UNIVERSITY OF TORONTO

MAT 1350F **Problems**

1. Show that $\mathbb{Q}$ is not rojective in the category of abelian groups.

2. Give an example of an additive category with a morphism in that category which has no kernel.

3. Let R be a commutative Rng. Must R be projective in the category of R-modules?

4. Give an example to show that (in spite of the fact that homology always with direct limits) homology need not commute with arbitrary colimits.

5. Give an example of a category and objects X and Y such that the canonical map $X \coprod Y \to X \prod Y$ is not a monomorphism.

6. Show that the category of torsion abelian groups has enough injectives but no nonzero projectives.

7. Give a version of the portion of the proof of the Snake Lemma in the notes (i.e. the construction of the connecting homomophism and exactness at $\operatorname{Ker} f''$) which is valid in any abelian category. That is, give the construction and the proof using universal properties rather than elements and diagram-chasing.

8. Show that in the category of abelian groups, any diagram

$$\begin{array}{ccc} A & \twoheadrightarrow & B \\ & & \downarrow \\ & & C \end{array}$$

can always be embeded in a bicartesian square (i.e. both pullback and pushout). Does this property remain valid in the category of R-modules over an arbitrary ring R?

(This problem might be more easily done using techniques discussed later in the course.)

9. Give an example of a module over some ring R which has no projective cover (in the category of R-modules).
   Note: The definition of projective cover of X is an epimorphism $p : P \to X$ such that P is projective, and $\ker p$ is a "superfluous submodule" of P, meaning that there is no proper submodule H of P such that $\ker p + H = P$. A consequence of the definition is that, given an epimorphism $p : P \to X$, if P contains a submodule N such that the composition $N \to P \to X$ is surjective, then p cannot be a projective cover. (This is the sense in which a projective cover is the "smallest" projective presentation of X.)

10. Give an example of chain complexs with the same homology which are not chain homotopy equivalent.

11. (Hilton Stambach: P46; #2.3)
Let $Z(G)$ denote the centre of the group G. Show that the association $G \mapsto Z(G)$ from Groups to Groups does not extend to morphisms to produce a functor in some "obvious" way.

12. Give a proof of the Algebraic Mapping Cylinder Theorem in the notes. Specifically: Let C and D be chain complexes and let $\phi : C \to D$ be a chain map. Show that there exists a chain complex $\tilde{D}$ and a chain homotopy equivalence $j : D \to \tilde{D}$ together with an injection $i : C \to \tilde{D}$ such that $i \simeq j \circ \phi$ and $\phi - k \circ i$, where k is a chain homotopy inverse to j.

13. Descibe how to grade and put an appropriate differential on $\mathrm{Hom}(C_*, D_*)$ to make it into a chain complex is a "useful" way.

14. Let $0 \to A' \to A \to A'' \to 0$ and $0 \to B' \to B \to B'' \to 0$ be short exact sequences of abelian groups. Give an example in which
   (i) $A' \cong B'$, $A \cong B$; but $A'' \ncong B''$
   (ii) $A' \cong B'$, $A'' \cong B''$; but $A \ncong B$
   (iii) $A \cong B$, $A'' \cong B''$; but $A' \ncong B'$

15. Prove the Mittag-Leffler Theorem. Specifically: Suppose A is an inverse system in which for each n there exists $k(n) \leq n$ such that $\mathrm{Im}\big(A_i \to A_n\big)$ equals $\mathrm{Im}\big(A_{k(n)} \to A_n\big)$ for all $i \leq k(n)$. Show that $\varprojlim_n{}^1 A = 0$.

16. Find the indecomposable projectives in the category of finite type chain complexes.
Note: A graded group is said to have"finite type" if each gradation is finitely generated.

17. Show that if $0 \to N \to P \to A \to 0$ and $0 \to M \to Q \to A \to 0$ are exact with P and Q projective, then $P \oplus M \cong Q \oplus N$.

18. Give an example of a sequence $T' \to T \to T''$ of functors and natural transformations between abelian categories which has the property that $T'(P) \to T(P) \to T''(P)$ is exact for every projective object P, but $T'(X) \to T(X) \to T''(X)$ is not exact for every object X.

19. Verify that the derived couple of an exact couple is an exact couple. Is the category of exact couples an abelian category?

20. Use the spectral sequence an appropriate double complex to very that $\mathrm{Tor}(M, N)$ is independent of which variable we take the projective resolution of.

21. Compute the cohomology with integer coefficients pf the group G given by the presentation $G = \langle x, y \rangle$ subject to the relation $x^2 = y^3$.