UNIVERSITY OF TORONTO

MAT 1345F **Problems**

1. (Milnor, p11; #1Bb) For $x = [x_1, x_2, \ldots, x_{n+1}] \in \mathbb{R}P^n$, define
$$f_{ij} := \frac{x_i x_j}{\sum_{k=1}^{n+1} x_k^2} \in \mathbb{R}.$$
Show that f is a diffeomorphism from $\mathbb{R}P^n$ to the set of symmetric matrices $n+1 \times n+1$ matrices A of trace 1 which satisfy $AA = A$.

2. (Milnor, p23; #2C) Using a partition of unity, show that any vector bundle over a paracompact base space can be given a Euclidean metric.

3. Show that the collection of isomorphism classes of line bundles over a space X forms a group under tensor product. Explain why in the case of real line bundles every element of this group always has order 2, but that this is not true in the case of complex line bundles.
Note: A direct proof is expected, not one using a classification theorem.

4. Is the squaring map $\mathrm{sq} : SU(2) \to SU(2)$ given by $A \mapsto A^2$ a principal $\mathbb{Z}/(2\mathbb{Z})$ bundle?

5. (Husemoller; p22; #1) Prove that $T(S^{n+q})\big|_{S^n} \cong T(S^n) \oplus \epsilon^q$, where $T(M)$ denotes the tangent bundle to M, and ϵ^q denotes a trivial bundle of dimension q.

6. Let $U_1 = \{[a, b, c] \in \mathbb{C}P^2 \mid a \neq 0\}$, $U_2 = \{[a, b, c] \in \mathbb{C}P^2 \mid b \neq 0\}$, and $U_3 = \{[a, b, c] \in \mathbb{C}P^2 \mid c \neq 0\}$. Then $\{U_1, U_2, U_3\}$ form a cover of $\mathbb{C}P^2$ by Euclidean charts. (This is standard; you do not need to prove it.) Describe the transition functions for the canonical line bundle γ_1^2 over $\mathbb{C}P^2$ in terms of its local trivialization coming from these charts.

7. Let γ_1^n be the canonical line bundle over $\mathbb{R}P^n$. Show that the pullback of γ_1^n to S^n under the covering projection $q : S^n \to \mathbb{R}P^n$ is a trivial bundle.

8. Prove the following isomorphisms of topological groups.
   a) $SO(2) \cong S^1$
   b) $SO(3) \cong \mathbb{R}P^3$
   c) $U(1) \cong S^1$
   d) $SU(2) \cong S^3$

   Prove that $U(2)$ is isomorphic to $S^1 \times S^3$ as topological spaces (i.e. homeomorphic) but not as topological groups.
   Note: The group structures on S^1 and S^3 are defined as the restriction to the unit ball of the multiplications in the complexes $\mathbb{C}$ and quaternions $\mathbb{H}$ respectively. The

multiplication on $\mathbb{R}P^3$ is defined to be that induced from the multiplication on S^3 via the quotient map $S^3 \to \mathbb{R}P^3$.

9. Calculate the cohomology ring $H^*\big(SU(3)\big)$.

10. a) Show that $\mathbb{R}P^{19}$ cannot be immersed in $\mathbb{R}^{30}$.
    b) When using the method of calculating characteristic classes for the normal bundle to give a lower bound for N such that $\mathbb{R}P^n$ can be immerse in $\mathbb{R}^N$, you might try to be clever and make use of the fact that an immersion of $\mathbb{R}P^{n+k}$ in $\mathbb{R}^N$ automatically gives one for $\mathbb{R}P^n$. If you were to get lucky then perhaps you could show all your friends how smart you are by getting better bounds for $\mathbb{R}P^n$ than they are getting by applying the method to $\mathbb{R}P^{n+k}$ for some $k > 0$ and then restricting back to $\mathbb{R}P^n$ as above. Show that you never get lucky!

11. Let $p : E \to B$ be a fibration. Suppose that p has a homotopy cross-section. That is, suppose there exists $s : B \to E$ such that $p \circ s \simeq 1_B$. Show that there exists $s' : B \to E$ such that $p \circ s' = 1_B$.

12. Let X be an H-space. Suppose Y is a pointed homotopy retract of X. That is, suppose that there exist basepoint-preserving maps $f : X \to Y$ and $s : Y \to X$ such that $f \circ s \simeq 1_Y$ via basepoint-preserving homotopies. Show that Y is an H-space.
    Note: If X happens to be homotopy associative, or in fact even if X is a topological group, it does not imply that Y is homotopy associative.

13. Show that $SU(3)$ is not homotopy equivalent to $SU(2) \times S^5$.