimple** if it is a direct sum of simple modules. R is called a **semisimple ring** if R is semisimple as a (left) R-module.

**Proposition 4.2.2.** If V is a semisimple module and U ⊂ V then V ≅ U ⊕ W for some W.

**Proof.** Consider $S = \{\text{submodules } U' \subset V \text{ s.t. } U \cap U' = 0\}$.

By Zorn’s lemma, let $W \subset V$ be maximal s.t. $U \cap W = 0$. If $U \oplus W \nsubseteq V$, choose $v \notin U \oplus W$. Write $v = v_1 + \cdots + v_n$, where $v_i \in V_j$ and $V_j \subset V$ is simple. Then $v_j \notin U \oplus W$ for some $j$. So

$V_j \cap (U \oplus W) \nsubseteq V_j$

and since $V_j$ is simple,

$V_j \cap (U \oplus W) = 0$.

But then $U \cap (W \oplus V_j) = 0$, so W is not maximal. Thus, $V = U \oplus W$. □

**Definition 4.2.3.** V is **completely splittable** if $U \subset V \Rightarrow V = U \oplus W$ for some W.

So V is semisimple $\Rightarrow$ V is completely splittable. Recalling Proposition 2.4.6, we see that V is completely splittable $\iff$ whenever $U \subset V$, if $i : U \hookrightarrow V$ is the inclusion then $\exists \sigma : V \twoheadrightarrow U$ a homo. s.t. $\sigma i = 1_U$; $\sigma$ is called a splitting of $i$.

**Example 4.2.4.** Let K be a field, $R = M_{nn}(K)$,

\[
V = \begin{pmatrix} * & 0 & \cdots & 0 \\ : & : & : & : \\ * & 0 & \cdots & 0 \end{pmatrix}.
\]

**Claim.** V is a simple R-module.

**Proof of claim.** Suppose $0 \nsubseteq W \subset V$. Let

\[
0 \neq x = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ : & : & : & : \\ x_n & 0 & \cdots & 0 \end{pmatrix} \in W,
\]

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and suppose $x_j \neq 0$. Then $W$ contains
\[
\begin{pmatrix}
0 & \cdots & 1 & \cdots & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \cdots & 0
\end{pmatrix}
x = \begin{pmatrix}
x_j & 0 & \cdots & 0 \\
0 & \vdots & & \vdots \\
0 & \cdots & 0
\end{pmatrix}.
\]

By dividing by $x_j$, $W$ contains
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \vdots & & \vdots \\
0 & \cdots & 0
\end{pmatrix}.
\]

Thus, $W$ contains
\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
1 & \vdots \\
\vdots \\
0 & \cdots & 0
\end{pmatrix},
\]

which is a basis of $V$. Hence $W = V$. □

Now,
\[
R = \begin{pmatrix}
* & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
* & 0 & \cdots & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & * & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & * & 0 & \cdots & 0
\end{pmatrix} \oplus \cdots \oplus \begin{pmatrix}
0 & \cdots & 0 & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & *
\end{pmatrix}.
\]

So $R$ is semisimple.

**Proposition 4.2.5.** $Z(M_{n\times n}(K)) = KI$.

**Proof.** Exercise. □

Let $R$ be a ring, $x, y \in R$. The **commutator** of $x$ and $y$ is
\[
[x, y] := xy - yx \in R.
\]

The **commutator subspace** is
\[
[R, R] = \{[x, y] \mid x, y \in R\}.
\]

Note: $[R, R]$ is not an $R$-submodule of $R$, in general.
Example 4.2.6. Let $R = M_{n \times n}(K)$. Then

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
- 
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

= 
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}

Similarly, denoting by $e^{ij}$ the matrix with $1$ in the $(i, j)^{th}$ position and $0$ elsewhere,

$e^{ij} \in [R, R] \ \forall i \neq j$.

Also,

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
- 
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

= 
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
\end{pmatrix}

Similarly,

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & 0 & \cdots \\
\vdots & \ddots & -1 & \cdots \\
0 & \cdots & 0 & \ddots \\
\end{pmatrix}
\in [R, R].
\]

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These matrices generate $\{ M \mid \text{Tr}M = 0 \}$ as a vector space. That is,
\[ [R, R] = \ker \text{Tr} : R \mapsto K. \]
In particular, $\dim [R, R] = n^2 - 1$ and $\dim (R/\ker \text{Tr}) = 1$.

**Lemma 4.2.7.** If $V$ is completely splittable and $U \subset V$ then $U$ is completely splittable.

**Proof.** Let $T \subset U$. Then

\[ i \quad T \quad \xrightarrow{j} \quad U \quad \xrightarrow{1_T} \quad V \quad \xrightarrow{\exists \sigma} \quad T \]

Since $V$ is completely splittable, $\exists \sigma$ s.t. $\sigma \circ ji = 1_T$, as in the diagram. So $\sigma \circ j$ is a splitting of $i$. □

**Theorem 4.2.8.** If $V$ is completely splittable then every submodule of $V$ is semisimple.

**Corollary 4.2.9.** $V$ is semisimple $\iff V$ is completely splittable.

**Corollary 4.2.10.** If $V$ is semisimple then every submodule of $V$ is semisimple.

**Proof of Theorem.** Let $U \subset V$. Consider sets $\{S_i\}_{i \in I}$ of simple $U$-submodules which are “linearly independent”, ie.
\[ \langle S_i \rangle_{i \in I} = \bigoplus_{i \in I} S_i. \]

By Zorn’s lemma, there is a maximal such set, $\{S_i\}_{i \in I}$. Let
\[ S = \bigoplus_{i \in I} S_i. \]

By the lemma, $S$ is completely splittable, so $\exists T$ s.t. $U = S \oplus T$. If $T \neq 0$, pick $0 \neq x \in T$. By Zorn’s lemma,
\[ \{T' \subset T \mid x \notin T'\} \]
has a maximal element, $T_0$. By the lemma, let $T_1$ be s.t.
\[ T = T_0 \oplus T_1. \]

If $T_1$ is not simple then let $T_1 = A \oplus B$. $x$ can’t be in both $T_0 \oplus A$ and $T_0 \oplus B$ since their intersection is $T_0$. This contradicts the maximality of $T_0$, so $T_1$ is simple.

But $T_1$ can be added to $\{S_i\}_{i \in I}$ to get a still-linearly independent set of simple submodules. This contradicts the maximality of $\{S_i\}_{i \in I}$.

So $T = 0$ and $U = S = \bigoplus_{i \in I} S_i$ is semisimple. □
Corollary 4.2.11. Let

$$0 \to U \to V \to W \to 0$$

be a short exact sequence of $R$-modules. If $V$ is semisimple then $U, W$ are semisimple.

Proof. $V \cong U \oplus W$, so $U \subset V$ and $V$ has a submodule isomorphic to $W$. So by the theorem, $U$ and $W$ are semisimple. □

Theorem 4.2.12 (Maschke). If $K$ is a field, $G$ a finite group s.t. $\text{char } K \nmid |G|$ then $K[G]$ is semisimple.

Proof. Write $V = K[G]$, as a module over itself. Suppose $U$ is a $K[G]$-submodule, and show that there exists a $K$-module splitting $p : V \mapsto U$.

As vector spaces, $\exists U_0$ s.t. $V \cong U \oplus U_0$ ($U_0$ is not necessarily a $K[G]$-module). This yields a linear map $p_0 : V \mapsto U$ ($p_0$ is not necessarily a $K[G]$-homomorphism).

Define $p : V \mapsto U$ by

$$p(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gv).$$

Then for $g' \in G$,

$$p(g'v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gg'v)$$

$$= \frac{1}{|G|} \sum_{f \in G} g' f^{-1} p_0(fv)$$

$$= g' \frac{1}{|G|} \sum_{f \in G} f^{-1} p_0(fv)$$

$$= g' p(v).$$

So $p$ is a $K[G]$-homomorphism.

Also, if $u \in U$ then

$$p(u) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p_0(gu)$$

$$= \frac{1}{|G|} \sum_{g \in G} g^{-1}(gu)$$

$$= \frac{1}{|G|} \sum_{g \in G} u$$

$$= u.$$

∴ $p$ is a splitting. □
Note: If $K$ is not a field then the same proof works, provided $|G|$ is invertible in $R$ and $\exists$ an $R$-module splitting $p_0 : V \mapsto U$.

Let $R$ be a semisimple ring,

$$R \cong \bigoplus_{i \in I} V_i$$

where each $V_i$ is simple.

**Proposition 4.2.13.** Every simple $R$-module appears (up to isomorphism) as $V_i$ for some $i$.

**Proof.** Let $W$ be a simple $R$-module and $0 \neq w \in W$. Then

$$R \xrightarrow{\phi} W$$

$$1 \mapsto w$$

is not zero, so it is onto (since $W$ is simple).

So $W$ is a summand of $R$. Furthermore,

$$0 \neq \phi \in \text{hom}_R(R, W) \cong \bigoplus_{i \in I} \text{hom}_R(V_i, W),$$

so $\text{hom}_R(V_i, W) \neq 0$ for some $i$. But any non-zero homo. between simple $R$-modules is an isomorphism, so $W$ is isomorphic to some $V_i$. \qed

**Proposition 4.2.14.** Let $R$ be semisimple, $I \subset R$ a left ideal. Then $\exists$ an idempotent $e \in R$ s.t. $I = Re$.

**Note:** If $V$ is a simple $R$-module then $Rv = V$, for any $v \in V$. Moreover, since $R$ is semisimple, $R \mapsto Rv$ splits, so $V$ is isomorphic to a left ideal of $R$.

**Proof.** Since $R$ is semisimple, $\exists J$ s.t. $R = I \oplus J$. Write $1 = e + f$ where $e \in I, f \in J$.

$e \in I \implies Re \subset I$. Conversely, given $x \in I$, $x = xe + xf$. $xf = x - xe \in I$ and since $f \in J, xf \in J$.

Thus

$$xf \in I \cap J = 0 \implies x = xe.$$  

\therefore $I = Re$.

Now, $x = xe \ \forall x \in I$, and in particular, $e = e^2$. \qed

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4.3 Artinian Rings

Recall that an $R$-module $V$ is Noetherian if for any chain
\[ V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \]
of submodules, $\exists N$ s.t. $V_n = V_N \ \forall n \geq N$. Likewise:

**Definition 4.3.1.** An $R$-module $V$ is **Artinian** if for any chain
\[ V_0 \supset V_1 \supset \cdots \supset V_n \supset \cdots \]
of submodules, $\exists N$ s.t. $V_n = V_N \ \forall n \geq N$. $R$ is an **Artinian ring** if $R$ is Artinian as a (left) $R$-module.

**Example 4.3.2.** $\mathbb{Z}$ is Noetherian (in fact, it is a PID) but not Artinian, since we have:
\[ 2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset \cdots \supset 2^n\mathbb{Z} \supset \cdots . \]

When $G$ is finite and $K$ is a field, $K[G]$ is both Noetherian and Artinian (by counting dimensions, can’t have a strictly increasing chain longer than $|G| + 1$).

**Proposition 4.3.3.** Let
\[ 0 \to U \to V \to W \to 0 \]
be a short exact sequence of $R$-modules. Then $V$ is Noetherian (respectively Artinian) $\iff U, W$ are Noetherian (resp. Artinian).

**Corollary 4.3.4.** If
\[ V = \bigoplus_{i=1}^n V_i \]
then $V$ is Noetherian (resp. Artinian) $\iff V_i$ is Noetherian (resp. Artinian) $\forall i$.

**Proposition 4.3.5.** If
\[ V = \bigoplus_{i \in I} V_i \]
with $V_i \neq 0$ and $V$ is finitely generated then $|I| < \infty$.

**Proof.** Each generator has only finitely many non-zero components. \hfill $\Box$

**Corollary 4.3.6.** If $V$ is finitely generated and semisimple then $V$ is both Noetherian and Artinian.
Proof. By the hypothesis,
\[ V = \bigoplus_{i=1}^{n} V_i \]
where each \( V_i \) is simple. So for each \( i \), the only chain is \( 0 \subset V_i \). Thus \( V_i \) is both Noetherian and Artinian.

\[ \square \]

**Corollary 4.3.7.** If \( R \) is semisimple then \( R \) is both Noetherian and Artinian.

**Proof.** As an \( R \)-module, \( R \) is generated by the single element 1.

\[ \square \]

**Proposition 4.3.8.** Let \( G \) be finite, \( K \) Noetherian (resp. Artinian). Then \( K[G] \) is Noetherian (resp. Artinian).

**Proof.** If \( K \) is Noetherian (or Artinian) then, as a \( K \)-module, so is \( K^{|G|} \), which is isomorphic, as a \( K \)-module, to \( K[G] \). But every \( K[G] \)-submodule of \( K[G] \) is a \( K \)-submodule, so if \( K[G] \) is Noetherian (or Artinian) as a \( K \)-module then it has the same property as a \( K[G] \)-module.

\[ \square \]

**Lemma 4.3.9** (Schur). Let \( V \) be a simple \( R \)-module. Then:

1. \( \text{End}_R(V) \) forms a division ring.

2. If \( R \) is a finite dimensional algebra (eg. \( R = K[G] \) with \( G \) finite) over an algebraically closed field \( K \) then \( \text{End}_R(V) \cong K \).

**Proof.**

1. If \( f : V \mapsto V \) is nonzero then \( \text{Im} f = V \) so \( V \) is onto. Also, since \( f \neq 0 \), \( \ker f \neq V \), so \( \ker f = 0 \). Hence \( f \) is an isomorphism, so it has an inverse. ie. \( \text{End}_R(V) \) is a division ring.

2. Let \( f \in \text{End}_R(V) \), and show \( f = \lambda I \) for some \( \lambda \in K \). For any \( 0 \neq x \in V \), \( Rx \) forms a finite dimensional subspace of \( V \) (its dimension is \( \leq \dim R \)). Since \( V \) is simple, \( Rx = V \), so \( V \) is finite dimensional.

So \( \exists \) an eigenvector \( 0 \neq v \in V \) for \( f \), so that \( fv = \lambda v \). Since \( V \) is simple, \( Rv = V \). Hence \( \forall w \in V, w = rv \) so
\[ f(w) = rf(v) = \lambda rv = \lambda w. \]

ie. \( f = \lambda I \).

\[ \square \]
4.4 Wedderburn’s Theorem

Let $R$ be semisimple. Then

$$R \cong n_1 V_1 \oplus n_2 V_2 \oplus \cdots \oplus n_k V_k$$

where $V_1, \ldots, V_k$ is the list of simple $R$-modules (one from each isomorphism class). (This is a finite decomposition by Proposition 4.3.5)

For a ring $A$,

$$A \xrightarrow{\phi} \text{End}_A(A)$$

is a bijection, since every endomorphism $f$ is equal to $\phi_{f(1)}$. Then

$$\phi_a \phi_b(1) = \phi_a(b) = ba = \phi_{ba}(1).$$

$\therefore \phi$ is a ring isomorphism

$$\phi : A^{\text{opp}} \mapsto \text{End}_A(A),$$

where $A^{\text{opp}}$ is the ring with the same group structure as $A$ but $a(\cdot^{\text{opp}})b = ba$.

Set $D_j = \text{End}_R(V_j)$, a division ring. Then

$$R \cong (\text{End}_R(R))^{\text{opp}}$$

$$\cong \prod_{j=1}^{k} (\text{End}_R(n_j V_j))^{\text{opp}}$$

$$\cong \prod_{j=1}^{k} (M_{n_j \times n_j}(\text{End}_R(V_j))^{\text{opp}}$$

$$\cong \prod_{j=1}^{k} (M_{n_j \times n_j}(D_j))^{\text{opp}}$$

$$\cong \prod_{j=1}^{k} M_{n_j \times n_j}(D_j^{\text{opp}})$$

where on the last line, the isomorphism is given by the transpose map.
Under this isomorphism, $V_j \subset n_j V_j \subset R$ corresponds to
\[
\text{hom}_R(n_j V_j, V_j) \cong \begin{pmatrix}
* & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
* & 0 & \cdots & 0
\end{pmatrix} \subset M_{n_j \times n_j}(D_j^{\text{opp}}).
\]

In particular, suppose $R$ is an algebra over a field $K$ and $D_j \cong K$ (e.g. if $K$ is algebraically closed). Then:

1. $\dim V_j = n_j$ for each $j$.
2. $\dim R = \sum_{j=1}^{k} n_j^2$.

**Example 4.4.1.**

1. $R = \mathbb{C}(S_2)$

   By Maschke’s Theorem, $\operatorname{char} K = 0 \Rightarrow K[G]$ is semisimple. Here,

   \[
   2 = 1^2 + 1^2
   \]

   and there are no other possibilities, so $R$ has 2 indecomposable reps., each on a 1-dimensional space.

   They are: Let $\dim V = 1$ with basis $v$. $S_2 = \{e, T\}$, with $T^2 = e$. The trivial rep. is:

   \[
   e \cdot v = v, \quad T \cdot v = v.
   \]

   The sign rep. is:

   \[
   e \cdot v = v, \quad T \cdot v = -v.
   \]

2. $R = \mathbb{C}(S_3)$

   Either

   \[
   6 = 1^2 + 1^2 + \cdots + 1^2 \quad \text{or} \quad 6 = 1^2 + 1^2 + 2^2.
   \]

   Easy to see that the trivial rep. and the sign rep. ($\sigma \cdot v = (-1)^{\operatorname{sgn} \sigma} v$, $\operatorname{sgn}$ is the homomorphism $\epsilon : S_n \mapsto \{1, -1\}$ used to define $A_n$ in section 1.6.2) are the only possible reps. of $R$ on a 1-dim. $V$.

   Hence $R$ has 3 indecomposable reps.: trivial rep., sign rep., a 2-dim. rep.
The 2-dim. rep. is:

\[(1 \ 2) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \]
\[(1 \ 3) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
\[(1 \ 2 \ 3) = (1 \ 2)(1 \ 3) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.\]

3. \( R = \mathbb{C}(S_4) \)

\[24 = 1^2 + 1^2 + (a)^2 + \cdots + ( )^2, \quad a \geq 2. \]

Looking at congruence \( \mod 4 \), need \( 3^2 \) (\( 2^2, 4^2 \) are divisible by \( 4 \), but \( 24 - 1^2 - 1^2 \) is not). Hence, the only possibility is

\[24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2.\]

ie. two 1-dim. reps., one 2-dim. rep., two 3-dim. reps.

**Theorem 4.4.2.** Let \( G \) be a finite group, \( K \) an algebraically closed field of characteristic 0. Then the number of isomorphic simple \( K[G] \)-modules is equal to the number of conjugacy classes of \( G \).

**Proof.** As seen earlier,

\[Z(K[G]) = \text{Free } K\text{-module on } \left\langle \sum_{g \in C} g \mid C \text{ a conj. class} \right\rangle.\]

So the number of conjugacy classes is equal to \( \dim Z(K[G]) \). Also,

\[K[G] = \prod_{j=1}^{k} M_{n_j \times n_j}(K).\]

Now, \( Z(M_{n \times n}(K)) = KI \), which has dimension 1. Thus,

\[\dim Z(K[G]) = k = \# \text{ nonisomorphic simple } K[G]\text{-modules}.\]

\( \square \)
4.5 Changing the Ground Ring

Example 4.5.1. Let $G = C_3 = \{e, t, t^2\}$, $K = \mathbb{R}$. Then using $V = \mathbb{R}^2$,

$$
\rho : K[G] \hookrightarrow M_2(\mathbb{R})
$$

$$
t \mapsto \begin{pmatrix}
\cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} \\
\sin \frac{2\pi}{3} & \cos \frac{2\pi}{3}
\end{pmatrix}.
$$

($\rho$ is rotation by $\frac{2\pi}{3}$.) Then $\rho$ is indecomposable. But if we use $K = \mathbb{C}$, and $\tilde{\rho} : \mathbb{C}[G] \hookrightarrow M_2(\mathbb{C})$ induced by the same representation, then $\tilde{\rho}$ is decomposable since over $\mathbb{C}$, we can change basis and diagonalize:

$$
\tilde{\rho}(t) = \begin{pmatrix}
e^{\frac{2\pi i}{3}} & 0 \\
0 & e^{\frac{2\pi i}{3}}
\end{pmatrix}.
$$

in an appropriate basis.

Given $f : R \mapsto S$ a ring homomorphism, $f$ induces a functor

$$
\{\text{(left) } R\text{-mods.}\} \mapsto \{\text{(left) } S\text{-mods.}\}
$$

$$
V \mapsto V_S := S \otimes_R V.
$$

The map $f$ makes $S$ a two-sided $R$-module (and in particular, a right module), so $S \otimes_R V$ makes sense. $S \otimes_R V$ is an $S$-module via the action

$$
s'(s \otimes v) = (s's) \otimes v.
$$

If

$$
0 \to U \to V \to W \to 0
$$

is a short exact sequence of left $R$-modules and $M$ is a right $R$-module then

$$
M \otimes_R U \to M \otimes_R V \to M \otimes_R W \to 0
$$

is exact, although the first map may not be injective. However, if $M$ is a free $R$-module, $M \cong R^n$ then $M \otimes_R N \cong N^n$, and so

$$
M \otimes_R U \hookrightarrow M \otimes_R V
$$

in this case.

In particular, if $f : R \mapsto S$ makes $S$ into a free $R$-module then when

$$
0 \to U \to V \to W \to 0
$$

is exact, although the first map may not be injective.
is exact, so is
\[0 \to U_S \to V_S \to W_S \to 0,\]

ie. \((V/U)_S \cong V_S / U_S\).

In particular, if \(K \subset M\) is a field extension then \(M\) is a free \(K\)-module.

\(f : K \mapsto M\) induces \(K[G] \mapsto M[G]\), and thus

\[K[G]\text{-mods.} \mapsto M[G]\text{-mods.},\]

\[V \mapsto V_M\]

ie. \((mg)(m' \otimes v) = (mm' \otimes gv)\), defines \(M[G]\)-action on \(V_M\).

Note: If \(v = kv'\) then \(m' \otimes v = m' f(k) \otimes v'\), but then

\[mg(m' \otimes v) = mm' \otimes gv = mm' \otimes kgv' = mm' f(k) \otimes gv' = mg(m' f(k) \otimes v'),\]

so the action is well-defined. Also,

\[g(m(m' \otimes v)) = g(mm' \otimes v) = mm' \otimes gv = m(m' \otimes gv) = mg(m' \otimes v),\]

so the action of \(g\) is \(M\)-linear.

If \(K \subset M\) is a field extension and \(n = \dim V < \infty\),

\[\rho : K[G] \mapsto \End_K(V) \cong M_{n \times n}(K)\]

then \(\dim V_M = n\) and for the induced map

\[\tilde{\rho} : M[G] \mapsto \End_M(V_M),\]

the matrix \(\tilde{\rho}(g)\) for the action of \(g\) is just \(\rho(g)\), regarded as a matrix in \(M\) (whose entries happen to lie in \(K\)).

As we have seen, \(V\) simple \(\not\Rightarrow V_M\) is simple.
4.6 Composition Series

Let $V$ be an $R$-module.

**Definition 4.6.1.** A *composition series* for $V$ consists of a chain of submodules

$$0 = V_n \subset V_{n-1} \subset \cdots \subset V_1 \subset V_0 = V$$

s.t. $V_{j-1}/V_j$ is simple $\forall j = 1, \ldots, n$.

The composition series

$$0 = V_n \subset \cdots \subset V_0 = V$$

and

$$0 = W_m \subset \cdots \subset W_0 = V$$

are called equivalent if $n = m$ and $\exists \sigma \in S_n$ s.t.

$$V_{j-1}/V_j \cong W_{\sigma(j)-1}/W_{\sigma(j)} \quad \forall j.$$

ie. the list of “composition factors” (including multiplicities) is the same, although the order may be different.

**Proposition 4.6.2.** $V$ has a composition series $\iff V$ is both Artinian and Noetherian. In this case, any series can be refined to a composition series.

**Proof.**

$\Leftarrow$: Suppose $V$ is Artinian and Noetherian. Let $V_0 = V$. Since $V$ is Noetherian, $V$ contains a maximal (proper) submodule, $V_1$ (by Theorem 2.6.2). Continuing, so long as $V_j \neq 0$, get

$$V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_j \supsetneq \cdots$$

s.t. $V_{j+1}$ is maximal in $V_j$, ie. $V_j/V_{j+1}$ is simple. Since $V$ is Artinian, the chain must terminate.

$\Rightarrow$: Suppose

$$0 = V_n \subset \cdots \subset V_0 = V$$

is a composition series. Then we have the exact sequence

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

Since $V_1$ is simple, $V_1$ is Artinian and Noetherian. $V/V_1$ has a composition series of length $n - 1$, so by induction, $V/V_1$ is Artinian and Noetherian. Thus, $V$ is Artinian and Noetherian.
Finally, given any series

\[ 0 = V_n \subset \cdots \subset V_0 = V, \]

each \( V_{i-1}/V_i \) is Noetherian and Artinian, so it has a composition series. Using each of these series, we may refine the given series to a composition series. \( \square \)

**Theorem 4.6.3.** Any two comp. series for \( V \) are equivalent.

**Proof.** Let

\[ 0 = V_n \subset \cdots \subset V_0 = V \]

and

\[ 0 = W_m \subset \cdots \subset W_0 = V \]

be comp. series. For \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), set

\[ V_{ij} := V_i + (V_{i-1} \cap W_j) \quad \text{and} \quad W_{ji} := W_j + (W_{j-1} \cap W_j). \]

**Claim.**

\[ \frac{V_{i,j-1}}{V_{ij}} \cong \frac{V_{i-1} \cap W_{j-1}}{(V_i \cap W_{j-1}) + (V_{i-1} \cap W_j)} \cong \frac{W_{j-1}}{W_{ji}}. \]

**Proof of claim.** Consider

\[ \phi : V_{i-1} \cap W_{j-1} \hookrightarrow V_i + (V_{i-1} \cap W_j - 1) \nrightarrow \frac{V_i + (V_{i-1} \cap W_{j-1})}{V_i + (V_{i-1} \cap W_j)} = \frac{V_{i,j-1}}{V_{ij}}. \]

\( V_i \subset V_{ij}, \) so every element of \( V_{i,j-1} \) is congruent modulo \( V_{ij} \) to one in \( V_{i-1} \cap W_{j-1}. \) ie. \( \phi \) is surjective.

Clearly, \( V_{i-1} \cap W_j \subset V_{ij}, \) so

\[ V_{i-1} \cap W_j \subset \ker \phi. \]

Also, \( V_j \cap W_{j-1} \subset V_i \subset V_{ij}, \) so

\[ V_i \cap W_{j-1} \subset \ker \phi. \]

Hence,

\( (V_{i-1} \cap W_j) + (V_i \cap W_{j-1}) \subset \ker \phi. \)

Conversely, suppose \( x \in V_{i-1} \cap W_{j-1} \) lies in

\[ \ker \phi = (V_{i-1} \cap W_{j-1}) \cap (V_i + (V_{i-1} \cap W_j)). \]

Write \( x = y + z \) where \( y \in V_i \) and \( z \in V_{i-1} \cap W_j. \) Since \( x \in W_{j-1} \) and \( z \in W_j \subset W_{j-1}, \) it follows that \( y \in W_{j-1}. \) So

\[ x = y + z \]

exhibits \( x \) as an elt. of \( (V_i \cap W_{j-1}) + (V_{i-1} \cap W_j). \) \( \square \)
Notice that since $V_{i-1}/V_i$ and $W_{j-1}/W_j$ are simple,

\[
\frac{V_{i-1} \cap W_{j-1}}{(V_i \cap W_{j-1}) + (V_{i-1} \cap W_j)}
\]

is either 0 or simple.

So we have

\[
V = V_0 = V_1 \supset V_2 \supset \cdots \supset V_m = V_{n-1} \supset \cdots \supset V_n = 0. \quad (*)
\]

and similarly,

\[
V = W_0 = W_1 \supset W_2 \supset \cdots \supset W_m = W_{n-1} \supset \cdots \supset W_n = 0. \quad (**)
\]

Notice that both chains have the same length, and by the claim, there is a bijection between the quotient modules, each of which is either simple or 0. So by shortening the chains by deleting entries which equal their predecessors, all the 0-quotient modules are deleted, and what is left are composition series. The number of 0-quotients deleted is the same (they are paired), so the resulting comp. series have the same length and same quotients, i.e. they are equivalent.

But (*) reduces to

\[
V_n \subset \cdots \subset V_0
\]

and (**) reduces to

\[
W_m \subset \cdots \subset W_0,
\]

since they are respectively refinements of these series, and you can’t refine a comp. series any further. So these two comp. series are equivalent. □
4.7 Characters

Let

\[ \rho : K[G] \rightarrow \text{End}_K(V) \]

be a rep. of \( K \) on a free \( K \)-module \( V \). Define

\[ \chi_\rho : K[G] \rightarrow K, \]

the character of \( \rho \) by \( \chi_\rho = \text{Tr}(\rho(x)) \).

Since \( \text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B) \), \( \chi_\rho \) is determined by its values on the basis \( G \) for \( K[G] \), so sometimes write \( \chi_\rho : G \rightarrow K \).

Recall that \( \text{Tr} \) is preserved under change of basis, since

\[ \text{Tr}(A^{-1}BA) = \text{Tr}(AA^{-1}B) = \text{Tr}(B). \]

So, if \( h = x^{-1}gx \) then

\[ \rho(h) = \rho(x)^{-1}\rho(g)\rho(x) \]

and thus, \( \chi_\rho(h) = \chi_\rho(g) \).

**Proposition 4.7.1.** Let

\[ 0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0 \]

be a short exact sequence of \( K[G] \)-modules, each of which is free as a \( K \)-module. Then

\[ \chi_V = \chi_U + \chi_W. \]

**Proof.** Since \( U \) is a \( K[G] \)-submodule, for all \( g \in U \), the matrix for \( \rho(g) \) has the form

\[ \rho_V(g) = \begin{pmatrix} \rho_U(g) & * \\ 0 & \rho_W(g) \end{pmatrix}. \]

\[ \square \]

**Proposition 4.7.2.** \( \chi_{V \otimes W} = \chi_V \chi_W \).

**Proof.** Let \( \{e_i\}, \{f_j\} \) be bases for \( V, W \) respectively. Then \( \{e_i \otimes f_j\} \) is a basis for \( V \otimes W \), and

\[ (A \otimes B)(e_i \otimes f_j) = a_{ij}b_{jj}(e_i \otimes f_j) + \text{other terms}. \]

So,

\[ \text{Tr}(A \otimes B) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{jj} = (\text{Tr}A)(\text{Tr}B). \]

\[ \square \]
Proposition 4.7.3. Viewing \( K[G] \) as a left \( K[G] \)-module,

\[
\chi_{K[G]}(g) = \begin{cases} |G|, & g = e, \\ 0, & g \neq e. \end{cases}
\]

Proof. In the basis \( \{ g \}_{g \in G} \) for \( K[G] \), the action of any elt. of \( G \) is given by a permutation matrix. So, by the definition of the trace,

\[
\chi_{K[G]}(g) = |\{ x \in G \mid gx = x \}|
\]

\[
= \begin{cases} |G|, & g = e, \\ 0, & g \neq e. \end{cases}
\]

\( \square \)

Corollary 4.7.4. Suppose

\( K[G] \cong V_1 \oplus \cdots \oplus V_r \)

and \( K \) is a field s.t. \( \text{char } K \nmid |G| \). Thus \( \chi_{K[G]} = \sum_{i=1}^r \chi_i \) where \( \chi_i = \chi_{V_i} \).

Let \( y = \sum_{g \in G} c_g g \in K[G] \). Then for any \( g \),

\[
c_g = \frac{1}{|G|} \sum_{i=1}^r \chi_i(y^{-1}).
\]

Proof. Pick \( g \in G \).

\[
y = \sum_{h \in G} c_h h = c_g g + \sum_{h \neq g} c_h h.
\]

\[
\therefore y^{-1} = c_g e + \sum_{h \neq g} c_h h^{-1}. \text{ Applying } \chi_{K[G]} = \sum_{i=1}^r \chi_i,
\]

\[
\sum_{i=1}^r \chi_i(y^{-1}) = \chi_{K[G]}(y^{-1})
\]

\[
= c_g \chi_{K[G]}(e) + \sum_{h \neq g} c_h \chi_{K[G]}(h^{-1})
\]

\[
= |G|c_g + 0.
\]

\( \square \)

Set \( \text{CF}_K(G) := \{ f : G \rightarrow K \mid f(y^{-1}xy) = f(x) \forall x, y \in G \} \). \( \text{CF}_K(G) \) is a ring using addition and multiplication of functions. It is called the ring of **class functions**.
$R_K(G)$ is the abelian group generated by iso. classes of f.d. reps. of $G$, with the relation

$$[V] = [V'] + [V'']$$

for every short exact sequence

$$0 \to V' \to V \to V'' \to 0.$$ 

Define multiplication on $R_K(G)$ by

$$[V][W] = [V \otimes W].$$

Then the preceeding implies that

$$\theta : R_K(G) \to \text{CF}_K(G)$$

$$[\rho] \mapsto \chi_{\rho}$$

is a ring homomorphism.

Set $\text{Ch}_K(G) := \text{Im} \theta$, the “ring of generalized $K$-characters of $G$”, or simply the “character ring of $G$ over $K$”.

**Lemma 4.7.5.** Let $V, W$ be $K[G]$-modules and let $f \in \text{hom}_K(V, W)$. Define $\tilde{f} : V \to W$ by

$$\tilde{f}(v) = \sum_{g \in G} g^{-1}f(gv).$$

Then $\tilde{f} \in \text{hom}_{K[G]}(V, W)$.

If $V = W$ then $\text{Tr}(\tilde{f}) = |G|\text{Tr}(f)$.

**Proof.** For $x \in G$,

$$\tilde{f}(xv) = \sum_{g \in G} g^{-1}f(gxv) = \sum_{h \in G} xh^{-1}f(hv) = x\tilde{f}(v).$$

Now suppose $V = W$. Then,

$$\tilde{f} = \sum_{g \in G} M_g^{-1}fM_g$$

where $M_g$ represents the action of $g$ on $V$. Hence,

$$\text{Tr}(\tilde{f}) = \sum_{g \in G} \text{Tr}(M_g^{-1}fM_g)$$

$$= \sum_{g \in G} \text{Tr}(f)$$

$$= |G|\text{Tr}(f).$$

$\square$
Let $K$ be a field.

**Lemma 4.7.6.** Let $\alpha : G \mapsto \text{Aut}_K(V)$, $\beta : G \mapsto \text{Aut}_K(W)$ be non-isomorphic simple reps. Pick bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ for $V$ and $W$. Let $[\alpha_{ij}(g)]$ and $[\beta_{ij}(g)]$ denote matrices for $\alpha(g), \beta(g)$ in these bases. Then for any $i, j, k, t, 1 \leq i, j \leq n, 1 \leq k, t \leq m$,

$$\sum_{g \in G} \beta_{ij}(g^{-1}) \alpha_{kt}(g) = 0.$$

**Proof.** Let $f : V \mapsto W$ be the linear transformation which in chosen bases for $V$ and $W$ is given by the matrix $E$ which is 1 in the $(j, k)$th position and 0 elsewhere. By the previous lemma, $\tilde{f} \in \text{hom}_K(G)(V, W) = 0$ (since $V, W$ are non-isomorphic and simple). The $(i, t)$th position of the matrix for $\tilde{f}$ is

$$0 = \sum_{g \in G} \sum_{r, s} \beta_{ir}(g^{-1}) E_{rs} \alpha_{st}(g) = \sum_{g \in G} \beta_{ij}(g^{-1}) \alpha_{kt}(g),$$

since $E_{rs} = 0$ except when $r = j, s = k$. □

**Corollary 4.7.7.** Let $V, W$ be non-isomorphic simple $K[G]$-modules. Then

$$\sum_{g \in G} \chi_V(g) \chi_W(g^{-1}) = 0.$$

**Proof.** Let $\alpha(g), \beta(g)$ be the matrices for the reps. Then

$$\sum_{g \in G} \chi_V(g) \chi_W(g^{-1}) = \sum_{g} \sum_{t} \sum_{i} \alpha_{it}(g) \beta_{it}(g^{-1}) = 0.$$

□

**Theorem 4.7.8.** Let $\alpha : G \mapsto \text{Aut}_K(V)$ be a simple $G$-rep. If $K$ is algebraically closed and $\text{char } K = 0$ then $\dim Z | |G|$ and

$$\sum_{g \in G} \alpha_{ij}(g^{-1}) \alpha_{kt}(g) = \delta_{jk} \delta_{it} \frac{|G|}{\dim V}.$$

**Proof.** Let $f : V \mapsto V$ be the linear transformation which in a chosen basis for $V$ is given by the matrix $E$ which is 1 in the $(j, k)$th position and 0 elsewhere. So $\tilde{f} \in \text{hom}_Z(G)(V, V)$. Since $V$ is simple, $\text{hom}_K(G)(V, V) = K$, and thus, $\text{hom}_Z(G)(V, V) = Z$. That is, $\tilde{f} = cI$ for some $c \in Z$. 204
As above, the \((i, t)\)th entry of the matrix for \(\tilde{f}\) is
\[
\sum_{g \in G} \alpha_{ij}(g^{-1})\alpha_{ij}(g^{-1})\alpha_{kt}(g).
\]

Now, \(\text{Tr}(\tilde{f}) = \text{Tr}(cI) = c \dim V\). On the other hand, by the earlier lemma, \(\text{Tr}(\tilde{f}) = |G|\text{Tr}(E)\). Thus,
\[
c \dim V = |G|\text{Tr}(E), \quad c = \frac{|G|\text{Tr}(E)}{\dim V}.
\]

If \(j \neq k\) then \(\text{Tr}(E) = 0\), so \(c = 0\). Also, if \(i \neq t\) then the \((i, t)\)th entry of \(\tilde{f}\) is 0, regardless of \(c\). Hence,
\[
\sum_{g \in G} \alpha_{ij}(g^{-1})\alpha_{kt}(g) = 0 \quad \text{unless } i = t \text{ and } j = k.
\]

When \(i = t\) and \(j = k\), \(\text{Tr}(E) = 1\) so
\[
\frac{|G|}{\dim V} = c = \sum_{g \in G} \alpha_{ij}(g^{-1})\alpha_{kt}(g),
\]
and in particular, \(\dim V | |G|\). □

**Corollary 4.7.9.** Let \(V\) be a simple \(K[G]\)-module where \(K\) is algebraically closed and \(\text{char } K = 0\). Then
\[
\sum_{g \in G} \chi_V(g)\chi_V(g^{-1}) = |G|.
\]

**Proof.** Let \(\alpha(g)\) be the matrix for \(V\). Set \(s := \dim V\). Then
\[
\sum_{g \in G} \chi_V(g)\chi_V(g^{-1}) = \sum_{g \in G} \sum_{t=1}^{s} \sum_{i=1}^{s} \alpha_{ti}(g)\alpha_{ii}(g^{-1})
\]
\[
= \sum_{i=1}^{s} \sum_{t=1}^{s} \alpha_{ti}(g)\alpha_{ii}(g^{-1})
\]
\[
= \sum_{i=1}^{s} \delta_{ii} |G| \frac{s}{s}
\]
\[
= |G|.
\]

□
If \( \text{char } K = 0 \), can define an inner product on \( \text{Ch}_K(G) \) via
\[
\langle \chi_V, \chi_W \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g^{-1}).
\]

If \( K \) is algebraically closed, have just shown that \( \{ \chi_V \mid V \text{ simple} \} \) forms an orthonormal set in \( \text{Ch}_K(G) \). Since \( K[G] \) is semisimple, every rep. is a sum of simple ones, so this is in fact a basis.

In particular:

**Corollary 4.7.10.** If \( \text{char } K = 0 \) then
\[
\text{R}_K(G) \cong \text{Ch}_K(G).
\]

If \( K \) is algebraically closed, then
\[
\text{Ch}_K(G) = \text{CF}_K(G).
\]

**Proof.** We have just shown that \( \{ \chi_V \mid V \text{ simple} \} \) is an orthonormal set in \( \text{Ch}_K(G) \subset \text{CF}_K(G) \) (the inner product extends in the obvious way to \( \text{CF}_K(G) \)). Thus, this set is linearly independent, so \( \theta : \text{R}_K(G) \mapsto \text{Ch}_K(G) \) is injective. By construction, \( \text{R}_K(G) \mapsto \text{Ch}_K(G) \) is onto, so \( \text{R}_K(G) \cong \text{Ch}_K(G) \).

Let \( C_1, \ldots, C_r \) be the set of conj. classes of \( G \). \( \text{CF}_K(G) \) has a basis \( \{ f_j : G \mapsto K \} \) where
\[
f_j(g) = \begin{cases} 1 & g \in C_j, \\ 0 & g \notin C_j. \end{cases}
\]

Hence, the dimension of \( \text{CF}_K(G) \) is the number of conj. classes of \( G \), which, we have seen, is the number of simple \( K[G] \)-modules, ie. the dimension of \( \text{R}_K(G) \). \( \square \)
4.8 Change of Group - Induction and Restriction

Let $H \leq G$, so $K[H] \subset K[G]$. Let $N$ be a rep. of $G$, $G \times N \mapsto N$. Restricting to $H$ produces an action $H \times N \mapsto N$. Denote the resulting rep. of $H$ by $N_H$.

Conversely, let $M$ be a rep. of $G$. Define the induced representation of $G$, denoted $M^G$, via

$$M^G := K[G] \otimes_{K[H]} M.$$  

ie. $M^G$ is generated as a $K$-module by

$$\{ g \otimes m \mid g \in G, m \in M \}$$

where $gh \otimes m \sim g \otimes hm$. The $G$-action on $M^G$ is defined by

$$g'(g \otimes m) = g'g \otimes m.$$  

Let $g_1, \ldots, g_r$ be a set of representatives for the left cosets $\{gH\}$. Then $\{g_j \otimes m\}$ generates $M^G$. In fact, if $K$ is a field and $m_1, \ldots, m_k$ is a basis for $M$ then

$$\{g_j \otimes m_i \mid 1 \leq j \leq r, 1 \leq i \leq k\}$$

forms a basis for $M^G$. In particular,

$$\dim M^G = \frac{|G|}{|H|} \dim M,$$

whereas $\dim N_H = \dim N$.

This is a special case of a ground-ring change. A ring homo. $f : R \mapsto S$ induces

$$\{S\text{-modules}\} \overset{P}{\longrightarrow} \{R\text{-modules}\}$$

$$N \overset{f}{\longrightarrow} N,$$

where $N$ (on the right) is regarded as an $R$-module via the action through $f$. $f$ also induces

$$\{R\text{-modules}\} \overset{Q}{\longrightarrow} \{S\text{-modules}\}$$

$$M \overset{f \otimes id}{\longrightarrow} M \otimes_R M.$$  

$Q$ and $P$ are adjoint functors, ie.

$$\hom_S(QM, N) = \hom_R(M, PN) \quad \forall R\text{-mods. } M, S\text{-mods. } N.$$  

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To see this, given $\alpha : QM = S \otimes_R M \mapsto N$, define $\beta : M \mapsto PN = N$ by

$$\beta(m) = \alpha(1 \otimes m).$$

Then

$$\beta(rm) = \alpha(1 \otimes rm) = \alpha(r \otimes m) = r\alpha(1 \otimes m) = r\beta(m),$$

$: \beta$ is an $R$-mod. homo.

Conversely, given $\beta : M \mapsto PN$, define $\alpha : QM \mapsto N$ by

$$\alpha(s \otimes m) = s\beta(m).$$

Then

$$\alpha(s'(s \otimes m)) = \alpha(ss' \otimes m) = ss'\beta(m) = s'\alpha(s \otimes m).$$

$: \alpha$ is an $S$-mod. homo.

In our special case,

$$\text{hom}_{K[G]}(M^G, N) = \text{hom}_{K[H]}(M, N_H).$$

This is called Frobenius Reciprocity.

Also, if $A \leq B \leq C$ then

$$M^C \cong (M^B)^C$$

and

$$N_A \cong (N_B)_A.$$
Note that if \( i_g = i \) then \( h = g_i^{-1}gg_i \). So

\[
\chi^G_M(g) = \sum_{i,j} \begin{cases} 
0 & \text{if } g_i^{-1}gg_i \notin H \\
\alpha_H(g_i^{-1}gg_i) & \text{if } g_i^{-1}gg_i \in H
\end{cases}
\]

\[= \sum_i \chi_M(g_i^{-1}gg_i) \]

\[= \sum_i \chi_M(g_i^{-1}gg_i), \quad \text{using the convention } \chi_M(x) = 0 \text{ if } x \notin H \]

\[= \frac{1}{|H|} \sum_{x \in G} \chi_M(x^{-1}gx). \]
4.9 Examples

4.9.1 $G = S_3$

We have $|G| = 6.$

<table>
<thead>
<tr>
<th>Conj. class</th>
<th># conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = (3), (1\ 2\ 3)$</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda = (2, 1), (1\ 2)$</td>
<td>3</td>
</tr>
<tr>
<td>$\lambda = (1, 1, 1), e$</td>
<td>1</td>
</tr>
</tbody>
</table>

We have 3 conjugacy classes, so 3 indecomposable reps. So our dimensions are determined:

$$6 = 1^2 + 1^2 + 2^2.$$ 

The reps. are:

1. $V_1 =$ trivial rep., $\dim V_1 = 1, \chi_1 = (3), (2, 1), (3).$ 
2. $V_2 =$ sign rep., $\dim V_2 = 1, \chi_2 = (1, -1, 1).$ 
3. $V_3 =$ natural rep., $\dim V_3 = 2.$ By orthogonality of characters, $\chi_2 = (2, 0, -1).$ This representation is given on the space 
   $\langle x_1, x_2, x_3 \rangle/\langle x_1 + x_2 + x_3 \rangle$

   by

   $$\sigma(x_i) = x_{\sigma(i)}.$$ 

Altogether, our character table is 

$$\chi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -1 \end{pmatrix}.$$ 

4.9.2 $G = D_8 = \langle a, b \mid a^4 = b^2 = e, bab^{-1} = a^{-1} \rangle$

$|G| = 8.$ We view $G \subset S_4$ via $a = (1\ 2\ 3\ 4), b = (1\ 2).$

<table>
<thead>
<tr>
<th>Conj. class</th>
<th># conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = (1\ 3\ 2\ 4)$</td>
<td>2</td>
</tr>
<tr>
<td>$a^2 = (1\ 2)(3\ 4)$</td>
<td>1</td>
</tr>
<tr>
<td>$b = (1\ 2)$</td>
<td>2</td>
</tr>
<tr>
<td>$ab = (1\ 3)(2\ 4)$</td>
<td>2</td>
</tr>
<tr>
<td>$e$</td>
<td>1</td>
</tr>
</tbody>
</table>
The dimensions of the irred. reps. are determined:

\[ 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2. \]

The 1-dim. reps. are given by \( \text{hom}(D_8, S^1) \) where

\[ S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}. \]

We have

\[ \text{hom}(D_8, S^1) = \text{hom}((D_8)_{ab}, S^1) = \text{hom}(C_2 \times C_2, S^1) = \text{hom}(C_2, S^1) \times \text{hom}(C_2, S^1). \]

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Rep.</th>
<th>Character ( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( V_1 )</td>
<td>trivial,</td>
</tr>
<tr>
<td>1</td>
<td>( V_2 )</td>
<td>( a \cdot v = -v, b \cdot v = v )</td>
</tr>
<tr>
<td>1</td>
<td>( V_3 )</td>
<td>( a \cdot v = v, b \cdot v = -v )</td>
</tr>
<tr>
<td>1</td>
<td>( V_4 )</td>
<td>( a \cdot v = -v, b \cdot v = -v )</td>
</tr>
<tr>
<td>2</td>
<td>( V_5 )</td>
<td>Find character by ( \chi_{K[G]}(\sigma) = \begin{cases}</td>
</tr>
</tbody>
</table>

Our character table (columns indexed by \( e, a, a^2, b, ab \)) is:

\[
\chi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 \\
2 & 0 & -2 & 0 & 0
\end{pmatrix}.
\]

Set \( IC_{(a)} := (\text{Triv}C_{(a)})^{D_8} \), that is, the 2-dimensional representation of \( D_8 \) obtained by induction from the trivial representation of the cyclic subgroup generated by \( a \). If \( v \) is a basis for the 1-dimensional vector space \( V \) for the trivial 1-dimensional vector representation of \( C_{(a)} \), then a basis for \( V^{D_8} = K[D_8] \otimes_{K[D_{(a)}]} V \) is given \( \langle 1 \otimes v, b \otimes v \rangle \), since \( 1 \) and \( b \) are a set of coset representatives. Since left multiplication by \( e, a, \) or \( a^2 \) preserve the cosets, in \( IC_{(a)} \) they are mapped to the identity matrix, while left multiplication by \( b \) or \( ab \) switches the cosets. Thus the traces are 2 and 0 respectively so \( \chi_{IC_{(a)}} = (2 \ 2 \ 2 \ 0 \ 0) \). Comparing this with the character table gives \( IC_{(a)} = V_1 + V_3 \).

Let \( \alpha^{3,1} \) denote the natural 3-dimensional representation of \( S_4 \) on \( \langle x_1, x_2, x_3, x_4 \rangle / \langle x_1 + x_2 + x_3 + x_4 \rangle \). \( \alpha^{3,1}|_{D_8} \) splits as \( W \oplus \alpha^{3,1}|_{D_8}/W \), where \( W = \langle w \rangle \) where \( w = x_1 + x_2 \). Since \( a \cdot w = x_3 + x_4 = -w \) and \( b \cdot w = x_2 + x_1 = w \), \( W \cong V_2 \). and, it is easy to see (using characters or otherwise) that \( W \oplus \alpha^{3,1}|_{D_8}/W \cong V_5 \), so \( \alpha^{3,1}|_{D_8} = V_2 + V_5 \).
4.9.3 $G = \mathbb{H}_8$

$\mathbb{H}_8$ is the group of Quaternions. It consists of 8 elements, 
\[ \pm i, \pm j, \pm k, \pm 1 \]
such that $(-1)^2 = 1, -1 \in \mathbb{Z}(\mathbb{H}_8)$ and 
\[ i^2 = j^2 = k^2 = -1, \]
\[ ij = k, \ jk = i, \ ki = j. \]

<table>
<thead>
<tr>
<th>Conj. class</th>
<th># conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i \sim -i$</td>
<td>2</td>
</tr>
<tr>
<td>$j \sim -j$</td>
<td>2</td>
</tr>
<tr>
<td>$k \sim -k$</td>
<td>2</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$1$</td>
<td>1</td>
</tr>
</tbody>
</table>

The dimensions of the irreducible representations are determined:
\[ 8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2. \]

$\mathbb{H}_8/\langle -1 \rangle \cong C_2 \times C_2$ is abelian, so $(\mathbb{H}_8)_{ab} = C_2 \times C_2$, and thus,
\[ \text{hom}(\mathbb{H}_8, S^1) = \text{hom}(C_2, S^1) \times \text{hom}(C_2, S^1). \]

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Rep.</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$V_1$</td>
<td>trivial</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$V_2$</td>
<td>$i \cdot v = -v, j \cdot v = v$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$V_3$</td>
<td>$i \cdot v = v, j \cdot v = -v$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$V_4$</td>
<td>$i \cdot v = -v, j \cdot v = -v$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$V_5$</td>
<td>Find character by $\chi_{K[G]}(\sigma) = \begin{cases}</td>
<td>G</td>
</tr>
</tbody>
</table>

Our character table (columns indexed by $1, -1, i, j, k$) is:
\[ \chi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 \\
2 & -2 & 0 & 0 & 0
\end{pmatrix}. \]

The “natural” representation of $\mathbb{H}_8$ on $W = \langle x_1, x_i, x_j, x_k \rangle$ is given by $i \cdot x_1 = x_i, i \cdot x_i = -x_1, i \cdot x_j = x_k, i \cdot x_k = -x_j$, etc. By inspection $\chi_W = (4 - 4 0 0 0)$, which, from the character table is recognized as $2V_5$. The subspace $\langle x_1 + x_i, x_j + x_k \rangle \subset W$ is closed under the action of $\mathbb{H}_8$ and provides a natural description of $V_5$. 

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4.9.4 \( G = C_7 \rtimes C_3 = \langle a, b \mid a^7 = e, b^3 = e, bab^{-1} = a^2 \rangle \)

<table>
<thead>
<tr>
<th>Conj. class</th>
<th># conjugates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>3</td>
</tr>
<tr>
<td>( a^3 \sim a^{-1} )</td>
<td>3</td>
</tr>
<tr>
<td>( b )</td>
<td>7</td>
</tr>
<tr>
<td>( b^2 )</td>
<td>7</td>
</tr>
<tr>
<td>( 1 )</td>
<td>1</td>
</tr>
</tbody>
</table>

\( G_{ab} = C_{(b)} = C_3 \), and thus,

\[
\text{hom}(G, S^1) = \text{hom}(C_3, S^1).
\]

yielding three 1-dimensional representatives.

The dimensions of the irred. reps. are determined:

\[
21 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2.
\]

Let \( \omega = e^{2\pi i/3} \).

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Rep.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( V_1 ) trivial</td>
</tr>
<tr>
<td>1</td>
<td>( V_2 ) ( a \cdot v = v, b \cdot v = \omega v )</td>
</tr>
<tr>
<td>1</td>
<td>( V_3 ) ( i \cdot v = v, b \cdot v = \omega^2 v )</td>
</tr>
<tr>
<td>3</td>
<td>( V_4 )</td>
</tr>
<tr>
<td>3</td>
<td>( V_5 )</td>
</tr>
</tbody>
</table>

Our character table (columns indexed by \( 1, a, a^3, b, b^2 \)) looks like:

\[
\chi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & \omega & \omega^2 \\
1 & 1 & \omega^2 & \omega \\
3 & x & y & s & t \\
3 & x' & y' & s' & t'
\end{pmatrix}.
\]

for some \( x, y, s, t, x', y', s', t' \).

Using \( \chi_K[G](\sigma) = \begin{cases} |G|, & \sigma = e \\ 0, & \sigma \neq e \end{cases} \) we find that \( x' = -(x + 1), y' = -(y + 1), s' = -s, t' = -t \).

Orthogonality of \( \chi_1 \) and \( \chi_4 \) gives \( 7(s + t) = -3 - 3x - 3y \) while orthogonality of the pairs \( \chi_2, \chi_4 \) and \( \chi_3, \chi_4 \) give \( 7(\omega s + \omega^2 t) = -3 - 2x - 3y \) and \( 7(\omega^2 s + \omega t) = -3 - 2x - 3y \) respectively. Thus \( 7(s + t) = 7(\omega s + \omega^2 t) = 7(\omega^2 s + \omega t) \), from which we deduce that \( s = t = 0 \) and so \( (x + y + 1) = -\frac{3}{7}(s + t) = 0 \). The inner product of \( \chi_4 \) with itself gives \( |G| = 21 = 9 + 3xy + 3xy \), which combined with \( x + y + 1 = 0 \) gives \( x^2 + x + 2 = 0 \), which determines \( x \). Notice that the solution of \( x^2 + x + 2 = 0 \) satisfies \( x = \zeta + \zeta^2 + \zeta^4 \), where \( \zeta = e^{2\pi i/7} \) and \( 1 - x = \zeta^3 + \zeta^5 + \zeta^6 \).
Thus our character table is (columns indexed by $1, a, a^3, b, b^2$) looks like:

$$
\chi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & \omega & \omega^2 & \\
1 & 1 & \omega^2 & \omega & \\
3 & x & y & 0 & 0 \\
3 & y & x & 0 & 0
\end{pmatrix}.
$$

where $x = \zeta + \zeta^2 + \zeta^4$ and $y = \zeta^3 + \zeta^5 + \zeta^6$.

The representation $V_4$ is given explicitly by $a \mapsto \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ while $V_5$ is given by $a \mapsto \begin{pmatrix} \zeta^3 & 0 & 0 \\ 0 & \zeta^6 & 0 \\ 0 & 0 & \zeta^5 \end{pmatrix}$, $b \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. 
4.10 Symmetric Polynomials

For a free $K$-module $V$, let

$$T^K(V) := \bigoplus_{n=0}^{\infty} V^\otimes n,$$

called the tensor algebra on $V$. Multiplication is defined on $T^K(V)$ by

$$(x_1 \otimes \cdots \otimes x_K)(x_{K+1} \otimes \cdots \otimes x_\ell) = x_1 \otimes \cdots \otimes x_K \otimes x_{K+1} \otimes \cdots \otimes x_\ell.$$

$S_n$ acts on $V^\otimes n$ by permuting factors (called the position action), ie.

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Let

$$S(V) := T(V)/\sim$$

where $(x_1 \otimes \cdots \otimes x_n) \sim (x_1 \otimes \cdots \otimes x_n)$. This is called the polynomial (symmetric) algebra on $V$.

**Example 4.10.1.** If $x_1, \ldots, x_m$ form a basis for $V$ then

$$S(V) \cong K[x_1, \ldots, x_m]$$

$$x_{i_1} \otimes \cdots \otimes x_{i_n} \mapsto x_{i_1} \cdots x_{i_n}.$$

Likewise, the exterior algebra on $V$ is

$$\Lambda(V) := T(V)/\sim$$

where

$$x_1 \otimes \cdots \otimes x_n \sim (-1)^{\ell} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

If $x_1, \ldots, x_m$ is a basis for $V$ then $S_m$ acts on $V$ (on the right) by

$$x_j \cdot \sigma = x_{\sigma^{-1}(j)}.$$

∴ Get induced action of $S_m$ on $T(V), S(V)$, and $\Lambda(V)$. This is called the internal action. Let

$$\Sigma(V) = \text{Fix}^{S_m}(S(V)) = \{a \in S(V) \mid a = a \cdot \sigma \forall \sigma \in S_m\}.$$

When $K$ is a field, the isomorphism $S(V) \cong K[x_1, \ldots, x_m]$ takes $\Sigma(V)$ to the ring of symmetric polynomials over $K$, as defined in Section 3.9. Recall the definition in that section of the elementary symmetric polynomials $s_1, \ldots, s_m$:

$$s_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}x_{i_2} \cdots x_{i_k}.$$

By identifying $S(V)$ with $K[x_1, \ldots, x_m]$, we have $s_j \in \Sigma(V) \forall j$. 

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Theorem 4.10.2. \( \Sigma(V) \cong K[s_1, \ldots, s_m] \).

If \( \text{rank} V = m \), write \( \Sigma^K[m] = \Sigma^K(V) \).

\[
\begin{align*}
K[x_1, \ldots, x_{m+1}] &\mapsto K[x_1, \ldots, x_m] \\
x_j &\mapsto x_j \quad j \leq m \\
x_{m+1} &\mapsto 0
\end{align*}
\]

induces the map

\[
\rho_{m+1} : \Sigma[m + 1] \mapsto \Sigma[m] \\
s_k(x_1, \ldots, x_{m+1}) &\mapsto s_k(x_1, \ldots, x_m).
\]

Set \( \Sigma := \lim_{\leftarrow m} \Sigma[m] \), the inverse limit of graded rings. That is,

\[
\Sigma = \{(a_m \in \Sigma[m])_{m=1}^{\infty} \mid \rho_{m+1}(a_{m+1}) = a_m \ \forall m\}.
\]

\( \Sigma \) is a graded ring; the elements of \( \Sigma_n \) are sequences \( (f[m])_{m=1}^{\infty} \), where \( f[m] \) is a degree \( n \) symmetric poly. in \( m \) variables, and

\[
f[m](x_1, \ldots, x_m) = f[m+1](x_1, \ldots, x_m, 0).
\]

\( \therefore \) \( f[m] \) determines \( f[k] \) for all \( k \leq m \). However, since each \( f[m] \) is of degree \( n \), \( f[n] \) determines \( f[m] \) \( \forall m \). ie. Given

\[
f[n] = \rho(s_1(x_1, \ldots, x_n), \ldots, s_n(x_1, \ldots, x_n)),
\]

we then have, for any \( m \geq n \),

\[
f[m] = \rho(s_1(x_1, \ldots, x_m), \ldots, s_n(x_1, \ldots, x_m)).
\]

Equivalently, \( f[m] \) is obtained from \( f[n] \) by “symmetrizing over the \( m \) variables”.

So, we may identify the sequence \( (f[m]) \) with the single element \( f[n] \). ie. \( \Sigma_n \) has a basis consisting of the symmetric polynomials of degree \( n \) in \( n \) variables. (Alternatively, \( \Sigma_n \) has a basis consisting of the symmetric polynomials of degree \( n \) in \( m \) variables, for any \( m \geq n \).) So

\[
\Sigma \cong K[s_1, s_2, \ldots, s_k, \ldots].
\]

**Definition 4.10.3.** A **partition** of \( n \) is a sequence \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of non-negative integers s.t.

\[
n = \lambda_1 + \cdots + \lambda_r.
\]

\( \lambda \vdash n \) means that \( \lambda \) is a partition of \( n \).
Pick \( n \geq 0 \), let \( K \) be a field and let \( V \) be the free module with basis \( x_1, \ldots, x_r \). For an unordered partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \) of \( n \), set \( V^\lambda \) to be the \( K[S_n] \)-submodule of \( V^{\otimes n} \) (position action) generated by 
\[
x_1^{\otimes \lambda_1} \otimes \cdots \otimes x_r^{\otimes \lambda_r}.
\]
That is, \( V^\lambda \) is the subspace of \( V^{\otimes n} \) with basis 
\[
\{x_i \otimes \cdots \otimes x_{i_n} \mid \{i_1, \ldots, i_n\} \text{ contains } \lambda_j \text{ copies of } j\}.
\]

Given \( A \subset V^{\otimes n} \) a subspace, the **characteristic polynomial** of \( A \) is 
\[
\text{Ch}(A) := \sum_{\lambda \vdash -n} d_\lambda x_\lambda
\]
where \( d_\lambda = \dim(A \cap V^\lambda) \) and \( x_\lambda = x_1^{\lambda_1} \cdots x_r^{\lambda_r} \).

It is clear from the definition that 
\[
\text{Ch}(A \oplus B) = \text{Ch}(A) + \text{Ch}(B)
\]
\[
\text{Ch}(A \otimes B) = \text{Ch}(A)\text{Ch}(B).
\]

If \( A \) is closed under the internal action of \( S_r \) on \( V \) then \( \text{Ch}(A) \) is symmetric.

Let \( P \) be a projective \( K[S_n] \)-module, so that \( P = K[S_n]e \) for some idempotent \( e \in K[S_n] \). For any right \( K[S_n] \)-module \( N \), 
\[
N \cong Ne \oplus N(1 - e)
\]
as vector spaces. Applying this in particular to \( V^{\otimes n} \) with the position action, 
\[
V^{\otimes n} = V^{\otimes n} e \oplus V^{\otimes n} (1 - e).
\]
Set \( P(V) := V^{\otimes n} e \). Then 
\[
K\text{-vector spaces} \mapsto K\text{-vector spaces}
\]
\[
V \mapsto P(V)
\]
is a functor.

**Example 4.10.4.** Suppose \( p \nmid n! \). Then letting \( P \) be the trivial 1-dimensional rep. of \( S_n \), \( P \) is an indecomposable proj. module with idempotent 
\[
e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma.
\]

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We have:

\[ P(V) = \text{span} \left\{ \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \mid v_1, \ldots, v_n \in V \right\} \subset V^\otimes n \]

\[ \cong S(V). \]

Let \( V = \langle x_1, \ldots, x_m \rangle \), i.e. \( V \) is the vector space with basis \( x_1, \ldots, x_m \). Then \( \text{Ch}(P(V)) \) is a symmetric polynomial in \( x_1, \ldots, x_m \) of degree \( n \). In fact, if we let

\[ P[m] = \text{Ch}(P(\langle x_1, \ldots, x_m \rangle)) \]

then

\[ \langle x_1, \ldots, x_{m+1} \rangle \mapsto \langle x_1, \ldots, x_m \rangle \]

\[ x_j \mapsto x_j \quad j \leq m \]

\[ x_{m+1} \mapsto 0 \]

induces by functoriality a map

\[ P(\langle x_1, \ldots, x_{m+1} \rangle) \mapsto P(\langle x_1, \ldots, x_m \rangle) \]

so by applying \( \text{Ch}(\cdot) \), we get

\[ P[m + 1] \mapsto P[m] \]

\[ x_j \mapsto x_j \quad j \leq m \]

\[ x_{m+1} \mapsto 0 \]

ie. \( (P[m]) \) forms an elt. of \( \Sigma^\mathbb{Z} \) (symmetric polys. with coeffs. in \( \mathbb{Z} \)). We write \( \text{Ch}(P) \) for this elt. of \( \Sigma^\mathbb{Z} \). It is determined by the degree \( n \) symmetric polynomial \( \text{Ch}(P(V)) \) in \( n \) vars. obtained from

\[ V = \langle x_1, \ldots, x_n \rangle. \]

For an arbitrary \( K[S_n] \)-module \( P \), we can write

\[ P = \sum n_j P_j \]

where each \( P_j \) is an indecomposable proj. module and \( n_j \geq 0 \). Set \( \text{Ch}(P) := \sum n_j \text{Ch}(P_j) \).

More generally, elements of \( K_0(K[S_n]) \) are sums

\[ \sum n_j P_j \]

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with \( n_j \in \mathbb{Z} \). So by extending the definition to this case via

\[
\text{Ch}(\sum n_j P_j) = \sum n_j \text{Ch}(P_j)
\]
yields a homomorphism

\[
\text{Ch} : K_0(K[S_n]) \mapsto \Sigma \mathbb{Z}[n].
\]

We shall show, for \( \text{char } K = 0 \), that \( \text{Ch}(P) \) determines \( P \).

For \( n \geq 0 \), set

\[
R_n := \text{Underlying group of the representation ring } R(S_n)
\]

\[
= K_0(K[S_n])
\]

\[
= \text{span}_\mathbb{Z}\{\text{simple } K[S_n]-\text{modules}\}
\]

\[
\cong \text{span}_\mathbb{Z}\{\text{simple characters of } S_n\}
\]

and set \( R = \bigoplus_n R_n \). The map \( \text{Ch} : R_n \mapsto \Sigma \mathbb{Z} \) for each \( n \) yields \( \text{Ch} : R_n \mapsto \Sigma \mathbb{Z} \).

Define a ring structure on \( R \) as follows: Let \( M \) be a \( K[S_m] \)-module, in \( R_m \), and \( N \) a \( K[S_n] \)-module, in \( R_n \). Set

\[
M \cdot N = (M \otimes N)^{S_{m+n}} \in R_{m+n}.
\]

ie. \( M \otimes N \) is a \( (S_m \times S_n) \)-module in an obvious way, and \( S_m \times S_n \subset S_{m+n} \); \( M \cdot N \) is the induced \( S_{m+n} \)-module.

**Theorem 4.10.5.** \( \text{Ch} : R \mapsto \Sigma \) is a ring isomorphism.

**Proof.** We know that \( \text{Ch}(M \oplus n) = \text{Ch}(M) + \text{Ch}(N) \). We must show that \( \text{Ch}(M \cdot N) = \text{Ch}(M)\text{Ch}(N) \).

It suffices to consider the case where \( M \) is a simple \( K[S_m] \)-module and \( N \) is a simple \( K[S_n] \)-module.

Write \( M = K[S_m] \cdot e, N = K[S_n] \cdot f \). Then \( e \otimes f \in K[S_m] \otimes K[S_n] = K[S_m \times S_n] \), and

\[
M \otimes N = K[S_m \times S_n] \cdot (e \otimes f).
\]

Now \( K[S_m \times S_n] \subset K[S_{m+n}] \) and so

\[
M \cdot N = K[S_{m+n}] \cdot (e \otimes f).
\]

For any \( V \),

\[
\text{Ch}(M \cdot N(V)) = \text{Ch}(V^{(m+n)} \cdot (e \otimes f))
\]

\[
= \text{Ch}(V^{\otimes m} \cdot e \otimes V^{\otimes n} \cdot f)
\]

\[
= \text{Ch}(V^{\otimes m} \cdot e)\text{Ch}(V^{\otimes n} \cdot f)
\]

\[
= \text{Ch}(M(V))\text{Ch}(N(V))
\]
\[ Ch(MN) = Ch(M)Ch(N). \] Thus, \( Ch \) is a ring homomorphism.

Since \( \Sigma = \mathbb{Z}[s_1, s_2, \ldots] \), to show \( Ch \) is onto, it suffices to show that \( s_n \in \text{Im}(Ch) \) \( \forall n \). Let \( P \) be the one-dimensional sign rep. of \( S_n \) ie. \( P = \langle w \rangle \) with \( \sigma \cdot w = (-1)^{\text{sgn} \sigma} w \). Then \( P = K[S_n] \cdot e \) with

\[
e = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sgn} \sigma} \sigma,
\]

an idempotent. For any vector space \( V \),

\[ P(V) = (V^0) \cdot e = \Lambda(V). \]

\[ \therefore \ Ch(P(V)) = Ch(\Lambda(V)) = s_n. \] Thus, \( Ch \) is onto.

**Claim.** For each \( n \), \( R_n \) and \( \Sigma_n \) are free abelian groups whose rank equals the number of partitions of \( n \) (into positive integers).

**Proof of claim.** The rank of \( R_n \) is equal to the number of non-isomorphic simple \( K[S_n] \)-reps, which is equal to the number of conjugacy classes in \( S_n \). Each conjugacy class is determined by its cycle type, which is a partition of \( n \) (by Corollary 1.6.3). Moreover, it is obvious that every partition of \( n \) is the cycle type of some element in \( S_n \). Thus, the rank of \( R_n \) is equal to the number of partitions of \( n \).

For \( \Sigma_n \), this follows from the fact that \( \Sigma = \mathbb{Z}[s_1, s_2, \ldots] \) and the degree of \( s_k \) is \( k \). ie. A basis for \( \Sigma_n \) consists of monomials in \( \{s_k\} \) of total degree \( n \), and since \( \deg s_k = k \), each such monomial corresponds to a partition of \( n \) via

\[(A_1, \ldots, A_r) \leftrightarrow s_{A_1} \cdots s_{A_r}.\]

\[ \square \]

Since \( Ch \) is one-to-one, this claim shows that \( Ch \) is also onto, whence an isomorphism. \( \square \)

### 4.10.1 Other Bases for \( \Sigma_n \)

There are 6 bases for \( \Sigma_n^Q \) in “common” use, of which 5 form bases in \( \Sigma_n^\mathbb{Z} \). All bases are indexed by partitions \( \lambda \) of \( n \).

1. **Elementary Symmetric Functions**

\[ s_{\lambda} = s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_r}. \]

eg.

\[
s_{(2)} = x_1 x_2, \quad s_{(1,2)} = (x_1 + x_2)^2.\]
2. Monomial Basis

\[ m_\lambda = \text{symmetrization of } x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_r^{\lambda_r}. \]

eg.
\[ m_{(2)} = x_1^2 + x_2^2, \quad m_{(1,1)} = x_1 x_2. \]

3. Homogeneous Functions

Let

\[ h_\lambda = \sum_k \text{monomials of degree } k. \]

Then

\[ h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}. \]

eg.
\[ h_{(2)} = x_1^2 + x_1 x_2 + x_2^2, \quad h_{(1,1)} = (x_1 + x_2)^2. \]

4. Power Functions

Let

\[ \psi_\lambda = x_1^{\mu_1} + x_2^{\mu_2} + \cdots + x_n^{\mu_n}. \]

Then

\[ \psi_\lambda = \psi_{\lambda_1} \psi_{\lambda_2} \cdots \psi_{\lambda_r}. \]

eg.
\[ \psi_{(2)} = x_1^2 + x_2^2, \quad \psi_{(1,1)} = (x_1 + x_2)^2. \]

5. Schur Functions

For \( \mu = (\mu_1, \ldots, \mu_n), \) with \( \mu_j \geq 0 \ \forall \ j, \) let

\[ V_\mu := \sum_{\sigma \in S_n} (-1)^{\text{sign} \sigma} x_1^{\mu_{\sigma(1)}} \cdots x_n^{\mu_{\sigma(n)}} \]

\[ = \det \begin{pmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{pmatrix}. \]

In particular,

\[ V_{(n-1,n-2,\ldots,1,0)} = \prod_{i<j} (x_i - x_j), \]
called the Vandermonde determinant. For the partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of \( n \) (with \( \lambda_j = 0 \) allowed),

\[
F_\lambda := \frac{V_{\lambda+(n-1,\ldots,0)}}{V_{(n-1,\ldots,0)}}.
\]

eg.

\[
F_{(2)} = F_{(2,0)} = \begin{vmatrix}
x_3 & 1 \\
x_2 & 1 \\
x_1 & 1
\end{vmatrix}
= \begin{vmatrix}
x_3 - x_2 \\
x_1 - x_2
\end{vmatrix}
= \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2,
\]

\[
F_{(1,1)} = \begin{vmatrix}
x_3 & x_1 \\
x_2 & x_2
\end{vmatrix}
= \begin{vmatrix}
x_1 & 1 \\
x_2 & 1
\end{vmatrix}
= x_1 x_2.
\]

Note:

(a) \( x_i = x_j \Rightarrow V_\mu = 0 \). Thus, \( V_{\lambda+(n-1,\ldots,0)} \) is divisible by \( V_{(n-1,\ldots,0)} \), and so \( F_\lambda \) is a polynomial.

(b) Interchanging \( x_i, x_j \) multiplies both numerator and denominator by \(-1\), so \( F_\lambda \) is symmetric.

6. Forgotten Basis

Let \( m_\lambda = p(s_1, \ldots, s_k) \) be the expansion for \( m_\lambda \) in the elem. symmetric polys. Then

\[
f_\lambda = p(h_1, \ldots, h_k).
\]

eg. For \( \lambda = (2) \),

\[
m_{(2)} = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = s_1^2 - 2s_2,
\]

\[
\therefore m_{(2)} = h_1^2 - 2h_2 = (x_1 + x_2)^2 - 2(x_1^2 + x_1 x_2 + x_2^2) = -x_1^2 - x_2^2.
\]
For \( \lambda = (1, 1) \),

\[
m_{(1,1)} = x_1 x_2 = s_2
\]

\[
\therefore f_{(1,1)} = h_2 = x_1^2 + x_1 x_2 + x_2^2.
\]

We know that 1 forms a basis for \( \Sigma_n^Q \) and it is trivial to see that 2 does. We have to prove that the others do.

Note: \( \{\psi_\lambda\} \) does not form a basis for \( \Sigma_n^Z \). e.g. \( s_2 = \frac{1}{2}(\psi_{(1,1)} - \psi_{(2)}) \) in \( \sigma_n^Q \), so

\[
s_2 \notin \mathbb{Z}[\psi_1, \psi_2, \psi_3, \ldots].
\]

**Generating Functions for** \( s_n, h_n, \psi_n \)

The first three of our bases are defined as monomials in some other symmetric functions. Set

\[
S(t) := \sum_{n=0}^{\infty} s_n t^n
\]

\[
H(t) := \sum_{n=0}^{\infty} h_n t^n
\]

\[
\Psi(t) := \sum_{n=0}^{\infty} \psi_n t^n.
\]
By expanding and examining the coefficient of $t^n$, we see that

$$S(t) = \prod_{j=1}^{\infty} (1 + x_j t),$$

$$H(t) = \prod_{j=1}^{\infty} (1 + x_j t + x_j^2 t^2 + x_j^3 t^3 + \cdots) = \prod_{j=1}^{\infty} \frac{1}{1 - x_j t},$$

$$\Psi(t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} x_j^n t^{n-1}$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} x_j^n t^{n-1}$$

$$= \sum_{j=1}^{\infty} \frac{x_j}{1 - x_j t}$$

$$= \sum_{j=1}^{\infty} -\frac{d}{dt} \log(1 - x_j t)$$

$$= \frac{d}{dt} \log \left( \prod_{j=1}^{\infty} \frac{1}{1 - x_j t} \right)$$

$$= \frac{d}{dt} \log(H(t))$$

$$= \frac{H'(t)}{H(t)}.$$ 

Thus,

$$S(t)H(-t) = 1 \quad (1)$$

$$\Psi(t) = \frac{H'(t)}{H(t)} \quad (2)$$

$$\Psi(-t) = \frac{H'(-t)}{H(-t)} = \frac{S'(t)}{S(t)} \quad (3)$$

(1) implies that

$$s_0 h_0 = 1$$

$$\sum_{j=0}^{n} (-1)^j s_j h_{n-j} = 0 \quad n > 0 \quad (1')$$
Define $\omega : \Lambda^Z = \mathbb{Z}[s_1, s_2, \ldots ] \mapsto \Lambda^Z$ by $\omega(s_j) = h_j$. Since (1′) is symmetrical in $h, s$, we get that $\omega$ is an isomorphism. In particular, $\Lambda = \mathbb{Z}[h_1, h_2, \ldots ]$, and so the homogeneous functions form a basis. Applying $\omega$ to (1′) gives

$$0 = \sum_{j=0}^{n} (-1)^j h_j \omega(h_{n-j})$$

$$= \sum_{j=0}^{n} (-1)^{n-j} h_{n-j} \omega(h_j)$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^j \omega(h_j) h_{n-j} \quad \forall n > 0.$$ 

Comparing with (1′), we see that $\omega(h_n) = s_n$, ie. $\omega^2 = 1$ ($\omega$ is an involution).

By (2),

$$\sum_{n=1}^{\infty} n h_n t^{n-1} = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \psi_j h_{n-j} t^{n-1}$$

$$= \sum_{j=1}^{n} \psi_j h_{n-j} = nh_n \quad \forall n.$$ (2′)

Similarly,

$$\sum_{j=1}^{n} (-1)^{j-1} \psi_j s_{n-j} = ns_j \quad \forall n.$$ (3′)

Using (3′), each $s_n$ can inductively be written as a polynomial in $\mathbb{Q}[\psi_1, \ldots, \psi_n]$, so the power functions form a basis for $\Lambda^Z$.

Since $\omega$ interchanges $h, s$, comparing (2′) and (3′) gives

$$\omega(\psi_n) = (-1)^{n-1} \psi_n.$$ 

To see that the Schur Functions form a basis for $\Lambda^Z_n$, set $V_n := V_{(n-1, \ldots, 0)}$. Let $A_k$ be the set of skew symmetric polynomials of degree $k$ in $n$ variables. Then we have an isomorphism

$$\Lambda_n \mapsto A_{n+\left(\frac{n}{2}\right)}$$

$$f \mapsto fV_n$$

Since $\{F_A V_n\}$ is the “monomial” basis for $A_{n+\left(\frac{n}{2}\right)}$ (ie. the basis obtained by skew symmetrizing each monomial), $\{F_A\}$ forms a basis for $\Lambda_n$. 

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