Contents

Chapter 1

Sets

Notation:
$f : X \to Y \qquad A \subset X \qquad B \subset Y$
$f(A) := \{f(a) \mid a \in A\} \subset Y$
$f^{-1}(B) := \{x \in X \mid f(x) \in B\}$

Note: $f^{-1}(\cap_{\alpha\in I} V_\alpha) = \cap_{\alpha\in I} f^{-1}(V_\alpha)$
$f^{-1}(\cup_{\alpha\in I} V_\alpha) = \cup_{\alpha\in I} f^{-1}(V_\alpha)$
$f(P \cup Q) = f(P) \cup f(Q)$ but in general $f(P \cap Q) \neq f(P) \cap f(Q)$

Theorem 1.0.1 *The following are equivalent (assuming the other standard set theory axioms):*

1. *Axiom of Choice*
2. *Zorn's Lemma*
3. *Zermelo well-ordering principle*

where the definitions are as follows.

Axiom of Choice: Given sets A_α for $\alpha \in I$, $A_\alpha \neq \emptyset \Rightarrow \prod_{\alpha\in I} \neq \emptyset$
(i.e. may choose $a_\alpha \in A_\alpha$ for each $\alpha \in I$ to form an element of the product)

To state (2) and (3):

Definition 1.0.2 *A* partially ordered set *consists of a set X together with a relation $\leq$ s.t.*

1. $x \leq x \qquad \forall x \in X \qquad$ *reflexive*
2. $x \leq y, y \leq z \Rightarrow x \leq z \qquad$ *transitive*

3. $x \le y, y \le x \Rightarrow x = y$ *(anti)symmetric*

Notation: $b \ge a$ means $a \le b$.
If X is a p.o. set:

Definition 1.0.3 1. *m is* maximal *if $m \le x \Rightarrow m = x$.*

2. *For $Y \subset X$, an element $b \in X$ is called an* upper bound*for Y if $y \le b$ $\forall y \in Y$. an element $b \in X$ is called an* lower bound*for Y if $y \ge b$ $\forall y \in Y$.*

3. *X is called* totally ordered *if $\forall x, y \in X$, either $x \le y$ or $y \le x$. A totally ordered subset of a p.o. set is called a* chain.

4. *X is called* well ordered *if each $Y \ne \emptyset$ has a least element. i.e. if $\forall Y \ne \emptyset$, $\exists y_0 \in Y$ s.t. $y_0 \le y$ $\forall y \in Y$.*

Remark 1.0.4 *In contrast to "well-ordered", which requires the element y_0 to lie in Y, a "lower bound" is an elt. of X which need not lie in Y.*

Note: well ordered $\Rightarrow$ totally ordered (given x, y apply defn. of well ordered to the subset $\{x, y\}$), but totally ordered $\not\Rightarrow$ well ordered (e.g. $X = \mathbb{Z}$).

Zorn's Lemma: A partially ordered set having the property that each chain has an upper bound (the bound not necessarily lying in the set) must ahve a maximal element.

Zermelo's Well-Ordering Principle: Given a set X, $\exists$ relation $\le$ on X such that $(X, \le)$ is well-ordered.

Proof of Theorem:

2$\Rightarrow$ 3:

Given X, let $\mathcal{S} := \{(A, \le_A) \mid A \subset X \text{ and } (A, \le_A) \text{ well ordered }\}$
Define order on $\mathcal{S}$ by:

$$(A \le_A) \le (B, \le_B) \text{ if } A \subset B \text{ and } \begin{cases} a \le_B a' \Leftrightarrow a \le_A a' & \forall a, a' \in B \\ a \le b & \forall a \in A, b \in B - A \end{cases}$$

(i) This is a partial order

Trivial. e.g. Symmetry: If $(A, \le_A) \le (B, \le_B) \le (A, \le_A)$ then $A \subset B \subset A$ so $A = B$ and defns. imply order is the same.

(ii) If $\mathcal{C} = \{(A, \le_A)\}$ is a chain in $\mathcal{S}$ then $(Y := \cup_{A \in \mathcal{C}}, \le_Y)$ is an upper bound for $\mathcal{C}$ where $\le_Y$ is defined by:

If $y, y' \in Y$, find $A, A' \in \mathcal{C}$ s.t. $y \in A, y' \in A'$.

$\mathcal{C}$ chain$\Rightarrow$ A, A comparable$\Rightarrow$ larger (say A) contains both y, y'.

So define $y \leq_Y y' \Leftrightarrow y \leq_A y'$.

To qualify as an upper bound for $\mathcal{C}$, must check that $Y \in \mathcal{S}$. i.e. Show Y is well-ordered.

Proof: For $\emptyset \neq W \subset Y$, find $A_0 \in \mathcal{C}$ s.t. $W \cap A_0 \neq \emptyset$.

$A_0 \in \mathcal{S} \Rightarrow A_0$ well-ordered$\Rightarrow W \cap A_0$ has a least elt. w_0.

$\forall w \in W,\ \exists A \in \mathcal{C}$ s.t. $w \in A$.

$\mathcal{C}$ chain$\Rightarrow A_0, A$ comparable in $\mathcal{S}$.

If $A \subset A_0$ then $w \in A_0$ so $w_0 \leq w$ ($w_0 =$ least elt. of A_0).

If $A_0 \subset A$ then $w_0 \leq w$ by defn. of ordering on $\mathcal{S}$.

Therefore $w_0 \leq w\ \forall w \in W$ so every subset of Y has a least elt.

Therefore Y is well-ordered.

Hence $(Y, \leq_Y)$ belongs to $\mathcal{S}$ and forms an upper bound for $\mathcal{C}$.

So Zorn applies to $\mathcal{S}$. Therefore $\mathcal{S}$ has a maximal elt. $(M \leq_M)$.

If $M \neq X$, let $x \in X - M$ and set $M' := M \cup \{x\}$ with $x \geq a\ \forall a \in M$.

Then $(M', \leq) \not\leq (M, \leq)$. $\Rightarrow\Leftarrow$

Therefore $M = X$.

Hence $\leq_M$ is a well-ordering on X.

3$\Rightarrow$ 1:

Well order $\cup_\alpha A_\alpha$. For each α, let $a_\alpha :=$ least elt. of A_α. Then $(a_\alpha)_{\alpha \in A}$ is an elt. of $\prod_\alpha A_\alpha$.

$\square$

Standard consequences of Zorn's Lemma:

1. Every vector space has a basis. (Choose a maximal linearly independent set)
2. Every proper ideal of a ring is contained in a maximal proper ideal
3. There is an injection from $\mathbb{N}$ to every infinite set.

1.1 Ordinals

Definition 1.1.1 *If W is well ordered, an* ideal *in W is a subset W' s.t. $a \in W', b \leq a \Rightarrow b \in W'$.*

Note: Ideals are well-ordered.

Lemma 1.1.2 *Let W' be an ideal in W. Then either $W' = W$ or $W' = \{w \in W \mid w < a\}$ for some $a \in W$.*

Proof: If $W' \neq W$, let a be least elt. of $W - W'$. If $x < a$ then $x \in W'$.
Conversely if $x \in W'$:
If $a \leq x$ then $a \in W' \Rightarrow\Leftarrow$
Therefore $x < a$. □

Corollary 1.1.3 *If I, J are ideals of W then either $I \subset J$ or $J \subset I$.*

Notation: $\mathrm{Init}_a := \{w \in W \mid w < a\}$ called an *initial interval*

Proof of Cor. If $I = \mathrm{Init}_a$ and $J = \mathrm{Init}_b$, compare a and b. □

Theorem 1.1.4 *Let X, Y be well ordered. Then*

either a) $Y \cong X$

or b) $Y \cong$ an initial interval of X

or c) $X \cong$ an initial interval of Y

The relevant iso. is always unique.

Lemma 1.1.5 *A, B well-ordered. Suppose $\zeta : A \to B$ is a morphism of p.o. sets mapping A isomorphically to an ideal of B. Let $f : A \to B$ be an injection of p.o. sets. Then $\zeta(a) \leq f(a)$ $\forall a \in A$.*

Proof: If non-empty $\{a \in A \mid \zeta(a) > f(a)\}$ has a least elt. a_0.
$\zeta(a_0) > f(a_0)$.
Since $\mathrm{Im}\,\zeta$ is an ideal, $f(a_0) = \zeta(a)$ for some $a \in A$.
$\zeta(a_0) > \zeta(a) \Rightarrow a_0 > a$ (ζ p.o set injection)
Choice of $a_0 \Rightarrow f(a_0) = \zeta(a) \leq f(a)$
$\Rightarrow a_0 \leq a$ (f p.o. set injection)
$\Rightarrow\Leftarrow$
Therefore $\zeta(a) \leq f(a)$ $\forall a$. □

Proof of Thm. From Lemma, if ζ_1, ζ_2 are both isos. from A onto ideals of B (not necess. the same ideal)

$\forall a$, $\zeta_1(a) \le \zeta_2(a) \le \zeta_1(a) \Rightarrow \zeta_1(a) = \zeta_2(a)$.

Therefore $\zeta_1 = \zeta_2$. So uniqueness part of thm. follows.

Claim: $X \ncong$ an initial interval of itself

Proof: If $g : X \cong I$ where $I = \mathrm{Init}_a$,

$I \xrightarrow{j} X$ and $I \xrightarrow{j} X \overset{g}{\cong} I \xrightarrow{j} X$ map I isomorphically onto an ideal of X. (i.e. If $b \le jgjx = g(jx) \in I$ then $b \in I$ since I ideal $\Rightarrow b = g(y)$ some y. Then $g(y) \le g(jx) \Rightarrow y \le jx \Rightarrow y \in I \Rightarrow y = j(y)$, so $b \in \mathrm{Im}\, jgj$.)

Therefore $j = jgj$ (Lemma).

Impossible since $jg(a) \in \mathrm{Im}\, j$ whereas $g(a) \notin \mathrm{Im}\, jgj$ (i.e. $a > jc\ \forall x \in I \Rightarrow jg(a) > jgj(x) \Rightarrow jg(a) \notin \mathrm{Im}\, jgj$)

Therefore at most one of (a), (b), (c) holds.

Let $\Sigma :=$ set of ideals of X which are isomorphic to some ideal of Y, ordered by inclusion.

$(K := \Sigma = \cup_{I \in \Sigma} I)$ is an ideal in X

For each $I \in \Sigma$, let $\zeta_I : I \to Y$ be the unique map taking I isomorphically onto an ideal of Y.

Therefore If $J \subset I$, $\zeta_J = \zeta_I|_J$.

So the ζ_I's induce a map $\zeta : K \to Y$ which takes K isomorphically to an ideal of Y. (i.e. If $y < \zeta(K)$, find I s.t. $k \in I$. Then ζ_I iso. to its image $\Rightarrow y = \zeta_I(l)$ for some l. Therefore $\mathrm{Im}\, \zeta$ is an ideal. And ζ is an injection: Remember, given two elts. $a \in I$, $a' \in I'$ either $a \le a'$ in which case $a \in I'$ or reverse is true.)

Therefore $K \in \Sigma$.

If (both) $K \ne X$ and $\zeta(K) \ne Y$, let x, y be least elts. of $X - K$, $Y - \zeta(K)$ respectively. Extend ζ by defining $\zeta(x) = y$ to get an iso. from $K \cup \{x\}$ to the ideal $\zeta(K) \cup \{y\}$ of Y. Contradicts defn. of $K \Rightarrow\Leftarrow$

So either $K = X$ or $\zeta(K) = Y$ or both, giving the 3 cases. □

Corollary 1.1.6 *Let $g : X \to Y$ be an injective poset morphism between between well-ordered posets. Then*

either a) $X \cong Y$

or b) $X \cong$ *initial interval of* Y

(i.e. $Y \ncong$ initial interval of X in previous thm.)

Proof: If $h : Y \cong$ initial interval of X then

$f : Y \xrightarrow{h}$ initial interval of $X = \mathrm{Init}(a) \hookrightarrow X \xrightarrow{g} Y$ is an injection from Y to Y. Applying earlier Lemma with $\zeta = 1_Y$ gives $g \le f(y)\ \forall y \in Y$.

But $\operatorname{Im} f \subset \operatorname{Init}\big(g(a)\big)$ so $y < g(a)\ \forall y \in Y$ (i.e. $y \le f(y) < g(a)$)
$\Rightarrow\Leftarrow$ (letting $y = g(a)$) $\square$

Definition 1.1.7 *An* ordinal *is an isomorphism class of well ordered sets.*

(Generally we refer to an ordinal by giving a representative set.)

Example 1.1.8

1. $\underline{\underline{n}} := \{1, \ldots, n\}$ *standard order*

2. $\omega := \mathbb{N}$ *standard order*

3. $\omega + \underline{\underline{n}} := \mathbb{N} \amalg \underline{\underline{n}}$ *with the ordering* $x < y$ *if* $x \in \mathbb{N}$ *and* $y \in \underline{\underline{n}}$ *and standard ordering if both* $x, y \in \mathbb{N}$ *or both* $x, y \in \underline{\underline{n}}$

   Note: $\amalg :=$ *disjoint union (i.e. union of* $\mathbb{N}$ *with a set isomorphic to* $\underline{\underline{n}}$ *containing no elts. of* $\mathbb{N}$*.)*

4. $2\omega = \mathbb{N} \amalg \mathbb{N}$*)*

Note: For any ordinal *gamma* there is a "next" ordinal $\gamma + 1$, but there is not necessarily an ordinal τ such that $\gamma = \tau + 1$.

Transfinite induction principle: Suppose W is a well ordered set and $\{P(x) \mid x \in W\}$ is a set of propositions such that:
(i) $P(x_0)$ is true where x_0 is the least elt. of W
(ii) $P(y)$ true for $\forall\ y < x \Rightarrow P(x)$ true
Then $P(x)$ is true $\forall\ x$.

1.2 Cardinals

Theorem 1.2.1 *(Shroeder-Bernstein). Let* X, Y *be sets. Then*

1. *Either* $\exists$ *injection* $X \hookrightarrow Y$ *or* $\exists$ *injection* $Y \hookrightarrow X$.

2. *If both injections exist then* $X \cong Y$

Proof: 1. Choose well ordering for X and for Y. Then use iso. of one with other or with ideal of other to define injection.

2. Suppose $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow X$. Choose well ordering for X and Y. If $\exists x \in X$ s.t. X is bijection with $\text{Init}(x)$, let x_0 be least such x. So in this case X is bijective with $\text{Init}(x_0)$ but not with any ideal of $\text{Init}(x_0)$. Replacing X by $\text{Init}(x_0)$ we may assume that X is not bijective with any of its ideals. (And in the case where $\nexists x \in X$ s.t. X is bijective with $\text{Init}(x)$ then this is clearly also true.) Similarly may assume that Y well ordered such that it is not bijective with any of its ideals. Assuming $X \not\cong Y$, one is iso. to an ideal of the other. Say $Y \cong \text{Init}(x)$. The inclusion $i : X \hookrightarrow Y$ induces a new well-order $(X, \prec)$ on X. from that on Y. By earlier Corollary, either $\exists$ iso. $\zeta : (X, \prec) \to Y$ or $\exists$ iso. $\zeta : (X \prec) \to \text{Init}(y)$ for some $y \in Y$. In the former case we are finished, so suppose the latter. $(X, \prec)) \overset{\zeta}{\cong} \text{Init}(y) \hookrightarrow Y \xrightarrow{\cong} \text{Init}(x)$ gives a bijection from X to an initial interval of X. (Note: Image of init interval under iso. is an init interval, and an init interval within an init interval is an init interval.)

$\Rightarrow\Leftarrow$

Therefore X is bijective with Y. $\square$

Definition 1.2.2 *A* cardinal *is an isomorphism class of sets. (In this context "isomorphism" means "bijection".)*

$\text{card}\, X = \text{card}\, Y$ *means* $\exists$ *bijection from* X *to* Y*.*

$\text{card}\, X \leq \text{card}\, Y$ *means* $\exists$ *injection from* X *to* Y*.*

(Thus previous Thm. says: $\text{card}\, X \leq \text{card}\, Y$ and $\text{card}\, Y \leq \text{card}\, X \Rightarrow \text{card}\, X = \text{card}\, Y$)

1.3 Countable and Uncountable Sets

Definition 1.3.1 *A set is called* countable *if either finite or numerically equivalent (i.e.* $\exists$ *a bijection) to the nature numbers* $\mathbb{N}$*. A set which is not countable is called* uncountable.

Example 1.3.2 *1. Even natural numbers*

2. Integers

3. Positive rational numbers $\mathbb{Q}^+$*.* **Proof:** *Define an ordering on* $\mathbb{Q}^+$ *by* $a/b \prec c/d$ *if* $\big(a + b < c + d$ *or* $(a + b = c + d$ *and* $a < c)\big)$ *where* a/b*,* c/d *are written in reduced form.*

e.g. 1, 1/2, 2, 1/3, 3, 1/4, 2/3, 3/2, 4, 1/5, 5, ...

For $f \in \mathbb{Q}^+$*, let* $S_r = x \in \mathbb{Q}^+ \mid x \leq r\}$*. This set is finite for each* r *so define* $f(r) = \|S_r|$*.*

Proposition 1.3.3 *A subset of a countable set is countable.*

Proof: Let A be a subset of X and let $f : X \to \mathbb{N}$ be a bijection. Define $g : A \to \mathbb{N}$ by $g(a) := \big|\{b \in A \mid f(b) \leq f(a)\}\big|$. $\square$

Proposition 1.3.4 *Let $g : X \to Y$ be onto. If X is countable then Y is.*

Proof: Let $f : X \to \mathbb{N}$ be a bijection. For $y \in Y$, set $h(y) := \min\{f(x) \mid g(x) = y\}$. Then h is a bijection between Y and some subset of $\mathbb{N}$ so apply prev. prop. $\square$

Proposition 1.3.5 *X, Y countable $\Rightarrow X \times Y$ countable.*

Proof: Use diagonal process as in pf. that rationals are countable. (Exercise.) $\square$

Theorem 1.3.6 *(Cantor). $\mathbb{R}$ is uncountable.*

Proof: Suppose $\exists$ bijection $f : \mathbb{R} \to \mathbb{N}$. Let $g : \mathbb{N} \to \mathbb{R}$ be the inverse bijection. For each $n \in \mathbb{N}$ define

$$a_n := \begin{cases} 1 & \text{if } n\text{th interger after decimal pt. in decimal expansion of } g(n) \text{ is not } 1 \\ 2 & \text{if } n\text{th interger after decimal pt. in decimal expansion of } g(n) \text{ is } 1 \end{cases}$$

Therefore $a_n \neq n$th integer after dec. pt. in the dec. expansion of $g(n)$. Let a be the real number represented by the decimal $0.a_1a_2a_3\cdots$. (i.e. a is defined as the limit of the convergent series $a_1/10 + a_2/100 + a_3/1000 + \ldots + a_n/(10^n) + \ldots$.) Let $f(a) = m$ or equivalently $g(m) = a$. Then $a_m = m$th integer after dec. pt. in dec. expansion of $g(m)$, contradicting defn. of a_m.

$\Rightarrow\Leftarrow$

Therefore no such bijection f exists. $\square$

Chapter 2

Topological Spaces

2.1 Metric spaces

Definition 2.1.1 *A* **metric space** *consists of a set X together with a function $d : X \times X \to \mathbb{R}^+$ s.t.*

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x) \qquad \forall x, y$
3. $d(x, z) \leq d(x, y) + d(y, z) \qquad \forall x, y, z \qquad$ *triangle inequality*

Example 2.1.2 Examples

1. $X = \mathbb{R}^n$
2. $X = \{\text{continuous real-valued functions on } [0, 1]\}$
   $d(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$
3. $X = \{\text{bounded linear operators on a Hilbert space } H\}$
   $d(f, g) = \sup_{x \in H} ||A(x) - B(x)|| =: ||A - B||$
4. X *any*
   $$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Notation:

$$N_r(a) = \{x \in X \mid d(x,a) < r\} \qquad \text{open } r\text{-ball centred at } a$$
$$N_r[a] = \{x \in X \mid d(x,a) \leq r\} \qquad \text{closed } r\text{-ball centred at } a$$

Definition 2.1.3 *A map $\phi : X \to Y$ is* **continuous at** *a if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $d(x,a) < \delta \Rightarrow d\big(\phi(x), \phi(a)\big) < \epsilon$. ϕ is called* **continous** *if ϕ is continuous at a for all $a \in X$.*

Equivalently, ϕ is continuous if $\forall \epsilon\ \exists \delta$ such that $\phi\big(N_\delta(a)\big) \subset N_\epsilon\big(\phi(a)\big)$.

Definition 2.1.4 *A sequence $(x_i)i \in \mathbb{N}$ of points in X* **converges** *to $\bar{x} \in X$ if $\forall \epsilon, \exists M$ s.t. $n \geq M \Rightarrow x_i \in N_\epsilon(\bar{x})$*

We write $(x_i) \to \bar{x}$.
Exercise: $(x_i) \to x$ in $X \Leftrightarrow d(x_i, x) \to 0$ in $\mathbb{R}$.

Proposition 2.1.5 *If (x_i) converges to $\bar{x}$ and (x_i) converges to $\bar{y}$ then $x = y$.*

Proof: Show $d(x,y) < \epsilon\ \forall \epsilon$. □

Proposition 2.1.6 *$f : X \to Y$ is continuous $\Leftrightarrow \big((x_i) \to \bar{x} \Rightarrow \big(f(x_i)\big) \to f(\bar{x})\big)$*

Proof: $\Rightarrow$ Suppose f continuous. Let $(x_i) \to \bar{x}$.
Given $\epsilon > 0, \exists \delta$ s.t. $f\big(N_\delta(\bar{x})\big) \subset N_\epsilon(\bar{x})$
Since $(x_i) \to \bar{x}$, $\exists M$ s.t. $n \geq M \Rightarrow x_i \in N_\delta(\bar{x}) \therefore n \geq M \Rightarrow f(x_i) \in N_\epsilon\big(f(\bar{x})\big)$.
$\Leftarrow$ Suppose that $\big((x_i) \to \bar{x} \Rightarrow \big(f(x_i)\big) \to f(\bar{x})\big)$
Assume f not cont. at a for some $a \in X$. Then $\exists \epsilon > 0$ s.t. there is no δ s.t. $f\big(N_\delta(\bar{x})\big) \subset N_\epsilon(\bar{x})$. Thus $\exists \epsilon > 0$ s.t. for every δ there is an $x \in N_\delta(\bar{x})$ s.t. $f(x) \notin N_\epsilon(\bar{x})$ Therefore we can select, for each integer n, an $x_n \in N_{1/n}(\bar{x})$ s.t. $f(x_n) \notin N_\epsilon(\bar{x})$. Then $(x_n) \to x$ but $f(x_n) \not\to f(\bar{x})$. $\Rightarrow\Leftarrow$

Definition 2.1.7 *An* **open set** *is a subset U of X s.t. $\forall x \in U$ existsϵ s.t. $N_\epsilon \subset U$.*

Proposition 2.1.8

1. *U_α open $\forall \alpha \Rightarrow \cup_{\alpha \in I} U_\alpha$ is open*
2. *U_α open $\forall \alpha, |I| < \infty \Rightarrow \cup_{\alpha \in I} U_\alpha$ is open*

Proof:

1. Let $x \in V = \cup_{\alpha \in I} U_\alpha$. So $x \in U_\alpha$ for some α.
   $\therefore N_\epsilon \subset U_\alpha \subset V$ for some ϵ.

2. Number the sets $U_1, \ldots, U_n$.
   Let $x \in V = \cap_{j=1}^{n} U_j$. So $\forall j$ $\exists \epsilon_j$ s.t. $N_{\epsilon_j}(x) \subset U_j$.
   Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $N_\epsilon(x) \subset V$.

$\square$

Note: An infinite intersecion of open sets need not be open. For example, $\cap_{n \geq 1}(-1/n, 1/n) = \{0\}$ in $\mathbb{R}$.

Lemma 2.1.9 *$N_r(x)$ is open $\forall x$ and $\forall r > 0$.*

Proof: Let $y \in N_r(x)$. Set $d = d(x, y)$. Then $N_{r-d}(y) \subset N_r(x)$ (and $r - d > 0$ since $y \in N_r(x)$). $\square$

Corollary 2.1.10 *U is open $\Leftrightarrow U = \cup N_\alpha$ where each N_α is an open ball*

Proof: $\Leftarrow$ N_α open $\forall \alpha$ so $\cup N_\alpha$ is open. $\Rightarrow$ If U open then for each $x \in U, \exists \epsilon_x$ s.t. $N_{\epsilon_x} \subset U$. $U = \cup_{x \in U} N_{\epsilon_x}(x)$. $\square$

Proposition 2.1.11 *$f : X \to Y$ is continuous $\Leftrightarrow \forall$ open $U \subset Y$, $f^{-1}(U)$ is open in X*

Proof: $\Rightarrow$ Suppose f continuous. Let $U \subset Y$ be open.

Given $x \in f^{-1}(U)$, $f(x) \in U$ so $\exists \epsilon > 0$ s.t. $N_\epsilon\big(f(x)\big) \subset U$. Find $\delta > 0$ s.t. $f\big(N_\delta(x)\big) \subset N_\epsilon\big(f(x)\big)$. Then $N_\delta(x) \subset f^{-1}\Big(N_\epsilon\big(f(x)\big)\Big) \subset f^{-1}(u)$.

$\Leftarrow$ Suppose that the inverse image of every open set is open.

Let $x \in X$ and assume $\epsilon > 0$.

Then $x \in f^{-1}\Big(N_\epsilon\big(f(x)\big)\Big)$ and $f^{-1}\Big(N_\epsilon\big(f(x)\big)\Big)$ is open so $\exists \delta$ s.t. $N_\delta(x) \subset f^{-1}\Big(N_\epsilon\big(f(x)\big)\Big)$ That is, $f\big(N_\delta(x)\big) \subset N_\epsilon\big(f(x)\big)$

$\therefore f$ continuous at x. $\square$

Note: Although the previous Prop. shows that knowledge of the open sets of a metric space is sufficient to determine which functions are cont., it is not sufficient to determine the metric. That is, different metrics may give rise to the same collection of open sets.

2.2 Norms

Let V be a vector space of F where $F = \mathbb{R}$ or $F = \mathbb{C}$.

Definition 2.2.1 *A* **norm** *on V is a function $V \to \mathbb{R}$, written $x \mapsto ||x||$, which satisfies*

1. $||x|| \geq 0$ *and* $||x|| = 0 \Leftrightarrow x = 0.$
2. $||x_y|| \leq ||x|| + ||y||$
3. $||\alpha x|| = |\alpha|\,||x|| \qquad \forall \alpha \in F, x \in V$

Given a normed vector space V, define metric by $d(x, y) = ||x - y||$.

Proposition 2.2.2 *(V, d) is a metric space*

Proof: Check definitions. □

2.3 Topological spaces

Definition 2.3.1 *A topological space consists of a set X and a set $\mathcal{T}$ of subsets of X s.t.*

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$
2. *For any index set I, if $U_\alpha \in \mathcal{T} \forall \alpha \in I$, then $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$.*
3. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$.

Definition 2.3.2 Open sets
The subsets of X which belong to $\mathcal{T}$ are called open.

If $x \in U$ and U is open then U is called a *neighbourhood* of X.

If $\mathcal{S} \subset \mathcal{T}$ has the property that each $V \in \mathcal{T}$ can be written as a union of sets from $\mathcal{S}$, then $\mathcal{S}$ is called a *basis* for the topology $\mathcal{T}$.

If $\mathcal{S} \subset \mathcal{T}$ has the property that each $V \in \mathcal{T}$ can be written as the union of finite intersections of sets in $\mathcal{S}$ then $\mathcal{S}$ is called a *subbasis*, in other words $V = \cup_\alpha \left(\cap_{i_1,\ldots,i_\alpha} S_{i_\alpha}\right)$

Given a set X and a set $\mathcal{S} \subset 2^X$ (the set of subsets of X), $\exists!$ topology $\mathcal{T}$ on X for which $\mathcal{S}$ is a subbasis. Namely, $\mathcal{T}$ consists of all sets formed by taking arbitrary unions of finite intersections of all sets in $\mathcal{S}$.

(Have to check that the resulting collection is closed under unions and finite intersections — exercise)

In contracts, a set $\mathcal{S} \subset 2^X$ need not form a basis for any topology on X. $\mathcal{S}$ will form a basis iff the intersection of 2 sets in $\mathcal{S}$ can be written as the union of sets in $\mathcal{S}$.

Definition 2.3.3 Continuous *Let $f : X \to Y$ be a function between topological spaces. f is* continuous *if U open in $Y \Rightarrow f^{-1}(U)$ open in X.*

Note: In general f(open set) is not open. For example, $f =$ constant map : $\mathbb{R} \to \mathbb{R}$.

Proposition 2.3.4 *Composition of continuous functions is continuous.*

Proof: Trivial □

Proposition 2.3.5 *If $\mathcal{S}$ is a subbasis for the topology on Y and $f^{-1}(U)$ is open in X for each $U \in \mathcal{S}$ then f is continuous.*

Proof: Check definitions. □

2.4 Equivalence of Topological Spaces

Recall that a *category* consists of objects and morphisms between the objects.

For example, sets, groups, vector spaces, topological spaces with morphisms given respectively by functions, group homomorphisms, linear transformations, and continuous functions.

(We will give a precise definition of category later.)

In a any category, a morphism $f : X \to Y$ is said to have a **left inverse** if $\exists g : Y \to X$ s.t. $g \circ f = 1_X$.

A morphism $f : X \to Y$ is said to have a **right inverse** if $\exists g : Y \to X$ s.t. $f \circ g = 1_Y$.

A morphism $g : Y \to X$ is said to be an **inverse** to f is it is both a left and a right inverse. In this case f is called **invertible** or an **isomorphism**.

Proposition 2.4.1

1. *If f has a left inverse g and a right inverse h then $g = h$ (so f is invertible)*

2. *A morphism has at most one inverse.*

Proof:

1. Suppose $g \circ f = 1_x$ and $f \circ h = 1_Y$.

   Part of the definition of category requires that composition of morphisms be associative.

   Therefore $h = 1_X \circ h = g \circ f \circ h = g \circ 1_Y = g$.

2. Let g, h be inverses to f. Then in particular g is a left inverse and h a right inverse so $g = h$ by (1).

□

Intuitively, isomorphic objects in a category are equivalent with regard to all properties in that category.

Some categories assign special names to their isomorphisms. For example, in the category of Sets they are called "bijections". In the category of topological spaces, the isomorphisms are called "homeomorphisms".

Definition 2.4.2 Homeomorphism *A continuous function $f : X \to Y$ is called a* homeomorphism *if there is a continuous function $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.*

Remark 2.4.3 *Although the word "homeomorphism" looks similar to "homomorphism" it is more closely analogous to "isomorphism".*

Note: In groups, the set inverse to a bijective homomorphism is always a homomorphism so a bijective homomorphism is an isomorphism. In contrast, a bijective continuous map need not be a homeomorphism. That is, its inverse might not be continuous. For example

$X = [0, 1) \qquad Y = \text{unit circle in } \mathbb{R}^2 = \mathbb{C}$

$f : X \to Y$ by $f(t) = e^{2\pi it}$.

2.5 Elementary Concepts

Definition 2.5.1 Complement *If $A \subset X$, the complement of A in X is denoted $X \smallsetminus A$ or A^c.*

Definition 2.5.2 Closed *A set A is closed if its complement is open.*

Definition 2.5.3 Closure *The* closure *of A (denoted $\overline{A}$) is the intersection of all closed subsets of X which contain A.*

Proposition 2.5.4 *Arbitrary intersections and finite unions of closed sets are closed.*

Definition 2.5.5 Interior *The* interior *of A (denoted $\overset{\circ}{A}$ or $\operatorname{Int} A$) is the union of all open subsets of X which contained in A.*

Proposition 2.5.6 *$x \in \overset{\circ}{A} \Leftrightarrow \exists U \subset A$ s.t. U is open in X and $x \in U$.*

Proof: $\Rightarrow$ If $x \in \overset{\circ}{A}$, let $U = \overset{\circ}{A}$.

$\Leftarrow$ $x \in U \subset A$. Since U is open, $U \subset \overset{\circ}{A}$, so $x \in A$. $\square$

Proposition 2.5.7 $\left(\overline{A}\right)^c = (\mathring{A^c})$

Proof: Exercise $\square$

Corollary 2.5.8 *If* $x \notin \overline{A}$ *then* $\exists$ *open* U *s.t.* $x \in U$ *and* $U \cap A = \emptyset$. $\square$

Definition 2.5.9 Dense *A subset* A *of* X *is called* dense *if* $\bar{A} = X$.

Definition 2.5.10 Boundary *Let* X *be a topological space and* A *a subset of* X*. The* boundary *of* A *(written* ∂A*) is*

$\{x \in X \mid$ *each open set of* X *containing* x *contains at least one point from* A *and at least one from* $A^c\}$

Proposition 2.5.11 *Let* $A \subset X$

1. $\partial A = \overline{A} \cap \overline{A^c} = \partial(A^c)$
2. ∂A *is closed*
3. A *is closed* $\Leftrightarrow \partial A \subset A$

Proof:

1. Suppose $x \in \partial A$.
   If $x \notin \overline{A}$ then $\exists$ open U s.t. $x \in U$ and $U \cap A = \emptyset$.
   Contradicts $x \in \partial A \Rightarrow\Leftarrow$
   $\therefore \partial A \subset \overline{A}$.
   Similarly $\partial A \subset \overline{A^c}$.
   $\therefore \partial A \subset \overline{A} \cap \overline{A^c}$.
   Conversely suppose $x \in \overline{A} \cap \overline{A^c}$.
   If U is open and $x \in U$ then
   $x \in \overline{A} \Rightarrow U \cap A \neq \emptyset$ and
   $x \in \overline{A^c} \Rightarrow U \cap A^c \neq \emptyset$
   True $\forall$ open U so $x \in \partial A$.
   $\therefore \overline{A} \cap \overline{A^c} \subset \partial A$.

2. By (1), ∂A is the intersection of closed sets

3. $\Rightarrow$ Suppose A closed
   $\partial A = \overline{A} \cap \overline{A^c} \subset \overline{A} = A$ (since A closed)
   $\Leftarrow$ Suppose A closed.
   Let $x \in \overline{A}$. Then every open U containing x containe a point of A.
   If $x \notin A$ then every open U containing x also contains a point of A^c, namely x.
   In this case $x \in \partial A \subset A \Rightarrow\Leftarrow$
   $\therefore \overline{A} \subset A$ so $A = \overline{A}$ and so A is closed.

$\square$

2.6 Weak and Strong Topologies

Given a set X, topological space $Y, \mathcal{S}$ and a collection of functions $f_\alpha : X \to Y$ then there is a 'weakest topology on X s.t. all f_α are continuous':

namely intersect all the topologies on X under which all f_α are continuous.

Given a set X, a topological space W and functions $g_\alpha : W \to X$ we can form $\mathcal{T}$, the strongest topology on X s.t. all g_α are continuous. Define $\mathcal{T}$ by $U \in \mathcal{T} \Leftrightarrow g_\alpha^{-1}(U)$ is open in W $\forall\alpha$.

Strong and weak topologies Given X, a topology on X is 'strong' if it has many open sets, and is 'weak' if it has few open sets.

Extreme cases:

(a) $\mathcal{T} = 2^X$ is the strongest possible topology on X. With this topology any function $X \to Y$ becomes continuous.

(b) $\mathcal{T} = \{\emptyset, X\}$ is the weakest possible topology on X. With this topology any function $W \to X$ becomes continuous.

Proposition 2.6.1 *If $\mathcal{T}_\alpha$ are topologies on X then so is $\cap_{\alpha\in I}\mathcal{T}_\alpha$.*

Common application: Given a set X, a topological space $(Y, \mathcal{S})$ and a collection of functions $f_\alpha : X \to Y$ then there is a 'weakest topology on X s.t. all f_α are continuous'. Namely, intersect all the topologies on X under which all f_α are continuous.

Similarly, given a set X, a topological space $(W, \mathcal{P})$ and functions $g_\alpha : W \to X$, we can form $\mathcal{T}$ which is the strongest topology on X s.t. all g_α are continuous. Explicitly, define $\mathcal{T}$ by $U \in \mathcal{T} \Leftrightarrow g_\alpha^{-1}(U)$ is open in W $\forall\alpha$.

Example: $\mathcal{H}$ = Hilbert space.

$B(\mathcal{H})$ = bounded linear operators on $\mathcal{H}$

Some common topologies on $B(\mathcal{H})$:

(a) Norm topology: Define

$$||A|| = \sup_{x\in\mathcal{H}, ||x||=1} ||A(x)||$$

A norm determines a metric, which determines a topology.

(b) Weak topology: For each $x, y \in \mathcal{H}$, define a function $f_{x,y} : B(\mathcal{H}) \to \mathbb{C}$ by

$$A \mapsto (Ax, y)$$

The weak topology on $B(\mathcal{H})$ is the weakest topology s.t. $f_{x,y}$ is continuous $\forall x, y$.

(c) Strong topology: For each $x \in \mathcal{H}$ define a function $g_x : B(\mathcal{H}) \to \mathbb{R}$ by

$$A \mapsto ||A(x)||$$

The strong topology is the weakest topology on $B(\mathcal{H})$ s.t. g_x is continuous $\forall x \in \mathcal{H}$.

Definition 2.6.2 Subspace topology

Let X be a topological space, and A a subset of X. The subspace topology on A is the weakest topology on A such that the inclusion map $A \to X$ is continuous.

Explicitly, a set V in A will be open in A iff $V = U \cap A$ for some open U of X.

Definition 2.6.3 Quotient spaces

If X is a topological space and $\sim$ an equivalence relation on X, the quotient space $X/\!\sim$ consists of the set $X/\!\sim$ together with the strongest topology such that the canonical projection $X \to X/\!\sim$ is continuous.

Special case: A a subset of X. $x \sim y \Leftrightarrow x, y \in A$. In this case $X/\!\sim$ is written X/A.

For example, if $X = [0, 1]$ and $A = \{0, 1\}$ then $X/A \cong$ circle.

(Exercise: Prove this homeomorphism between X/A and the circle.)

Example 2.6.4 Examples

1. *Spheres:*

$$S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$$

2. *Projective spaces:*

(a) *Real projective space* $\mathbb{R}P^n$*: Define an equivalence relation on* S^n *by* $x \sim -x$*. Then*

$$\mathbb{R}P^n = S^n/\sim$$

with the quotient topology.

Thus points in $\mathbb{R}P^n$ *can be identified with lines through* 0 *in* $\mathbb{R}^{n+1}$*, in other words identify the equivalence class of* x *with the line joining* x *to* $-x$*.*

Similarly

(b) *Complex projective space* $\mathbb{C}P^n$*:*

$$S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}.$$

Define an equivalence relation $x \sim \lambda x$ *for every* $\lambda \in S^1 \subset \mathbb{C}$ *where* λx *is formed by scalar multiplication of* $\mathbb{C}$ *on* $\mathbb{C}^{n+1}$*. Then*

$$\mathbb{C}P^n = S^{2n+1}/\sim$$

with the quotient topology. The points correspond to complex lines through the origin in $\mathbb{C}^{n+1}$*.*

(c) *Quaternionic projective space* $\mathbb{H}^n$

$$S^{4n+3} \subset \mathbb{R}^{4n+4} = \mathbb{H}^{n+1}$$

Define $x \sim \lambda x$ *for every* $\lambda \in S^3 \subset \mathbb{H}$ *where* λx *is formed by scalar multiplication of* $\mathbb{H}$ *on* $\mathbb{H}^{n+1}$*.*

$$\mathbb{H}P^n = S^{4n+3}/\sim$$

with the quotient topology.

3. *Zariski topology:*

(This is the main example in algebraic geometry.)

R *is a ring.*

$\operatorname{Spec} R = \{$*prime ideals in* $R\}$

Define Zariski topology on $\operatorname{Spec} R$ *as follows: Given an ideal* I *of* R*, define* $V(I) = \{P \in \operatorname{Spec}(R) \mid I \subset P\}$*.*

Specify the topology by declaring the sets of the form $V(I)$ *to be closed.*

To show that this gives a topology, we must show this collection is closed under finite unions and arbitrary intersections.

This follows from

Lemma 2.6.5

(a) $V(I) \cup V(J) = V(IJ)$

(b) $\cap_{\alpha \in K} V(I_\alpha) = V(\sum_{\alpha \in K} I_\alpha$.

4. *Ordinals:*

   Let γ *be an ordinal.*

   Define $X = \{\text{ordinals } \sigma \mid \sigma \leq \gamma\}$, *where for ordinals* σ *and* γ, $\sigma < \gamma$ *means that the well-ordered set representing* σ *is isomorphic to an initial interval of that representing* γ.

   Recall the Theorem: For two well-ordered sets X *and* Y *either* $X \cong Y$ *or* $X \cong$ *initial interval of* Y *or* $Y \cong$ *initial interval of* X. *Thus all ordinals are comparable.*

   Define a topology on X *as follows.*

   For $w_1.w_2 \in X$ *define* $U_{w_1,w_2} = \{\sigma \in X | w_1 < \sigma < w_2\}$. *Here allow* w_1 *or* w_2 *to be* ∞.

   Take as base for the open sets all sets of the form U_{w_1,w_2} *for* $w_1, w_2 \in X$ *Note that this collection of sets is the base for a topology since it is closed under intersection,*

   in other words $U_{w_1,w_2} \cap U_{w'1,w'_2} = U_{\max\{w_1.w'_1\}\min\{w_2,w'_2\}}$.

Definition 2.6.6 Product spaces
The product of a collection $\{X_\alpha\}$ *of topological spaces is the set* $X = \prod_\alpha X_\alpha$ *with the topology defined by: the weakest topology such that all projection maps* $\pi_\alpha : X \to X_\alpha$ *are continuous.*

Proposition 2.6.7 *In* $\prod_\alpha X_\alpha$ *sets of the form* $\prod_\alpha U_\alpha$ *for which* $U_\alpha = X_\alpha$ *for all but finitely many* α *form a basis for the topology of* X.

Proof: Let $\mathcal{S} \subset 2^X$ be the collection of sets of the form $\prod_\alpha U_\alpha$.
Intersection of two sets in $\mathcal{S}$ is in $\mathcal{S}$ so $\mathcal{S}$ is the basis for some topology $\mathcal{T}$.
Claim: In the topology $\mathcal{T}$ on X, each π_α is continuous.
Proof: Let $U \subset X_{\alpha_0}$ be open.
Then $\pi_{\alpha_0}^{-1}(U) = U \times \prod_{\alpha \neq \alpha_0} X_\alpha \in \mathcal{S} \subset \mathcal{T}$
$\therefore \pi_{\alpha_0}$ is continuous.
Claim: If $\mathcal{T}'$ is any topology s.t. all π_α are continuous then $\mathcal{S} \subset \mathcal{T}$ (and thus $\mathcal{T} \subset \mathcal{T}'$)
Proof Let $V = U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1,\ldots,\alpha_n} X_\alpha \in \mathcal{S}$.
Then $V = \pi_{\alpha_1}^{-1} U_{\alpha_1} \cap \pi_{\alpha_2}^{-1} U_{\alpha_2} \cap \cdots \cap \pi_{\alpha_n}^{-1} U_{\alpha_n}$ which must be in any topology in which all π_α are cont.
$\therefore \mathcal{T} =$ weakest topology on X s.t. all π_α are cont. $\square$

Note: A set of the form $\prod_\alpha U_\alpha$ in which $U_\alpha \neq X_\alpha$ for infinitely many α will not be open.

Proposition 2.6.8 *Let $X = \prod_{\alpha \in I} X_\alpha$. Then $\pi_\alpha : X \to X_\alpha$ is an open map $\forall$ α.*

Proof: Let $U \subset X$ be open, and let $y \in \pi_\alpha(U)$.
So $y = \pi_\alpha(x)$ for some $x \in U$.
Find basic open set $V = \prod_\beta V_\beta$ (with $V_beta = X_\beta$ for almost all β) s.t. $x \in V \subset U$.
Then $y \in V_\alpha = \pi_\alpha(V) \subset \prod_\alpha(U)$.
$\therefore$ every pt. of $\prod_\alpha(U)$ is interior, so $\prod_\alpha(U)$ is open.
$\therefore \pi_\alpha$ is an open map. $\square$

Proposition 2.6.9 *If F_α is closed in X_α $\forall\alpha$ then $\prod_\alpha F_\alpha$ is closed in $\prod_\alpha X_\alpha$.*

$\prod_\alpha F_\alpha = \cap_\alpha \left(F_\alpha \times \prod_{\beta \neq \alpha} X_\beta\right)$
$F_\alpha \times \prod_{\beta \neq \alpha} X_\beta$ is closed (compliment is $F_\alpha^c \times \prod_{\beta \neq \alpha} X_\alpha$).
$\Rightarrow \prod_\alpha F_\alpha$ is closed $\square$

Theorem 2.6.10 *X_1, X_2, …, X_k, … metric $\Rightarrow X = \prod_{i \in \mathbb{N}} X_i$ metrizable*

Proof: Let $x, y \in X$.
Define $d(x, y) = \sum_{n=1}^\infty d_n(x_n, y_n)/2^n$.
Let X denote X with the product topology and let (X, d) denote X with the metric topology.
Clear that $\pi_n : (X, d) \to X_n$ is continous $\forall n$.
$\therefore 1_X : (X, d) \to X$ is continous.
Conversely, let $N_r(x)$ be a basic open set in (X, d).
To show $N_r(x)$ open in X, let $y \in N_r(x)$ and show y interior.
Find $\tilde{r}$ such that $N_{\tilde{r}}(y) \subset N_r(x)$.
Find M s.t. $1/2^{(M-1)} < \tilde{r}$.
$y \in U := \prod_{k \leq M} N_{1/2^M}(y_k) \times \prod_{k > M} X_k$, which is open in X
For $z \in U$,

$$d(y, z) \leq \frac{1}{2^M}\left(\frac{1}{2} + \ldots + \frac{1}{2^M}\right) + \frac{1}{2^{M+1}} + \frac{1}{2^{M+2}} + \ldots < \frac{1}{2^M} + \frac{1}{2^M} = \frac{1}{2^{M-1}} < \tilde{r}.$$

$\therefore U \subset N_{\tilde{r}}(y) \subset N_r(X)$ so y is interior.
$\therefore N_r(x)$ is open in X. $\square$

2.7 Universal Properties

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ X/\sim \arrow[ur, "\bar{f}", "\exists!"'] & \end{tikzcd}$$

A set function $\bar{f}$ making the diagram commute exists iff $\Big(a \sim b \Rightarrow f(a) = f(b)\Big)$

Proposition 2.7.1 *$\bar{f}$ is cont. $\Leftrightarrow$ f is cont.*

Proof: Check definitions. □

$$\begin{tikzcd} & W \arrow[dl, "f"', "\exists!"] \arrow[dr, "f_\alpha"] & \\ \prod X_\alpha & & X_\alpha \arrow[ll, "\pi_\alpha"'] \end{tikzcd}$$

A function into a product is determined by its projections onto each component.

Proposition 2.7.2 *f is continuous $\Leftrightarrow$ f_α is cont. $\forall\alpha$*

Proof: $\Rightarrow f_\alpha - \pi_\alpha \circ f$ so f cont.$\Rightarrow f_\alpha$ cont. $\Leftarrow$ Suppose f_α cont. $\forall\alpha$.
Let $V = U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha_1,\ldots,\alpha_n} X_\alpha \in \mathcal{S}$
Then

$$f^{-1}(V) = f^{-1}\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f^{-1}\pi_{\alpha_n}^{-1}(U_{\alpha_n}) = f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_n}^{-1}(U_{\alpha_n}) = \text{ open}$$

Since $\mathcal{S}$ is a basis, this implies f cont. □

2.8 Topological Algebraic Structures

Definition 2.8.1 *A* topological group *consists of a group G together with a topology on the underlying set G s.t.*

1. *multiplication* $G \times G \xrightarrow{\text{mult}} G$
2. *inversion* $G \to G$

are continuous (using the given topology on the set G and the product topology on $G \times G$)

Example 2.8.2

1. $\mathbb{R}^n$ with the standard topology (coming from the standard metric) and $+$ as the group operation

   $(x, y) \mapsto x + y$ is continuous

   $x \mapsto -x$ is continuous

2. $G = S^1 \subset \mathbb{R}^2 = \mathbb{C}$.

   Group operation is multiplication as elements of $\mathbb{C}$

   (a) $S^1 \times S^1 \to S^1$

   $(e^{it}, e^{iw}) \mapsto e^{i(t+w)}$ is continuous

   (b) $e^{it} \mapsto e^{-it}$ is continuous

   Similarly $G = S^3 \subset \mathbb{R}^4 = \mathbb{H}$

   S^3 becomes a topological group with multiplication induced from that on quaternions

3. $G = GL_n(\mathbb{C}) = \{$ invertible $n \times n$ matrices with entries in $\mathbb{C}$ $\}$

   Group operation: matrix multiplication

   Topology: subspace topology induced from inclusion into $\mathbb{C}^{n^2}$ (with standard metric on $\mathbb{C}^{n^2}$)

   In other words, the topology comes from the metric

$$d(A, B)^2 = \sum_{i,j} |a_{ij} - b_{ij}|^2$$

(a) $G \times G \xrightarrow{mult} G$ is continuous since the entries in the product matrix AB depend continuously on the entries of A and B

(b) the inversion map $G \to G$ is continuous since there is a formula for the entries of A^{-1} in terms of entries of A using only addition, multiplication and division by the determinant.

Similarly $SL_n(\mathbb{C})$, $U(n)$, $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$ and $O(n)$ are topological groups.

4. Let G be any group topologized with the discrete topology.

Lemma 2.8.3 *If X and Y have discrete topology then the product topology on $X \times Y$ is also discrete.*

For $(x, y) \in X \times Y$ the subset consisting of the single element (x, y) is open (a (finite) product of open sets).

Every set is a union of such open sets so is open.

Hence multiplication and inversion are continous. (Any function is continuous if the domain has the discrete topology.)

Similarly one can define topological rings, topological vector spaces and so on.

A *topological ring* R consists of a ring R with a topology such that addition, inversion and multiplication are continuous.

A *topological vector space* over $\mathbb{R}$ consists of a vector space V with a topology such that the following operations are continuous: addition, multiplication by -1 and

$$\mathbb{R} \times V \to V$$

$t, v \mapsto tv$ where $\mathbb{R}$ has its standard topology and $\mathbb{R} \times V$ the product topology.

Exercise: The standard topology on $\mathbb{R}^n$ is the only one which gives it the structure of a topological vector space over $\mathbb{R}$.

2.9 Manifolds

A Hausdorff (see Definition 4.1.1) topological space M is called an *n-dimensional manifold* if $\exists$ a collection of open sets $U_\alpha \subset M$ such that $M = \bigcup_{\alpha \in I} U_\alpha$ with each U_α homeomorphic to $\mathbb{R}^n$.

This is usually known as a "topological" manifold. One can also define differentiable or C^∞ manifold or complex analytic manifold, by requiring the functions giving the homeomorphisms to be differentiable, C^∞ or complex analytic respectively. (The last concept only makes sense when n is even.)

Example 2.9.1 *S^n is an n-dimensional manifold.*

Lemma 2.9.2 $S^n \smallsetminus \{\text{pt}\} \cong \mathbb{R}^n$.

Proof: Stereographic projection:

Place the sphere in $\mathbb{R}^{n+1}$ so that the south pole is located at the origin. Let the missing point be the north pole (or N), located at $(0, \ldots, 0, 2)$. (Note that we also introduce the notation S for the south pole.)

Define $f : S^n \smallsetminus \{N\} \to \mathbb{R}^n$ by joining N to x and $f(x)$ be the point where the line meets $\mathbb{R}^n$ (the plane where the z coordinate is 0).

Explicitly $f(x) = x + \lambda(x - a)$ for the right λ.

$$0 = f(x) \cdot a = x \cdot a + \lambda(x - a) \cdot a$$

so

$$\lambda = -\frac{x \cdot a}{(x - a) \cdot a}.$$

Hence $f(x) = x - \frac{x \cdot a}{(x-a)\cdot a}.(x - a)$. This is a continuous bijection.

The inverse map $g : \mathbb{R}^n \to S^n \smallsetminus \{N\}$ is given by $y \mapsto$ the point on the line joining y to N which lies on S^{n+1}.

Explicitly, $g(y) = ty + (1 - t)a$ where t is chosen s.t. $||g(y)|| = 1$.

Hence $\big(ty + (1 - t)a\big) \cdot \big(ty + (1 - t)a\big) = 1$ so

$$t^2||y||^2 + 2t(1 - t)y \cdot a + (1 - t)^2||a||^2 = 1$$

The solution for t depends continuously on y.

Write $S^n = \big(S^n \smallsetminus \{N\}\big) \cup \big(S^n \smallsetminus \{S\}\big)$ which is a union of open sets homeomorphic to $\mathbb{R}^n$.

$\square$

Lemma 2.9.3 *$\forall r > 0$ and $\forall x \in \mathbb{R}^n$, $N_r(x)$ is homeomorphic to $\mathbb{R}^n$.*

Proof: It is clear that translation gives a homeomorphism $N_r(x) \cong N_r(0)$ so we may assume $x = 0$.

Define $f : N_r(0) \to \mathbb{R}^n$ by $f(y) = \frac{y}{r-||y||}$ and $g : \mathbb{R}^n \to N_r(0)$ by $g(z) = \frac{r}{1+||z||}z$. It is clear that f and g are inverse homeomorphisms.

Corollary 2.9.4 *Let X be a topological space having the property that each point in X has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$. Then X is a manifold.*

Proof: Let $x \in X$. $\exists U_x$ with $x \in U_x$ and a homeomorphism $h_x : U_x \to V$.

If V is open, $\exists r_x$ s.t. $N_r\big(h_x(x)\big) \subset V$.

The restriction of h_x to $h_x^{-1}(N_r(z))$ gives a homeomorphism $W_x \to N_r(z)$.

(By definition of the subspace topology, the restriction of a homeomorphism to any subset is a homeomorphism.)

Hence $X = \cup_{x \in X} W_x$ and each W_x is homeomorphic to $N_{r_x}(Z)$ for some Z which is in turn homeomorphic to $\mathbb{R}^n$. □

Example 2.9.5 $\mathbb{R}P^n$

Let $\pi : S^n \to \mathbb{R}P^n$ be the canonical projection.

Let $x \in S^n$ represent an element of $\mathbb{R}P^n$.

Let $U = \{y \in \mathbb{R}P^n \mid \pi^{-1}(y) \cap N_r(x) \neq \emptyset\}$

$\pi^{-1}(U) = N_r(x) \cup N_r(-x)$ which is open. Hence U is open in $\mathbb{R}P^n$ by definition of the quotient topology.

Because $r < 1/2$, $N_r(x) \cap N_r(-x) = \emptyset$.

So $\forall y \in U$ $\pi^{-1}(y)$ consists of two elements, one in $N_r(x)$ and the other in $N_r(-x)$.

Define $f_x : U \to N_r(x)$ by $y \mapsto$ unique element of $\pi^{-1}(y) \cap N_r(x)$.

Claim: f_x is a homeomorphism.

Proof: For any open set $V \subset N_r(x)$ $\pi^{-1} f_x^{-1}(V) = V \cup -V$ which is open in S^n.

Hence $f_x^{-1}(V)$ is open in $\mathbb{R}P^n$ by definition of the quotient topology.

Hence f_x is continuous.

The restriction of π to $N_r(x)$ gives a continuous inverse to f_x so f_x is a homeomorphism.

Let $h_x : S^n \setminus \{-x\} \to \mathbb{R}^n$ be a homeomorphism. So $h_x\big(N_r(x)\big)$ is open in $\mathbb{R}^n$. So we have homeomorphisms

$$U \xrightarrow{f_x} N_r(x) \xrightarrow{h_x|_{N_r(x)}} h_x\big(N_r(x)\big)$$

giving a homeomorphism from U to an open subset of $\mathbb{R}^n$.

Since every point of $\mathbb{R}P^n$ is $\pi(x)$ for some $x \in S^n$ we have shown that every point of $\mathbb{R}P^n$ has a neighbourhood homeomorphic to a neighbourhood of $\mathbb{R}^n$. So $\mathbb{R}P^n$ is a manifold by the previous Corollary. □

Definition 2.9.6 *A topological group which is also a manifold is called a* Lie group*.*

Examples: $\mathbb{R}^n$, S^1, S^3, $GL_n(\mathbb{R})$.

To check the last example, we must show $GL_n(\mathbb{R})$ is a manifold.

Since the topology on $GL_n(\mathbb{R})$ is that as a subspace of $\mathbb{R}^{n^2}$, by Corollary 2.9.4 it suffices to show that $GL_n(\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$.

Let $M_n(\mathbb{R}) = \{n \times n \text{ matrices over } \mathbb{R}\}$ with topology coming from the identification of $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$.

So by construction $M_n(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2}$.

det : $M_n(\mathbb{R}) \to \mathbb{R}$ is continuous (it is a polynomial in the entries of A).

$$\det : A \mapsto \det A$$

$GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$

0 is closed in $\mathbb{R}$ so $\mathbb{R} \setminus \{0\}$ is open. Hence $GL(n, \mathbb{R})$ is open in $\mathbb{R}^{n^2}$. $\square$

Chapter 3

Compactness

Definition 3.0.7 *A topological space X is called compact if it has the property that every open cover of X has a finite subcover.*

Theorem 3.0.8 Heine-Borel *A subset $X \subset \mathbb{R}^n$ is closed and bounded if and only if every open cover of X has a finite cover.*

Proposition 3.0.9 *Given a basis for the topology on X, X is compact $\Leftrightarrow$ every open cover of X by sets from the basis has a finite subcover.*

$\Rightarrow$ Obvious
$\Leftarrow$ Let U_α be an open cover of X.
Write each U_α as a union of sets in the basis to get a cover of X by basic open sets.
Select a finite subcover $V_1, \dots, V_n$ from these.
By construction $\forall j\ \exists \alpha_j$ s.t. $U_{\alpha_1}, \dots, U_{\alpha_n}$ cover X $\square$

Theorem 3.0.10 *Given a subbasis for X, X is compact $\Leftrightarrow$ every open cover of X by sets from the subbasis has a finite subcover.*

$\Rightarrow$ Obvious

$\Leftarrow$ Consider the basis formed by taking finite intersections of sets in $\{U_\alpha\}_{\alpha \in I}$. By Proposition 3.0.9, it suffices to show that any open cover by sets in this basis has a finite subcover.

Let $\{V_\alpha\ \}$ be such an open cover. So WLOG each V_β is a finite intersection of sets from $\{U_\alpha\}$.

Suppose $\{V_\beta\}_{\beta \in J}$ has no finite subcover.

Well-order I and J.

Define $f : J \to I$ as follows so that for each β, $V_\beta \subset U_{f(\beta)}$ and $\{U_{f(\gamma)}\}_{\gamma \le \beta} \cup \{V_\gamma\}_{\gamma > \beta}$ has no finite subcover.

Step 1: Define $f(j_0)$:

Write $V_{j_0} = U_{\sigma_1} \cap U_{\sigma_2} \cap \cdots \cap U_{\sigma_n}$.

Claim 1: For some $i = 1, \ldots, n$, $\{U_{\sigma_i}\} \cup \{V_\gamma\}_{\gamma > j_0}$ has no finite subcover.

Proof: Suppose not. Then $\exists$ a finite collection of the V_γ s.t. $\forall i$ $X = U_{\sigma_i} \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r}$.

So

$$
\begin{aligned}
X &= \cap_{i=1}^n U_{\sigma_i} \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r} \\
&= (\cap_{i=1}^n U_{\sigma_i}) \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r} \\
&= V_{j_0} \cup V_{\gamma_1} \cup \cdots \cup V_{\gamma_r}.
\end{aligned}
$$

This contradicts our earlier assertion that X does not have a finite subcover by a finite collection of the V_γ. □

Choose i such that $\{U_{\sigma_i}\} \cup \{V_\gamma\}_{\gamma > j_0}$ has no finite subcover, and define

$$f(j_0) = \sigma_i. \tag{3.1}$$

Suppose now that f has been defined for all $\gamma < \beta$.

Claim 2:

$$\{U_{f(\gamma)}\}_{\gamma < \beta} \cup \{V_\gamma\}_{\gamma \ge \beta}$$

has no finite subcover.

Proof: Such a subcover would contradict the definition of $f(\hat{\gamma})$ where $\hat{\gamma}$ is the largest index occurring in the sets $\{U_{f(\gamma)}\}$ used in the subcover.

In other words, if $U_{f(\beta_1)}, \ldots, U_{f(\beta_k)}, V_{\beta'_1}, \ldots, V_{\beta'_r}$ is a subcover then it is also a subcover of $\{U_{f(\gamma)}\}_{\gamma \le \beta_k} \cup \{V_\gamma\}_{\gamma > \beta_k}\}$. This contradicts the definition of $f(\hat{\gamma})$ where $\hat{\gamma} = \beta_k$.

Write $V_\beta = U_{\sigma_1} \cap \cdots \cap U_{\sigma_n}$.

Claim 3. For some $i = 1, \ldots, n$ $\{U_{f(\gamma)}\}_{\gamma < \beta} \cup \{U_{\sigma_i}\} \cup \{V_\gamma\}_{\gamma > \beta}$ has no finite subcover.

Proof: If not, we get a contradiction to the previous claim as in the proof of the definition of $f(j_0)$.

So choose i as in the previous claim and set $f(\beta) = \sigma_i$.

Now that f has been defined,

Claim 4. $\{U_{f(\beta)}\}$ has no finite subcover.

Proof: If $U_{f(\beta_1)} \cup \cdots \cup U_{f(\beta_k)}$ is a subcover then it is also a subcover of $\{U_{f(\gamma)}\}_{\gamma \le \beta_k} \cup \{V_\gamma\}_{\gamma > \beta_k}$, contradicting the definition of $f(\beta_k)$.

But Claim 4 contradicts the definition of $\{U_\alpha\}$.

So $\{V_\beta\}_{\beta \in J}$ has a finite subcover and thus X is compact.

□

Theorem 3.0.11 (Tychonoff) *If X_α is compact for all α then $\prod_{\alpha \in I} X_\alpha$ is compact.*

Proof: Sets of the form

$$V_\alpha = U_\alpha \times \prod_{\gamma \neq \alpha} X_\gamma$$

(with U_α open in X_α) form a subbasis for the topology of X.

Let $\{V_\beta\}_{\beta \in J}$ be an open cover of X by sets in this subbasis.

Suppose $\{V_\beta\}$ has no finite subcover.

Let $F_\beta = (V_\beta)^c$.

Then

$$\cap_\beta F_\beta = \emptyset \tag{3.2}$$

but

$$\cap\{\text{any finite subcollection} F_\beta\} \neq \emptyset \tag{3.3}$$

where $V_\beta = U_{\alpha_0} \times \prod_{\gamma \neq \alpha_0} X_\gamma$

Note that for any β, the image of each of the projections of F_β is closed. That is, if $V_\beta = U_{\alpha_0} \times \prod_{\gamma \neq \alpha_0} X_\gamma$ then $\pi_{\alpha_0} F_\beta = (\pi_{\alpha_0} V_\beta)^c$ which is closed and for all other α, $\pi_\alpha F_\beta = X_\alpha$ which is closed.

So for any α, if $\cap_\beta(\pi_\alpha F_\beta) = \emptyset$ then $\pi_\alpha F_{\beta_1} \cap \cdots \cap \pi_\alpha(F_{\beta_r}) = \emptyset$ for some $\beta_1, \ldots, \beta_r$, since X_α is compact. This implies $F_{\beta_1} \cap \cdots \cap F_{\beta_r} = \emptyset$. This is a contradiction to (3.3). So there exists an $x_\alpha \in \cap_\beta \Pi_\alpha F_\beta$.

This is true for all α. So let $x = (x_\alpha)$.

Then $x \in \cap_\beta F_\beta$. This contradicts (3.2).

So $\{V_\beta\}$ has a finite subcover. Hence X is compact.

$\square$

Chapter 4

Separation

4.1 Separation Axioms; Urysohn's Lemma; Stone-Cech Compactification

Let X be a topological space.

Definition 4.1.1 *X has the following names if it has the following properties:*

1. X is T_0 if $\forall x \neq y \in X$ either $\exists$ open U s.t. $x \in U, y \notin U$ $x \in U, y \notin U$ or $\exists$ open U s.t. $x \notin U, y \in U$

2. X is T_1 if $\forall x \neq y \in X$ $\exists$ open U s.t. $x \in U$, $y \notin U$ and $\exists$ open V s.t. $y \in V, x \notin V$.

3. X is T_2 or Hausdorff if $\forall x \neq y \in X$ $\exists$ open U, V with $U \cap V = \emptyset$ s.t. $x \in U$ and $y \in V$

4. X is T_3 or regular if X is T_1 and given $x \in X$ and a closed set $F \subset X$ with $x \notin F$, $\exists$ open U and V s.t. $x \in U$, $F \subset V$ and $U \cap V = \emptyset$

5. X is $T_{3\frac{1}{2}}$ or completely regular if X is T_1 and also given $x \in X$ and a closed set $F \subset X$ with $x \notin F$, $\exists f : X \to [0,1]$ s.t. $f(x) = 0$ and $f(F) = 1$.

6. X is T_4 or normal if X is T_1 and also given closed $F, G \subset X$ s.t. $F \cap G = \emptyset$ $\exists$ open U, V s.t. $F \subset U$, $G \subset V$ and $U \cap V = \emptyset$.

We say U and V *separate* A and B if $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.
Some reformulations:

Proposition 4.1.2

1. *X is T_1 $\Leftrightarrow$ the points of X are closed subsets of X*

2. *X is Hausdorff*

   (a) $\quad \Leftrightarrow \{x\} = \bigcap\limits_{\substack{U \text{open} \\ x \in U}} \bar{U}$

   (b) $\quad \Leftrightarrow \Delta(X)$ *is closed in* $X \times X$ *(where $\Delta(X)$ means the diagonal subset $\{(x,x) \mid x \in X\}$ of $X \times X$)*

3. *X is regular $\Leftrightarrow$ X is Hausdorff and given $x \in U$, $\exists$ open V s.t. $x \in V \subset \bar{V} \subset U$*

4. *X is normal*

   (a) $\quad \Leftrightarrow$ *X is Hausdorff and given $x \in U$ $\exists$ open V s.t. $F \subset V \subset \bar{V} \subset U$*

   (b) $\quad \Leftrightarrow$ *X Hausdorff and given closed F, G with $F \cap G = \emptyset$ $\exists$ open U, V s.t. $F \subset U$, $G \subset V$ and $\bar{U} \cap \bar{V} = \emptyset$.*

Proof:
1: $(\Rightarrow)$ X T_1. Let $x \in X$. $\forall y \in X$ $\exists$ open V_y s.t. $x \notin V_y$ and $y \in V_y$. Hence $X \setminus \{x\} = \cup_{y \neq x} V_y$ is open so $\{x\}$ is closed.

$(\Leftarrow)$ Suppose points closed. Let $x, y \in X$. $U = X \setminus \{y\}$ is open. $x \in U, y \notin U$. Similarly the reverse.

2a: $(\Rightarrow)$ X is Hausdorff. Let $x \in X$. $\forall y \neq x$ $\exists U_y, V_y$ s.t. $x \in U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. $U_y \subset (V_y)^c \Rightarrow \bar{U_y} \subset (V_y)^c \Rightarrow y \notin \bar{U_y} \Rightarrow y \notin \bigcap\limits_{\substack{U \text{open} \\ x \in U}} \bar{U}$.

$(\Leftarrow)$ Let $x \neq y \in X$. $\{x\} = \bigcap\limits_{\substack{U \text{open} \\ x \in U}} \bar{U}$. Find open U s.t. $x \in U$ and $y \notin \bar{U}$. Let $V = \bar{U}^c$, which is open.

2b: $(\Rightarrow)$ Suppose X is Hausdorff.

If $(x,y) \in (\Delta(x))^c$ find U, V s.t. $x \in U, y \in V$, $U \cap V = \emptyset$

Then $(x,y) \in U \times V$ but $U \times V \subset \big(\Delta(X)\big)^c$. Since $U \times V$ is open, $(x,y) \in$ interior of $\big(\Delta(X)\big)^c$. This is true $\forall (x,y) \in \big(\Delta(X)\big)^c$, so $\big(\Delta(X)\big)^c$ is open, and $\big(\Delta(X)\big)$ is closed.

$(\Leftarrow)$ Suppose $\Delta(X)$ is closed.

If $x \neq y$ then $(x,y) \in \big(\Delta(X)\big)^c$. Since $U \times V$ is open, $(x,y) \in$ interior of $\big(\Delta(X)\big)^c$. This is true $\forall (x,y) \in \big(\Delta(X)\big)^c$, so $\big(\Delta(X)\big)^c$ is open, and $\big(\Delta(X)\big)$ is closed.

$(\Leftarrow)$ Suppose $\Delta(X)$ is closed.

If $x \neq y$ then $(x,y) \in (\Delta_c(X))^c$ which is open so there exists a basic open set $U \times V$ s.t. $(x,y) \in U \times V \subset (\Delta(X))^c$. Hence $x \in U, y \in V$, $U \cap V = \emptyset$.

3: ($\Rightarrow$) Suppose X is regular. Then X is T_1 so points are closed. Hence given $x \neq y \in X$ let $F = \{y\}$ and apply defn. of regular to see that X is Hausdorff. Given $x \in U$, $x \cap U^c = \emptyset$ and U^c is closed so $\exists$ open V, W s.t. $x \in V$, $U^c \subset W$ and $V \cap W = \emptyset$.

$x \in V \subset W^c \subset U$.

Since W^c is closed, $\bar{V} \subset W^c$

($\Leftarrow$) Hausdorff $\Rightarrow T_1$.

Let $x \in X$, $F \subset X$ with $x \notin F$.

Then $x \in F^c$, which is open, so $\exists$ open U s.t. $x \in U \subset \bar{U} \subset F^c$. Let $V = (\bar{U})^c$. Then $F \subset V$ and $U \cap V = \emptyset$.

4a: $\Leftrightarrow$ similar to (3.)

4b: ($\Leftarrow$) trivial

($\Rightarrow$) Given closed F, G s.t. $F \cap G = \emptyset$. Then $F \subset G^c$ so $\exists$ open U s.t. $F \subset U \subset \bar{U} \subset G^c$.

$G \subset (\bar{U})^c$ so $\exists$ open V s.t. $G \subset V \subset \bar{V} \subset (\bar{U})^c$.

Hence $\bar{U} \cap \bar{V} = \emptyset$. $\square$

Proposition 4.1.3 *Let $f, g : X \to Y$, with Y Hausdorff. Suppose $A \subset X$ is dense and $f\big|_A = g\big|_A$. Then $f = g$.*

Proof: Define $h : X \to Y \times Y$ by $h(x) = \big(f(x), g(x)\big)$. Then h is continous (since its projections are).

Let $F = \{x \in X \mid f(x) = g(x)\}$.

$F = h^{-1}\big(\Delta(Y)\big)$ which is closed since Y is Hausdorff.

$A \subset F \Rightarrow X = \overline{A} \subset F$

Hence $f(x) = g(x)$ $\forall x \in X$. $\square$

Theorem 4.1.4 *metric* $\Rightarrow T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_2 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

Proof: $T_2 \Rightarrow T_1 \Rightarrow T_0$ is trivial. $T_3 \Rightarrow T_2$ by definition, and part (1) of the previous proposition.

$T_{3\frac{1}{2}} \Rightarrow T_3$: Given x, F let $f : X \to [0,1]$ s.t. $f(X) = 0$, $f(F) = 1$, as in the definition of T_3. Set $U = f^{-1}\big([0, 1/2)\big)$ and $V = f^{-1}\big((1/2, 1]\big)$ which are open in $[0,1]$. Then U, V separate x, F in X.

metric$\Rightarrow T_4$: Let F, G be closed in metric space X s.t. $F \cap G = \emptyset$.

For $x \in F$, let $d_x = \inf_{y \in G}\{d(x,y)\}$.

Claim: $d_x \neq 0$.

Proof:

If $d_x = 0$ then $\forall n\ \exists y_n \in G$ s.t. $d(x, y_n) < 1/n$.
Hence $(y_n) \to x$. Hence $x \in G$.
(Exercise: G closed, $y_n \in G$, $(y_n) \to x \Rightarrow x \in G$)
$\Rightarrow\Leftarrow$

Let $Y = \cup_{x \in F} N_{d_x/2}(x)$ which is open with $F \subset U$.
Claim: $\overline{U} \cap G = \emptyset$
Proof:
Let $y \in \overline{U} \cap G$.
Then $\exists$ sequence $(u_n) \to y$ with $u_n \in U$.
$\forall n$ find $x_n \in F$ s.t. $u_n \in N_{d_{x_n}/2}(x)$
$d_{x_n} \leq d(x_n, y) \leq d(x_n, u_n) + d(u_n, y) < d_{x_n}/2 + d(u_n, y)$.
Hence $d_{x_n}/2 < d(u_n, y)$.
$(u_n) \to y \Rightarrow d(u_n, y0) \to 0 \Rightarrow d_{x_n}/2 \to 0$.
Hence $d(x_n, y) < d_{x_n}/2 + d(u_n, y) \Rightarrow d(x_n, y) \to 0 \Rightarrow (x_n) \to y$.
So $y \in F \Rightarrow\Leftarrow$.
Hence $\overline{U} \cap G = \emptyset$.

So let $V = (\overline{U})^c \supset G$.

$T_4 \Rightarrow T_3\frac{1}{2}$: Corollary of

Theorem 4.1.5 (Urysohn's Lemma) *Suppose X is normal, and F and G are closed subsets of X with $F \cap G = \emptyset$. Then $\exists f : X \to [0, 1]$ s.t. $f(F) = 0$ and $f(G) = 1$.*

Proof:
Apply 4(b) of Proposition 4.1.2 to $F \subset G^c$. Then $\exists$ open $U_{1/2}$ s.t. $F \subset U_{1/2} \subset \bar{U_{1/2}} \subset G^c$.
Two more applications of Proposition 4.1.2:
4(b) $\Rightarrow \exists$ open $U_{1/4}, U_{3/4}$ s.t. $F \subset U_{1/4} \subset \bar{U}_{1/4} \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_{3/4} \subset \bar{U}_{3/4} \subset G^c$.
Continuing, construct an open set U_t for all t of the form $m/2^n$ for some m and n. For $x \in X$ define

$$f(x) = \begin{cases} 0 & x \in U_t\ \forall t \\ sup(\{t | x \notin U_t\} & \text{otherwise} \end{cases} \tag{4.1}$$

It is clear that $f(F) = 0$ and $f(G) = 1$. We show that f is continuous.
Intervals of the form $[0, a)$ and $(a, 1]$ form a subbasis for $[0, 1]$.
$f(x) < a \Leftrightarrow x \in U_t$ for some $t < a$.
Hence $f^{-1}([0, a)) = \{x | f(x) < a\} = \cup_{t<a} U_t$, which is open.
Similarly $f(x) > a \Leftrightarrow x \notin U_t$ for some $t > a$. which is true iff $x \notin \bar{U}_s$ for some $s > a$.
Hence $f^{-1}((a, 1]) = \cup_{s>a} \left(\bar{U}_s\right)^c$, which is open.
We conclude that f is continuous. $\square$

Lemma 4.1.6 *Suppose X is Hausdorff. Suppose $x \in X$ and $Y \subset X$ is compact s.t. $x \notin Y$. Then $\exists$ open U, V separating x and Y.*

Proof: $\forall y \in Y$ $\exists$ open U_y, V_y s.t. $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. $Y = \cup_{y \in Y} V_y$ is a cover of Y by open sets in X so $\exists$ a finite subcover $V_{y_1}, \ldots, V_{y_n}$.

Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$ and $V = V_{y_1} \cup \cdots \cup V_{y_n}$. Then
(i) $x \in U_{y_j}$ $\forall j \Rightarrow x \in U$
(ii) $V_{y_1}, \ldots, V_{y_n}$ cover $Y \Leftrightarrow Y \subset V$.
(iii) $U \cap V = \emptyset$.
(Proof: If $z \in U \cap V$ then $z \in V_{y_j}$ for some j and $z \in U_{y_j}$ $\forall j$. But $U_{y_j} \cap V_{y_j} = \emptyset$. Contradiction.)

□

Corollary 4.1.7 *A compact subspace of a Hausdorff space is closed.*

Proof: Suppose $A \subset X$ where A is compact and X is Hausdorff. By Lemma, $\forall y \in A^c$ $\exists$ open U_y, V_y separating y and A so $y \in U_y \subset A^c$. Hence y is an interior point of A^c. This is true for all y so A^c is open (equivalently A is closed).

□

Theorem 4.1.8 *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof: Let $f : X \to Y$ where f is compact and Y is Hausdorff. We must show that the inverse to f is continuous, which is equivalent to showing that for any closed set B, $f(B)$ is closed. If $B \subset X$ is closed, then by our earlier Theorem, B is compact, so by another earlier Theorem, $f(B)$ is compact. By a previous Corollary, this implies $f(B)$ is closed. □

Theorem 4.1.9 *A compact Hausdorff space is normal.*

Proof: Suppose X is a compact Hausdorff space. Suppose A and B are closed subsets of X with $A \cap B = \emptyset$. Since A and B are closed and X is compact, we conclude that A and B are also compact.

By the Lemma, $\forall a \in A$ $\exists$ open sets U_a, V_a s.t. $a \in U_a$, $b \in V_a$ and $U_a \cap V_a = \emptyset$.

$\cup_a U_a$ is a cover of A by open sets in X so by compactnss there is a finite subcover $U_{a_1}, \ldots, U_{a_n}$. Let $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cup \cdots \cup V_{a_n}$.

Then as in the proof of the Lemma
(i) $A \subset U$
(ii) $B \subset V$
(iii) $U \cap V = \emptyset$

□

Proposition 4.1.10 *Suppose $A \subset X$.*
If X is T_j for $j < 4$ then so is A.
If A is closed and X is T_4 then A is T_4.

Proof:
$j = 0, 1, 2$: Trivial

$j = 3$: Let $a \in A$ and let $F \subset A$ be closed in A with $a \notin F$.
Let $\overline{F}$ denote the closure of F within X.
Then $a \notin \overline{F}$.
(Proof: $\overline{F} = \bigcap\limits_{\substack{G \supset F \\ G \text{ closed in } X}} G$. Therefore

$$(G \cap A) = \bigcap_{\substack{G' \supset F \\ G' \text{ closed in } X}} (G \cap A) = \bigcap_{\substack{G' \supset F \\ G' \text{ closed in } X}} G' = (\text{closure of } F \text{ in } A) = F.$$

Hence $a \in A, \notin F \Rightarrow a \notin \overline{F}$.)
So $\exists$ open U, V in X s.t. $a \in U$, $\overline{F} \subset V$ and $U \cap V = \emptyset$.
But then $U' = U \cap A$ and $V' = V \cap A$ are open in A and satisfy:
(i) $a \in U \cap A$
(ii) $F = \overline{F} \cap A \subset V \cap A = V'$
(iii) $U' \cap V' = \emptyset$

$j = 3\frac{1}{2}$: Let $a \in F$, $F \subset A$ with F closed in A, $a \notin F$.
$F = \overline{F} \cap A$ with $\overline{F}$ as above.
Since, as above, $a \notin \overline{F}$, $\exists f : X \to [0, 1]$ s.t. $f(a) = 0$, $f(\overline{F}) = 1$.
The composition $\hat{f} : A \hookrightarrow X \xrightarrow{f} [0, 1]$ is continuous and satisfies $\hat{f}(a) = 0$ and $(\hat{F}) = 1$ (since $F \subset \overline{F}$).

$j = 4$: $A \subset X$ closed.
Let F, G be closed in X. As in previous two cases, $F = \overline{F} \cap A$ and $\overline{F} \cap A = \overline{F}$ since A is closed in X. So F is closed in X and similarly G is closed in X.
Therefore $\exists U, V$ open in X separating F, G in X.
So $U \cap A$ and $V \cap A$ separate F, G in A. □

Proposition 4.1.11 *Let $X = \prod_{\alpha \in I} X_\alpha$ with $X_\alpha \neq \emptyset$ $\forall \alpha$.*
For $j < 4$, X is $T_j \Leftrightarrow X_\alpha$ is T_j $\forall \alpha$. X is $T_4 \Rightarrow X_\alpha$ is $T_j \forall \alpha$.

Proof:
$\Rightarrow$ Suppose X is T_j. Show X_{α_0} is T_j.

For $\alpha \neq \alpha_0$, select $x_\alpha \in X_\alpha$. (Axiom of Choice)

Define $i : X_\alpha \to X$ by $\pi_\alpha\big(i(a)\big) = \begin{cases} a & \alpha = \alpha_0; \\ x_\alpha & \alpha \neq \alpha_0. \end{cases}$

$$\begin{tikzcd} X_{\alpha_0} \arrow[r, "i"] \arrow[dr, "1_{X_{\alpha_0}}"'] & X \arrow[d, "\pi_{\alpha_0}"] \\ & X_{\alpha_0} \end{tikzcd}$$

Note: Provided X_α is T_1 for $\alpha \neq \alpha_0$, i(closed) =closed (since a product of closed sets is closed).

If $a \neq b \in X_{\alpha_0}$ then $i(a) \neq i(b)$ in X.

$j = 0$: If $i(a) \in U$, $i(b) \notin U$, find basic open U' s.t. $i(a) \in U' \subset U$. So $i(b) \notin U'$.

But $a = \pi_{\alpha_0} i(a) \in \pi_{\alpha_0}(U')$ (open since projections maps are open maps)

Claim: $\notin \pi_{\alpha_0}(U')$

Proof: Since U' basic, $U' = \prod_\alpha \pi_\alpha(U')$

For $\alpha \neq \alpha_0$, $\pi_\alpha(ib) = x_\alpha = \pi_\alpha(ia) \in \pi_\alpha(U')$.

Therefore $ib \notin U'$ so $b = \pi_{\alpha_0} b \notin \pi_{\alpha 0}(U')$

$j = 1$: Similar

$j = 2$: Begining with open U, V, separating ia, ib, find basic U', V' separating ia, ib.

Claim: $\pi_{\alpha_0}(U')$ and $\pi_{\alpha_0}(V')$ (which are open) separate a and b.

Proof: $\pi_{\alpha_0} \circ i = 1_{X_{\alpha_0}}$ so $a \in \pi_{\alpha_0}(U')$ and $b \in \pi_{\alpha_0}(V')$.

If $c \in \pi_{\alpha_0}(U') \cap \pi_{\alpha_0}(U')$ then $ic \in U' \cap V'$ since U', V' basic and $\pi_\alpha(ic) = x_\alpha \in \pi_\alpha(U') \cap \pi_\alpha(V')$ for $\alpha \neq \alpha_0$.

Contradiction.

$j = 3$: X_{α_0} is T_1 by above.

Let $a \in X_{\alpha_0}$, B closed $\subset X_{\alpha_0}$ with $a \notin B$.

$i(a) \notin i(B)$ (closed because $(x_\alpha)_{\alpha \neq \alpha_0}$ is closed in $\Pi_{\alpha \neq \alpha_0} X_\alpha$ by $j = 1$ case and so $i(B) = B \times \prod_{\alpha \neq \alpha_0} X_\alpha$=closed)

Find U, V separating $i(a)$, $i(B)$ in $X/$

Find basic U' with $i(a) \in U' \subset U$.

$\forall z \in i(B)$, $\exists$ basic open V_z s.t. $z \in V_z \subset V$.

Let $\tilde{V} = \cup_{z \in i(B)} \pi_{\alpha_0}(V_z)$ open in X_α

Therefore $B \subset \tilde{V}$ (i.e. $b \in \pi_{\alpha_0}(V_{i(b)})$)

Claim: $\pi_{\alpha_0}(U')$, $\tilde{V}$ is a separation of a and B.

Proof: $c \in \pi_{\alpha_0}(U') \cap \tilde{V} \Rightarrow \pi_{\alpha_0}(ic) \in \pi_{\alpha_0}(U')$ and $\pi_{\alpha_0}(ic) \in \pi_{\alpha_0}(V_z)$ for some $z \in i(B)$.
For $\alpha \neq \alpha_0$, $\pi_\alpha(ic) = x_\alpha = \pi_\alpha(a) \in \pi_\alpha(U')$ and $\pi_\alpha(ic) = x_\alpha = \pi_\alpha(z) \in \pi_\alpha(V_z)$
That is, $ic \in U' \cap V_z \subset U \cap V$. $\Rightarrow\Leftarrow$.
Therefore case $j = 3$ follows.

$j = 3\frac{1}{2}$: X_{α_0} is T_1 by above.
Let $a \in X_{\alpha_0}$, B closed $\subset X_{\alpha_0}$, $a \notin B$.
$i(a) \notin i(B)$ (which is closed) implies $\exists g : X \to 0, 1$ s.t. $g(ia) = 0$, $g(oB) = 1$.
Let $f = g \circ i$.

$j = 4$: X_{α_0} is T_1 as above. Find separating function as in previous case, using Urysohn.

$\Leftarrow$ Suppose X_α is T_j for all α.
First consider cases $j < 3$.
Let $x, y \in X$ with $x_{\alpha_0} \neq y_{\alpha_0}$ for some α_0.

$j = 0$: If $x_{\alpha_0} \in U_0$, $y_{\alpha_0} \notin U_{\alpha_0}$ then $U = U_0 \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is open in X and $x \in U$, $y \notin U$.

$j = 0$: Similar

$j = 2$: If U_0, V_0 separate x_{α_0}, y_{α_0} in X_{α_0} then $U = U_0 \times \prod_{\alpha \neq \alpha_0} X_\alpha$ and $V = V_0 \times \prod_{\alpha \neq \alpha_0} X_\alpha$ separate x and y in X.

$j = 3$: By above X is T_1.
Let $x \in U$ (open)
Find basic open U' s.t. $U' \subset U$. Write $U' = \prod_\alpha U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \ldots, \alpha_n$.
For $j = 1, \ldots, n$ find V_{α_j} s.t. $x_{\alpha_j} \in V_{\alpha_j} \subset \overline{V_{\alpha_j}} \subset U_{\alpha_j}$
Let $V = V_{\alpha_1} \times \cdots \times V_{\alpha_r} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha$ closed
Therefore $\overline{V} \subset W$.
Hence $x \in V \subset \overline{V} \subset W \subset U' \subset U$
Therefore X is T_3.

$j = 3\frac{1}{2}$: A corollary of the Stone-Cech Compactification Thm (below) is

Corollary 4.1.12 *X is completely regular $\Leftrightarrow$ X is homeomorphic to a subspace of a compact Hausdorff space.*

Proof of Case $j = 3\frac{1}{2}$ (Given Corollary)
By Corollary, $\forall \alpha$, find compact Hausdorff Y_α s.t. X_α homeomorphic to a subspace of Y_α.
Hence X is homeomorphic to a subspace of $Y := \prod_\alpha Y_\alpha$.
By Tychonoff, Y is compact and by case $j = 2$, Y is Hausdorff. Hence X is homeomorphic a subspace of a compact Hausdorff space so is completely regular by the Corollary.
Proof of Corollary:

$\Leftarrow$: By earlier theorems, a compact Hausdoff space is normal and thus completely regular and a subpace of a completely regular space is completely regular.

$\Rightarrow$ Follows from:

Theorem 4.1.13 (Stone-Cech Compactification) *Let X be completely regular. Then there exists a compact Hausdorff space $\beta(X)$ together with a (continuous) injection $X \hookrightarrow \beta(X)$ s.t.*

1. *$i : X \hookrightarrow \beta(X)$ is a homeomorphism*
2. *X is dense in $\beta(X)$*
3. *Up to homeomorphism $\beta(X)$ is the only space with these properties*
4. *Given a compact Hausdorff space W and $h : X \to W$ there is a unique $\bar{h}$ s.t. $h = \bar{h} \circ i$*

Definition 4.1.14 *$\beta(X)$ is called the* Stone-Cech *compactification of X.*

Example 4.1.15 *Let $X = (0, 1]$. Let $f : X \to [-1, 1]$ by $f(x) = \sin(1/x)$. Then f is a continuous function from X to the compact Hausdorff space $[-1, 1]$, but f does not extend to $[0, 1]$. Thus although $[0, 1]$ is a compact Hausdorff space containing $(0, 1]$ as a dense subspace, it is not the Stone-Cech compactification of $(0, 1]$.*

Proof of Theorem: Let $J = \{f : X \to \mathbb{R} \mid f \text{ bounded and continuous}\}$.

For $f \in J$, let I_f be the smallest closed interval containing $\mathrm{Im}(f)$. As f is bounded, I_f is compact.

Let $Z = \prod_{f \in J} I_f$. It is compact Hausdorff.

Define $i : X \to Z$ by $(ix)_f = f(x)$. Since X is completely regular, $x \neq y \Rightarrow \exists f : X \to [0, 1]$ s.t. $f(x) \neq f(y)$. Thus i is injective.

Claim: $i : X \xrightarrow{\cong} i(X)$.

Proof: Use the injection i to define another topology on X – the subspace topology as a subset of Z.

The Claim is equivalent to showing the subspace topology is equals to the original topology.

Since i is continuous (because its projections are), if U is open in the subspace topology then U is open in the original topology.

Conversely suppose U is open in the original topology.

Let $x \in U$. To show x is interior (in the subspace topology):

By definition of the subspace and product topologies, the subspace topology is the weakest topology s.t. $f : X \to \mathbb{R}$ is continuous $\forall f \in J$.

Because X is completely regular, $\exists f : X \to [0, 1]$ s.t. $f(x) = 0$, $f(U^c) = 1$

$f \in J \Rightarrow f^{-1}([0,1))$ is open in the subspace topology.
$f^{-1}([0,1)) \subset U$ since $f(x) = 1\ \forall x$ not in U.
Therefore $x \in \text{Int}(U)$ (in the subspace topology).
True $\forall x \in U$, so U is open in the subspace topology.

Let $\beta(X) = \overline{i(X)}$.

Then $\beta(X)$ is compact Hausdorff, as it is a closed subspace of a compact Hausdorff space and $X \cong i(X)$ is dense in $\beta(X)$ by construction.

To show the extension property and uniqueness of $\beta(X)$ up to homeomorphism,

Lemma 4.1.16

1. *Given $g : X \to Y$, $\exists!\ \hat{g} : \beta(X) \to \beta(Y)$ s.t.*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ \beta(X) \arrow[r, "\hat{g}"] & \beta(Y) \end{tikzcd}$$

2. *If X is compact Hausdorff then $X \to \beta(X)$ is a homeomorphism.*

Proof:

1. *Uniqueness:* Since $\beta(Y)$ is Hausdorff and X is dense in $\beta(X)$ any two maps from $\beta(X)$ agreeing on X are equal. So $\hat{g}$ is unique.

   Existence: Let $\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f$ is bounded and continuous$\}$, and let $\mathcal{C}(Y) = \{f : Y \to \mathbb{R} \mid f$ is bounded and continuous$\}$.

   Let $z \in \beta(X)$.

   To define $\hat{g}(z)$: For $f \in \mathcal{C}_Y$, define $\Pi_f(\hat{g}z) = \Pi_{f \circ g}(z)\ \forall x \in X$, and $\forall f \in \mathcal{C}_Y$. Each projection is continous so $\hat{g}$ is continuous.

   $\forall x \in X$ and $\forall f \in \mathcal{C}(Y)$:

   $\Pi_f(i_Y g x) = f\big(g(x)\big)$ while $\pi_f(\hat{g} i_X x) = \pi_{f \circ g}(i_X x) = f \circ g(x)$. Therefore $i_Y \circ g = \hat{g} \circ i_X$ which also shows that $\hat{g}\big(\beta(X)\big) \subset \overline{\hat{g}\big(i(X)\big)} \subset \overline{i(Y)} = \beta(Y)$.

   Hence $\hat{g}$ is the desired extension of g.

2. $i : X \hookrightarrow \beta(X)$ is continuous, and X is compact $\Rightarrow i(X)$ is compact $\Rightarrow i(X)$ is closed in $\beta(X)$ since $\beta(X)$ is Hausdorff.

   But $i(X)$ is dense in $\beta(X)$ so $i(X) = \beta(X)$. Hence i is a bijective map from a compact space to a Hausdorff space and is thus a homeomorphism.

Proof of Theorem (continued): Let $h : X \to Y$ where Y is compact Hausdorff. Then

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"', "\cong"] \\ \beta(X) \arrow[r, "\hat{h}"] & \beta(Y) \end{tikzcd}$$

So $i_Y^{-1} \circ \hat{h}$ is the desired extension of h to $\beta(X)$. If W is another space with these properties then $X \cong W$ by the standard category theory proof. $\square$

4.2 1st and 2nd countability

Definition 4.2.1 *X is called* 2nd countable *if $\exists$ a countable basis for the open sets of X.*

e.g. $X = \mathbb{R}^n$. Basis $= \{N_r(X) \mid r$ rational and all coordinates of X are rational $\}$

Definition 4.2.2 *X is called* 1st countable *if each $x \in X$ has a countable basis for its neighbourhoods.*

e.g. $X = metric$. $\{N_r(X) \mid r$ rational$\}$ is a basis for the neighbourhoods of X.

Definition 4.2.3 *X is called* separable *if it has a countable dense subset*

Proposition 4.2.4 *$2nd$ countable implies $1st$ countable and separable.*

Proof: $2nd$ countable implies $1st$ countable is trivial.

Let $\{U_j\}$ be a countable basis of (non-empty) open sets. $\forall j$, select $x_j \in U_j$. Let $A = \{x_j\}$. A is countable. Any open set intersects A so A is dense. $\square$

Example 4.2.5 *Compact subspace which is not closed.*

Let $X := \mathbb{R}$ as a set.

Specify the topology on X to be the one coming from the subbasis;

$$\{U \cap \mathbb{Q} \mid U \text{open in standard topology on } \mathbb{R}\} \cup \{V \mid V \text{is the complement of a finite set of rationals}\}$$

Observe: In corresponding basis, any basis set containing an irrational can be obtained only by intersecting the second type of sets, yielding another set of this type. Therefore any open set in X containing an irrational is the complement of a finite set of rationals.

Hence if $S \subset X$ contains an irrational then S is compact because in any open cover of S at least one set contains all but finitely many points of S, so S can be covered by that set together with one set for each of the missing points. In particular, if y is irrational, $\mathbb{Q} \cup \{y\}$ is compact but not closed. (Its complement contains irrationals, so it can't be open since any open set containing an irrational contains all irrationals.)

4.3 Convergent Sequences

Definition 4.3.1 *A sequence (x_n) in X* converges *to x, written $(x_n) \to x$, if $\forall$ open U, $\exists N$ s,t, $n \geq N \Rightarrow x_n \in U$.*

Proposition 4.3.2 *X Hausdorff, $(x_n) \to x$, $(x_n) \to y$ implies that $x = y$.*

Proof: If $x \neq y$ separate x, y by open sets and apply definition to give contradiction. $\square$

Proposition 4.3.3 *Suppose $A \subset X$. If $(a_n) \to x$ where $a_n \in A$ $\forall n$ then $x \in \overline{A}$. Conversely, if X is 1st countable and $x \in \overline{A}$ then $\exists$ sequence (a_n) in A s.t. $(a_n) \to x$ in X.*

Proof: Supppose $(a_n) \to x$. Then $\forall$ open U s.t. $x \in U$, $U \cap A \neq \emptyset$ so $x \notin \overline{A}$.

Conversely, suppose X is 1*st* countable and $x \in \overline{A}$.

Then any open neighbourhood of x intersects A.

Let $\{U_1, U_2, \ldots, U_n, \ldots\}$ be a basis for the open neighbourhoods of x.

Select $a_1 \in U$, $a_2 \in U_1 \cap U_2$, ..., $a_n \in U_1 \cap U_2 \cdots \cap U_n$, ..., with $a_n \in A$ $\forall n$. So $a_n \in U_k$ $\forall n \geq k$.

Given open V s.t. $x \in V$ find basic open U_N s.t. $U_N \subset V$.

Then $\forall n \geq N$, $a_n \in U_N \subset V$ so $(a_n) \to x$. $\square$

Definition 4.3.4 *If $A \subset X$ and $(a_n) \to x$ where $a_n \in A$ then x is called a* limit point *of A.*

Thus previous proposition says that in a 1 countable space, as set is closed if and only if it contains its limit points.

Proposition 4.3.5 *Let $f : X \to Y$ be a (set) function. f is continuous if and only if ($(x_n) \to x \Rightarrow f(x_n) \to f(x)$).*

Proof Suppose f is continous and $(x_n) \to x$.
Given U s.t. $f(x) \in U$ then $x \in f^{-1}(U)$ so $\exists N$ s.t. $nn \geq N \Rightarrow x_n \in f^{-1}(U)$.
Therefore $n \geq N \Rightarrow f(x_n) \in U$ so $f(x_n) \to f(x)$.
Conversely, suppose X $1st$ countable and ($(x_n) \to x \Rightarrow f(x_n) \to f(x)$).
Let $A \subset Y$ be closed. Show $f^{-1}(A)$ is closed.
Let $x \in \overline{f^{-1}(A)}$. Find sequence (x_n) in $f^{-1}(A)$ s.t. $(x_n) \to x$.
Then for all n, $f(x_n) \in A$ and hypothesis implies $\big(f(x_n)\big) \to f(x)$. So A closed implies $f(x) \in A$. Therefore $x \in f^{-1}(A)$.
Thus $\overline{f^{-1}(A)} = f^{-1}(A)$ and hence $f^{-1}(A)$ is closed.
Therefore f is continuous. □

Definition 4.3.6 *X is called* sequencially compact *if every sequence has a convergent subsequence.*

Definition 4.3.7 *Suppose X is Hausdorff and $1st$ countable. Then X compact implies X sequentially compact.*

Proof: Let X be Hausdorff, $1st$ countable and compact.

Let (x_n) be a sequence in X. If any element appeas infinitely many times in (x_n) then (x_n) has a constant (thus convergent) subsequence, so suppose not. Then discarding repeated elements gives us a subsequence so we may assume that (x_n) has no repetitions.

Claim: $\exists x \in X$ s.t. $\forall$ open U containing x, $U \cap \{x_n\}$ is infinite.
Proof: Suppose not. That is, suppose that $\forall x$, $\exists$ open U_x s.t. $x \in U_x$ and $U_x \cap \{x_n\}$ is finite.
Then $\{U_x\}$ is an open cover so ha s a finite subcover $U_x^{(1)}$, $U_x^{(2)}$, …, $U_x^{(k)}$.
Since $\forall j$, $U_x^{(j)} \cap \{x_n\}$ is finite, $\{x_n\}$ is finite.
$\Rightarrow\Leftarrow$.

Choose x as in claim and let $\{V_1, V_2, \ldots, V_k, \ldots\}$ be a basis for the neighbourhoods of x.
Choose $x_{n(1)} \in V_1 \cap \{x_n\}$.
Choose $x_{n(2)} \in V_1 \cap V_2 \cap \{x_n \mid n > n(1)\}$.

$\vdots$

Choose $x_{n(k)} \in V_1 \cap \cdots \cap V_k \cap \{x_n \mid n > n(k-1)\}$.

$\vdots$

Then $(x_{n(1)}, x_{n(2)}, \ldots, x_{n(k)}, \ldots)$ is a subsequence of (x_n) and converges to x. $\square$

Chapter 5

Metric Spaces

5.1 Completeness

Definition 5.1.1 *Let (x_n) be a sequence in (X, d). Then (x_n) is called a* Cauchy sequence *if $\forall \epsilon > 0\ \exists N$ s.t. $n, m > N \Rightarrow d(x_n, x_m) < \epsilon$.*

Proposition 5.1.2 *$(x_n) \to x \Rightarrow (x_n)$ Cauchy.*

Proof: Obvious.

Definition 5.1.3 *A* complete metric space *is one in which $\forall$ Cauchy sequences (x_n) $\exists\, x \in X$ s.t. $(x_n) \to x$.*

Definition 5.1.4 *A complete normed vector space is called a* Banach space.

Proposition 5.1.5 *Suppose (X, d) is complete, and $Y \subset X$. Then Y is complete $\Leftrightarrow$ Y is closed.*

Proof: Exercise.

Theorem 5.1.6 Cantor intersection theorem *Let (X, d) be a complete metric space. Let (F_n) be a decreasing sequence of nonempty closed subsets of X s.t. $\operatorname{diam}(F_n) \to 0$ in $\mathbb{R}$. Then $\cap_n F_n$ contains exactly one point.*

Proof: Let $F = \cap_n F_n$. If F contains two points x and y then we have a contradiction when $\operatorname{diam}(F_n) < d(x, y)$. Hence $|F| \leq 1$.

$\forall n$ choose $x_n \in F_n$. $\operatorname{diam}(F_n) \to 0 \Rightarrow (x_n)$ is Cauchy.

Hence $\exists x \in X$ s.t. $(x_n) \to x$. We show that $x \in F_n\ \forall n$. If $\{x_n\}$ is finite then $x_n = x$ for infinitely many n, so that $x \in F_n$ for infinitely many n. Since $F_{n+1} \subset F_n$ this implies $x \in F_n\ \forall n$. So suppose $\{x_n\}$ is infinite. $\forall m$, $(x_m, x_{m+1}, \ldots, x_{m+k}, \ldots)$ is a sequence in F_m converging to x. Since $\{x_n\}_{n \geq m}$ is infinite, this implies x is a limit point of F_m. But F_m is closed, so $x \in F_m$. $\square$

Theorem 5.1.7 *Let (X, d) be a metric space. Then $\exists!$ metric space $(\tilde{X}, \tilde{d})$ together with an isometry $\imath : X \to \tilde{X}$ s.t.*

1. *$(\tilde{X}, \tilde{d})$ is complete.*
2. *Given any complete (Y, d') and an isometry $j : X \to Y$, $\exists!$ isometry $\tilde{j}\colon \tilde{X} \to Y$ s.t.*

Note: An isometry $f : X \to Y$ is a map s.t. $d\big(f(a), f(b)\big) = d(a, b)\ \forall a, b \in X$.

Definition 5.1.8 *$\tilde{X}$ is called the completion of X.*

Sketch of Proof:
Let $C = \{$ Cauchy sequences in $X\}$.
Impose an equivalence relation $(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0$ in $\mathbb{R}$.
Let $\tilde{X} = C/\sim$. Define $\tilde{d}\big((x_n), (y_n)\big) = \lim_{n\to\infty} d(x_n, y_n)$.
Define $\imath : X \to \tilde{X}$ by $x \mapsto (x, x, \ldots, x, \ldots)$ Check that it works. (Exercise) $\square$

Proposition 5.1.9 *X is dense in $\tilde{X}$.*

Proof: $\bar{X}$ is closed in $\tilde{X}$, so complete. It also satisfies the universal property of completion so $\bar{X} = \tilde{X}$. $\square$

Definition 5.1.10 *$f : X \to Y$ is called* uniformly continuous *if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d(a, b) < \delta \Rightarrow d\big(f(a), f(b)\big) < \epsilon$.*

Proposition 5.1.11 *$f : X \to Y$ is uniformly continuous, (x_n) is Cauchy in $X \Rightarrow \big(f(x_n)\big)$ is Cauchy in Y.*

Proof: Exercise.

Definition 5.1.12 *Let (f_n) be a sequence of functions $f_n : X \to Y$. We say f_n* converges uniformly *to $f : X \to Y$ if $\forall \epsilon > 0\ \exists N$ s.t. $n > N \Rightarrow d\big(f(x), f(y)\big) < \epsilon\ \forall x \in X$.*

Proposition 5.1.13 *Suppose f_n converges uniformly to f and f_n is continuous $\forall n$. Then f is continuous.*

Proof: Let $a \in X$. Show f is continuous at a. Given $\epsilon > 0$, choose N_0 s.t. $n \geq N_0 \Rightarrow d\big(f(x), f_n(x)\big) < \epsilon/3 \; \forall x \in X$.

Choose δ s.t. $d(x,a) < \delta \Rightarrow d\big(f_{N_0}(x), f_{N_0}(a)\big) < \epsilon/3$. Then $d(x,a) < \delta \Rightarrow d\big(f(x), f(a)\big) \leq d\big(f(x), f_{N_0}(x)\big) + d\big(f_{N_0}(x), f_{N_0}(a)\big) + d\big(f_{N_0}(a), f(a)\big) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. $\square$

Example 5.1.14 *Sequence of continuous functions whose pointwise limit is not continuous:*

$f_n : [0,1] \to [0,1]$, $f_n(x) = x^n$. $f(x) = \begin{cases} 0 & x < 1; \\ 1 & x = 1. \end{cases}$

Notation: Let X be a topological space (not necessarily metric).

$\mathcal{C}(X, \mathbb{R})$, resp. $\mathcal{C}(X, \mathbb{C})$ are real-valued (resp. complex-valued) bounded continuous functions on X.

Proposition 5.1.15 $\mathcal{C}(X, \mathbb{R})$ *and* $\mathcal{C}(X, \mathbb{C})$ *are Banach spaces.*

Proof: Let $Y = \mathcal{C}(X, \mathbb{R})$, or $\mathcal{C}(X, \mathbb{C})$.

For $f \in Y$, setting $||f|| = \sup_{x \in X} |f(x)|$ makes Y into a normed vector space. Let (f_n) be a Cauchy sequence in Y. Then $\forall x \in X$, $\big(f_n(x)\big)$ is a Cauchy sequence in $\mathbb{R}$ (resp. $\mathbb{C}$) so set $f(x) = \lim_{n \to \infty} f_n(x)$.

Must show f is bounded and continuous, and show $(f_n) \to f$ in Y.

Given $\epsilon > 0$, find N s.t. $n, m > N \Rightarrow ||f_n - f_m|| < \epsilon/2$.

Given $x \in X$ find $n_x > N$ s.t. $|f_{n_x}(x) - f_n(x)| < \epsilon/2$.

Then $n > N \Rightarrow |f(x) - f_n(x)| \leq |f(x) - f_{n_x}(x)| + |f_{n_x}(x) - f_n(x)| < \epsilon/2 + \epsilon/2 = \epsilon$ Hence (f_n) converges uniformly to f so f is continuous. $||f|| \leq ||f - f_N|| + ||f_N|| < ||f_N|| + \epsilon < \infty$ so f is bounded. Therefore $f \in Y$, and $\{f\} \to f$ in Y since $||f - f_N|| \to 0$.

Theorem 5.1.16 (Tietze extension theorem) *Let X be normal and $A \subset X$ is closed. Let $f : A \to [p,q]$. Then there exists $F : X \to [p,q]$ s.t. $F|_A = f$.*

Proof: If $p = q$ then f is constant and the theorem is trivial so suppose $p < q$. Let $c = \max(p,q)$.

Claim: $\exists\, h : X \to [-c/3, c/3]$ s.t. $|h(a) - f(a)| \leq 2/3c \; \forall a \in A$.

Proof: Set $A_- = f^{-1}[-c, -c/3]$ and $A_+ = f^{-1}[c/3, c]$. By Urysohn, $\exists g : X \to [0,1]$ s.t. $g(A_-) = 0$ and $g(A_+) = 1$.

Composing with a homeomorphism of $[0,1]$ with $[-c/3, c/3]$ gives a function $h : X \to [-c/3, c/3]$ s.t. $h(A_-) = -c/3$ and $h(A_+) = c/3$. If $a \in A$ then $|h(a) - f(a)| \leq 2/3c$.

Apply the Claim to f. This implies $\exists h_1 : X \to [-c/3, c/3]$ s.t. $|f(a) - h_1(a)| \leq 2/3c$. Apply the Claim to $f - h_1$. This implies $\exists h_2 : X \to [-2c/3^2, 2c/3^2]$ s.t. $|f(a) - h_1(a) -$

$h_2(a)| \leq (2/3)^2 c$. By induction, we apply the Claim to $f - h_1 - \cdots - h_{n-1}$. This implies $\exists h_n : X \to [-2^{n-1}c/3^n, 2^n c/3^n]$ s.t. $|f(a) - h_1(a) - \cdots - h_{n-1}(a)| \leq (2/3)^n c$.

Let $G(x) = \sum_{n=1}^{\infty} h_n(x)$.

$\forall x \in X$,

$$|G(x)| \leq \sum_{n=1}^{\infty} |\,|h_n(x)|\,| \leq \sum_{n=1}^{\infty} |\,|h_n|\,| = c/3(1 + 2/3 + (2/3)^2 + \dots) = c/3(\frac{1}{1-2/3}) = c.$$

The partial sums of G are a Cauchy sequence in $\mathcal{C}(X, \mathbb{R})$.

Hence by completeness of $\mathcal{C}(X, \mathbb{R})$ their pointwise limit $G : X \to [-c, c]$ is continuous.

Define F by

$$F(x) = \begin{cases} G(x) & \text{if } p \leq G(x) \\ p & \text{if} G(x) < p \\ q & \text{if} G(x) > q \end{cases}$$

$F|_A = G|_A$ since $p \leq f(a) \leq q \ \forall a \in A$. □

5.2 Compactness in Metric Spaces

Proposition 5.2.1 *A sequentially compact metric space is complete.*

Proof: Suppose X is sequentially compact, and (x_n) is Cauchy in X.

Some convergent subsequence of (x_n) converges to $x \in X$ so since (x_n) is Cauchy, with $(x_n) \to x$. That is, given $\epsilon > 0$, $\exists N$ s.t. $m, n \geq N \Rightarrow d(x_n, x_m) < \epsilon/2$. Therefore since some subsequence of (x_n) converges, $N_{\epsilon/2}(x)$ contains x_m for infinitely many m, so $\exists m > N$ s.t. $x_m \in N_{\epsilon/2}(x)$ and therefore $n \geq N \Rightarrow d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) < \epsilon/2 + \epsilon/2 = \epsilon$.

Definition 5.2.2 *Given $\epsilon > 0$, a finite subset T of X is called an* ϵ-net *if $\{N_\epsilon(t)\}_{t \in T}$ forms an open cover of X.*

X is called totally bounded *if $\forall \epsilon > 0$, $\exists$ an ϵ-net for X.*

Note: X totally bounded $\Rightarrow \text{diam}(X) < \text{diam}(T) + 2\epsilon$ and $\text{diam}(T) < \infty$ since T finite, so totally bounded implies bounded.

Example 5.2.3 *Suppose X is infinite with*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Then X is bounded but $\nexists$ an ϵ -net for any $\epsilon < 1$.

Theorem 5.2.4 *For metric X, the following are equivalent:*

1. *X compact*
2. *X sequentially compact*
3. *X is complete and totally bounded.*

Proof:
(1) $\Rightarrow$ (2)
Already showed: metric $\Rightarrow$ first countable and Hausdorff
and first countable and Hausdorff and compact $\Rightarrow$ sequentially compact.

(2) $\Rightarrow$ (3):
Suppose X is sequentially compact.
We already showed this implies X is complete.
Given $\epsilon > 0$: Pick $a_1 \in X$.
Having chosen $a_1, \ldots, a_{n-1}$ if $N_\epsilon(a_1) \cup \ldots N_\epsilon(a_{n-1})$ covers X, we are finished.
If not, choose $a_n \in X - \big(N_\epsilon(a_1) \cup \ldots N_\epsilon(a_{n-1})\big)$.
So either we get an ϵ-net $\{a_1, \ldots, a_n\}$ for some n, or we get an infinite sequence $(a_1, a_2, \ldots, a_n, \ldots)$.
If the latter: By construction $d(a_k, a_n) \geq \epsilon \ \forall k, n$ so (a_n) has no convergent subsequence. This is a contradiction. So the former holds. $\square$

(2) $\Rightarrow$ (1):

Definition 5.2.5 *Let $\{G_\alpha\}_{\alpha \in J}$ be an open cover of the metric space X. Then $a > 0$ is called a* Lebesgue number *for the cover if* $\operatorname{diam}(A) < a \Rightarrow A \subset G_\alpha$ *for some α.*

Theorem 5.2.6 (*Lebesgue's Covering Lemma*) *If X is sequentially compact, then every open cover has a Lebesgue number.*

Proof: Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover.
Say $A \subset X$ is "big" if A is not contained in any U_α.
If $\nexists$ big subsets then any $a > 0$ is a Lebesgue number, so assume $\exists$ big subsets.
Let $a = \inf\{\operatorname{diam}(A) \mid A \text{ big}\}$
If $a > 0$, a is a Lebesgue number, so we assume $a = 0$.
Hence $\forall n > 0$, $\exists$ a big B_n s.t. $\operatorname{diam}(B_n) < 1/n$.
$\forall n$, pick $x_n \in B_n$. Find x s.t. a subsequence of (x_n) converges to x.
Find α_0 s.t. $x \in U_{\alpha_0}$.
U_{α_0} is open, so $\exists r > 0$ s.t. $N_r(x) \subset U_{\alpha_0}$.
For infinitely many n, $x_n \in N_{r/2}(x)$.

Find N s.t. $N > 2/r$ and $x_N \in N_{r/2}(x)$.

$\operatorname{diam}(B_N) < 1/N < r/2$ and $B_N \cap N_{r/2}(x) \neq \emptyset$ $\Big($since $x \in B_N \cap N_{r/2}(x)$ $\Big)$ so $B_N \subset N_r(x) \subset U_{\alpha_0}$. This is a contradiction, since B_N is big.

Hence $a > 0$ so X has a Lebesgue number.

Proof that (2) $\Rightarrow$ (1):

Given an open cover $\{U_\alpha\}_{\alpha \in J}$, find a Lebesgue number a for $\{U_\alpha\}$.

Let $\epsilon = a/3$ and using (2) $\Rightarrow$ (3) from the above, pick an ϵ-net $T = \{t_1, t_2, \ldots, t_n\}$. For $k = 1, \ldots, n$ $\operatorname{diam} N_\epsilon(t_k) = 2\epsilon < a$ so $N_\epsilon(t_k) \subset U_{\alpha_k}$ for some α_4.

Since $\{N_\epsilon(t_1), N_\epsilon(t_2), \ldots, N_\epsilon(t_n)\}$ covers X (by definition of ϵ-net), so does $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$.

$3 \Rightarrow 2$:

Suppose X is complete and totally bounded.

Let $S^{(0)} = (x_1, x_2, \ldots, x_m, \ldots)$ be a sequence in X.

Since X is complete, to show $S^{(0)}$ has a convergent subsequence, it suffices to show $S^{(0)}$ has a Cauchy subsequence.

Choosing an ϵ-net for $\epsilon = 1/2$, cover X with finitely many balls of radius $1/2$. Since $S^{(0)}$ is infinite, some ball contains infinitely many x_m so discard the x_n outside that ball to get a subsequence $S^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)}, \ldots)$ with $d(x_m^{(1)}, x_p^{(1)}) < 2\epsilon = 1$ $\forall m, p$. Repeating this procedure with $\epsilon = 1/4, 1/6, \ldots, 1/(2n), \ldots$ gives for each n a subsequence of $S^{(n-1)}$.

$S^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}, \ldots)$ s.t. $d(x_m^{(n)}, x_p^{(n)}) < 1/n$ $\forall m, p$.

Let $S^{(n)} = (x_1^{(1)}, x_2^{(2)}, \ldots, x_n^{(n)}, \ldots)$

If $m, p \geq n$ then since $S^{(m)}$ and $S^{(p)}$ are subsequences of $S^{(n)}$, $d(x_m^{(m)}, x_p^{(p)}) < 1/n$ so S is a Cauchy subsequence of $S^{(0)}$ as desired. $\square$

Theorem 5.2.7 *If X and Y are metric spaces, and $f : X \to Y$ is a continuous function with X compact, then f is uniformly continuous.*

Proof: Given $\epsilon > 0$, $x \in f^{-1}\big(N_{\epsilon/2}(f(x)\big)$, so $\Big\{f^{-1}\big(N_{\epsilon/2}(f(x)\big)\Big\}_{x \in X}$ is an open cover of X.

Let δ be a Lebesgue number for this cover.

$\forall a, b \in X$: $d(a, b) < \delta \Rightarrow \operatorname{diam}\{a, b\} < \delta \Rightarrow \{a, b\} \subset f^{-1}\big(N_{\epsilon/2}(f(x)\big)$ for some x. Hence $d\big(f(a), f(b)\big) \leq d\big(f(a), f(x)\big) + d\big(f(x), f(b)\big) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence f is uniformly continuous. $\square$

Corollary 5.2.8 *A compact metric space is second countable.*

Lemma 5.2.9 *For metric spaces second countable $\Leftrightarrow$ separable.*

Proof: Second countable $\Rightarrow$ separable in general.
$\Longleftarrow$ Suppose X is a separable metric space. Let $\{x_1, \ldots, x_n, \ldots\}$ be a countable dense subset. Then $\{N_r(x_j)|$ r rational $\}$ forms a countable basis for X. (That is: Given $N_{r'}(x)$, find x_n s.t. $d(x_n, x) < r'/3$. Choose rational r s.t. $r < r'/3$. Then $N_r(x_n) \subset N_{r'}(x)$.) $\square$

Proof of Corollary: Suppose X is a compact metric space. Show X is separable.

For each $\epsilon = 1/n$, choose an ϵ-net $T_n = \{x_1^{(n)}, \ldots, x_{k_n}^{(n)}\}$. Let $S = \cup_n T_n$. Then S is a countable dense subset of X.

$\square$

Example 5.2.10 *Normal but not metric:*

Let $X = \prod_{t \in R} I_t$ where $I_t = [0, 1]$ $\forall t$. X is compact by Tychonoff and is Hausdorff so X is normal.

If X were metric, then being compact, it would be second countable.

Let $\mathcal{S} = \{U_1, \ldots, U_n, \ldots\}$ will be a countable basis.

Since $\mathbb{R}$ is uncountable, $\exists t_n \in \mathbb{R}$ s.t. $\pi_{t_0}(U_n) = I_{t_0}$ $\forall n$. But then $\mathcal{S}$ is not a basis. (e.g. The set $(1/4, 3/4) \times \prod_{t \neq t_0} I_t$ is not a union of sets in $\mathcal{S}$.

This is a contradiction. So X is not metric.

$\square$

Chapter 6

Paracompactness

Let $\{W_\alpha\}_{\alpha \in I}$ be a cover of X. (We do not assume W_α is open.)

Definition 6.0.11 *A cover* $\{T_\beta\}_{\beta \in J}$ *is called a* refinement *of* $\{W_\alpha\}_{\alpha \in I}$ *if* $\forall \beta \in J$, $\exists \alpha \in I$ *s.t.* $T_\beta \subset W_\alpha$.

Definition 6.0.12 *A collection* $\{W_\alpha\}_{\alpha \in I}$ *of subsets of* X *is called* locally finite *if each* $x \in X$ *has an open neighbourhood whose intersection with* W_α *is non-empty for only finitely many* α.

Proposition 6.0.13 $\{W_\alpha\}_{\alpha \in I}$ *is locally finite* $\Rightarrow \cup_\alpha \overline{W}_\alpha = \overline{\cup_\alpha W_\alpha}$

Proof: $\overline{W_\alpha} \subset \overline{\cup_\alpha W_\alpha} \;\Rightarrow\; \cup_\alpha \overline{W_\alpha} \subset \overline{\cup_\alpha W_\alpha}$

Conversely suppose $y \notin \cup \overline{W}_\alpha$.

Find open U s.t. $y \in U$ and $U \cap W_\alpha = \emptyset$ for $\alpha \neq \alpha_1, \dots \alpha_n$.

$y \notin \overline{W_{\alpha_1}}, \dots, \overline{W_{\alpha_n}}$.

Therefore $y \in V := U \cap (\overline{W_{\alpha_1}})^c \cap \dots \cap (\overline{W_{\alpha_n}})^c$ open

$V \cap W_\alpha = \emptyset \; \forall \alpha$

Therefore $V^c \subset \overline{\cup_\alpha W_\alpha}$ (since V^c closed)

Therefore $V \cap (\overline{\cup_\alpha W_\alpha}) = \emptyset$.

Hence $y \notin \overline{\cup W_\alpha}$ □

Definition 6.0.14 *A topological space* X *is called* paracompact *if every open cover of* X *has a locally finite refinement.*

Note: Compact $\Rightarrow$ paracompact. (A subcover is also a refinement.)

Proposition 6.0.15 *If* A *is closed* $\subset X$ *and* X *is paracompact, then* A *is paracompact.*

Proof: Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of A. For all α write $U_\alpha = V_\alpha \cap A$ with V_α open in X. Then $\{V_\alpha\} \cup \{A^c\}$ is an open cover of X so it has a locally finite refinement $\{W_\beta\}_{\beta\in I}$.

Then $\{W_\beta \cap A\}_{\beta\in I}$ is a locally finite refinement of $\{U_\alpha\}_{\alpha\in J}$. $\square$

Proposition 6.0.16 *X paracompact Hausdorff $\Rightarrow$ X normal*

Proof:
First show that X is regular:

Let $a \in X$ and let $B \subset X$ be closed with $a \notin B$.
$\forall b \in B$ $\exists$ open nbhd. U_b s.t. $a \notin \overline{U_b}$ (X Hausdorff)
$\{U_b\}_{b\in B} \cup B^c$ is an open cover of X.
Let $\{W_\alpha\}_{\alpha\in J}$ be a locally finite refinement.
Let $I = \{\alpha \in J \mid W_\alpha \cap B \neq \emptyset\}$ Therefore $\{W_\alpha\}_{\alpha\in I}\}$ covers B.
Set $V := \cup_{\alpha\in I} W_\alpha \supset B$.
$\forall\alpha$ $\exists b \in B$ s.t. $W_\alpha \subset U_b$, and so $\overline{W_\alpha} \subset \overline{U_\beta} \Rightarrow a \notin \overline{W_\alpha}$
Therefore $a \notin \cup_{\alpha\in I}\overline{W_\alpha} = \overline{\cup_{\alpha\in I} W_\alpha} = \overline{V}$.
Therefore X is regular.

Now given closed A, B, s.t. $A \cap B = \emptyset$
$\forall b \in B \exists$ open U_b s.t. $A \cap \overline{U_b} = \emptyset$.
$\{U_b\}_{b\in B} \cup B^c$ covers X.
Let $\{W_\alpha\}_{\alpha\in J}$ be a locally finite refinement.
Let $I = \{\alpha \in J | W_\alpha \cap B \neq \emptyset\}$. Then $\{W_\alpha\}_{\alpha\in I}$ covers B. Set $V = \cup_{\alpha\in I} W_\alpha$.
For all α $\exists b \in B$ s.t. $W_\alpha \subset U_b$ so $\overline{W_\alpha} \subset \bar{U}_b \Rightarrow A \cap \bar{W}_\alpha = \emptyset$. Hence $\emptyset = A \cap (\cup_{\alpha\in I}\overline{W_\alpha}) = A \cap \overline{\cup_{\alpha\in I} W_\alpha} = A \cap \bar{V}$.
Hence X is normal. $\square$

Definition 6.0.17 *Let X be a topological space and let $\{U_j\}_{j\in J}$ be an open cover of X. A* partition of unity *relative to the cover $\{U_j\}_{j\in J}$ consists of a set of functions $f_j : X \to [0,1]$ such that:*

1. $\overline{f_j^{-1}\big((0,1]\big)} \subset U_j$ $\forall j \in J$.
2. $\overline{f_j^{-1}\big((0,1]\big)}_{j\in J}$ *is locally finite.*
3. $\sum_{j\in J} f_j(x) = 1$ $\forall x \in X$.

Note: (2) implies that if $x \in X$, $f_j(x) = 0$ for all but finitely many j so the sum in (3) makes sense.

$\{f_j\}_{j\in J}$ is a partition of unity implies that $\left\{ f_j^{-1}\big((0,1]\big)_{j\in J} \right\}$ is a locally finite refinement of $\{U_j\}$.

Hence if every open cover of X has a partition of unity then X is paracompact.

Conversely

Theorem 6.0.18 *If X is paracompact Hausdorff, then for every open cover $\{U_\alpha\}_{\alpha\in J}$ of X there is a partition of unity relative to $\{U_\alpha\}_{\alpha\in J}$.*

Proof: Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of X where X is paracompact Hausdorff.

Let $\{V_\beta\}_{\beta\in I}$ be a locally finite refinement.

Then $\exists \phi : I \to J$ s.t. $V_\beta \subset U_{\phi(\beta)}\ \forall \beta \in I$.

Given $\alpha \in J$ set $W_\alpha = \cup_{\{\beta|\phi(\beta)=\alpha\}} V_\beta$. Then $W_\alpha \subset U_\alpha$.

Claim: $\{W_\alpha\}$ is locally finite.

Proof of Claim: Let $x \in X$. Then $\exists U_x$ s.t. $U_x \cap V_\beta = \emptyset$ for all but $\beta_1, \ldots, \beta_n$. Hence $U_x \cap W_\alpha = \emptyset$ unless $\phi(\beta_j) = \alpha$ for some $j = 1, \ldots, n$.

Therefore $U_x \cap W_\alpha = \emptyset$ unless $\phi(\beta_j) = \alpha$, some $j = 1, \ldots, n$.

i.e. $U_x \cap W_\alpha = \emptyset$ for all but $\phi(\beta_1), \ldots, \phi(\beta_n)$ which is a finite set (although it might contain duplicate entries).

Therefore $\{W_\alpha\}$ locally finite. $\surd$

Proof of Thm. (cont.) Suff. to show $\exists$ partition of unity relative to $\{W_\alpha\}$ since this gives functions $f_\alpha : X \to [0,1]$ s.t. $\overline{f^{-1}((0,1])} \subset W_\alpha \subset U_\alpha$.

Lemma 6.0.19 *Let $\{U_\alpha\}_{\alpha\in J}$ be a locally finite open cover of X where X normal. Then $\exists$ locally finite open cover $\{V_\alpha\}_{\alpha\in J}$ s.t. $V_\alpha \subset \overline{V_\alpha} \subset U_\alpha \quad \forall \alpha \in J$.*

Proof of Thm. (concluded; given Lemma):

Apply Lemma to $\{W_\alpha\}_{\alpha\in J}$ to get cover $\{V_\alpha\}_{\alpha\in J}$ s.t. $V_\alpha \subset \overline{V_\alpha} \subset W_\alpha \quad \forall \alpha$.

$\{W_\alpha\}$ locally finite $\Rightarrow \{V_\alpha\}$ locally finite.

Do it again to get locally finite cover $\{T_\alpha\}_{\alpha\in J}$ s.t. $T_\alpha \subset \overline{T_\alpha} \subset V_\alpha \subset \overline{V_\alpha} \subset W_\alpha \quad \forall \alpha$.

X paracompact Hausdorff $\Rightarrow X$ normal $\Rightarrow \exists g_\alpha : X \to [0,1]$ s.t. $g_\alpha(\overline{T_\alpha}) = 1$, $g_\alpha(V_\alpha^c) = 0$.

$g_\alpha^{-1}(0,1] \subset V_\alpha \Rightarrow \overline{g_\alpha^{-1}(0,1]} \subset \overline{V_\alpha} \subset W_\alpha$.

Define $g(x) = \sum_\alpha g_\alpha(x)$ (finite sum since $f_\alpha(x) = 0$ unless $x \in V_\alpha$ and $\{V_\alpha\}$ locally finite so x in only finitely many V_α)

Set $f_\alpha(x) = g_\alpha(x)/g(x)$.

Then $\{f_\alpha\}_{\alpha\in J}$ is the desired partition of unity.

Proof of Lemma: To help prove Lemma:

Lemma 6.0.20 *(Sublemma). Let X be normal. Suppose $X = U \cup V \qquad U, V$ open. Then $\exists$ open W s.t. $W \subset \overline{W} \subset U$ and $X = W \cup V$.*

Proof:(Exercise)

Proof of Lemma (cont.): Well order J.
$X = U_{j_0} \cup W_{j_0}$ where $j_0 =$ least elt. of J and $W_{j_0} = \bigcup_{j>j_0} U_j$.
SubLemma $\Rightarrow \exists$ open V_{j_0} s.t. $V_{j_0} \subset \overline{V_{j_0}} \subset U_\alpha$ and $X = V_{j_0} \cup W_{j_0}$.
Suppose that for all $\gamma < \beta$ we have found open V_γ s.t. $V_\gamma \subset \overline{V_\gamma} \subset U_\gamma$ and

$$X = \bigcup_{j \le \gamma} V_j \cup \bigcup_{j > \gamma} U_j.$$

Claim: $X = \bigcup_{j<\beta} V_j \cup \bigcup_{j \ge \beta} U_j$.

Proof of Claim: Let $x \in X$.
If $x \in U_j$ some $j \ge \beta$, then $x \in$ RHS.
Otherwise, let M be max. s.t. $x \in U_M$. $\qquad$ ($\{U_j\}$ locally finite $\Rightarrow \exists$ such max.)
Since $M < \beta$, applying induction hypoth. with $\gamma = M$:

$$X = \bigcup_{j \le M} V_j \cup \bigcup_{j > M} U_j.$$

$x \notin U_j$ any $j > M$ so $x \in V_j$ some $j \le M$.
i.e. $x \in$ RHS.

Proof of Lemma (cont.): By Claim, $X = U_\beta \cup W_\beta$ where

$$W_\beta = \bigcup_{j \le \beta} V_j \cup \bigcup_{j > \beta} U_j.$$

SubLemma$\Rightarrow \exists$ open V_β s.t. $V_\beta \subset \overline{V_\beta} \subset U_\beta$ and $X = V_\beta \cup W_\beta$. i.e.

$$X = \bigcup_{j \le \beta} V_j \cup \bigcup_{j > \beta} U_j.$$

completing induction step.
Therefore $\exists$ open V_j s.t. $V_j \subset \overline{V_j} \subset U_j$ and

$$X = \bigcup_{j \le \gamma} V_j \cup \bigcup_{j > \gamma} U_j \qquad \forall \gamma.$$

Claim: $X = \cup_j V_j$.

Proof: Given $x \in X$ find max. M s.t. $x \in U_M$.
Apply above with $\gamma = M$ to see that $x \in V_j$ some $j \le \gamma$.

Proof of Lemma (concluded): $V_j \subset U_j \quad \forall j$, $\{U_j\}$ locally finite $\Rightarrow \{V_j\}$ locally finite.
$\{V_j\}$ is the required cover. $\square$

Theorem 6.0.21 *Let X be regular. Suppose that every open cover of X has a countable refinement. Then X is paracompact.*

Lemma 6.0.22 *Let $\{B_\beta\}_{\beta\in J}$ be a locally finite cover of X by closed sets. Suppose $\{E_\alpha\}_{\alpha\in I}$ is a collection of sets (arbitrary — not necessarily open, closed, ...) s.t. $\forall \beta$, $B_\beta \cap E_\alpha = \emptyset$ for almost all α. Then $\forall \alpha \in I$ we can choose open U_α s.t. $E_\alpha \subset U_\alpha$ and $\{U_\alpha\}$ locally finite.*

Note: $\{E_\alpha\}$ must be locally finite.
i.e. $\forall x \exists Q_x$ s.t. Q_x intersects only finite many B_β and each such B_β intersects only finitely many E_α.

Proof of Lemma: Set $C_\alpha := \bigcup_{B_\beta\cap E_\alpha=\emptyset} B_\beta$.
$\{B_\beta \mid B_\beta \cap E_\alpha = \emptyset\} \subset \{B_\beta\}$ which is locally finite.
Therefore $\overline{C_\alpha} = \bigcup_{B_\beta\cap E_\alpha=\emptyset} \overline{B_\beta} = \bigcup_{B_\beta\cap E_\alpha=\emptyset} B_\beta = C_\alpha$.
Therefore C_α is closed.
Set
$$U_\alpha := (C_\alpha)^c = \bigcup_{\substack{B_\beta\cap E_\alpha=\emptyset \\ \Leftrightarrow \\ E_\alpha \subset B_\beta^c}} B_\beta^c \supset E_\alpha.$$

Show $\{U_\alpha\}$ locally finite.
Let $x \in X$.
Find open V s.t. $x \in V$ and $V \cap B_\beta = \emptyset$ for $\beta \ne \beta_1, \ldots, \beta_n$.
Therefore $V \subset B_{\beta_1} \cup \ldots \cup B_{\beta_n}$.
$\forall j$, $B_{\beta_j} \cap E_\alpha = \emptyset$ for all but finitely many α.
Let $\{\alpha_1, \ldots, \alpha_k\}$ be the set of all such α for all $j = 1, \ldots, n$.
For $\alpha \ne \alpha_1, \ldots, \alpha_k$:
$V \subset B_{\beta_1} \cup \ldots \cup B_{\beta_n} \subset \bigcup_{B_\beta\cap E_\alpha=\emptyset} B_\beta = C_\alpha$
Therefore $V \cap U_\alpha = \emptyset$ for $\alpha \ne \alpha_1, \ldots, \alpha_k$. $\square$

Proof of Thm. Let $\{U_j\}_{j\in J}$ be an open cover of X.
$\forall x \in X$, $x \in U_{j(x)}$ for some $j(x)$.

X regular $\Rightarrow \exists W_x$ s.t. $x \in W_x \subset \overline{W_x} \subset U_{j(x)}$

$\{W_x\}$ is an open cover refining $\{U_\alpha\}_{\alpha\in J}$

Applying hypothesis to $\{W_x\}$ gives a countable refinement of $\{W_x\}$ (thus a refinement of $\{U_\alpha\}_{\alpha\in J}$) $V_1, V_2, \ldots, V_n, \ldots$, where $\forall j\ V_j \subset \overline{V_j} \subset U_{\alpha(j)}$ for some $\alpha(j)$

Set

$$\begin{aligned}
E_1 &:= \overline{V_1} \\
E_2 &:= \overline{V_2 - V_1} \\
&\vdots \\
E_n &:= \overline{V_n - \textstyle\bigcup_{j=1}^{n-1} V_j} \subset \overline{V_n} \subset U_{\alpha(n)} \\
&\vdots
\end{aligned}$$

For $x \in X$:

$\exists$ least n s.t. $x \in V_n$.

$x \in E_n$ for this n.

Therefore $\{E_n\}$ covers X.

If $k > n$, $V_n \cap \left(V_k - \bigcup_{j=1}^{k-1} V_{k-1}\right) = \emptyset$

Since E_k is the closure of $V_k - \bigcup_{j=1}^{k-1} V_{k-1} = \emptyset$, V open $\Rightarrow V_n \cap E_k = \emptyset$.

Therefore $\{E_k\}$ locally finite (since each $x \in V_n$ for some n.)

$\{E_k\}$ is a locally finite refinement of $\{U_\alpha\}$.

Repeat procedure on cover $\{V_n\}$ to get a locally finite closed refinement $\{B_\beta\}$ of $\{V_n\}$.

By construction $\forall\beta$, $B_\beta \subset V_n$ for some n so $B_\beta \cap E_k = \emptyset$ for almost all k.

Therefore Lemma $\Rightarrow \forall k\ \exists$ open W_k s.t. $E_k \subset W_k$ and $\{W_k\}$ locally finite.

Set $W'_k := W_k \cap U_{\alpha(k)} \subset U_{\alpha(k)}$ open.

$E_k \subset W_k$ and $E_k \subset U_{\alpha(k)} \Rightarrow E_k \subset W'_k$.

$\{E_k\}$ covers so $\{W'_k\}$ covers.

$W'_k \subset U_{\alpha(k)} \Rightarrow \{W'_k\}$ is a refinement.

$W'_k \subset W_k$, $\{W_k\}$ locally finite $\Rightarrow \{W'_k\}$ locally finite. □

Corollary 6.0.23 *X regular and 2nd countable $\Rightarrow X$ paracompact.*

Proof: Let $\{U_\alpha\}$ be an open cover of X.

Let $W_1, W_2, \ldots, W_n, \ldots$ be a countable basis.

If $x \in X$ then $x \in U_\alpha$ some α so $\exists$ basic open $W_{n(x)}$ s.t. $x \in W_{n(x)} \subset U_\alpha$.

Therefore $\{W_{n(x)}\}$ is a refinement of $\{U_\alpha\}$ which covers X and is countable (subcollections of a countable collection)

Therefore Thm. $\Rightarrow X$ paracompact. □

Chapter 7

Connectedness

Definition 7.0.24 *A pair of nonempty open subsets A and B of a topological space X is called a* disconnection *of X if $A \cap B = \emptyset$ and $A \cup B = X$.*

Note: If A, B is a disconnection of X then A and B are also closed since $A = B^c$ and $B = A^c$.

Proposition 7.0.25 *A subspace of $\mathbb{R}$ is connected $\Leftrightarrow$ it is an interval. In particular $\mathbb{R}$ is connected.*

Proof: Exercise.

Proposition 7.0.26 *Suppose $f : X \to Y$ is continuous. If X is connected then $f(X)$ is connected.*

Proof: Exercise.

Proposition 7.0.27 *Suppose $f : X \to Y$ is continuous. If X is connected then $f(X)$ is connected.*

Proof: Assume there is a disconnection G, H of $f(X)$. Then $f^{-1}(G)$, $f^{-1}(H)$ is a disconnection of $f(X)$. This is a contradiction, so $f(X)$ must be connected. $\square$

Proposition 7.0.28 *Suppose $A \subset X$. If A is connected then $\bar{A}$ is also connected.*

Proof: Suppose G, H is a disconnection of $\bar{A}$. Then $G \cap A, H \cap A$ is a disconnection of A. (Note that $G \cap \bar{A} \neq \emptyset \Rightarrow G \cap A \neq \emptyset$. Similarly for H.) $\square$

Proposition 7.0.29 *If X_α is connected $\forall \alpha$, and $\cap_\alpha X_\alpha \neq \emptyset$, then $\cup_\alpha X_\alpha$ is connected.*

Proof: Suppose G, H is a disconnection of $\cup_\alpha X_\alpha$. Then $\forall \alpha$, $X_\alpha = (G \cap X_\alpha) \cup (H \cap X_\alpha)$. Hence either $G \cap X_\alpha = \emptyset$ or $H \cap X_\alpha = \emptyset$. If $H \cap X_\alpha = \emptyset$, then $X_\alpha = G \cap X_\alpha$ so $X_\alpha \subset G$. Otherwise $X_\alpha \subset H$. In other words, each X_α is in one of the sets G, H. Since $\cap_\alpha X_\alpha \neq \emptyset$ and $G \cap H = \emptyset$, each X_α is in the same set, say G. But then $\cup_\alpha X_\alpha \subset G$ so that $H = G^c = \emptyset$, which is a contradiction. Hence $\cup_\alpha X_\alpha$ is connected. $\square$

Lemma 7.0.30 *Let X be disconnected. Then $\exists f : X \to \{0,1\}$ which is onto.*

Proof: Let A, B be a disconnection. Define $f(x) = 0, x \in A$ and $f(x) = 1, x \in B$. $\square$

Theorem 7.0.31 *Let X $\prod_{\alpha \in J} X_\alpha$. Then X is connected $\Leftrightarrow$ X_α is connected $\forall \alpha$.*

Proof: ($\Longrightarrow$) Suppose X is connected. Then $X_\alpha = \pi_\alpha(X)$ is connected.

($\Longleftarrow$) Suppose X_α is connected $\forall \alpha$. Assume X is disconnected. Let $f : X \to \{0,1\}$ be onto. Pick $x_\alpha \in X_\alpha$. (The theorem is trivial if $X_\alpha = \emptyset$ for some α.)

For $\alpha \in J$ and $x \in X$, define $\imath_{\alpha_0} : X_{\alpha_0} \to X$ by

$$\pi_\alpha(\imath_{\alpha_0}(w)) = w \text{ for } \alpha = \alpha_0$$

and

$$\pi_\alpha(\imath_{\alpha_0}(w)) = x_\alpha \text{ for } \alpha \neq \alpha_0.$$

Then

$$X_{\alpha_0} \xrightarrow{\imath_{\alpha_0}} X \xrightarrow{f} \{0,1\}$$

is continuous, so X_{α_0} is connected $\Rightarrow f\imath_{\alpha_0}(X_{\alpha_0})$ is connected.

Then $f\imath_{\alpha_0}$ must not be onto since $\{0,1\}$ is disconnected.

Therefore $\forall w \in X_{\alpha_0}$, $f \circ \imath_{\alpha_0}(w) = f \circ \imath_{\alpha_0}(x_{\alpha_0}) = f(x)$.

In other words, if $x, y \in X$ and $x_\alpha = y_\alpha$ for $\alpha \neq \alpha_0$ then $f(x) = f(y)$.

This is true $\forall \alpha_0$ so $f(x) = f(y)$ whenever x and y differ in only one coordinate.

By induction, $f(x) = f(y)$ whenever x, y differ in only finitely many coordinates.

Claim: Given $z \in X$, $\{y \in X | y_\alpha = z_\alpha \text{ for almost all } \alpha\}$ is dense in X.

Proof (of Claim): Every open set V contains a basic open set $U = \prod_\alpha U_\alpha$ with $U_\alpha = X_\alpha$ for almost all α. Hence $\exists y \in U^c$ s.t. $y_\alpha = z_\alpha$ for almost all α. $\surd$

Since $\{0,1\}$ is Hausdorff, $f(y) = f(z)$ $\forall y$ in a dense subset $\Rightarrow f(y) = f(z)$ $\forall y \in X$. Hence f is constant. Since f is onto, this is a contradiction. So X is connected. $\square$

7.1 Components

Definition 7.1.1 *A (connected) component of a space X is a maximal connected subspace.*

Theorem 7.1.2

1. Each nonempty connected subset of X is contained in exactly one component. In particular each point of X is in a unique component so X is the union of its components.

2. Each component of X is closed.

3. Any nonempty connected subspace of X which is both open and closed is a component.

Proof:

1. Let $\emptyset \neq Y \subset X$ be connected. Let $C = \displaystyle\bigcup_{A \text{ connected}, Y \subset A} A$.

Since $Y \subset \displaystyle\bigcap_{A \text{ connected}, Y \subset A} A$, this intersection is non-empty, so by the earlier Proposition, C is connected. C is a component containing Y. If C' is another component containing Y then by construction $C' \subset C$ so $C' = C$ by maximality.

2. If C is a component then $\bar{C}$ is connected by the earlier Proposition, and $C \subset \bar{C}$ so $C = \bar{C}$ by maximality. Hence C is closed.

3. Suppose $\emptyset \neq Y$ with Y connected, and both closed and open. Let C be the component of X containing Y. Let $A = C \cap Y$ and $B = C \cap Y^c$. Since Y and Y^c are open, we must have $C \cap Y^c = \emptyset$ so that A, B is not a disconnection of C. Hence $C = C \cap Y$ so $C \subset Y$. So $Y = C$ is a component.

□

Note: A component need not be open. For example, in $\mathbb{Q}$ the components are single points.

7.2 Path Connectedness

Notation: Let $I = [0, 1]$.

Definition 7.2.1 *X is called* path connected *if $\forall x, y \in X\ \exists w : I \to X$ s.t. $w(0) = x$, $w(1) = y$.*

Proposition 7.2.2 *Path connected $\Rightarrow$ connected.*

Proof: Suppose X is path connected. If X is not connected, then X has at least two components C_1, C_2. Pick $x \in C_1$, $y \in C_2$ and find $w : I \to X$ s.t. $w(0) = x$, $w(1) = y$. I is connected, so $w(I)$ is connected, so by an earlier Proposition, $w(I)$ is contained in a single component. This is a contradiction, so X is connected. □

Example: A connected space need not be path connected.

Let $Y = \{(0, y) \in \mathbb{R}^2\}$ (the y-axis)

$Z = \{\big(x, \sin(1/x)\big) \mid 0 < x \leq 1\}$ the graph of $y = \sin(1/x)$ on $(0, 1]$

$X = Y \cup Z$.

(a) X is connected:

Proof: The map $(0, 1] \xrightarrow{f} \mathbb{R}^2$ given by $t \mapsto (t, \sin(1/t))$ is continuous so $Z = \mathrm{Im}(f)$ is connected.

Hence $\overline{Z}$ is connected.

$0 \in \overline{Z}$. But $0 \in Y$ so $Y \cap \bar{Z} \neq \emptyset$ and Y is connected. Hence $Y \cap \overline{Z}$ is connected. But $Y \cap Z = Y \cap \overline{Z}$ since the limit points of Z are in Y.

(b) X is not path connected:

Proof: Suppose $w : I \to X$ s.t. $w(0) = (0, 0)$ and $w(1) = \big(1, \sin(1)\big)$.

Let $t_0 = \inf\{t | w(t) \in Z\}$.

$t < t_0 \Rightarrow w(t) \in Y$ and Y is closed so by continuity $w(t_0) \in Y$.

By definition of inf, $\forall \delta > 0i\ \exists 0 < r > \delta$ s.t. $\omega(t_0 + r) = \big(a, \sin(1.a)\big) \in Z$ for some a. Then $\pi_x\omega[t_0, t_0 + r]$ contains 0 and a and is connected so it contains all x in $[0, a]$. In particular, $\omega[t_0, t_0 + \delta) \supset \omega[t_0, t_0 + r]$ contains points of the form $(*, 0)$ and points of the form $(*, 1)$. This is true for all δ, so w is not continuous at t_0. This is a contradiction, so X is not path connected.

$\square$

Note that from this example, $A \subset X$ is path connected does not always imply $\bar{A}$ is path connected. (Let $A = Z$ in the above example.)

Proposition 7.2.3 *If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is path connected.*

Proof: Given $f(x_1), f(x_2) \in f(X)$ let w be a path connecting x_1 and x_2. Then $f \circ w : I \to Y$ connects $f(x_1)$ and $f(x_2)$.

$\square$

Proposition 7.2.4

1. *If X_α is path connected $\forall\alpha$, then $\cap_\alpha X_\alpha \neq \emptyset \Rightarrow \cup_\alpha X_\alpha$ is path connected.*
2. *$\prod_\alpha X_\alpha$ is path connected $\Leftrightarrow X_\alpha$ is path connected $\forall\alpha$.*

Proof:

1. Let $a \in \cap_\alpha X_\alpha$. Given $x, y \in \cup_\alpha X_\alpha$, connect them to each other by connecting each to a.
2. Let $X = \prod_\alpha X_\alpha$.

($\Longrightarrow$) Suppose X is path connected. Then $X_\alpha = \pi_\alpha(X)$ is path connected.

($\Longleftarrow$) Suppose X_α is path connected $\forall \alpha$. Given $x = (x_\alpha), y = (y_\alpha) \in X$, $\forall \alpha$ select $w_\alpha : I \to X_\alpha$ s.t. $w_\alpha(0) = x_\alpha$, $w_\alpha(1) = y_\alpha$.

Define $w : I \to X$ by $\pi_\alpha \circ w = w_\alpha$. Then w is continuous since each projection is, and $w(0) = x$ and $w(1) = y$.

$\square$

Definition 7.2.5 *A* path component *of a space X is a maximal path connected space.*

Proposition 7.2.6 *Each path connected subset of X is contained in exactly one path component. In particular each point of X is in a unique path component, so X is the union of its path components.*

Proof: Insert "path" before "connected" and before "component" in the earlier proof, since it used only that $\cap_\alpha X_\alpha \neq \emptyset$ with X_α connected implies $\cup_\alpha X_\alpha$ connected.

$\square$

Chapter 8

Local Properties

Definition 8.0.7 *A space X is called* locally compact *if every point has a neighbourhood whose closure is compact.*

Example: $\mathbb{R}^n$ is locally compact, but not compact.

Proposition 8.0.8 *If a space X is compact, then it is locally compact.*

(The proof is obvious.)

Theorem 8.0.9 *Let X be a locally compact Hausdorff space. Then $\exists$ a compact Hausdorff space X and an inclusion $\imath : X \longrightarrow X_\infty$ s.t. $X_\infty \smallsetminus X$ is a single point.*

Proof: Let ∞ denote an element not in the set X and define $X_\infty = X \cup \{\infty\}$ as a set. Topologize X_∞ by declaring the following subsets to be open:
(i) $\{U \mid U \subset X \text{ and } U \text{ open in } X\}$
(ii) $\{V \mid V^c \subset X \text{ and } V^c \text{ is compact}\}$
(iii) the full space X_∞
Exercise: Check this is a topology.
Claim: X_∞ is compact.
Proof: Let $\{U_\alpha\}$ be an open cover of X_∞. If some U_α is X_∞ itself, it is a finite subcover so we are finished. Suppose not. Find U_{α_0} s.t. $\infty \in U_{\alpha_0}$. U_{α_0} must be a set of type (ii) so $U^c_{\alpha_0}$ is a compact subset of X.

$\{U_\alpha \cap X\}$ covers $U^c_{\alpha_0}$ so there is a finite subcover $\{U_{\alpha_1} \cap X, \ldots, U_{\alpha_n} \cap X\}$. But then $\{U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ covers X_∞.

I claim that X_∞ is Hausdorff.

Proof: Let $x \neq y \in X_\infty$. If $x, y \in X$, we can separate them using the open sets from X, so say $y = \infty$.

Since X is locally compact, $\exists U$ s.t. $x \in U$ and $\bar{U}$ is a compact subset of X. Hence $X_\infty \setminus \bar{U}$ is open in X_∞ and $\infty \in X_\infty \setminus \bar{U}$.

Definition 8.0.10 *Given a locally compact Hausdorff space X, the space X_∞ formed by the above construction is called the* one point compactification *of X.*

Example: If $X = \mathbb{R}^n$ then X_∞ is homeomorphic to S^n. (The inverse homeomorphism is given by stereographic projection.)

Corollary 8.0.11 *Suppose X is locally compact and Hausdorff, and $A \subset X$ is compact. If U is open s.t. $A \subset U$ and $U \neq X$, then $\exists f : X \to [0,1]$ s.t. $f(A) = 0$ and $f(U^c) = 1$.*

Proof: X_∞ is normal so $\exists$ such an f on X_∞ by Urysohn. Restrict f to X. $\square$

Definition 8.0.12 *A space X is called* locally [path] connected *if the [path] components of open sets are open.*

Proposition 8.0.13 *X is locally [path] connected $\Leftrightarrow$ $\forall x \in X$ and $\forall$ open U containing x, $\exists$ a [path] connected open V s.t. $x \in V \subset U$.*

Proof: ($\Longrightarrow$) Given $x \in U$, Let V be the [path] component of U containing x.

($\Longleftarrow$) Let U be open. Let C be a [path] component of U and let $x \in C$. There exists an open [path] connected V s.t. $x \in V \subset U$ so by maximality of [path] components, $V \subset C$.

Hence $x \in \overset{\circ}{C}$. This is true $\forall x \in C$ so C is open. $\square$

Note:

1. Locally [path] connected does not imply [path] connected.
For example, $[0,1] \cup [2,3]$ is locally [path] connected but not [path] connected.
2. Conversely [path] connected does not imply locally [path] connected.
For example, the *comb space*

$$X = \{(1/n, y) \mid n \geq 1,\ 0 \leq y \leq 1\} \cup \{(0,y) \mid 0 \leq y \leq 1\} \cup \{(x,0) \mid 0 \leq y \leq 1\}$$

X is [path] connected but not locally [path] connected.

Another example is the union of the graph of $\sin(1/x)$ with the y-axis and a path from the y-axis to $(1, \sin(1))$. Without this path, the space is not path connected.

Proposition 8.0.14 *If X is locally path connected, then X is locally connected.*

Proof: $\forall U$ and $\forall x \in U$ $\exists$ a path connected V s.t. $x \in V \subset U$. But V is connected since path connected implies connected. $\square$

Proposition 8.0.15 *If X is connected and locally path connected, then X is path connected.*

Proof: Let C be a path component of X. Hence C is open (by definition of locally path connected applied to the open set X).

Let $x \in \bar{C}$.

X is locally path connected $\Rightarrow$ $\exists$ a connected open set U containing x. (Apply the definition of locally path connected to the open set X. The component of X containing x is open.)

$x \in \bar{C} \Rightarrow U \cap C \neq \emptyset \Rightarrow C \cup U$ is path connected.

So $C \cup U = C$ (by maximality of components)

Hence $x \in U \subset C$ and therefore $C = \bar{C}$, in other words C is closed.

Since C is both open and closed, by theorem 7.1.2, C is a connected component.

Since X is connected, $C = X$.

Hence X is path connected.

$\square$

Chapter 9

CW complexes

9.1 Attaching Maps

Given $A \subset X$ with $f : A \to Y$, we define "the space obtained from Y by attaching X by means of f" (written $X \cup_f Y$) as

$$X \cup_f Y = (X \amalg Y) /\!\sim$$

where $a \sim f(a)$ $\forall a \in A$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook, "i_Y"] \arrow[ddr] & \\
X \arrow[r, "j_X"] \arrow[drr] & X \cup_f Y \arrow[dr, "\exists!"] & \\
 & & Z
\end{tikzcd}$$

is a pushout in the category of topological spaces.

i_Y is always an injection.

j_X is an injection iff f is.

Example 9.1.1 $Y = *$ $\quad f : A \to *$.

Then $X \cup_f * = X/A$.

Associativity: $A \subset X$, $B \subset Y$.
$f : A \to Y$, $g : B \to Z$.
Then

$$X \cup_{j_Y \circ f} (Y \cup_g Z) \cong (X \cup_f Y) \cup_g Z = \frac{X \amalg Y \amalg Z}{\sim}.$$

Assume A is closed in X.

Proposition 9.1.2

1. *In $X \cup_f Y$, $i_Y(Y)$ is closed, $j_X(X \setminus A)$ is open.*
2. *(a)* $i_Y : Y \cong i_Y(Y)$,
   (b) $j_X : X \setminus A \cong j_X(X \setminus A)$.

Proof:

1. $X \cup_f Y = i_Y(Y) \cup j_X(X \setminus A)$ and $i_Y(Y) \cap j_X(X \setminus A) = \emptyset$
   $\pi : X \amalg Y \to X \cup_f Y$
   $\pi^{-1}\big(j_X(X \setminus A)\big) = X \setminus A \qquad$ open in $X \amalg Y$
   Therefore $j_X(X \setminus A)$ open in $X \cup_f Y$
   Therefore $i_Y(Y)$ closed
2. (a) Show Y open in $Y \Rightarrow i_Y(U)$ open in $i_Y(Y)$
   Notice that $i_Y(Y) = A \cup_f Y \subset X \cup_f Y$
   $\pi^{-1}\big(i_Y(U)\big) = f^{-1}(U) \amalg U \qquad$ open in $A \amalg Y$.
   Therefore $i_Y(U)$ open in $A \cup_f Y = i(Y)$
   (b) Show V open in $X \setminus A \Rightarrow j_X(V)$ open in $j_X(X)$
   $\pi^{-1}\big(j_X(V)\big) = V \qquad$ open in $A \amalg Y$
   Therefore $j_X(V)$ open in $X \cup_f Y$
   Therefore $j_X(V)$ open in $j_X(X)$ (since it is even open in entire space)

□

From now on we think of Y as the subset $i_Y(Y)$ of $X \cup_f Y$.

Corollary 9.1.3 *$F \subset X \cup_f Y$ is closed $\Leftrightarrow$ $F \cap i_Y(Y)$ and $F \cap \overline{j_X(X \setminus A)}$ are closed.*

Proof: Since $X \cup_f Y = i_Y(Y) \cup \overline{j_X(X \setminus A)}$ this follows from the fact that $i_Y(Y)$ is closed.

□

Proposition 9.1.4 *If X and Y are compact, then $X \cup_f Y$ is compact.*

Proof: X, Y compact $\Rightarrow X \amalg Y$ compact $\Rightarrow X \cup_f Y = \pi(X \amalg Y)$ compact $\square$

Proposition 9.1.5 *If X and Y are normal, then $X \cup_f Y$ is also normal.*

Proof: Suppose $B, C \subset X \cup_f Y$ with $B \cap C = \emptyset$ where B, C are either closed or singletons. (We don't assume singletons are closed — have to show T_1 as well)

Then $B \cap Y$, $C \cap Y$ are disjoint closed subsets of Y so $\exists g : Y \to I$ s.t. $g(B \cap Y) = 0$, $g(C \cap Y) = 1$.

Define $h : j_X^{-1}(B) \cup j_X^{-1}(C) \cup A \to I$ by $h|_{j_X^{-1}(B)} = 0$, $h|_{j_X^{-1}(C)} = 1$, $h_A = g \circ f$.

This agrees on overlaps (which are closed) so yields a well-defined cont. function. Domain of h closed in X, X normal $\xRightarrow{\text{(Tietze)}} \exists H : X \to I$ extending h.

$$\begin{tikzcd} X \amalg Y \arrow[rr, "H \amalg g"] \arrow[dr] & & I \\ & X \cup_f Y \arrow[ur, dotted, "\phi"] & \end{tikzcd}$$

$\leftarrow$ universal property of quotient

$\phi(B) = 0$, $\phi(C) = 1$,

Therefore $\exists$ open sets separating B and C. Applied to singletons gives Hausdorff (thus T_1) and then applied again to closed sets gives normal. $\square$

Proposition 9.1.6 *If Y is Hausdorff and X is metric, then $X \cup_f Y$ is Hausdorff.*

Proof:

1. $x \neq w \in X \setminus A$

   Separation in $X \setminus A$ gives a separation in $X \cup_f A$ since $X \setminus A$ is open.

2. $X \in X \setminus A$, $y \in Y$

   Find $\epsilon > 0$ s.t. $N_{2\epsilon}(x) \subset X \setminus A$

   Then $x \in N_\epsilon(x) \subset \overline{N_\epsilon(x)} \subset X \setminus A$, (where the closure can be taken either in $X \setminus A$ or in $X \cup_f Y$ — it's the same)

   Then $N_\epsilon(x), \left(\overline{N_\epsilon(x)}\right)^c$ separate x and y.

3. $y_1, y_2 \in Y$

Lemma 9.1.7 *X metric. $A \subset X$. V open in A.*

Then $\exists$ open U in X s.t. $U \cap A = V$ and $\overline{U} \cap A =$ closure of V in A

Proof:

See Problem Set I. $\square$

Proof of Prop. (cont):

Let U', V' be a separation of y_1, y_2 in Y (with $y_1 \in U'$, $y_2 \in V'$)

$f^{-1}(U')$ open in A. X metric so by Lemma, $\exists$ open U in X s.t. $U \cap A = f^{-1}(U)$, $\overline{U} \cap A =$closure of $f^{-1}(U')$ in $A = \overline{f^{-1}(U')}$ (since A closed).

Let $W = (j_X U) \cup U' \subset X \cup_f Y$

$\pi^{-1}(\text{any}) = j_X^{-1}(\text{any}) \amalg i_Y^{-1}(\text{any})$

Since $j_X^{-1}\big(U' \cup j_X(U)\big) = f^{-1}(U') \cup U = U$ and $i_Y^{-1}\big(U' \cup j_X(U)\big) = U' \cup \big(j_X(U) \cap Y\big) = U' \cup f(U \cap A) = U'$ we get $\pi^{-1}(W) = U \amalg U'$ in $X \amalg Y$ so W is open in $X \cup_f Y$.

Claim: $\overline{W} = j_X(\overline{U}) \cup \overline{U'}$

Proof: $B \subset f^{-1}\big(\overline{f(B)}\big) \Rightarrow \overline{B} \subset f^{-1}\big(\overline{f(B)}\big) \Rightarrow f(\overline{B}) \subset \overline{f(B)}$ in general, and so $W \subset \overline{U'} \cup j_X(\overline{U}) \subset \overline{U'} \cup \overline{j_X(U)} = \overline{W}$.

Therefore sufficient to show that $\overline{U'} \cup j_X(\overline{U})$ is closed.

SubClaim: $j_X^{-1}\big(j_X(\overline{U} \cup \overline{U'})\big) = \overline{U} \cup j_X^{-1}(\overline{U'})$

Proof: $\overline{U} \subset j_X^{-1} j_X(\overline{U})$ so RHS$\subset$ LHS.

Conversely, suppose that $a \in$LHS.

If $a \in j_X^{-1}(\overline{U'})$ then $a \in$RHS and if $a \in \overline{U}$ then $a \in$RHS.

So suppose $a \in (j_X^{-1} j_X \overline{U}) \setminus \overline{U}$.

Then $\exists b \in \overline{U}$ s.t. $j_X(a) = j_X(b)$. Since $a \neq b$ this implies $a, b \in A$. Hence $b \in \overline{U} \cap A =$ closure of $f^{-1}(U')$ in A.

If Z is a nbhd. of $j_X(b)$ then $j_X^{-1}(Z)$ is a nbhd. of b, so $j_X^{-1}(Z)$ contains pts. of V. Hence Z contains pts. of $j_X\big(f^{-1}(U')\big) \subset U'$. True $\forall$ nbhds. of $j_X(b)$, so $j_X(a) = j_X(b) \in \overline{U'}$.

Therefore $a \in j_X^{-1}(\overline{U'}) \in$ RHS.

Proof of Claim (cont.):

SubClaim $\Rightarrow j_X^{-1}\big(j_X(\overline{U}) \cup \overline{U'}\big) = \overline{U} \cup j_X^{-1}(\overline{U'}) \qquad$ closed in X

$$i_Y^{-1}\big(j_X(\overline{U}) \cup \overline{U'}\big) = \big(j_X(\overline{U}) \cap Y\big) \cup (\overline{U'} \cap Y) \tag{9.1}$$

$j_X(\overline{U}) \cap Y = j_X(\overline{U} \cap A) = f(\overline{U} \cap A) = f\big(\text{closure of } f^{-1}(U') \text{ in } A\big) \subset \text{closure of } \; f\big(f^{-1}(U')\big)$ in $Y \subset \overline{U'} \cap Y$, and so (9.1) $\Rightarrow i_Y^{-1}\big(j_X(\overline{U}) \cup \overline{U'}\big) = \overline{U'} \cap Y$ which is closed in Y.

Therefore we have shown that $\pi^{-1}\big(j_X(\overline{U}) \cup \overline{U'}\big) = \text{closed} \amalg \text{closed}$ so $j_X(\overline{U} \cup \overline{U'})$ closed, as desired.

Proof of Prop. (cont.):

$y_1 \in U' \subset W$

Show $y_2 \notin \overline{W}$ so that W, $(\overline{W})^c$ is the desired separation.

Suppose $y_2 \in \overline{W}$. Then $y_2 \in \overline{W} \cap Y = i_Y^{-1}(\overline{W}) \subset \overline{U'} \cap Y = \text{closure of } U'$ in Y. But $y_2 \in V'$ and $V' \cap (\text{closure of } U' \text{ in } Y) = \emptyset$

$\Rightarrow\Leftarrow$

So $y_2 \notin \overline{W}$, as desired. $\square$

9.2 Coherent Topologies

Let $X_1 \subset X_2 \subset \cdots \subset X_n \subset$ be topological spaces.

Let $X = \cup_n X_n$

The *coherent topology* on X defined by the subspaces X_n is the topology whose closed sets are $\{A \subset X \mid A \cap X_n \text{ is closed in } X_n \ \forall n\}$. (Clearly this collection is closed under intersections and finite unions.) This is the weakest topology on X s.t. all the inclusion maps are continuous.

Notation: Write $X = \varinjlim_n X_n$ for $\cup_n X_n$ with this topology.

Proposition 9.2.1 *Given $f_n : X_n \to Y$ s.t. $f_n\big|_{X_k} = f_k$ for $k < n$, $\exists!\ f : X \to Y$ s.t. $f\big|_{X_n} = f_n$.*

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[drrr] & X_1 \arrow[r, hook] \arrow[drr] & \cdots \arrow[r, hook] & X_n \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & X \arrow[dll, dotted, "\exists!"] \\
& & & Y & &
\end{tikzcd}$$

Proof: Let f be the unique set map on X restricting to f_n on X_n. Given closed A in Y, $f^{-1}(A) \cap X_n = f_n^{-1}(A)$ which is closed in X_n. Hence $f^{-1}(A)$ is closed in X. Therefore f is continuous. □

Proposition 9.2.2 *Suppose $\forall n$ that X_n is normal and X_n is closed in X. Then X is normal.*

Proof:

$\forall x \in X, \{x\} \cap X_n = \big\{\{x\} \text{ or } \emptyset\big\} = \text{closed in } X_n$

Hence $\{x\}$ closed.

So X is T_1.

Suppose A, B closed in X with $A \cap B = \emptyset$.

X_1 normal $\Rightarrow \exists g_1 : X_1 \to I$ s.t. $g_1(X_1 \cap A) = 0$, $g_1(X_1 \cap B) = 1$.

Suppose $g_n : X_n \to I$ has been defined s.t. $g_n(X_n \cap A) = 0$, $g_n(X_n \cap B) = 1$, $g_n\big|_{X_k} = g_k$ for $k < n$.

To define g_{n+1}:

Define $f_n : Y_n := X_n \cup A \cup B \to I$ by $f_n(X_n) = g_n$, $f_n(A) = 0$, and $f_n(B) = 1$.

A, B, X_n closed and f_n agrees on the overlaps, so f_n is continuous.

Y_n closed in $X \Rightarrow Y_n \cap X_{n+1}$ closed in X_{n+1}, so by Tietze (using X_{n+1} normal) $\exists g_{n+1} : X_{n+1} \to I$ extending $f\big|_{Y \cap X_{n+1}}$.

Hence $g_{n+1}\big|_{X_n} = f_n\big|_{X_n} = g_n$, $g_{n+1}(X_{n+1} \cap A) = 0$, $g_{n+1}(X_{n+1} \cap B) = 1$.

By universal property of $\varinjlim$, $\exists!\ g : X \to I$ extending $g_n\ \forall n$.

Then $g(A) = 0$ and $g(B) = 1$. $\square$

9.3 CW complexes

Motivation: Finite CW complexes:

A finite 0-dimensional CW complex consists of a finite set with the discrete topology.

A finite $(n+1)$-dimensional CW complex is a space of the form $\left(\coprod_{\alpha\in J} D^{n+1}\right) \cup_f X$ where

(1) X is a finite k-dimensional CW complex for some $k \leq n$

(2) D^{n+1} denotes $[0,1]^{n+1}$. $\coprod_{\alpha\in J} D^{n+1}$ has the "disjoint union" topology: U is open if its intersection with each D^{n+1} is open.

(3) $f : \coprod \partial D^{n+1} \to X$, where $S^n \cong \partial D^{n+1} \subset D^{n+1}$

Examples:

(1) $I = [0,1]$

(2) S^n which is homeomorphic to $D^n \cup_f \text{pt} = D^n/\partial D^n$.

Definition of *CW* complex which follows is more general and allows for infinite *CW*-complexes as well.

Terminology:

Spaces homeomorphic to D^m will be called *m-cells*.

Spaces homeomorphic to the interior of D^m will be called *open m-cells*.

m is called the *dimension* of the cell.

Definition 9.3.1 *A CW-structure on a Hausdorff space X consists of a collection of disjoint open cells $\{e_\alpha\}_{\alpha\in J}$ and a collection of maps $f_\alpha : D^m \to X$ s.t.*

1. $X = \cup_{\alpha\in J} e_\alpha$ *(disjoint as a set)*
2. $\forall\alpha$:
   (a) $f_\alpha\big|_{\overset{\circ}{D^m}} : \overset{\circ}{D^m} \cong e_\alpha$
   (b) $f_\alpha(\partial D^m) \subset \{$union of finitely many of the cells e_α having dimension less than m $\}$
3. $A \subset X$ *is closed* $\Leftrightarrow A \cap \overline{e_\alpha}$ *is closed in* $\overline{e_\alpha}$ *for all* α

A space with a CW-structure is called a CW-complex*.*

To see that this generalizes the above description:

Suppose $Y = X \cup \left(\coprod_{\beta\in K} D_\beta^{n+1}\right) \cup_g X$ where $X = \cup_{\alpha\in J} e_\alpha$ is a CW complex with $\dim e_\alpha \leq n\ \forall\alpha$. Write $C = \coprod_{\beta\in K} D_\beta^{n+1}$ and $\partial C = \coprod_{\beta\in K} \partial D_\beta^{n+1}$.

So $C \smallsetminus \partial C = \coprod_{\beta\in K} \overset{\circ}{D}{}_\beta^{n+1}$.

$$\begin{tikzcd}
\partial C \arrow[r, hook] \arrow[d] & C \arrow[d, "j"] \\
X \arrow[r, hook, "i"] & Y
\end{tikzcd}$$

Let $f_\beta = j\big|_{D_\beta^{n+1}} : D_\beta^{n+1} \to X$. (So X is a union of cells having dimension $< n+1$.)

Since $Y = \cup_{\alpha \in J} \overline{e_\alpha} \bigcup \cup_{\beta \in K} \overline{e_\beta}$ in the case of a finite CW complex (where the sets J and K are finite) the third condition is automatic.

Terminology:

$\cup \{e_\alpha \mid \dim e_\alpha \leq n\}$ is called the n-skeleton of X, written $X^{(n)}$.

The restrictions $f_\alpha|_{\partial D^m}$ are called the *attaching maps*.

Notice that we can recover X from knowledge of $X^{(0)}$ and the attaching maps as follows: Inductively define $X^{(n+1)}$ by $X^{(n+1)} = \left(\coprod_{\beta \in K_{n+1}} D_\beta^{n+1}\right) \cup_f X^{(n)}$ where $K_{n+1} = \{$all (n+1)-cells$\}$. (Knowledge of a map includes knowledge of its domain so we know the set K_{n+1}.)

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(n)} \ldots$$

Define $X = \cup_n X^{(n)} = \cup_{\alpha \in J} e_\alpha$ and topologize it by condition 3.

If $\exists M$ s.t. $X^{(M)} = X$ then X is called finite dimensional.

X is called *finite* if it has finitely many cells.

Note: A space can have more than one CW-structure giving the same topology.

e.g.

$$S^2 = e_0 \cup e_2$$

$$S^2 = e_0 \cup e_0 \cup e_1 \cup e_1 \cup e_2 \cup e_2$$

Note: The open n-cells comprising X are not necessarily open as subsets of X. Only the top dimensional open cells are actually open in X.

Lemma 9.3.2 $\overline{e_\alpha} = f_\alpha(D^m)$

Proof: D^m compact $\Rightarrow f_\alpha(D^m)$ compact $\Rightarrow f_\alpha(D^m)$ closed as X is Hausdorff. (In fact X is normal.)

$e_\alpha = f_\alpha(\mathring{D}^m) \subset f_\alpha(D^m) \Rightarrow \overline{e_\alpha} \subset f_\alpha(D^m)$.

Conversely $f_\alpha^{-1}(\overline{e_\alpha}) = \overline{f_\alpha^{-1}(e_\alpha)} = \overline{\mathrm{Int(D^m)}} = D^m$ so $f_\alpha(D^m) \subset \overline{e_\alpha}$. $\square$

Corollary 9.3.3 $\overline{e_\alpha} \subset X^{(m)}$.

Proof: $\overline{e_\alpha} = f_\alpha(D^m) = f_\alpha(\overset{\circ}{D^m}) \cup f_\alpha(\partial D^m)$ with $f_\alpha(\overset{\circ}{D^m}) = e_\alpha$ and $f_\alpha(\partial D^m) \subset X^{(m-1)}$, so $\overline{e_\alpha} \subset X^{(m)}$. $\square$

Corollary 9.3.4 *For any* α_0, $\overline{e_{\alpha_0}} \cap e_\alpha = \emptyset$ *for all but finitely many* α.

Proof: By definition $f_{\alpha_0}(\partial D^m) \cap e_\alpha = \emptyset$ for all but finitely many α. $\square$

Theorem 9.3.5 A *compact* $\subset X \Rightarrow A \cap e_\alpha = \emptyset$ *for all but finitely many* α.

Proof: $X = \cup_{\alpha \in J} e_\alpha$. Let $I = \{\alpha \in J \mid \alpha \cap e_\alpha \neq \emptyset\}$.
For all $\alpha \in I$, choose $y_\alpha \in A \cap e_\alpha$. Set $Y = \{y_\alpha\}_{\alpha \in I}$.
$\forall \beta$, $\{\alpha \mid \overline{e_\beta} \cap e_\alpha \neq \emptyset\}$ is finite, so $\overline{e_\beta} \cap Y$ is finite.
Suppose $S \subset Y$.
$\forall \beta \in J$, $S \cap \overline{e_\beta}$ is finite, thus closed in X, since X is T_1.
Hence S is closed in X. (Property 3)
In particular, Y is closed in X and every subset of Y is closed in Y.
So Y has the discrete topology.
But $Y \subset A$, A is compact, and Y is closed, hence Y is compact. Therefore Y is discrete implies Y is finite. Hence I is finite. $\square$

Corollary 9.3.6 *If* A *is a compact subset of* X*, then* $A \subset X^{(N)}$ *for some* N.

Corollary 9.3.7 X *is compact* $\Leftrightarrow$ X *is finite.*

Proof: $\Longrightarrow$ If X is compact then X intersects only finitely many e_α. But X intersects all e_α so X is finite.
$\Longleftarrow$ $X^{(n+1)} = C_{n+1} \cup_f X^{(n)}$ where $C_{n+1} = \coprod_{\beta \in K_{n+1}} D^{n+1}$.
If X is finite, then K_{n+1} is finite, and so C_{n+1} is compact, and hence $X^{(n+1)}$ is compact (by induction).
If X is finite, then $X = X^{(N)}$ for some N. $\square$

9.3.1 Subcomplexes

Let $X = \cup_{\alpha \in J} e_\alpha$ be a CW complex. Suppose $J' \subset J$.
$Y = \cup_{\alpha \in J'} e_\alpha$ is called a *subcomplex* of X if $\overline{e_\alpha} \subset Y$ $\forall \alpha \in J'$.

Example: $X^{(n)}$ is a subcomplex of X $\forall n$.

Proposition 9.3.8 *Let* Y *be a subcomplex of* X*. Then* Y *is closed in* X*.*

Proof: For $\beta \in J$ show $Y \cap \overline{e_\beta}$ is closed in $\overline{e_\beta}$.

$\{\alpha \in J \mid e_\alpha \cap \overline{e_\beta} \neq \emptyset\}$ is finite, so $\overline{e_\beta} = e_{\alpha_1} \cup \cdots \cup e_{\alpha_k}$.

The e_α are disjoint so $Y \cap e_\alpha = \emptyset$ unless $e_\alpha \subset Y$.

Discarding those α for which $Y \cap e_\alpha = \emptyset$, write

$$\begin{aligned} Y \cap \overline{e_\beta} &= \big((Y \cap e_{\alpha_1}) \cup \cdots \cup (Y \cap e_{\alpha_r})\big) \cap \overline{e_\beta} \qquad \text{with } e_{\alpha_1}, \ldots, e_{\alpha_r} \subset Y \\ &\subset \big(Y \cap \overline{e_{\alpha_1}}) \cup \cdots \cup (Y \cap \overline{e_{\alpha_r}})\big) \cap \overline{e_\beta} \\ &\subset (\overline{e_{\alpha_1}} \cup \cdots \cup \overline{e_{\alpha_r}}) \cap \overline{e_\beta} \\ &\subset Y \cap \overline{e_\beta} \end{aligned}$$

using $e_{\alpha_j} \subset Y$ and applying the definition of subcomplex.

Hence $Y \cap \overline{e_\beta} = (\overline{e_{\alpha_1}} \cup \cdots \cup \overline{e_{\alpha_r}}) \cap \overline{e_\beta}$ is closed in $\overline{e_\beta}$. Hence Y is closed in X. $\square$

Corollary 9.3.9 *A subcomplex of a CW complex is a CW complex.*

Proof: Let $Y \subset X$ be a subcomplex where $Y = \cup_{\alpha \in J'} e_\alpha$ and $X = \cup_{\alpha \in J} e_\alpha$.

For $\alpha \in J'$, $f_\alpha(D^m) = \overline{e_\alpha} \subset Y$ (thus it is in finitely many cells of Y since X is a CW-complex) so condition (2) is satisfied.

Check condition (3).

Suppose $A \cap \overline{e_\alpha}$ closed in $\overline{e_\alpha}$ for all $\alpha \in J'$.

Given $\beta \in J$, write $Y \cap \overline{e_\beta} = (\overline{e_{\alpha_1}} \cup \cdots \cup \overline{e_{\alpha_r}}) \cap \overline{e_\beta}$ with $\alpha_1, \ldots, \alpha_r \in J'$ as above.

Then $A \cap \overline{e_\beta} = \big((A \cap \overline{e_{\alpha_1}}) \cup \cdots \cup (A \cap \overline{e_{\alpha_r}})\big) \cap \overline{e_\beta}$.

$A \cap \overline{e_{\alpha_j}}$ is closed in $\overline{e_\alpha}$, thus compact, for $j = 1, \ldots, r$.

Therefore $A \cap \overline{e_\beta} = (\text{compact}) \cap \overline{e_\beta}$ =closed subset of $\overline{e_\beta}$.

Hence A is closed in X and thus closed in Y. $\square$

Corollary 9.3.10 $X^{(n)}$ *is closed in* X $\forall n$.

Corollary 9.3.11 $X = \varinjlim_n X^{(n)}$.

Proof: $X^{(n)}$ is closed in X for all n. If $A \subset X$ satisfies $A \cap X^{(n)}$ closed for all n, then $\forall \alpha$, $(A \cap X^{(n)}) \cap \overline{e_\beta} = A \cap \overline{e_\beta}$ closed, since $\overline{e_\alpha} \subset X^{(n)}$ for some n. $\square$

Proposition 9.3.12 $X^{(m)}$ *is normal* $\forall m$.

Proof: $X^{(n+1)} = C_{n+1} \cup_f X^{(n)}$ where $C_{n+1} = \coprod_\beta D^{n+1}$ is normal. Hence $X^{(m)}$ is normal $\forall m$ by induction. $\square$

Corollary 9.3.13 X *is normal.*

There is a stronger theorem which we won't prove which says

Theorem 9.3.14 (Mizakawa) X *is a CW-complex* $\Rightarrow$ X *is paracompact.*

9.3.2 Relative CW-complexes

Definition 9.3.15 *A* relative CW-structure (X, A) *consists of a Hausdorff space X, a subspace A of X, a collection of disjoint open cells $\{e_\alpha\}_{\alpha \in J}$ and maps $f_\alpha : D^m \to X$ s.t.*

1. $X = A \cup \bigcup_{\alpha \in J} e_\alpha$
2. $\forall \alpha$
   (a) $f(D^m) \subset e_\alpha$ *and* $f_\alpha|_{\overset{\circ}{D^m}} \cong e_\alpha$
   (b) $f_\alpha(\partial D^m) \subset A \cup \{$ *union of finitely many of the cells e_α having dimension less than* $m\}$
3. $B \subset X$ *is closed* $\Leftrightarrow B \cap A$ *is closed in A and $B \cap (A \cup \overline{e_\alpha})$ is closed in $A \cup \overline{e_\alpha}$* $\forall \alpha$.

A pair (X, A) with a relative CW-structure is called a relative CW-complex.

Define $X^{(n)} = A \cup \bigcup_{\dim e_\alpha \leq n} e_\alpha$. By convention, set $X^{(-1)} = A$.

Proposition 9.3.16 *Let (X, A) be a relative CW-complex.*

1. $X = \varinjlim_n X^{(n)}$.
2. *A is normal $\Rightarrow X$ is normal.*
3. *$X^{(n)}$ is closed in X $\forall n$.*
4. *$(X/A, *)$ is a relative CW complex.*

□

9.3.3 Product complexes

Let $X = \cup_{\alpha \in J} e^\alpha$ and $Y = \cup_{\beta \in K} e^\beta$ be CW complexes.

Then $X \times Y = \bigcup_{(\alpha,\beta) \in J \times K} (e_\alpha \times e_\beta)$.

Note: If e_α is an m-cell and e_β is an n-cell then $e_\alpha \times e_\beta$ is an $(m+n)$-cell.

Define $f_{\alpha,\beta}$ by $D^{m+n} = D^m \times D^n \xrightarrow{f_\alpha \times f_\beta} X \times Y$.

$\overset{\circ}{D}{}^{m+n} = \overset{\circ}{D}{}^m \times \overset{\circ}{D}{}^n \overset{f_\alpha \times f_\beta}{\to} X \times Y$ is a homeomorphism from $\overset{\circ}{D}{}^{m+n}$ to its image.

$$\partial D^{m+n} = (\partial D^m \times D^n) \cup (D^m \times \partial D^n) \hookrightarrow X \times Y$$

$f_{\alpha,\beta}(\partial D^{m+n}) \subset \Big\{\big((m-1) - \text{cells}\big) \times \big(n - \text{cells}\big)\Big\} \cup \Big\{\big(m - \text{cells}\big) \times \big((n-1) - \text{cells}\big)\Big\} = \big\{(m+n-1) - \text{cells}\big\}$.

So $X \times Y$ will be a CW-complex if condition 3 is satisfied. In general, it will not be satisfied.

9.4 Compactly Generated Spaces

In this section, all spaces will be assumed to be Hausdorff.

Definition 9.4.1 *A (Hausdorff) space X is called* compactly generated *(or a k-space) if it satisfies $A \subset X$ is closed $\Leftrightarrow A \cap K$ is closed in K for all compact subspaces K of X.*

Examples:

1. Compact spaces
2. CW-complexes

Given X we define a space $X_{\mathbf{k}}$ as follows.

As a set, $X_{\mathbf{k}} = X$. Topologize $X_{\mathbf{k}}$ by: closed sets $= \{A \subset X_{\mathbf{k}} | A \cap K$ is closed (in the original topology) in K for every $K \subset X$ which is compact in the original topology $\}$.

Note: Since X is Hausdorff, A closed in K is equivalent to A closed in X.

$A \subset X$ is closed in the original topology $\Rightarrow A$ is closed in $X_{\mathbf{k}}$.
Hence

Proposition 9.4.2 $X_{\mathbf{k}} \xrightarrow{\text{id}} X$ *is continuous.*

Thus the topology on $X_{\mathbf{k}}$ is finer. In particular $X_{\mathbf{k}}$ is Hausdorff.
Clearly X compact $\Rightarrow X_{\mathbf{k}} = X$.

Proposition 9.4.3 *$f : X \to Y$ continuous implies that f is continuous when considered as a map $X_{\mathbf{k}} \to Y_{\mathbf{k}}$.*

Proof: Suppose $B \subset Y_{\mathbf{k}}$ is closed. If $K \subset X$ is compact, then $f(K)$ is compact, so $B \cap f(K)$ is closed in Y

This implies $f^{-1}(B \cap f(K))$ is closed in X. Hence $f^{-1}(B \cap f(K)) = f^{-1}(B) \cap f^{-1}(f(K)) \supset f^{-1}(B) \cap K$. So $f^{-1}(B) \cap K = f^{-1}(B \cap f(K)) \cap K$ which is closed in K Hence $f^{-1}(B)$ is closed in $X_{\mathbf{k}}$. □

Proposition 9.4.4 *If A is closed in X, then $A_{\mathbf{k}}$ is the subspace topology from the inclusion $A \hookrightarrow X_{\mathbf{k}}$.*

Proof: $A \hookrightarrow X \Rightarrow A_{\mathbf{k}} \hookrightarrow X_{\mathbf{k}}$ is continuous so the $A_{\mathbf{k}}$ topology is finer than the subspace topology. Suppose that $B \subset A_{\mathbf{k}}$ is closed. So for all compact $K \subset A$, $B \cap K$ is closed in K. We show that B is closed in $X_{\mathbf{k}}$. Suppose $L \subset X$ is compact. A is closed, so $A \cap L$ is a compact subset of A. However $B \cap L = B \cap A \cap L$, so $B \cap L$ is closed. Hence B is closed in $X_{\mathbf{k}}$. □

Corollary 9.4.5 *K is compact in $X_{\mathbf{k}} \Leftrightarrow K$ is compact in X.*

Proof: K is compact in $X_{\mathbf{k}} \Rightarrow \mathrm{id}(K) = K$ is compact in X.

If K is compact in X, then K is closed in X which implies that $K_{\mathbf{k}}$ is the subspace topology as a subset of $X_{\mathbf{k}}$. Hence K is compact when regarded as a subspace of $X_{\mathbf{k}}$. $\square$

Corollary 9.4.6 *$X_{\mathbf{k}}$ is compactly generated.*

Proof: Suppose $A \subset X_{\mathbf{k}}$ is such that $A \cap K$ is closed for all compact K of $X_{\mathbf{k}}$. $\{$compact subspaces of $X_{\mathbf{k}}\}$ $= \{$ compact subspaces of $X\}$ so this implies A is closed in $X_{\mathbf{k}}$. Hence $X_{\mathbf{k}}$ is compactly generated. $\square$

Proposition 9.4.7 *If X is compactly generated, then $X_{\mathbf{k}} = X$. In particular $(X_{\mathbf{k}})_{\mathbf{k}} = X_{\mathbf{k}}$.*

Proof: If A is closed in X, then A is closed in $X_{\mathbf{k}}$. Conversely suppose A is closed in $X_{\mathbf{k}}$. Then $A \cap K$ is closed $\forall$ compact K of X. Hence A is closed in X. $\square$

Theorem 9.4.8 *Let X and Y be CW complexes. Then $(X \times Y)_{\mathbf{k}}$ is a CW complex.*

Proof: Write $X = \cup_{\alpha \in J} e_\alpha$, and $Y = \cup_{\beta \in K} e_\beta$. So as a set $Z = X \times Y = \cup_{J \times K} e_\alpha \times e_\beta$. Since D^{m+n} is compact, $f_{\alpha,\beta}(D^{m+n})$ is compact so its topology as a subspace of X is the same as that as a subspace of $X \times Y$. Hence $f_{\alpha,\beta}$ is continuous as a map from D^{m+n} to Z and $f_{\alpha,\beta}\big|_{\overset{\circ}{D^{m+n}}}$ is still a homeomorphism to its image in Z, so property (2) in the definition of CW-complex is satisfied. For property (3): Suppose $A \cap \overline{e_\alpha \times e_\beta}$ is closed for all α, β. For any compact K, $\pi_1(K)$ and $\pi_2(K)$ are compact so $\pi_1(K) \subset \cup_{j=1,\ldots,r} e_{\alpha_j}$, $\pi_2(K) \subset \cup_{k=1,\ldots,s} e_{\beta_k}$.

Hence

$$\begin{aligned} K &\subset \cup_{\substack{j=1,\ldots,r \\ k=1,\ldots,s}} e_{\alpha_j} \times e_{\beta_k} \\ &\subset \cup_{\substack{j=1,\ldots,r \\ k=1,\ldots,s}} \overline{e_{\alpha_j} \times e_{\beta_k}} \end{aligned}$$

Hence

$$\begin{aligned} A \cap K &= A \cap \left(\cup_{\substack{j=1,\ldots,r \\ k=1,\ldots,s}} \overline{e_{\alpha_j} \times e_{\beta_k}} \right) \cap K \\ &= \left(\cup_{\substack{j=1,\ldots,r \\ k=1,\ldots,s}} A \cap \overline{e_{\alpha_j} \times e_{\beta_k}} \right) \cap K \end{aligned}$$

which is closed. So A is closed in Z. $\square$

Chapter 10

Categories and Functors

Definition 10.0.9 *A* category $\mathbf{C}$ *consists of:*

E1) A collection of objects (which need not form a set) known as $\mathrm{Obj}(\mathbf{C})$

E2) For each pair X, Y *in* $\mathrm{Obj}(\mathbf{C})$*, a set (denoted* $\mathbf{C}(X,Y)$ *or* $\mathrm{Hom}_{\mathbf{C}}(X,Y)$*) called the morphisms in the category* $\mathbf{C}$ *from* X *to* Y

E3) For each triple X, Y, Z *in* $\mathrm{Obj}(\mathbf{C})$*, a set function* $\circ : \mathbf{C}(X,Y) \times \mathbf{C}(Y,Z) \to \mathbf{C}(X,Z)$ *called composition*

E4) For each X *in* $\mathrm{Obj}(\mathbf{C})$*, an element* $1_X \in \mathbf{C}(X,X)$ *called the identity morphism of* X

such that:

A1) $\forall f \in \mathbf{C}(X,Y)$, $1_Y \circ f = f$ *and* $f \circ 1_X = f$.

A2) $f \in \mathbf{C}(X,Y)$, $g \in \mathbf{C}(Y,Z)$, $h \in \mathbf{C}(Z,W) \Rightarrow h \circ (g \circ f) = (h \circ g) \circ f \ \in \ \mathbf{C}(X,W)$

Examples:

	Objects	Morphisms	$\circ$	id
1.	Sets	Set functions	comp. of functions	identity set map
2.	Groups	Group homomorphisms	”	”
3.	Top. spaces	conts. functions	”	”

4. “Topological pairs”
An object in $\mathbf{C}$ is a pair (X, A) of topological spaces with $A \subset X$.

Morphisms $(X, A) \mapsto (Y, B) = \{$ conts. $f : X \to Y \mid f(A) \subset B\}$

5. X p.o. set. Define $\mathbf{C}$ by $\mathrm{Obj}(\mathbf{C}) = X$.

$$\mathbf{C}(x, y) = \begin{cases} \text{set with one element} & \text{if } x \leq y; \\ \emptyset & \text{if } y \leq x \text{ or } x, y \text{ not comparable} \end{cases}$$

6. $\mathbf{C}$ any category. Define $\mathbf{C}^{\mathrm{op}}$ by

$$\mathrm{Obj}\,\mathbf{C}^{\mathrm{op}} = \mathrm{Obj}\,\mathbf{C}.$$
$$\mathbf{C}^{\mathrm{op}}(X, Y) = \mathbf{C}(Y, X).$$

$$g \circ_{\mathbf{C}^{\mathrm{op}}} f = f \circ_{\mathbf{C}} g.$$

Definition 10.0.10 *A* functor $F : \mathbf{C} \to \mathbf{D}$ *consists of:*

E1) For each object $\mathbf{X}$ *in* $\mathbf{C}$*, an object* $F(\mathbf{X})$ *in* $\mathbf{D}$

E2) For each morphism g *in* $\mathbf{C}(X, Y)$*, a morphism* $F(g)$ *in* $\mathbf{D}\big(F(\mathbf{X}), F(G)\big)$

such that:

A1) $F(1_{\mathbf{X}}) = 1_{F(\mathbf{X})}$

A2) $F(g \circ f) = F(g) \circ F(f)$

Examples:

1. "Forgetful" functor F : Top Spaces $\to$ Sets
   $F(X) =$ underlying set of top. space X
2. Sets $\to$ $\mathbf{k}$-vector spaces
   $S \mapsto$ "Free" vector space over $\mathbf{k}$ on basis S
   $\big(S \to TF(S)\big) \longmapsto F(T)$
3. Completely regular topological spaces and continuous maps $\to$ Compact topological spaces and conts. maps
   $X \longmapsto \beta(X)$

4. Top. spaces $\to$ Compactly generated top. spaces

   $X \longmapsto X_{\mathbf{k}}$

Definition 10.0.11 *If F and G are functors from $\mathbf{C}$ to $\mathbf{D}$, then a* natural transformation *$n : F \to G$ consists of:*

For all X in $\mathbf{C}$, a morphism $n_X \in D(F(X), G(G))$ s.t. $\forall f \in \mathbf{C}(X, Y)$,

$$\begin{tikzcd}
F(X) \arrow[r, "n_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "n_X"] & G(Y)
\end{tikzcd}$$

commutes.

Example: C =topological pairs

$D =$ topological spaces

$F : \mathbf{C} \to \mathbf{D}$ forget A. i.e. $(X, A) \longmapsto X$

$G : \mathbf{C} \to \mathbf{D}$ $(X, A) \longmapsto X/A$

$\big((X, A) \xrightarrow{f} (Y, B)\big) \longmapsto \big(X/A \xrightarrow{G(f)} Y/B\big)$.

$n : F \to G$ by $n_X : F(X, A) \to G(X, A)$ is the canonical projection, $X \to X/A$.

Then $(X, A) \xrightarrow{f} (Y, B)$ yields

$$\begin{tikzcd}
X \arrow[r, "F(f)"] \arrow[d, "n_X"] & Y \arrow[d, "n_Y"] \\
X/A \arrow[r, "G(f)"] & Y/B.
\end{tikzcd}$$

Chapter 11

Homotopy

11.1 Basic concepts of homotopy

Example:

$$\int_{\gamma_1} \frac{1}{z} dz = \int_{\gamma_2} \frac{1}{z} dz$$

but

$$\int_{\gamma_1} \frac{1}{z} dz \neq \int_{\gamma_3} \frac{1}{z} dz.$$

Why? The domain of $1/z$ is $\mathbb{C} \setminus \{0\}$. We can deform γ_1 continuously into γ_2 without leaving $\mathbb{C} \setminus \{0\}$.

Intuitively, two maps are homotopic if one can be continuously deformed to the other.

The value of $\int_\gamma \frac{1}{z} dz$ is an example of a situation where only the homotopy class is important.

Definition 11.1.1 *Let X and Y be topological spaces, and $A \subset X$, and $f, g : X \to Y$ with $f|_A = g|_A$. We say f* is homotopic *to g* relative to *A (written $f \simeq g \operatorname{rel} A$) if $\exists H : X \times I \to Y$ s.t. $H|_{X \times 0} = f$, $H|_{X \times 1} = g$, and $H(a, t) = f(a) = g(a)$ $\forall a \in A$. H is called a homotopy from f to g.*

In the example, $X = I$, $Y = \mathbb{C} \setminus \{0\}$, $A = \{0\} \cup \{1\}$, $f(0) = g(0) = p$, $f(1) = g(1) = q$.

Notation: For $t \in I$, $H_t : X \to Y$ by $H_t(x) = H(x, t)$. In other words $H_0 = f$, $H_1 = g$.

$f \overset{H}{\simeq} g$ rel A or $H : f \simeq g$ rel A mean H is a homotopy from f to g. We write $f \simeq g$ if A is understood.

Example: $Y = \mathbb{R}^n$, $f, g : X \to \mathbb{R}^n$. $f|_A = g|_A$. Then $f \simeq g$ rel A.
Proof: Define $H(x, t) = tg(x) + (1 - t)f(x)$

Proposition 11.1.2 *$A \subset X$. $j : A \to Y$. Then homotopy rel A is an equivalence relation on $\mathcal{S} = \{f : X \to Y | f|_A = j\}$.*

Proof: (i) reflexive: given $f \in \mathcal{S}$, define $H : f \simeq f$ by $H(x,t) = f(x)\ \forall t$.
(ii) Symmetric: Given $H : f \simeq g$ define $G : g \simeq f$ by $G(x,t) = H(x, 1-t)$.
(iii) Transitive: Given $F : f \simeq g$, $G : g \simeq h$ define $H : f \simeq h$ by

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } 0 \le t \le 1/2 \\ G(x,2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

□

Important special case: $A =$ pt x_0 of X.

Definition 11.1.3 *A* pointed space *consists of a pair $\{X, x_0\}$. $x_0 \in X$ is called the* basepoint*. A* map of pointed spaces *$f : (X, x_0) \to (Y, y_0)$ is a map of pairs, in other words $f : X \to Y$ s.t. $f(x_0) = y_0$.*

Note: Pointed spaces and basepoint-preserving maps form a category.
Notation: X, Y pointed spaces. $[X, Y] = \{$ homotopy equivalence classes of pointed maps $\}$.
Top (X, Y) is far too large to describe except in trivial cases (such as $X =$ pt). But $[X, Y]$ is often countable or finite so that a complete computation is often possible. For this case under certain hypotheses (discussed later) this set has a natural group structure.

Notation: $\pi_n(Y, y_0) \stackrel{\text{def}}{=} [S^n, Y]$ with basepoints $(1, 0, \ldots, 0)$ and y_0 respectively. In this special case $X = S^n$, this set has a natural group structure (described later). $\pi_n(Y, y_0)$ is called the *n-th homotopy group of* Y with respect to the basepoint y_0.

$\pi_1(Y, y_0)$ is called the *fundamental group of* Y with respect to the basepoint y_0.

11.1.1 Group Structure of $\pi_1(Y, y_0)$

Notation: $f, g : I \to Y$. Suppose $f(1) = g(0)$.
Define $f \cdot g : I \to Y$ by

$$f \cdot g(s) = \begin{cases} f(2s) & \text{if } 0 \le s \le 1/2 \\ g(2s-1) & \text{if } 1/2 \le s \le 1 \end{cases}$$

Lemma 11.1.4 *$f, g : I \to Y$ s.t. $f(1) = g(0)$. $A = \{0\} \cup \{1\} \subset I$. Then the homotopy class of $f \cdot g$ rel A depends only on the homotopy classes of f and g rel A. In other words $f \simeq f'$ and $g \simeq g' \Rightarrow f \cdot g \simeq f' \simeq g'$.*

$F : f \simeq f'$, $G : g \simeq g'$.
$H : I \times I \to Y$

$$H(s,t) = \begin{cases} F(2s,t) & \text{if } 0 \le s \le 1/2 \\ G(2s-1,t) & \text{if } 1/2 \le s \le 1. \end{cases}$$

$H : f \cdot g \simeq f' \cdot g'$. □

Let $f, g \in \pi_1(Y, y_0)$. So $f, g : S^1 \to Y$.

Thought of as maps $I \to Y$ for which $f(0) = f(1) = g(0) = g(1) = y_0$.

Define $f \star g$ in $\pi_1(Y, y_0)$ to be $f \cdot g$.

Theorem 11.1.5 *$\pi_1(Y, y_0)$ becomes a group under $[f][g] := [fg]$.*

Proof: The preceding lemma show that this multiplication is well defined.

Associativity:

Follows from:

Lemma 11.1.6 *Let $f, g, h : I \to Y$ such that $f(1) = g(0)$ and $g(1) = h(0)$. Then $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$,*

Proof: Explicitly $H(s,t) = \begin{cases} f(\frac{4s}{2-t}) & 4s \le 2-t; \\ g(4s+t-2) & 2-t \le 4s \le 3-t; \\ h(\frac{4s+t-3}{1+t}) & 3-t \le 4s. \end{cases}$ √

Identity: Given $y \in Y$, define $c_y : I \to Y$ by $c(s) = y$ for all s. Constant map.

Lemma 11.1.7 *Let $f : I \to Y$ be such that $f(0) = p$. Then $c_p \cdot f \simeq f \,\mathrm{rel}(\{0\} \cup \{1\})$.*

$$H(s,t) = \begin{cases} p & 2s \le t; \\ f(\frac{2s-t}{2-t}) & 2s \ge t. \end{cases}$$ □

Similarly if $f(1) = q$ then $f \cdot c_q \simeq f \,\mathrm{rel}\, A$. Applying this to the case $p = q = y_0$ gives that $[f][c_{y_0}] = [c_{y_0}][f] = [f]$. √

Inverse: Let $f : I \to Y$ Define $f^{-1} : I \to Y$ by $f^{-1}(s) := f(1-s)$.

Lemma 11.1.8 *. $f \cdot f^{-1} \simeq c_p \,\mathrm{rel}(\{0\} \cup \{1\})$.*

Proof: Intuitively:

$t = 1$ Go from p to q and return.
$0 < t < 1$ Go from p to $f(t)$ and then return.
$t = 0$ Stay put.

$$H(s,t) = \begin{cases} f(2st) & 0 \le s \le 1/2; \\ f\big(2(1-s)t\big) & 1/2 \le s \le 1. \end{cases}$$ □

Applying the lemma to the case $p = q = y_0$ shows $[f][f^{-1}] = [c_{y_0}]$ in $\pi_1(Y, y_0)$, $\checkmark$

This completes the proof that $\pi_1(Y, y_0)$ is a group under this multiplication. $\square$

Note: In general $\pi_1(Y, y_0)$ is nonabelian.

Proposition 11.1.9 *Let $f : X \to Y$ be a pointed map. Define $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_\#[\omega] := [f \circ \omega]$. Then $f_\#$ is a group homomorphism.*

($f_\#$ is called the map *induced* by f.)

Proof:
Show that $f_\#$ is well defined.

Lemma 11.1.10

$$(W, A) \underset{g'}{\overset{g}{\rightrightarrows}} (X, B) \underset{h'}{\overset{h}{\rightrightarrows}} (Y, C)$$

Suppose $g \simeq g' \operatorname{rel} A$ and $h \simeq h' \operatorname{rel} B$. Then $h \circ g \simeq h' \circ g' \operatorname{rel} A$.

Proof of Lemma:
Let $G : g \simeq g'$ and $H : h \simeq h'$ be the homotopies. Define $K : W \times I \to Y$ by $K(w, t) := H\big(G(w, t), t\big)$. Then $K : h \circ g \simeq h' \circ g' \operatorname{rel} A$. (i.e. $K(w, 0) = H\big(G(w, 0), 0\big) = H\big(g(w), 0\big) = h \circ g(w)$ and similarly $K(w, 1) = h' \circ g'(w)$ while for $a \in A$, $K(a, t) = H\big(G(a, t), t\big) = H\big(g(a), t\big) = h\big(g(a)\big) = h'\big(g'(a)\big)$.

Proof of Proposition (cont.) Thus $f_\#$ is well defined (applying the lemma with $W := S^1$, $A = \{w_0 := (1, 0)\}$, $B := \{x_0\}$, $C := \{y_0\}$, $g := w$, $g' := w'$, and $h = h' := f$). $\checkmark$

$f \circ (w \cdot \gamma) = (f \circ \omega) \cdot (f \circ \gamma)$ Therefore $f_\#([\omega][\gamma]) = f_\#([\omega \cdot \gamma]) = [f \circ (\omega \cdot \gamma)] = [(f \circ \omega) \cdot (f \circ \gamma)] = [f \circ \omega][f \circ \gamma] = f_\#([\omega]) f_\#([\gamma])$. $\square$

Corollary 11.1.11 *The associations $(X, x_0) \longmapsto \pi_1(X, x_0)$ with $f \longmapsto f_\#$ defines a functor from the category of pointed topological spaces to the category of groups.* $\square$

To what extent does $\pi_1(Y, y_0)$ depend on y_0?

Proposition 11.1.12

1. *Let Y' be the path component of Y containing y_0. Then $\pi_1(Y', y_0) \simeq \pi_1(Y, y_0)$.*

2. *If y_0 and y_1 are in the same path component then $\pi_1(Y, y_0) \simeq \pi_1(Y, y_1)$*

Proof:

1. Any curve of Y beginning at y_0 lies entirely in Y' (since curves are images of a path connected set and thus path connected).

2. Pick a path α joining y_0 to y_1. Define $\phi : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ by $[f] \mapsto [\alpha^{-1} \cdot f \cdot \alpha]$ (where α^{-1} denotes the path which goes backwards along α).

   Check that ϕ is a homorphism:

   $\phi([f][g]) = [\alpha^{-1} f \alpha][\alpha^{-1} g \alpha] = [\alpha^{-1} f \alpha \alpha^{-1} g \alpha] = [\alpha^{-1} f g \alpha]$ since $f \alpha \alpha^{-1} g \simeq f c_{y_1} g \simeq fg$. Thus $\phi([f][g]) = [\alpha^{-1} fg\alpha] = \phi([fg])$ $\surd$

   Show ϕ is injective:

   Suppose that $\phi([f]) = e$. That is $[\alpha^{-1} f \alpha] = [c_{y_1}]$. Then $\alpha^{-1} f \alpha \simeq c_{y_1}$. Hence $f \simeq c_{y_0} f c_{y_0} \simeq \alpha\alpha^{-1} f \alpha\alpha^{-1} \simeq \alpha c_{y_1} \alpha^{-1} \simeq \alpha\alpha^{-1} \simeq c_{y_0}$. Thus $[f] = [e]$ in $\pi_1(Y, y_0)$. $\surd$

   Check that ϕ is onto:

   Given $[g] \in \pi_1(Y, y_1)$, set $f := \alpha \cdot g \cdot \alpha^{-1}$. Then $\phi[f] = [\alpha^{-1} f \alpha] = [\alpha^{-1} \alpha g \alpha^{-1} \alpha] = [g]$. $\surd$

$\square$

In algebraic topology, path connected is a more important concept than connected. From now on, we will use the term "connected" to mean "path connected" unless stated otherwise.

Notation: If Y is (path) connected, write $\pi_1(Y)$ for $\pi_1(Y, y_0)$ since up to isomorphism it is independent of y_0. The constant function $(X, x_0) \to (Y, y_0)$ taking x to y_0 for all $x \in X$ is often denoted $*$. Also the basepoint itself is often denoted $*$.

If $f \simeq *$ then f is called *null homotopic*. So for $f : S^1 \to Y$, f is null homotopic if and only if $[f] = e$ in $\pi_1(Y)$.

Theorem 11.1.13 *Let $X = \prod_{j \in I} X_j$. Let $* = (x_j)_{j \in I} \in X$. Then $\pi_1(X, *) = \prod_{j \in I} \pi_1(X_j, x_j)$.*

Proof:

Let $p_j : X \to X_j$ be the projection. The homomorphisms $p_{j\#} : \pi_1(X, *) \to \pi_1(X_j, x_j)$ induce $\phi := (p_{j\#}) : \pi_1(X, *) \to \prod_{j \in I} \pi_1(X_j, x_j)$.

To show ϕ injective:

Suppose that $\phi([\omega]) = 1$. Then $\forall j \in I$, $\exists$ a homotopy $H_j : p_j \circ \omega \simeq c_{x_j}$. Put these together to get $H : \omega \simeq c_*$. (i.e. for $z = (z_j)_{j \in I} \in X$, define $H(z, t) := \big(H_j(z_j, t)\big)_{j \in I}$ Hence $[\omega] = 1$ in $\pi_1(X, *)$.

To show ϕ surjective:

Given $([\omega_j])_{j \in I}$ where $[\omega_j] \in \pi_1(X_j, x_j)$:

Define ω to be the path whose jth component is ω_j. (That is, $\omega(t) = \big(\omega_j(t)\big)_{j \in I}$.) Then $\phi([\omega]) = ([\omega_j])_{j \in I}$. $\square$

Definition 11.1.14 *If X is (path) connected and $\pi_1(X) = 1$ (where 1 denotes the group with just one element) then X is called* simply connected*.*

11.2 Homotopy Equivalences and the Homotopy Category

Definition 11.2.1 *A (pointed) map $f : X \to Y$ of pointed spaces is called a* homotopy equivalence *if $\exists$ (pointed) $g : Y \to X$ s.t. $g \circ f \simeq 1_X \operatorname{rel} *$ and $f \circ g \simeq 1_Y \operatorname{rel} *$. If $\exists$ a homotopy equivalence between X and Y then X and Y are called* homotopy equivalent*.*

We write $X \simeq Y$ or $f : X \xrightarrow{\simeq} Y$.

Define the homotopy category (HoTop) by:
Obj HoTop = Topological Spaces
HoTop$(X, Y) = [X, Y]$ (pointed homotopy classes of pointed maps from X to Y)

Examples of homotopy equivalences:

1. Any homeomorphism
2. $\mathbb{R}^n \simeq *$

   Proof: : Let $f : * \to \mathbb{R}^n$ by $* \mapsto a$ (where a is some chosen basepoint) and $g : \mathbb{R}^n \to *$ by $x \mapsto *$ for all x. Then $g \circ f = 1_*$ and $f \circ g \simeq 1_{\mathbb{R}^n}$ since any two maps into $\mathbb{R}^n$ are homotopic and furthermore can do it leaving the basepoint fixed.

3. Inclusion of S^1 into $\mathbb{C} \setminus \{0\}$ is a homotopy equivalence.

   Proof: Intuitively widen the hole in $\mathbb{C} \setminus \{0\}$ and then squish everything to a single curve. Explicitly,

   $i : S^1 \to \mathbb{C} \setminus \{0\}$ inclusion

   Define $r : \mathbb{C} \setminus \{0\} \to S^1$ by $z \mapsto z/||z||$. Then $r \circ i = 1_{s^1}$. To show $i \circ r \simeq 1_{\mathbb{C} \setminus \{0\}}$, note that $ir * z) = z/||z||$ and define a homotopy $H : \mathbb{C} \setminus \{0\} \times I \to \mathbb{C} \setminus \{0\}$ via $(z, t) \mapsto \frac{z}{1+t(||z||-1)}$.

Definition 11.2.2 *A pointed space (X, x_0) is called* contractible *if $1_X \simeq c_{x_0} \operatorname{rel}\{x_0\}$.*

If (X, x_0) is contractible as a pointed space then we say that the (unpointed) spaces X is contractible to x_0. (Note: It is possible that a space X is contractible to some point x_0 but not contractible to some different point x_0'.)

Proposition 11.2.3 *Suppose Y contractible. Then any two maps from X to Y are homotopic.*

Proof: $1_Y \simeq c_{y_0}$. Hence $\forall f : X \to Y$, $f = 1_Y \circ f \simeq c_{y_0} \circ f = c_{y_0}$. □

Proposition 11.2.4 *X contractible $\Leftrightarrow X \simeq *$* $\square$

Example: Any convex subset of $\mathbb{R}^n$ is contractible to any point in the space. **Proof:** Let x_0 belong to X where X is convex. Define $H : X \times I \to X$ by $H(x,t) = tx_0 + (1-t)x$, which lies in X since X is convex. $\square$

The two most basic questions that homotopy theory attempts to answer are:

1. Extension Problems:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[dl, dotted, "\exists?"] \\ Y & \end{tikzcd}$$

2. Lifting Problems:

$$\begin{tikzcd} & E \arrow[d] \\ X \arrow[r] \arrow[ur, dotted, "\exists?"'] & B \end{tikzcd}$$

Lemma 11.2.5 *$f : S^n \to Y$. Then f extends to $\bar{f} : D^{n+1} \to Y \Leftrightarrow f \simeq c_{y_0}$.*

Proof:

$$\begin{tikzcd} S^n \arrow[rr, hook, "i"] \arrow[dr, "f"] & & D^{n+1} \arrow[dl, dotted, "\exists \bar{f}\,?"] \\ & Y & \end{tikzcd}$$

($\Rightarrow$) Suppose $\bar{f}$ exists. $f = \bar{f} \circ \imath$. D^{n+1} is contractible (as it is a convex subspace of $\mathbb{R}^{n+1}$) $\Rightarrow$ $i \simeq *$.

Hence $f = \bar{f} \circ \imath \simeq \bar{f} \circ * = *$.

($\Leftarrow$) Suppose $H : c_{y_0} \simeq f$. $H : S^n \times I \to Y$.

Define

$$\bar{f}(x) = \begin{cases} y_0 & 0 \leq ||x|| \leq 1/2 \\ H(x/||x||, 2||x|| - 1) & 1/2 \leq ||x|| \leq 1 \end{cases}$$

□

Corollary 11.2.6 *Suppose $f, g : I \to Y$ s.t. $f(0) = g(0)$, $f(1) = g(1)$. If Y simply connected, then $f \simeq g \operatorname{rel}(0, 1)$.*

Proof: To show $f \simeq g \operatorname{rel}(0, 1)$ we want to extend the map shown on $\partial(I \times I)$ to all of $I \times I$. Up to homeomorphism, $I \times I = D^2$ and $\partial(I \times I) = S^1$. By the Lemma, the extension exists $\Leftrightarrow$ the map on the boundary is null homotopic.

$\pi_1(Y) = 1 \Rightarrow$ any map $S^1 \to Y$ is null homotopic. □

Lemma 11.2.7 *$f : S^n \to Y$. Then f extends to $\bar{f} : D^{n+1} \to Y \Leftrightarrow f \simeq c_{y_0}$.*

Proof: ($\Rightarrow$) Suppose $\bar{f}$ exists. $f = \bar{f} \circ \imath$. D^{n+1} is contractible (as it is a convex subspace of $\mathbb{R}^{n+1}$) $\Rightarrow i \simeq \star$.

Hence $f = \bar{f} \circ \imath \simeq \bar{f} \circ * = *$.

($\Leftarrow$) Suppose $H : c_{y_0} \simeq f$.

Define

$$\bar{f}(x) = \begin{cases} y_0 & 0 \leq ||x|| \leq 1/2 \\ H(x/||x||, 2||x|| - 1) & 1/2 \leq ||x|| \leq 1 \end{cases}$$

□

Corollary 11.2.8 *Suppose $f, g : I \to Y$ s.t. $f(0) = g(0)$, $f(1) = g(1)$. If Y simply connected, then $f \simeq g \operatorname{rel}(0, 1)$.*

Proof: To show $f \simeq g \operatorname{rel}(0, 1)$ we want to extend the map shown on $\partial(I \times I)$ to all of $I \times I$. Up to homeomorphism, $I \times I = D^2$ and $\partial(I \times I) = S^1$. By the Lemma, the extension exists $\Leftrightarrow$ the map on the boundary is null homotopic.

$\pi_1(Y) = 1 \Rightarrow$ any map $S^1 \to Y$ is null homotopic. □

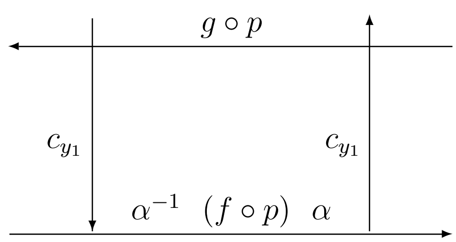
Theorem 11.2.9 *Suppose $H : f \simeq g \operatorname{rel} \emptyset$ where $f, g : X \to Y$. Let $y_0 = f(x_0), y_1 = g(x_1)$. Let*

α be the path $\alpha(t) = H(x_0, t)$ joining y_0 and y_1. Then

$$\begin{tikzcd}
 & \pi_1(Y, y_0) \arrow[dd, "\alpha_*"', "\cong"] \\
\pi_1(X, x_0) \arrow[ur, "f_\#"] \arrow[dr, "g_\#"'] & \\
 & \pi_1(Y, y_1)
\end{tikzcd}$$

commutes, where α_* denotes the isomorphism $\alpha_*([h]) = [\alpha^{-1} h \alpha]$.

Proof: Let $p : (S^1, *) \to (X, x_0)$ represent an element of $\pi_1(X, x_0)$. We must show $g \circ p \simeq \alpha^{-1} \cdot (f \circ p) \cdot \alpha \,\mathrm{rel}\, *$.



Thinking of S^1 as $I/(\{0\} \cup \{1\})$, show the map defined on $\partial(I \times I)$ as shown extends to $I \times I$. Hence show the map on $\partial(I \times I)$ is null homotopic. The boundary map under the homeomorphism $\partial(I \times I) \cong S^1 \cong I/(\{0\} \cup \{1\})$ becomes $[c_{y_1}^{-1} \cdot \alpha^{-1} \cdot (f \circ p) \cdot \alpha \cdot c_{y_1} \cdot (g \circ p)^{-1}] = [\alpha^{-1} \cdot (f \circ p) \cdot \alpha \cdot (g \circ p)^{-1}]$.

$$\begin{tikzcd}
X \times I \arrow[r, "H"] & Y \\
S^1 \times I \arrow[u, "p \times I"] &
\end{tikzcd}$$

(where, by convention, we sometimes write the name of a space to denote the identity map of that space).

$H : f \simeq g$

$$H \circ (p \times I) : f \circ p \simeq g \circ p$$

By the Lemma, since the extension exists, $\alpha^{-1} \cdot (f \circ p) \cdot \alpha \cdot (g \circ p)^{-1}$ is null homotopic.

$\square$

Corollary 11.2.10 *Let $f : X \to Y$ be a homotopy equivalence. Then $f_\# : \pi_1(X, x_0) \to \pi_1\big(Y, f(x_0)\big)$ is an isomorphism.*

Proof: Let $g : Y \to X$ be a homotopy inverse to f. Let $H : gf \simeq 1_X$. Let $\alpha(t) = H(x_0, t)$ joining x_0 to $gf(x_0)$.

By the Theorem:

$$\begin{tikzcd}
 & \pi_1(X, x_0) \arrow[dd, "\alpha_*"', "\cong"] \\
\pi_1(X, x_0) \arrow[ur, "(1_X)_\#"] \arrow[dr, "(gf)_\#"'] & \\
 & \pi_1\big(X, gf(x_0)\big)
\end{tikzcd}$$

Hence $g_\# f_\# = (gf)_\# = \alpha_*$ is an isomorphism. Similarly $f_\# g_\#$ is an isomorphism. It follows (from category theory) that $f_\#$ (and $g_\#$) are isomorphisms.

In other words,

Lemma 11.2.11 *$\phi : G \to H$, $\psi : H \to G$ s.t. $\psi\phi$ and $\phi\psi$ are isomorphisms. Then ϕ is an isomorphism.*

Proof: Let $a = (\psi\phi)^{-1} : G \to G$. Then $a\psi\phi = 1_G$ so $\phi a\psi\phi\psi = \phi 1_G \psi = \phi\psi$. Right multiplication by $(\phi\psi)^{-1}$ gives $\phi a\psi = 1_H$. $a\psi\phi = 1_G$, $\phi a\psi = 1_H \Rightarrow a\psi$ is inverse to ϕ so ϕ is an isomorphism. $\square$

Corollary 11.2.12 *X contractible $\Rightarrow$ X simply connected.*

Proof: Let $H : 1_X \simeq c_{x_0}$.

(1) Show X (path) connected.

Let $x \in X$. Define $I \overset{w}{\to} X$ by $w(t) = H(x,t)$. w joins x_0 to x_1. So all points are connected by a path to x_0. So X is connected.

(2) Show $\pi_1(X, x_0) = 1$:

By earlier Proposition, X is contractible $\Leftrightarrow$ $X \simeq *$. Hence $\pi_1(X, x_0) \simeq \pi_1(*, *)$ and it is clear from the definition that $\pi_1(*, *) = 1$. $\square$

Chapter 12

Covering Spaces and the Fundamental Group

12.1 Introduction to covering spaces

Covering spaces have many uses both in topology and elsewhere. Our immediate goal is to use them to help compute $\pi_1(X)$.

Definition 12.1.1 *A map $p : E \to X$ is called a* covering projection *if every point $x \in X$ has an open neighbourhood U_x s.t. $p^{-1}(U_x)$ is a (nonempty) disjoint union of open sets each of which is homeomorphic by p to U_x. E is called the* covering space*, X the* base space *of the covering projection.*

Remark: It is clear from the definition that a covering projection must be onto.

Example: $\mathbb{R} \overset{\exp}{\to} S^1$ by $t \mapsto e^{2\pi i t}$
$\exp^{-1}(U_x) = \coprod_{n=-\infty}^{\infty} V_n$.
$V_n \cong U_x \forall n$.

More generally: A (left) action of a topological group G on a topological space X consists of a (continuous) map $\phi : G \times X \to X$ s.t.

1. $ex = x \ \forall x$

2. $g_1(g_2 x) = (g_1 g_2)x \ \forall g_1, g_2 \in G, x \in X$.

Given action $\phi : G \times X \to X$, for each $g \in G$ we get a continuous map $\phi_g : X \to X$ sending x to gx. Each ϕ_g is a homeomorphism since $\phi_{g^{-1}} = (\phi_g)^{-1}$.

Note: Any group becomes a topological group if given the discrete topology. In the case where G has the discrete topology, ϕ is continuous $\Leftrightarrow$ ϕ_g is continuous $\forall g \in G$. (In general, ϕ_g continuous for all g is not sufficient to conclude that ϕ is continuous.)

Suppose G acts on X.

Define an equivalence relation on X by $x \sim gx$ $\forall x \in X$, $g \in G$. Write X/G for $X/\sim$ (with the quotient topology).

Remark: The notation is in conflict with the previously given notation that X/A means identify the points of A to a single point. Rely on context to decide which is meant.

Preceding example: $X = \mathbb{R}$, $G = \mathbb{Z}$. $\phi(n, x) = x + n$. Then $\mathbb{R}/\mathbb{Z} \cong S^1$. In this example X happens to also be a topological group and G a normal subgroup so X/G also has a group structure. The homeomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$ is an isomorphism of topological groups.

Theorem 12.1.2 *Suppose a group G acts on a space X s.t. $\forall x \in X$, $\exists$ an open neighbourhood V_x s.t. $V_x \cap gV_x = \emptyset$ for all $g \neq e$ in G. Then the quotient map $p : X \to X/G$ is a covering projection.*

Proof: Given $[x] \in X/G$, find V_x as in the hypothesis. Set $U_{[x]} = p(V_x)$. $p^{-1}(U_{[x]}) = \bigcup_{g \in G} g{\cdot}V_x$.

V_x open $\Rightarrow$ gV_x open $\forall g$ $\Rightarrow$ $p^{-1}(U_{[x]})$ open $\Rightarrow$ $U_{[x]}$ open.

$g_1 V_x \cap g_2 V_x = \emptyset$ so the union is a disjoint union.

$p : V_x \to U_{[x]}$ is a bijection and check that by definition of the quotient topology it is a homeomorphism.

$$\begin{tikzcd}
gV_x \arrow[rr, "\cong"] \arrow[dr, "p"] & & V_x \arrow[dl, "p"', "\cong"] \\
& U_{[x]} &
\end{tikzcd}$$

Both gV_x and V_x map to $U_{[x]}$ under p, and the map p composed with $g : V_x \to gV_x$ equals the map $p : V_x \to U_{[x]}$, which shows that $p|_{gV_x}$ is a homeomorphism $\forall g$.

Hence $p : X \to X/G$ is a covering projection.

Corollary 12.1.3 *Suppose H is a topological group and G a closed subgroup of H s.t. as a subspace of H, G has the discrete topology Then $p : H \to H/G$ is a covering projection.*

Example 2: $S^n \to \mathbb{R}P^n$ is a covering projection.

Proof: $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{-1, 1\}$ acts by $1x = x, -1x = -x$. Furthermore, the hypothesis of the previous theorem is satisfied.

Similarly $\mathbb{C}P^n = S^{2n+1}/S^1$ and $\mathbb{H}P^n = S^{4n+3}/SU(2)$, but these quotient maps are not covering projections (since the group is not discrete).

What have covering spaces got to do with $\pi_1(X)$?

Return to the example $\mathbb{R} \xrightarrow{\exp} S^1$.

Let w be a path in $\mathbb{R}$ which begins at 0 and ends at the integer n. w is not a closed curve in $\mathbb{R}$ (unless $n = 0$, where in this context "closed" means a curve which ends at the point at which it starts) but $\exp(w)$ is a closed curve in S^1 joining $*$ to $*$.

So $\exp(w)$ represents an element of $\pi_1(S^1)$.

We will show that the resulting element of $\pi_1(S^1)$ depends only on n (not on w) and that this correspondence sets up an isomorphism $\pi_1(S^1) \cong \mathbb{Z}$.

Terminology: Let $p : E \to X$ be a covering projection. Let $U \subset X$ be open. If $p^{-1}(U)$ is a disjoint union of open sets each homeomorphic to U, then we say that U is *evenly covered*. If $U \subset X$ is evenly covered, with $p^{-1}(U) = \coprod_i T_i$ with $T_i \cong U$, then each T_i is called a *sheet* over U.

Theorem 12.1.4 (Unique Lifting Theorem) *Let $p : (E, e_0) \to (X, x_0)$ be a map of pointed spaces in which $p : E \to X$ is a covering projection.*

Let $f : (Y, y_0) \to (X, x_0)$. If Y is connected, then there is at most one map $f' : (Y, y_0) \to (E, e_0)$ s.t.

$$\begin{tikzcd} & E \arrow[d, "p"'] \\ Y \arrow[ur, dotted, "f'"] \arrow[r, "f"] & X \end{tikzcd}$$

Remark 12.1.5 *: For this theorem it suffices to know that Y is connected under the standard definition, although in most applications we will actually know that Y is path connected, which is even stronger.*

Proof:

Suppose $f', f''(Y, y_0) \to (E, e_0)$ s.t. $pf' = f$ and $pf'' = f$. Let $A = \{y \in Y \mid f'(y) = f''(y)\}$, $B = \{y \in Y \mid f'(y) \neq f''(y)\}$. Then $A \cap B = \emptyset$, $A \cup B = Y$.

It suffices to show that both A and B are open because then one of them is empty. But $A \neq \emptyset$ since $y_0 \in A$, so this would imply that $B = \emptyset$ and $A = X$, in other words $f' = f''$.

To show A is open: Let $y \in A$. Let U be an evenly covered set in X containing $f(y)$. Let S be a sheet in $p^{-1}(U)$ containing $f'(y) = f''(y)$. Let $V = (f')^{-1}(S) \cap (f'')^{-1}(S)$, which is open in Y and contains y. $\forall v \in V$, $pf'(v) = f(v) = pf''(v) \Rightarrow f'(v) = f''(v)$ (since $p|_S$ is a homeomorphism). Hence $V \subset A$, so y is interior. So A is open.

To show B is open: Let $y \in B$. Let U be an evenly covered set containing $f(y)$. $f'(y) \neq f''(y)$ but $pf'(y) = f(y) = f''(y)$ so $f'(y)$ and $f''(y)$ lie in different sheets (say S', S'') over $p^{-1}(U)$.

Let $V = (f')^{-1}(S') \cap (f'')^{-1}(S'')$, which is open in Y. Since $S' \cap S = \emptyset$, $f'(V) \neq f''(V)$ $\forall v \in V$. Hence $V \subset B$. So y is interior. Therefore B is open. □

Theorem 12.1.6 *(**Path Lifting Theorem**) Let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering projection. Let $w : I \to X$ s.t. $w(0) = x_0$. Then w lifts uniquely to a path $w' : I \to E$ s.t. $w'(0) = e_0$.*

$$\begin{tikzcd} & E \arrow[d] \\ I \arrow[ur, dotted, "\omega'"] \arrow[r, "\omega"] & X \end{tikzcd}$$

Proof: Uniqueness follows from the previous theorem (since I is connected).

Existence: Cover X by evenly covered sets. Using a Lebesgue number for the inverse images under w in the compact set I, we can partition I into a finite number of subintervals $[t_i, t_{i+1}]$ $(0 = t_0 < t_1 < \cdots < t_n = 1)$ s.t. $\forall i$, $w([t_i, t_{i+1}]) \subset U_i$. Note that U_i is evenly covered.

Let S_0 = sheet in $p^{-1}(U_0)$ containing e_0. $p|_{S_0}$ is a homeomorphism $\Rightarrow \exists$ unique path in S_0 covering $w([t_0, t_1])$. Let e_1 denote the end of this path. $(p(e_1) = w(t_1))$

Let S_1 = sheet in $p^{-1}(U_1)$ containing e_1.

As above, $\exists$ unique path in S_1 covering $w([t_1, t_2])$.

Continuing: Build a path w' in E beginning at e_0 and covering w. □

Remark 12.1.7 *The procedure is reminiscent of analytic continuation. Notice that even through ω is closed ($\omega(0) = \omega(1)$), this need not be true for ω'. e.g. Consider $p = \exp : \mathbb{R} \to S^1$ and let $\omega(t) = e^{2\pi i t} : I \to S^1$. Then ω' is the line segment joining 0 to 1.*

We will show that under the right conditions (e.g. $\mathbb{R} \to S^1$) elements of $\pi_1(X, x_0)$ can be identified by the endpoint in E of the lifted representing path.

Need:

Theorem 12.1.8 *(Covering Homotopy Theorem) Let $p : (E, e_0) \to (X, x_0)$ be a covering projection. Let (Y, y_0) be a pointed space. Let $f : (Y, y_0) \to (X, x_0)$ and let $f' : (Y, y_0) \to (E, e_0)$ be a lift of f. Let $H : Y \times I \to X$ be a homotopy with $H - 0 = f$. Then H lifts to a homotopy $H' : Y \times I \to E$ s.t. $H'_0 = f'$.*

Before the proof, we examine the consequences.

Corollary 12.1.9 *Let $(E, e_0) \to (X, x_0)$ be a covering projection. Let $\sigma, \tau : I \to X$ be paths from x_0 to x_1 s.t. $\sigma \simeq \tau \operatorname{rel}\{0,1\}$. Let σ', τ' be lifts of σ, τ respectively, beginning at e_0. Then $\sigma'(1) = \tau'(1)$ and $\sigma' = \tau' \operatorname{rel}\{0,1\}$.*

Note in particular that this implies that the endpoint of a lift of a homotopy class is independent of the choice of representative for that class.

Proof of Corollary (assuming Theorem): Let $H : \sigma \simeq \tau \operatorname{rel}\{0,1\}$. Apply the theorem to get $H' : I \times I \to E$ which lifts H and s.t. $H'_0 = \sigma'$

The left vertical line of H' can be thought of as a path in E begining at $\sigma'(0) = e_0$ and lifting c_{x_0}. By uniqueness it must be c_{e_0}. Similarly the right must be c_{e_1}, where $e_1 = \sigma'(1)$. Also, the top is a lift of τ beginning at e_0 so it must be τ'. Thus $H' : \sigma' \simeq \tau' \operatorname{rel}\{0,1\}$ and $\tau'(1) =$ upper right corner $= e_1 = \sigma'(1)$. □

Proof of Theorem:
Technical remark: It is easy to define the required lift, but not so easy to show continuity. i.e. Given $y \in I$, $H\big|_{y \times I}$ is a path in X beginning at $f(y)$ so $H'\big|_{f'(y) \times I}$ is the unique lift beginning at $f'(y)$.

Step 1: $\forall y \in Y$, $\exists$ open neighbourhood V_y and a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of I (depending on y) s.t. $\forall i$, $H(V_y \times [t_i, t_{i+1}])$ is contained in an evenly covered set.
Proof: Given y:

$\forall t \in I$ find evenly covered neighbourhood U_t of $H(y,t)$ in X.

Find basic open $A_t \times B_t \subset H^{-1}(U_t) \subset Y \times I$ containing (y,t). Then $\cup_{t \in I} B_t$ covers I so choose a finite subcover $B_{t_1}, \ldots B_{t_{n-1}}$. Set $V_y := A_{t_1} \cap \cdots \cap A_{t_{n-1}} \cap A_0 \cap A_1$. Use V_y together with the partition $0 < t_1 < \ldots < t_{n-1} < 1$. $\surd$

Step 2: $\forall y$, $\exists$ continuous $H'_y : V_y \times I \to E$ lifting $H\big|_{V_y \times I}$ and extending $H'_y\big|_{V_y \times 0} = f'\big|_{V_y}$.
Proof: Use the same inductive argument as in the proof of the Path Lifting Theorem. $\surd$

Step 3: The various lifings H'_y from Step 2 combine to produce a well defined map of sets $H' : Y \times I \to E$.
Proof: Suppose $(y,t) \in (V_{t_1} \times I) \cap (V_{y_2} \times I)$. The restrictions $H'_{y_1}\big|_{y \times I}$ and $H'_{y_2}\big|_{y \times I}$ each produce paths in E beginning at $f'(y)$ and lifting $H\big|_{y \times I}$. So by unique path lifting, $H'_{y_1}(y,t) = H'_{y_2}(y,t)$. Hence the value of $H'(y,t)$ is independent of the set V_{y_i} used to compute it. i.e. H' is well defined. $\surd$

Step 4: The map H' defined in Step 3 is continuous.
Proof: Suppose $U \subset E$ is open.

$H'^{-1}(U) = \bigcup_{y \in U} (H'_y)^{-1}(U)$.

$\forall y \in U$, $H'_y : V_y \times I \to E$ is continuous which implies that $(H'_y)^{-1}(U)$ is open in $V_y \times I$. Since $V_y \times I$ is open in $Y \times I$, this implies that $(H'_y)^{-1}(U)$ is open in $H'^{-1}(U)$. Hence $H'^{-1}(U)$ is open and thus H' is continuous. □

Corollary 12.1.10 *Let $p : (E, e_0) \to (X, x_0)$ be a covering projection. Then $p_\# : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is a monomorphism.*

Proof: Let $[\omega] \in \pi_1(E, e_0)$. ω is a path in E beginning and ending at e_0. Suppose $p_\#([\omega]) = 1$. Then $p \circ \omega \simeq c_{x_0} \operatorname{rel}\{0,1\}$. By the Corollary 12.1.9, $(p \circ \omega)' \simeq c'_{x_0} \operatorname{rel}\{0,1\}$ where $(p \circ \omega)'$, c'_{x_0} are, respectively, the lifts of $p \circ \omega$, c_{x_0} beginning from e_0. Clearly these lifts are ω and c_{e_0} respectively. Hence $\omega \simeq c_{e_0} \operatorname{rel}\{0,1\}$, so $[\omega] = 1 \in \pi_1(E, e_0)$. □

Theorem 12.1.11 $\pi_1(S^1) \cong \mathbb{Z}$

Proof: Let $\omega : (S^1, *) \to (S^1, *)$ represent an element of $\pi_1(S^1, *)$. Regard ω as a path which begins and ends at $*$. By unique path lifting in $\exp : (\mathbb{R}, 0) \to (S^1, *)$ we get a path ω' in $\mathbb{R}$ lifting ω beginning at 0. Hence $\exp\big(\omega'(1)\big) = \omega(1) = *$ so $\omega'(1) = n \in \mathbb{Z}$. By Corollary 12.1.9 n is independent of the choice of representative for the class $[\omega]$. Thus we get a well defined $\phi : \pi_1(S^1) \to \mathbb{Z}$ given by $[\omega] \mapsto \omega'(1)$.

Claim: ϕ is a group homomorphism.

Let $\sigma, \tau : (S^1, *) \to (S^1, *)$ represent elements of $\pi_1(S^1)$. Let $\sigma', \tau' : I \to \mathbb{R}$ be lifts of σ, τ respectively beginning at 0. Let $n = \sigma'(1) = \phi([\sigma])$ and $m = \tau'(1) = \phi([\tau])$. Define τ'' by $\tau''(t) = \tau'(t) + n$. Then τ'' = lift of τ beginning at n, ending at $n + m$. The path $\sigma' \cdot \tau''$ in $\mathbb{R}$ makes sense (since $\sigma'(1) = n = \tau''(0)$). $\sigma' \cdot \tau''$ begins at 0 and ends at $n + m$. But $\exp(\sigma' \cdot \tau'') = \sigma \cdot \tau$ so it lifts $\sigma \cdot \tau$. Hence $\phi([\sigma][\tau]) = \phi([\sigma \cdot \tau]) = n + m = \phi([\sigma]) + \phi([\tau])$. Thus ϕ is a homomorphism. $\surd$

Claim: ϕ is injective

Suppose $\phi([\sigma]) = 0$. Let $\sigma' : I \to \mathbb{R}$ be the lift of σ beginning at 0. Then the definition of ϕ implies that σ' ends at 0 so σ' represents an element of $\pi_1(\mathbb{R})$ and $\exp_\#([\sigma']) = [\sigma]$. But $\mathbb{R}$ is simply connected ($\pi_1(\mathbb{R}) = 1$) and so $[\sigma'] = 1$ which implies $[\sigma] = 1$. $\surd$

Claim: ϕ is onto

Given $n \in \mathbb{Z}$, let ω' be any path in $\mathbb{R}$ joining 0 to n. Let $\omega = \exp \circ \omega' : I \to S^1$. Then ω is a closed path in S^1 and $\phi([\omega]) = n$. □

Corollary 12.1.12 $\pi_1(\mathbb{C} - \{0\}) \cong \mathbb{Z}$

Proof: $S^1 \to \mathbb{C} - \{0\}$ is a homotopy equivalence. □

We wish to apply the method used above to calculate $\pi_1(S^1)$ to calculate $\pi_1(X)$ for other spaces X. For this, we need a covering projection $E \to X$, called the universal covering projection of X with properties described in the next section. For reference, we note here the properties of $\mathbb{R} \to S^1$ which were needed in the calculation of $\pi_1(S^1)$.

1. $\mathbb{Z}$ acts on $\mathbb{R}$, $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, by $(n, x) \mapsto n + x$ s.t.

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "T_n"] \arrow[dr, "\exp"'] & & \mathbb{R} \arrow[dl, "\exp"] \\ & S^1 & \end{tikzcd}$$

   where T_n is the translation $T_n(X) = n + x$.

2. $\pi_1(\mathbb{R}) = 1$

We will return to this later. First some applications.

Theorem 12.1.13 *$\nexists f : D^2 \to S^1$ s.t.*

$$\begin{tikzcd} S^1 \arrow[rr, hook] \arrow[dr, "1_{S^1}"'] & & D^2 \arrow[dl, "f"] \\ & S^1 & \end{tikzcd}$$

commutes.

Proof: If f exists then, since D^2 is contractible, applying π_1 yields

$$\begin{tikzcd} \mathbb{Z} = \pi_1(S^1) \arrow[rr, hook] \arrow[dr, "1"'] & & \pi_1(D^2) = 0 \arrow[dl, "f_\#"] \\ & \mathbb{Z} = \pi_1(S^1) & \end{tikzcd}$$

This is a contradiction so f does not exist. $\square$

Corollary 12.1.14 *(Brouwer Fixed Point Theorem): Let $g : D^2 \to D^2$. Then $\exists x \in D^2$ such that $g(x) = x$.*

Proof: Suppose g has no fixed point. Define $f : D^2 \to S^1$ as follows:
$g(x) \neq x$ implies that $\exists$ a well defined line segment joining $g(x)$ to x. Follow this line until it reaches S^1 and call this point $f(x)$.

f is a continuous function of x (since g is) and if $x \in S^1$ then $f(x) = x$. This contradicts the previous theorem. Hence g has no fixed point. $\square$

12.2 Universal Covering Spaces

Definition 12.2.1 *Let $p : E \to X$ and $p' : E' : X$ be covering projections. A* morphism *of covering spaces over X consists of a map $\phi : E \to E'$ s.t.*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

commutes.

A morphism of covering spaces which is also a homeomorphism is called an equivalence of covering spaces.

Remark: Covering spaces over a fixed X together with this notion of morphism form a category. An equivalence is an isomorphism in this category.

Definition 12.2.2 *A covering projection $\tilde{p} : \tilde{X} \to X$ is called the* universal covering projection *of X (and $\tilde{X}$ is called the* universal covering space *of X) if for any covering projection $p : E \to X$ $\exists!$ morphism $f : \tilde{X} \to E$ of covering projections.*

i.e.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "\tilde{p}"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes.

Remark: This says $\tilde{p} : \tilde{X} \to X$ is an initial object in the category of covering spaces over X.

Proposition 12.2.3 *If X has a universal covering space then it is unique up to equivalence of covering spaces.*

Proof: Standard categorical argument. $\square$

Theorem 12.2.4 (**Lifting Theorem**) *Let $p : (E, e_0) \to (X, x_0)$ be a covering projection and let $f : (Y, y_0) \to (X, x_0)$ where Y is connected and locally path connected. Then $\exists\ f' : (Y, y_0) \to (E, e_0)$ lifting $f \Leftrightarrow f_\# \pi_1(Y, y_0) \subset p_\# \pi_1(E, e_0)$.*

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dotted, "f'"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Remark: X connected $\Rightarrow$ at most one such lift exists, by the Unique Lifting Theorem.
Proof: ($\Rightarrow$) Suppose f' exists. Then $f_\# = (pf')_\# = p_\# f'_\#$. Hence Im $f_\# \subset$ Im $p_\#$.

($\Leftarrow$) Suppose Im $f_\# \subset$ Im $p_\#$. For $y \in Y$ choose a path σ joining y_0 to y. Then $f \circ \sigma : I \to X$ joins x_0 to $f(y)$. Lift to a path $(f\sigma)'$ in E beginning at e_0 and define $f'(y) = (f\sigma)'(1)$.
Claim this gives a well-defined function of y:

Suppose $\tau : I \to Y$ also joins y_0 to y. Then $\sigma \cdot \tau^{-1}$ represents an element of $\pi_1(Y, y_0)$ so by hypothesis $\exists [w] \in \pi_1(E, e_0)$ s.t. $[p \circ \omega] = p_\#([w]) = f_\#([\sigma \cdot \tau^{-1}]) = [f \circ (\sigma \cdot \tau^{-1})]$. Since $p \circ w \simeq f \circ (\sigma \cdot \tau^{-1})$, lifting these paths to E beginning at e_0 results in paths with the same endpoint.

But w lifts $p \circ w$ and it ends at e_0 (it is a closed loop since it represents an element of $\pi_1(E, e_0)$). Hence the lift $\alpha : I \to E$ of $f \circ (\sigma \cdot \tau^{-1})$ beginning at e_0 also ends at e_0. Let $e_1 = \alpha(1/2)$.

The restriction of α to $[0, 1/2]$ lifts σ (beginning at e_0, ending at e_1).

The restriction of α to $[1/2, 1]$ lifts τ^{-1} (beginning at e_1, ending at e_0).

So the curve lifting τ beginning at e_0 ends at e_1. So using either σ or τ in the definition of $f'(y)$ results in $f'(y) = e_1$. Hence f' is well defined. $\surd$

To help show f' continuous:

Lemma 12.2.5 *Let $y, z \in Y$ and let γ be a path in Y from y to z. If the path $f \circ \gamma$ is contained in some evenly covered set U of X then $f'(y), f'(z)$ lie in the same sheet in $p^{-1}(U)$.*

Proof: Let $(f \circ \gamma)'$ be the lift of $f \circ \gamma$ beginning at $f'(y)$.
Claim: $(f \circ \gamma)'$ ends at $f'(z)$.

Proof of Claim: Use $\sigma \circ \gamma$ as the path joining y_0 to z in the definition of $f'(z)$. Then $(f \circ \sigma)' \cdot (f \circ \gamma)'$ is the lift of $f \circ (\gamma \cdot \sigma)$ which begins at e_0, so $f'(z)$ is the endpoint of $(f \circ \sigma)' \cdot (f \circ \gamma)'$, in other words the endpoint of $(f \circ \gamma)'$. $\surd$

Let S be the sheet of $p^{-1}(U)$ containing $f'(y)$.

$p|_S$ is a homeomorphism, which implies S contains the entire path $(f \circ \gamma)'$, so in particular it contains $f'(z)$. $\square$

Claim: f' is continuous.

Given $e \in E$, let $U_{p(e)} \subset X$ be an evenly covered set containing $p(e)$ and let S_e be the sheet in $p^{-1}(U_{p(e)})$ which contains e.

For an open set $V \subset E$, $V = \bigcup_{e \in V}(S_e \cap V)$, so to show f' is continuous, it suffices to show $f'^{-1}(W)$ is open whenever $W \subset E$ is open in some S_e.

Since $p|_{S_e}$ is a homeomorphism, $p(W)$ is open in X and is evenly covered (being a subset of the evenly covered set $U_{p(e)}$).

Set $A := f^{-1}\big(p(W)\big) \subset Y$. By continuity of f, A is open so its path components are open by hypothesis.

$(f')^{-1}(W) \subset A$. Show $(f')^{-1}(W)$ is open by showing $(f')^{-1}(W)$ is a union of path components of A.

Write $A = \bigcup_{i \in I} A_i$ where A_i is a path component of A.

Claim: $\forall i$, either $A_i \cap (f')^{-1}(W) = \emptyset$ or $A_i \subset (f')^{-1}(W)$.

Note: This shows $(f')^{-1}(W)$ is the union of those A_i which intersect it, thus completing the proof.

Proof of Claim: Suppose $y \in A_i \cap (f')^{-1}(W)$. Let $z \in A_i$. Show $z \in (f')^{-1}(W)$.

Let γ be a path joining y to z in A_i. (A_i is a path component so is path connected.)

Since $A_i \subset A = f^{-1}\big(p(W)\big)$, $f \circ \gamma$ is entirely contained in the evenly covered set $p(W)$, so by the Lemma, $f'(y)$ and $f'(z)$ lie in the same sheet of $p^{-1}\big(p(W)\big)$.

$y \in (f')^{-1}(W) \Rightarrow$ that sheet is W so $z \in (f')^{-1}(W)$. $\square$

Lemma 12.2.6 *A covering space of a locally path connected space is locally path connected.*

Proof: Let $E \xrightarrow{p} X$ be a covering projection, with X locally path connected.

Let V be open in E, let A be a path component of V and let $a \in A$.

Let $U \subset X$ be an evenly covered set containing $p(A)$ and let S be the sheet in $p^{-1}(U)$ containing a.

Replacing U by the smaller evenly covered set $p(S \cap V)$, we may assume $S \subset V$.

Let W be the path component of U containing $p(a)$. Hence W is open by hypothesis. $p|_S$ is a homeomorphism, so $B := p^{-1}(W) \cap S$ is a path connected open subset in E.

B is path connected, and $a \in B$, so $B \subset A$. Since B is open, $a \in \mathring{A}$ so A is open. $\square$

Corollary 12.2.7 *(***of Lifting Theorem***): A simply connected locally path connected covering space is a universal covering space.*

Proof: Let $\tilde{p} : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering projection s.t. $\tilde{X}$ is simply connected and locally path connected. Let $p : (E, e_0) \to (X, x_0)$ be a covering projection of X.

$\pi_1(\tilde{X}, \tilde{x_0}) = 1$ so the hypothesis $\tilde{p}_\# \pi_1(\tilde{X}, \tilde{x_0}) \subset p_\# \pi_1(E, e_0)$ of the Lifting Theorem is trivial. Hence $\exists f : \tilde{X} \to E$ s.t.

$$\begin{tikzcd} \tilde{X} \arrow[rr, dashed, "f"] \arrow[dr, "\tilde{p}"] & & E \arrow[dl, "p"'] \\ & X & \end{tikzcd}$$

The Unique Lifting Theorem shows f is unique. $\square$

Corollary 12.2.8 *(***of Lifting Theorem***:) Let W be simply connected and let $(E, e_0) \xrightarrow{p} (X, x_0)$ be a covering projection. Then $[(W, w_0), (E, e_0)] \xrightarrow{p_\#} [(W, w_0), (X, x_0)]$ is a set bijection.*

Proof: Essentially the same as the proof of Corollary 12.2.7. $\square$

12.2.1 Computing Fundamental Groups from Covering Spaces

Definition 12.2.9 *Let $p : E \to X$ be a covering projection. A self-homeomorphism $\phi : E \to E$ is called a* covering transformation *if*

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p"] & & E \arrow[dl, "p"'] \\ & X & \end{tikzcd}$$

commutes.

Remark: $p\phi = p$ guarantees that $\forall x \in X$, ϕ is a self-map of $p^{-1}(x)$. $p^{-1}(x)$ is often called the *fibre* over x.

$\{$ covering transformations of $E \xrightarrow{p} X$ $\}$ forms a group under composition.

Example 1: $\exp : \mathbb{R} \to S^1$. The group of covering transformations is $\mathbb{Z}$.

Example 2: $p : S^n \to \mathbb{R}P^n$. The group of covering transformations is $\mathbb{Z}_2$, because it is the collection of maps sending $x \to x$ or $x \to -x$ (for $x \in S^n$).

Notice that in each case $|G| = \text{card}\,\big(p^{-1}(x)\big)$.

Lemma 12.2.10 *Let $p : E \to X$ be a covering projection with E connected. Let $\phi, \phi' : E \to E$ s.t. $p\phi = p$, $p\phi' = p$. If $\phi(e) = \phi'(e)$ for some $e \in E$ then $\phi = \phi'$. In particular, a covering transformation is determined by its value at any point.*

Proof:

$$\begin{tikzcd}
& E \arrow[d, "p"] \\
E \arrow[ur, dotted, "{\phi, \phi'}"] \arrow[r, "p"] & X
\end{tikzcd}$$

Apply the Unique Lifting Theorem with $y_0 = e$ and $x_0 = \phi(e) = \phi'(e)$.

□

Theorem 12.2.11 *Let $p : E \to X$ be a covering projection s.t. E is simply connected and locally path connected (thus a universal covering space). Then $\pi_1(X)$ = group of covering transformations of p.*

(Since "simply connected" includes "path connected", notice that p onto implies that X is path connected, so $\pi_1(X)$ is well defined, i.e. independent of the choice of basepoint.)

Proof: Let G be the group of covering tranformations of p. Define $\psi : G \to \pi_1(X)$ as follows: Given $\phi \in G$, select a path w_ϕ joining e_0 to $\phi(e_0)$.

$p\phi(e_0) = pe_0 = x_0 \Rightarrow p \circ w_\phi$ is a closed loop in X so it represents an element of $\pi_1(X, x_0)$.

Define $\psi(\phi) = [p \circ w_\phi]$.

Claim: ψ is well-defined.

Proof: (of Claim:) If w'_ϕ is another path joining e_0 to $\phi(e_0)$ then E is simply connected $\Rightarrow w_\phi \simeq w'_\phi \operatorname{rel}\{0,1\}$.

Hence $p \circ w_\phi \simeq p \circ w_{\phi'} \operatorname{rel}\{0,1\}$. i.e. $[p \circ \omega_\phi] = [p \circ \omega'_\phi]$ in $\pi_1(X)$.

Claim: ψ is a group homomorphism.

Proof: (of Claim:) Let $\phi_1, \phi_2 \in G$. Pick paths w_{ϕ_1}, w_{ϕ_2} as above joining e_0 to $\phi_1(e_0)$ resp. , $\phi_2(e_0)$. Then $\phi_1 \circ w_{\phi_2}$ is a path joining $\phi_1(e_0)$ to $\phi_1\big(\phi_2(e_0)\big) = \phi_1\phi_2(e_0)$. So we use $w_{\phi_1}\dot{(}\phi_1 \circ w_{\phi_2})$ to define $\psi(\phi_1\phi_2)$.

ϕ is a covering transformation, so $p \circ \phi_1 \circ w_{\phi_2} = p \circ w_{\phi_2}$.

Hence $\psi(\phi_1\phi_2) = [p \circ (w_{\phi_1} \cdot (\phi_1 \circ w_{\phi_2})] = [p \circ w_{\phi_1}][p \circ \phi_1 \circ w_{\phi_2}]$
$= [p \circ w_{\phi_1}][p \circ w_{\phi_2}]$
$= \psi(\phi_1)\psi(\phi_2)$.

Claim: ψ is injective.

Proof: (of Claim:) $\psi(\phi_1) = \psi(\phi_2) \Rightarrow p \circ w_{\phi_1} \simeq p \circ w_{\phi_2}$. This implies the lifts of w_{ϕ_1} and w_{ϕ_2} beginning at e_0 must end at the same point.

Hence $\phi_1(e_0) = \phi_2(e_0)$ which implies $\phi_1 = \phi_2$.

Claim: ψ is surjective.

Proof: (of Claim:) Let $[\sigma] \in \pi_1(X, x_0)$.

Lift σ to a path σ' in E beginning at e_0.

Let $e = \sigma'(1)$.

It suffices to show there exists a covering transformation $\phi : E \to E$ s.t. $\phi(e_0) = e$.

Then we use σ' to define $\psi(\phi)$ to see that $\psi(\phi) = \sigma$.

$$\begin{tikzcd} & (E, e) \arrow[d, "p"] \\ (E, e_0) \arrow[ur, dotted, "\phi"] \arrow[r, "p"] & (X, x_0) \end{tikzcd}$$

Since E is connected and locally path connected and $1 = p_\#\pi_1(E, e_0) \subset p_\#\pi_1(E, e)$, the lifting theorem implies $\exists\phi$ s.t. $p \circ \phi = p$ and $\phi(e_0) = e$.

It remains to show ϕ is a homeomorphism.

But we may apply the lifting theorem again with the roles of e_0 and e reversed to get $\theta : (E, e) \to (E, e_0)$.

Then $p \circ \theta \circ \phi = p$ and $\theta \circ \phi(e_0) = e_0$ so by the previous Lemma, $\theta \circ \phi = 1_E$. Similarly $\phi \circ \theta = 1_E$. So ϕ is a homeomorphism. $\square$

Remark: We already used this to show that $\pi_1(S^1) = \mathbb{Z}$. Later we will show that S^n is simply connected for $n \geq 2$, so that the theorem applies to $S^n \to \mathbb{R}P^n$, giving $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

Note: The preceding proof showed a bijection between covering transformations and elements of $p^{-1}(x_0)$. Each point corresponds to a covering transformation taking e_0 to that point.

12.2.2 'Galois' Theory of Covering Spaces

Theorem 12.2.12 *Let $p : E \to X$ be a covering projection s.t. E is simply connected and locally path connected (thus a universal covering space). Then for every subgroup $H \subset \pi_1(X)$, $\exists$ a covering projection $p_H : E_H \to X$, unique up to equivalence of covering spaces, such that $(p_H)_\#(\pi_1(E_H)) = H$.*

Proof: $\{$covering transformations of $E\} \cong \pi_1(X)$ so H can be regarded as the set of covering transformations of E. Hence H acts on E. Let $E_H = E/H$.

If $e' = h \circ e$ for $h \in H$, since h is a covering transformation, $p(e') = p(e)$.

Hence p induces a well defined map $p_H : E/H \to X$.

For evenly covered U_x of $p : E \to X$, sheets $p^{-1}(U_x)$ correspond bijectively to elements of $\pi_1(X)$.

$p_H^{-1}(U_x)$ is what we get by identifying S, S' whenever S, S' correspond to group elements g, g' s.t. $g' = gh$ for some $h \in H$ (in other words g' and g are in the same coset of $G \pmod H$).

Hence p_H is a covering projection (with U_x as evenly covered set).

Also Theorem 12.1.2 implies $E \xrightarrow{f} E/H$ is a covering projection. To apply the theorem we need to know that $\forall e \in E$, $\exists V_e$ s.t. $V_e \cap hV_e = \emptyset$ unless $h = 1$. Set $V_e :=$ the sheet over $U_{p(e)}$ which contains e for some evenly covered $U_{p(e)} \subset X$. This works since h is a covering translation so hS is also a sheet and sheets are disjoint.

By inspection, the group of covering translations of $f_H \cong H \cong \pi_1(E/H)$. (In general, the group of covering translations of $Y \to Y/G$ is isomorphic to G.)

By Corollary 12.1.10, any covering projection induces a monomorphism on π_1.

Hence $(p_H)_\# : H = \pi_1(E_H) \hookrightarrow \pi_1(E)$.

In other words $(p_H)_\#\big(\pi_1(E_H)\big) = H$. $\square$

12.2.3 Existence of Universal Covering Spaces

Not every space has a universal covering space.

Example: Let $X = \prod_{j=1}^{\infty} S^1$.

Proof: Let $E_n = \prod_{j=1}^{n} \mathbb{R} \times \prod_{j=n+1}^{\infty} S^1$.

It's easy to check that $p_n = \exp \times \cdots \times \exp \times 1\big|_{\prod_{j=n+1}^{\infty} S^1}$ is a covering projection.

(In general a product of covering projections is a covering projection.)

Suppose X had a universal covering projection $\tilde{p} : \tilde{X} \to X$.

Then $\forall n$, we have

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "f_n"] \arrow[dr, "\tilde{p}"'] & & E_n \arrow[dl, "p_n"] \\
& X &
\end{tikzcd}$$

By uniqueness of f_n,

$$\begin{tikzcd}
 & E_{n+1} \arrow[dd, "e_{n+1}"] \\
\tilde{X} \arrow[ur, "f_{n+1}"] \arrow[dr] & \\
 & X
\end{tikzcd}$$

where e_{n+1} is exp on factor $(n+1)$ and the identity on the other factors.

Apply π_1 and use that $p_\#$ is a monomorphism to see that all maps on π_1 are monomorphisms.

$$\pi_1(\tilde{X}) \subset \cdots \subset \pi_1(E_{n+1}) \subset \pi_1(E_n) \subset \cdots \subset \pi_1(X).$$

$$\pi_1(X) = \prod_{j=1}^{\infty} \pi_1(S^1) = \prod_{j=1}^{\infty} \mathbb{Z}$$

and $\pi_1(E_n)$ is the subgroup $\prod_{j=n+1}^{\infty} \mathbb{Z}$. Hence $\pi_1(X) \subset \bigcap_{n=1}^{\infty} \pi_1(E_n) = 0$. So $\pi_1(X) = 0$.

Let $U \subset X$ be an evenly covered set for the covering projection $\tilde{X} \to X$.

Replace U by the basic open subset $U_1 \times U_2 \times \cdots \times U_n \times S^1 \times S^1 \times \ldots$

For $j = 1, \ldots, n$ select $u_j \in U_j$.

Define $\alpha : S^1 \to X$ by

$$\begin{cases} \alpha_j = c_{u_j} & j = 1, \ldots, n \\ \alpha_{n+1} = 1_{S^1} & \\ \alpha_j = c_* & j > n+1 \end{cases}$$

Notice that Im $(\alpha) \subset U$. $[\alpha] = (0, \ldots, 0, 1, 0, \ldots) \in \pi_1(X) = \prod_{j=1}^{\infty} \mathbb{Z}$ (where the '1' is in position $n+1$).

Let T be a sheet in $\tilde{p}^{-1}(U)$.

Im $(\alpha) \subset U$, $\tilde{p}|_T$ is a homeomorphism, so α has a lift α' which is a closed curve in T.

So α' represents a class in $\pi_1(\tilde{X})$ and $\tilde{p}_\#([\alpha']) = [\alpha]$. But $\pi_1(\tilde{X}) = 0$. This is a contradiction since $[\alpha] = (0, \ldots, 0, 1, 0, \ldots) \neq 0$.

Hence X has no universal covering space.

$\square$

Definition 12.2.13 *A space X is called* semilocally simply connected *if each point $x \in X$ has an open neighbourhood U_x s.t. $i_{\#} : \pi_1(U_x, x) \to \pi_1(X, x)$ is the trivial map of groups. (where $i : U_x \hookrightarrow X$ denotes the inclusion).*

Notice that $\prod_{n=1}^{\infty} S^1$ is not semilocally simply connected.

Theorem 12.2.14 *Let X be connected, locally path connected and semilocally simply connected. Then X has a universal convering space.*

Proof: Choose $x_0 \in X$.

For path α, β in X beginning at x_0, define equiv. reln.: $\alpha \sim B$ if $\alpha(1) = \beta(1)$ and $\alpha \simeq \beta$ rel $(0, 1)$.

Let $\tilde{X} = \{$equiv. classes$\} \leftarrow$ (*paths beginning at* x_0)

Define $\tilde{p} : \tilde{X} \to X$.

$[\alpha] \to \alpha(1)$.

Topologize $\tilde{X}$ as follow: Given $[\alpha] \in \tilde{X}$ and open $V \subset X$ containing $\alpha(1)$, define subset denoted $\langle \alpha, V \rangle$ of $\tilde{X}$ by $\langle \alpha, V \rangle = \{[w] \in \tilde{X} | [w] = [\alpha \cdot \beta]$ for some path β in $V\}$. $\leftarrow$ *(strictly speaking mean* Im$\beta \subset V$.*)*

Note: $\langle \alpha, V \rangle$ is independent of choice of representation for $[\alpha]$ used to define it.

Claim: $\{\langle \alpha, V \rangle\}$ form a base for a topology on $\tilde{X}$.

Proof: Show intersection of 2 such sets is $\emptyset$ or a union of sets of this form.

Suppose $[w] \in \langle \alpha, V \rangle \cap \langle \alpha', V' \rangle \neq \emptyset$

Suff. to show:

Claim: $\langle w, V \cap V' \rangle \subset \langle \alpha, V \rangle \cap \langle \alpha', V' \rangle$

Proof: Suppose $\gamma \in \langle w, V \cap V' \rangle$ $\quad \therefore$ $[\gamma] = [w \cdot \beta]$ some β in $V \in V'$.

$[w] \in \langle \alpha, V \rangle \Rightarrow \exists \beta_1$ in V s.t. $[w] = [\alpha \cdot \beta_1]$

$[w] \in \langle \alpha', V' \rangle \Rightarrow \exists \beta_2$ in V' s.t. $[w] = [\alpha', \beta_2]$

where $w' \equiv \alpha' \cdot \beta_2 \simeq w$.

$\beta_1 \cdot \beta$ in V, $[\alpha] = [\alpha \cdot \beta_1 \cdot \beta_2] \Rightarrow [\gamma] \in \langle \alpha, V \rangle$.

Similarly $[\gamma] \in \langle \alpha', V' \rangle$. $\therefore$ $\langle w, V \cap V' \rangle \subset \langle \alpha, V \rangle \cap \langle \alpha', V' \rangle$

Give $\tilde{X}$ the topology defined by this base.

Let $V \subset X$ be open.

Then $\tilde{p}^{-1}(V) = \{[w] \in \tilde{X} | w(1) \in V\} = \bigcup_{\{\alpha | \alpha(1) \in V\}} \langle \alpha, V \rangle$

$\therefore \tilde{p}$ cont.

For $x \in X$ find V_x s.t. $i_{\#} : \pi_1(V_x, x) \to \pi_1(X, x)$ is trivial. $i : V_x \mapsto X$

Let U_x = path component of V_x containing x. open since X locally path connected.

(A): Show $\tilde{p}^{-1}(U_x) = \coprod_{\{[\alpha] | \alpha(1) = x\}} \langle \alpha, U_x \rangle$.

1. $\supset$ $\alpha(1) = x$. $\tilde{p}([w]) = w(1) \in U_x$.
2. $\subset$ Suppose $[w] \in \tilde{X}$ s.t. $\tilde{p}[w] \in U_x$. *i.e.* $[w] \in p^{-1}(U_x)$
   Then $\exists$ path β in U_x joining x to $w(1)$.
   Let $\alpha = w \cdot \beta^{-1}$. $[\alpha \cdot \beta] = [w \cdot \beta^{-1} \cdot \beta] = [w]$.
   $\therefore [w] \in \langle \alpha, U_x \rangle \subset \bigcup_{\alpha(1) = x} \langle \alpha, U_x \rangle \leftarrow$ (*α ends where β begins – at x)*
3. union is disjoint Suppose $[w] \in \langle \alpha, U_x \rangle \cap \langle \alpha', U_x \rangle$
   $[\alpha' \cdot \beta'] = [w] = [\alpha \cdot \beta] \quad \beta, \beta'$ paths in U_x

   $U_x \subset V_x \Rightarrow$ path $\beta \cdot \beta'^{-1}$ reps. elt. of $\pi_1(V_x, x)$ so choice of $V_x \Rightarrow [\beta \cdot \beta'^{-1}] = [c_x]$ in $\pi_1(X, x)$.
   $\therefore\ [\alpha] = [\alpha \cdot \beta \cdot \beta'^{-1}] = [w \cdot \beta'^{-1}] = [\alpha' \cdot \beta' \cdot \beta'^{-1}] = [\alpha']$.

(B) Show $\forall\ [\alpha]$ s.t. $\alpha(1) = x$ that $\tilde{p}|_{\langle \alpha, U_x \rangle} : \langle \alpha, U_x \rangle \to U_x$ is a homeomorphism.

Any pt. in U_x can be joined to x by a path in U_x, hence q is onto.

Claim: q is $1 - 1$.

Suppose $[w]$, $[w'] \in \langle \alpha, U_x \rangle$ s.t. $q([w]) = q([w'])$.

Find paths β, β' in U_x s.t. $[w] = [\alpha \cdot \beta]$, $[w'] = [\alpha \cdot \beta']$.

β, β' each join x to $w(1) = w'(1)$ in U_x so as above $[\beta^{-1} \cdot \beta'] = [c_x]$ in $\pi_1(X, x)$.

$\therefore [w] = [\alpha \cdot \beta] = [\alpha \cdot \beta \cdot \beta^{-1} \cdot \beta'] = [\alpha \cdot \beta'] = [w']$.

Claim: q^{-1} is continuous.

Let $\langle\gamma, V\rangle$ be basic open set with $\langle\gamma, V\rangle \subset \langle\alpha, U_x\rangle$.

$q(\langle\gamma, V\rangle) =$ path component of x within $V \cap U_x$ open since X locally path connected.

Note: $q\langle\gamma, V\rangle = \langle\gamma$, path component of $\gamma(1)$ within $V\rangle$. This implies we may assume V is path connected.

$q(w) = \beta(1)$ where β in V, $\beta(1) \in U_x$, and $\beta(0) = \alpha(1) = x$ since $\beta \in \langle\gamma, V\rangle \subset \langle\alpha, U_x\rangle$.

$\Rightarrow\ q(\langle\gamma, V\rangle) \subset V \cap U_x$.

Conversely $V \cap U_x \subset q(\langle\gamma, V\rangle)$ since endpt. of γ can be joined to $\beta(1)$ by path in V.

$\therefore q^{-1}$ cont.

$\therefore \tilde{p} : \tilde{X} \to X$ covering proj.

$\therefore$ Suff. to show:

(C) $\tilde{X}$ is simply connected:

Pick $\tilde{x}_0 := [c_{x_0}] \in \tilde{X}$ as basept. of $\tilde{X}$.

1. $\tilde{X}$ is path connected:

   Given $[w] \in \tilde{X}$, define $I \xrightarrow{\phi_w} \tilde{X}$ by $\phi_w(s) = [w_s]$ where $w_s(t) = w(st)$.

   $w_0 = c_{x_0}$, $w_1 = w$.

   $$\therefore\ \phi_w(0) = [w_0] = [c_{x_0}] = \tilde{x}_0\ .$$

   Hence

   $$\phi_w \text{ joins } \tilde{x}_0 \text{ to } [w].$$

   $$\therefore\ \phi_w(1) = [w_1] = [w].$$

   $\therefore\ \tilde{X}$ path connected.

   Before showing $\pi_1(\tilde{X}, \tilde{x}_0) = 1$ need properties of ϕ_w.

   (a) $\tilde{p} \circ \phi_w(s) = \tilde{p}([w_s]) = w_s[1] = w(s) \Rightarrow \phi_w$ is the lift of w to $\tilde{X}$ beginning at $\tilde{x}_0$.

   (b) Claim: $[w] = [\gamma] \Rightarrow \emptyset_w \simeq \emptyset_\gamma$ rel $(0, 1)$.

      Proof: Follows from Covering Homotopy Thm.

2. Show $\pi_1(\tilde{X}, \tilde{x}_0) = 1$. Let σ rep. an elt. of $\pi_1(\tilde{X}, \tilde{x}_0)$. Then $\tilde{p} \circ \sigma$ is a path in X joining x_0 to itself. $\therefore \sigma$, $\phi_{\tilde{p}\circ\sigma}$ are both lifts of $\tilde{p} \circ \sigma$ to $\tilde{X}$ beginning at $\tilde{x}_0$. $\therefore$ Unique lifting $\Rightarrow\ \sigma = \phi_{\tilde{p}\circ\sigma} \Rightarrow \sigma(1) = \phi_{\tilde{p}\circ\sigma}(1)$ and $\tilde{x}_0 = \sigma(1)$ because σ represents an element of $\pi_1(X, \tilde{x}_0)$. $\pi_1(X, \tilde{x}_0)$.)

Therefore in $\tilde{X}$, $[\tilde{p} \circ \sigma] = [(\tilde{p} \circ \sigma)_1] = \phi_{\tilde{p}\circ\sigma}(1) = \tilde{x}_0 = [c_{x_0}]$.

Therefore $\sigma = \phi_{\tilde{p}\circ\sigma} \overset{\text{part (b) above}}{\simeq} \phi_{c_{x_0}} = c_{\tilde{x}_0}$ so $[\sigma] = 1$ in $\pi_1(\tilde{X}, \tilde{x}_0)$.

Therefore $\tilde{X}$ is simply connected. $\checkmark$

(So by Corollary 12.2.7, being a simple connected cover of a connected, path connected and locally path connected space, $\tilde{X}$ is a universal covering space.) $\square$

12.3 Van Kampen's theorem

Theorem 12.3.1 (Seifert-) Van Kampen *Let U and V be connected open subsets of X s.t. $U \cup V = X$ and $U \cap V$ is connected and nonempty. Let $i_1 : U \cap V \to U$, $i_2 : U \cap V \to V$, $j_1 : U \to X$ and $j_2 : V \to X$ be the inclusion maps. Choose a basepoint in $U \cap V$.*

Let $G = \pi_1(U)$, $H = \pi_1(V)$ and let $A = \pi_1(U \cap V)$. Then

$$\pi_1(X) = G *_A H$$

where $*$ denotes the amalgamated free product defined below.

Definition 12.3.2 Amalgamated free product

If A, G, H are groups, $\alpha : A \to G$, $\beta : A \to H$ group homomorphisms, define $G *_A H$ as follows. The elements are "words" $w_1 \ldots w_n$ where for each j either $w_j \in G$ or $w_j \in H$, modulo relations generated by $\big(g\alpha(a)\big)h = g\big(\beta(a)h\big)$

(Thus every element can be written as a word alternating between elements of G and H.)

Group multiplication is by juxtaposition.

Remark: $G *_A H$ is a pushout in the category of groups:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr] & \\
H \arrow[r] \arrow[drr] & G *_A H \arrow[dr, dotted] & \\
& & Z
\end{tikzcd}$$

If $A = 1$ then $G * H$ is called the *free product* of G and H.

Proof: (of Theorem): Pick a basepoint x_0 for X lying in $U \cap V$. By the universal property, there exists $\phi : G *_A H \to \pi_1(X)$.

(Map $G \hookrightarrow \pi_1(X)$, $H \hookrightarrow \pi_1(X)$ and map a word in $G *_A H$ to the product of images of the elements of the word.)

Lemma 12.3.3 *ϕ is onto.*

Proof: Let $f : I \to X$ represent an element of $\pi_1(X)$. $f^{-1}(U) \cup f^{-1}(V) = I$ so by compactness $\exists N$ s.t. $J \subset I$, $\operatorname{diam} J \leq 1/N \Rightarrow J \subset f^{-1}(U)$ or $J \subset f^{-1}(V)$. (i.e. $\frac{1}{N}$ is a Lebesgue number for the covering $f^{-1}(U)$, $f^{-1}(V)$.) Partition I into intervals of length $1/N$.

By discarding some division points, we may assume images of intervals alternate between U and V, so the (remaining) division points are in $U \cap V$.

Pick path α_i in $U \cap V$ joining x_0 to the i-th division point. In $\pi_1(X)$ $[f] = [f_1] \dots [f_q]$ where $f_i = \alpha_i \circ f|_{J_{i+1}} \circ \alpha_{i+1}^{-1}$. $\forall i$, $[f_i] \in G$ or $[f_i] \in H$ so $[f] \in \operatorname{Im} \phi$.

Lemma 12.3.4 *ϕ is injective.*

Proof: Notation: $A = V_0$, $U = V_1$, $V = V_2$.

Let $w = w_1 \dots w_q \in G *_A H$ s.t. $\phi(w) = 1$. For each $i = 1, \dots, q$, represent each w_i by a path f_i in either V_1 or V_2.

Reparametrize f_i so that $f_i : [(i-1)/q, i/q] \to V_1$ or V_2 in X.

Let $f : I \to X$ by $f\big|_{[(i-1)/q, i/q]} := f_i$.

$\phi(w) = 1 \Rightarrow f \cong * \operatorname{rel}\{0,1\}$ so $\exists F : I \times I \to X$ s.t. $F(s,0) = f(s)$, $F(s,1) = x_0$, $F(0,t) = F(1,t) = x_0$ $\forall t$.

By compactness $\exists$ a Lebesgue number ϵ s.t. $S \subset I \times I$ with $\operatorname{diam} S < \epsilon \Rightarrow$ either $F(S) \subset V_1$ or $F(S) \subset V_2$.

Choose partitions $0 = s_0 < s_1 < \dots < s_m = 1$ and $0 = t_0 < \dots < t_n = 1$ of I s.t. the diameter of each rectangle on the resulting grid on $I \times I$ is less than ϵ.

Include the points k/q among the s_i.

For each ij select $\lambda(ij) = 1$ or 2 s.t. $F(R_{ij}) \subset V_{\lambda(ij)}$. (If $F(R_{ij} \subset$ both, take your pick.)

For each vertex v_{ij}, $V_{ij} =$ intersection of $V_{\lambda(kl)}$ over the 4 (or fewer for edge vertices) rectangles having v_{ij} as vertex.

(So $\forall i, j$, $V_{ij} = V_0, V_1$ or V_2.)

$\forall i, j$ choose a path $g_{ij} : I \to V_{ij}$ joining x_0 to $F(v_{ij})$ in $V_{\lambda(ij)}$, using that V_0, V_1, and V_2 are path connected.

Choose these g_{ij} arbitrarily except:

If $s_i = k/q$ choose $g_{i0} = c_{x_0}$

Choose $g_{0j} = c_{x_0}$ and $g_{1j} = c_{x_0}$ $\forall j$.

Choose $g_{i1} = c_{x_0}$ $\forall i$.

Let $A_{ij} = F_{a_{ij}}$, $B_{ij} = F_{b_{ij}}$.

A_{ij}, B_{ij} are not closed paths, but from them form closed paths $\alpha_{ij} = g_{i-1,j} \circ A_{ij} \circ g_{ij}^{-1}$, $\beta_{ij} = g_{i-1,j} \circ B_{ij} \circ g_{ij}^{-1}$

$\forall i, j$ either $[\alpha_{ij}]$ and $[\beta_{ij}] \in G$, or $[\alpha_{ij}]$ and $[\beta_{ij}] \in H$.

$$w_1 = [A_{01} \cdot \dots \cdot A_{0i_1}] = [\alpha_{01} \cdot \dots \cdot \alpha_{0i_1}]$$

(since $g_{00} = g_{0,i_1} = c_{x_0}$, because the points s/q are among the s_i).

Similarly

$$w_2 = [A_{0(i_1+1)} \cdot \dots \cdot A_{0i_2}] = [\alpha_{0(i_1+1)} \cdot \dots \cdot \alpha_{0i_2}]$$

$$\vdots$$

$$w_q = [A_{0(i_q+1)} \cdot \dots \cdot A_{0m}] = [\alpha_{0(i_q+1)} \cdot \dots \cdot \alpha_{0m}]$$

Therefore $w = [\alpha_{01} \cdot \dots \cdot \alpha_{0m}][\alpha_{01}] \dots [\alpha_{0m}] \in G *_A H$.
By Lemma 11.2.5, each $R_{i,j}$ gives $A_{i,j-1}B_{ij} \simeq B_{i-1,j}A_{ij}$ rel$\{0,1\}$.
Hence $\alpha_{i,j-1}\beta_{ij} \cong \beta_{i-1,j}\alpha_{ij}$ rel$\{0,1\}$.
So the relation
$[\alpha_{i,j-1}][\beta_{ij}] = [\beta_{i-1,j}][\alpha_{ij}]$ holds in either G or H and thus in $G \times_A H$.
Also $[\beta_{0j}] = [\beta_{mj}] = 1 \ \forall j$ (again for each j it holds in one of G, H) and $[\alpha_{in}] = 1 \ \forall i$.
Hence $\forall j$

$$\begin{aligned} [\alpha_{1,j-1}] \dots [\alpha_{m,j-1}] &= [\alpha_{1,j-1}] \dots [\alpha_{m,j-1}][\beta_{m,j}] \\ &= [\alpha_{1,j-1}] \dots [\alpha_{m-1,j-1}][\beta_{m-1,j}][\alpha_{m,j}] \\ &= \dots \\ &= [\beta_{0,j}][\alpha_{1,j}] \dots [\alpha_{m-1,j}][\alpha_{m,j}] \\ &= [\alpha_{1,j}] \dots [\alpha_{m-1,j}][\alpha_{m,j}]. \end{aligned}$$

Hence $w_1 \dots w_q = \prod_{i=1}^m \alpha_{i_0} = \dots = \prod_{i=1}^m \alpha_{i_n} = 1$. □

Corollary 12.3.5 *If X can be written as the union of 2 simply connected open subsets whose intersection is connected then X is simply connected.*

Corollary 12.3.6 *S^n is simply connected for $n \geq 2$.*

Proof: Write S^n = slightly enlarged upper hemisphere $\cup$ slightly enlarged lower hemisphere. □

Example 1: $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ for $n \geq 2$.

(Our covering space argument to compute $\pi_1(\mathbb{R}P^n)$ required knowing that S^n is simply connected for $n \geq 2$.)

Example 2: X is the figure eight. Then $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$.
Proof: Circles comprising X are not open, but slightly enlarge to form U and V .Then $U \cong S^1$ and $V \cong S^1$. □

The space X is denoted $S^1 \vee S^1$. The *wedge* of pointed spaces $(Y, *)$ and $(Z, *)$ written $Y \vee Z$ is the space formed from the disjoint union of Y and Z by identifying respective basepoints and using the common basepoint as the basepoint of $Y \vee Z$. In other words, $Y \vee Z = \{(y, z) \in Y \times Z \mid y = *$ or $z = *\}$

$Y \simeq Y' \Rightarrow Y \vee Z \simeq Y' \vee Z$

In particular, if W is contractible then $Y \vee W \simeq Y$. So if $X \simeq Y \vee Z$ where $\exists$ contractible open $* \in U \subset Y$ and contractible open $* \in V \subset Y$ then $\pi_1(X) = \pi_1(Y) * \pi_1(Z)$.

Chapter 13

Homological Algebra

Introductory concepts of homological algebra

Definition 13.0.7 *A* chain complex (C, d) *of abelian groups consists of an abelian group* C_p *for each integer* p *together with a morphism* $d_p : C_p \to C_{p-1}$ *for each* p *such that* $d_{p-1} \circ d_p = 0$. *Maps* d_p *are called* boundary operators *or* differentials.

The subgroup $\ker d_p$ *of* C_p *is denoted* $Z_p(C)$. *Its elements are called* cycles.

The subgroup $\operatorname{Im} d_{p+1}$ *of* C_p *is denoted* $B_p(C)$. *Its elements are called* boundaries.

$d_p \circ d_{p+1} = 0 \Rightarrow B_p(C) \subset Z_p(C)$.

The quotient group $Z_p(C)/B_p(C)$ is denoted $H_p(C)$ and called the p-th *homology group* of C. Its elements are called *homology classes*.

$x, y \in C_p$ are called *homologous* if $x - y \in B_p(C)$.

Definition 13.0.8 *A* chain map $f : C \to D$ *consists of a group homomorphism* f_p $\forall p$ *s.t.*

$$\begin{tikzcd}
C_p \arrow[r, "d_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[d, "f_{p-1}"] \\
D_p \arrow[r, "d_p"] & D_{p-1}
\end{tikzcd}$$

Notation: The subscripts are often omitted, so we might write $d^2 = 0$ or $fd = df$.

Remark: The composition of chain maps is a chain map so chain complexes and chain maps form a category.

A chain map $f : C \to D$ induces a homomorphism $f_* : H_p(C) \to H_p(D)$ for all p, defined as follows:

Let $x \in Z_p(C)$ represent an element $[x] \in H_p(C)$.

Then $df(x) = fd(x) = f(0) = 0$ so $f(x) \in Z_p(D)$.

Define $f_*([x]) := [f(x)]$.

If x, x' represent the same element of $H_p(C)$ then $x - x' = dy$ for some $y \in C_{p+1}(C)$. Therefore $fx - fx' = fdy = d(fy)$ which implies $f(x), f(x')$ represent the same element of $H_p(D)$. So f_* is well defined.

Definition 13.0.9 *A composition of homomorphisms of abelian groups*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called exact *at Y if* $\ker g = \operatorname{Im} f$. *A sequence*

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow X_1 \xrightarrow{f_1} X_0$$

is called exact *if it is exact at X_i for all $i = 1, \dots, n-1$.*

Remark: An exact sequence can be thought of as a chain complex whose homology is zero. More generally, homology can be thought of as the deviation from exactness.

A chain complex whose homology is zero is called *acyclic*.

Definition 13.0.10 *A 5-term exact sequence of the form*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence.

Proposition 13.0.11 *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence. Then f is injective, g is surjective and $B/A \cong C$.

Proof:

Exactness at $A \Rightarrow$ Ker $f =$ Im $(0 \to A) = 0 \Rightarrow f$ injective

Exactness at $C \Rightarrow$ Im $g =$ Ker $(C \to 0) = C \Rightarrow g$ surjective

Exactness at $B \Rightarrow B/\ker g \cong \operatorname{Im} g = C \Rightarrow B/\operatorname{Im} f \cong B/A$.

Corollary 13.0.12

(a) $0 \to A \xrightarrow{f} B \to 0$ *exact* $\Rightarrow f$ *is an isomorphism.*
(b) $0 \to A \to 0$ *exact* $\Rightarrow A = 0$.

Definition 13.0.13

A map $i : A \to B$ *is called a* split monomorphism *if* $\exists s : B \to A$ *s.t.* $si = 1_A$.
A map $p : A \to B$ *is called a* split epimorphism *if* $\exists s : B \to A$ *s.t.* $ps = 1_B$.

Note: The splitting s (should it exist) is not unique.

It is trivial to check:
(1) A split monomorphism is a monomorphism
(2) A split epimorphism is an epimorphism

Proposition 13.0.14 *The following are three conditions (1a, 1b, and 2) are equivalent:*

1. $\exists$ *a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *s.t.*

   1a) i is a split monomorphism

   1b) p is a split epimorphism

2. $B \cong A \oplus C$.

Remark: The isomorphism in 2. will depend upon the choice of splitting s in 1a (respectively 1b).

Lemma 13.0.15 (Snake Lemma) *Let*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i'"] \arrow[d, "f'"] & A \arrow[r, "i''"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "j'"] & B \arrow[r, "j''"] & B'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in which the rows are exact. Then $\exists$ a long exact sequence

$$0 \to \ker f' \to \ker f \to \ker f'' \xrightarrow{\partial} \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f'' \to 0.$$

Proof:

Step 1. Construction of the map ∂ (called the "connecting homomorphism"):

Let $x \in \ker f''$. Choose $y \in A$ s.t. $i''(y) = x$. Since $j''fy = f''i''y = f''x = 0$, $fy \in \ker j'' = \mathrm{Im} j'$ so $fy = j'(z)$ for some $z \in B'$. Define $\partial x = [z]$ in coker f'.

Show ∂ well defined:

Suppose $y, y' \in A$ s.t. $i''y = x = i''y'$.

$i''(y - y') = 0 \Rightarrow y - y' = i'(w)$ for some $w \in A'$. Hence $fy - fy' = fi'w = j'f'w$.

Therefore if we let $fy = j'z$ and $fy' = j'z'$ then $j'(z - z') = j'f'w \Rightarrow z - z' = f'w$ (since j is an injection). So $[z] = [z']$ in Coker f'. $\surd$

Step 2: Exactness at Ker f'':

Show the composition $\ker f \xrightarrow{i''} \ker f'' \xrightarrow{\partial}$ Coker f' is trivial.

Let $k \in$ Ker f. Then $\partial(i''k) = [z]$ where $j'(z) = f(k) = 0$. So $z = 0$.

So $\partial \circ i'' = 0$. Hence Im $(i'') \subset$ Ker ∂.

Conversely let $x \in$ Ker ∂. Let $y \in A$ s.t. $i''y = x$. We wish to show that we can replace y by a $y' \in \ker f$ which satisfies $i''y' = x$.

Find $z \in B'$ s.t. $j'z = fy$. So $\partial x = [z]$. $\partial x = 0 \Rightarrow z \in$ Coker f'.

Hence $z = f'w$ for some $w \in A'$.

Set $y' := y - i'w$. Then $i''y = iy - i''i'w = iy = x$ and $fy' = fy - fi'w = fy - j'f'w = fy - j'z = 0$.

Hence $y' \in$ Ker f.

The rest of the proof is left as an exercise $\square$

Lemma 13.0.16 (5-Lemma)
Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows. If f, g, i, j are isomorphisms then h is also an isomorphism.

(Actually , we need only f mono and j epi with g and i iso.)

Definition 13.0.17 *A sequence*

$$0 \to \underline{C} \xrightarrow{f} \underline{D} \xrightarrow{g} \underline{E} \to 0$$

of chain complexes and chain maps is called a short exact sequence of chain complexes *if*

$$0 \to C_p \overset{f_p}{\to} D_p \overset{g_p}{\to} E_p \to 0$$

is a short exact sequence (of abelian groups) for each p.

Theorem 13.0.18 *Let*

$$0 \to \underline{P} \overset{f}{\to} \underline{Q} \overset{g}{\to} \underline{R} \to 0$$

be a short exact sequence of chain complexes. Then there is an induced natural (long) exact sequence

$$\cdots \to H_n(P) \overset{f_*}{\to} H_n(Q) \overset{g_*}{\to} H_n(R) \overset{\partial}{\to} H_{n-1}(P) \overset{f_*}{\to} H_{n-1}(Q) \to \dots$$

Remark 13.0.19 *Natural means:*

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] & Q' \arrow[r] & R' \arrow[r] & 0
\end{tikzcd}$$

implies

$$\begin{tikzcd}
\dots \arrow[r] & H_n(P) \arrow[r] \arrow[d] & H_n(Q) \arrow[r] \arrow[d] & H_n(R) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(P) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_n(P') \arrow[r] & H_n(Q') \arrow[r] & H_n(R') \arrow[r, "\partial"] & H_{n-1}(P') \arrow[r] & \dots
\end{tikzcd}$$

Proof:

1. Definition of ∂:
   Let $[r] \in H_n(R)$, $r \in Z_n(R)$. Find $q \in Q_n$ s.t. $g(q) = r$.
   $g(dq) = d(qd) = dr = 0$ (since $r \in Z_n(R)$), which implies $dg = fp$ for some $p \in P_{n-1}$.
   $f(dp) = dfp = d^2q = 0 \Rightarrow dp = 0$ (as f injective).
   So $p \in Z_{n-1}(p)$. Define $\partial[r] = [p]$.

2. ∂ is well defined:
   (a) Result is independent of choice of q:

Suppose $g(q) = g(q') = r$.
$g(q - q') = 0 \Rightarrow q - q' = f(p'')$ for some $p'' \in P_n$.
Find p' s.t. $dq' = fp'$.
$f(p - p') = d(q - q') = dfp'' = fdp'' \Rightarrow p - p' = dp'' \in B_{n-1}(P)$.
So $[p] = [p']$ in $H_{n-1}(P)$.

(b) Result is independent of the choice of representative for $[r]$:
Suppose $r' \in Z_n(R)$ s.t. $[r'] = [r]$.
$r - r' = dr''$ for some $r'' \in R_{n+1}$.
Find $q'' \in Q_{n+1}$ s.t. $gq'' = r''$.
$gdq'' = dgq'' = dr'' = r - r' = g(q) - r' \Rightarrow r' = g(q - dq'')$.
Set $q' := q - dq'' \in Q_n$.
$gq' = r'$ so we can use q' to compute $\partial[r']$.
$dq' = dq - d^2q'' = dq$ so the definition of $\partial[r']$ agrees with the definition of $\partial[r]$.

3. Sequence is exact at $H_{n-1}(P)$.

To show that the composition $H_n(R) \xrightarrow{\partial} H_{n-1}(P) \xrightarrow{f_*} H_{n-1}(Q)$ is trivial:
Let $[r] \in H_n(R)$. Find $q \in Q_n$ s.t. $gq = r$.
Then $\partial[r] = [p]$ where $fp = dq$.
So $f_*\partial[r] = [fp] = [dq] = 0$ since $dq \in B_{n-1}(Q)$.
Hence Im $\partial \subset$ Ker f_*.
Conversely let $[p] \in$ Ker f_*.
Since $[fp] = 0$, $fp = dq$ for some $q \in Q_n$.
Let $r = gq$. Then $\partial[r] = [p]$.
So Ker $f_* \subset$ Im ∂.

The proof of exactness at the other places is left as an exercise. $\square$

Definition 13.0.20 *Let $f, g : C \to D$ be chain maps.*

A collection of maps $s_p : C_p \to D_{p+1}$ is called a chain homotopy *from f to g if the relation $ds + sd = f - g : C_p \to D_p$ is satisfied for each p. If there exists a chain homotopy from f to g, then f and g are called* chain homotopic.

Proposition 13.0.21 *Chain homotopy is an equivalence relation.*

Proof: Exercise $\square$

Proposition 13.0.22 $f \simeq f'$, $g \simeq g' \Rightarrow gf \simeq g'f'$.

Proof: $\underline{C} \overset{f}{\underset{f'}{\rightrightarrows}} \underline{D} \overset{g}{\underset{g'}{\rightrightarrows}} \underline{E}$

Show $gf \simeq gf'$:

Let $s : f \simeq f'$. $s : C_p \to D_{p+1}$ s.t. $ds + sd = f' - f$.

$g \circ s : C_p \to E_{p+1}$ satisfies $dgs + gsd = gds + gsd = g(ds + sd) = g(f' - g) = gf' - gf$.

Similarly $g'f \simeq g'f'$. □

Definition 13.0.23 *A map $f : C \to D$ is a* chain (homotopy) equivalence *if $\exists g : D \to C$ s.t. $gf \simeq 1_C$, $fg \simeq 1_D$.*

Proposition 13.0.24 $f \simeq g \Rightarrow f_* = g_* : H_*(C) \to H_*(D)$.

Proof: Let $[x] \in H_p(C)$ be represented by $x \in Z_p(C)$. Let $s : f \simeq g$.

Then $fx - gx = sdx + dsx = dsx \in B_p(C)$. So $[fx] = [gx] \in H_p(D)$. □

Corollary 13.0.25 *$f : C \to D$ is a chain equivalence $\Rightarrow f_* : H_*(C) \to H_*(D)$ is an isomorphism.* □

Proposition 13.0.26 (Algebraic Mayer-Vietoris) *Let*

$$\begin{tikzcd}
\arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\alpha"] & B_n \arrow[r, "j"] \arrow[d, "\beta"] & C_n \arrow[r, "\partial"] \arrow[d, "\gamma"] & A_{n-1} \arrow[r, "i"] \arrow[d, "\alpha"] & B_{n-1} \arrow[r, "j"] \arrow[d, "\beta"] & C_{n-1} \arrow[r] \arrow[d, "\gamma"] & {} \\
\arrow[r] & A'_n \arrow[r, "i'"] & B'_n \arrow[r, "j'"] & C'_n \arrow[r, "\partial"] & A'_{n-1} \arrow[r, "i'"] & B'_{n-1} \arrow[r, "j'"] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

be a commutative diagram with exact rows. Suppose $\gamma : C_n \to C'_n$ is an isomorphism $\forall n$. Then there is an induced long exact sequence

$$\ldots \longrightarrow A_n \xrightarrow{\rho} B_n \oplus A'_n \xrightarrow{q} B'_n \xrightarrow{\Delta} A_{n-1} \longrightarrow B_{n-1} \oplus A'_{n-1} \longrightarrow B'_{n-1}$$

where

$\rho(a) = (ia, \alpha a)$

$q(b, a') = \beta b - i'a'$

$\Delta = \partial\gamma^{-1}j'$

Proof: Exercise □

Chapter 14

Homology

14.1 Eilenberg-Steenrod Homology Axioms

Historically:

1. Simplical homology was defined for simplicial complexes.
2. It was proved that the homology groups of a simplicial complex depend only on its geometric realization, not upon the actual triangulation.
3. Various other "homology theories" were defined on various subcategories of topological spaces. (e.g. singular homology, de Rham (co)homology, Čech homology, cellular homology,...) The subcollection of spaces on which each was defined was different, but they had similar properties, were all defined for polyhedra (i.e. realizations of finite simplicial complexes) and furthermore gave the same groups $H_*(X)$ for a polyhedron X.
4. Eilenberg and Steenrod formally defined the concept of a "homology theory" by giving a set of axioms which a homology theory should satisfy. They proved that if X is a polyhedron then any theory satisfying the axioms gives the same groups for $H_*(X)$.

Definition 14.1.1 (Eilenberg-Steenrod) *Let $\mathcal{A}$ be a class of topological pairs such that:*

1) (X, A) in $\mathcal{A} \Rightarrow (X, X)$, $(X, \emptyset)$, (A, A), $(A, \emptyset)$, and $(X \times I, A \times I)$ are in $\mathcal{A}$;

2) $(*, \emptyset)$ is in $\mathcal{A}$ (where $*$ denotes a space with one point).

A homology theory *on $\mathcal{A}$ consists of:*

E1) an abelian group $H_n(X, A)$ for each pair (X, A) in $\mathcal{A}$ and each integer n;

E2) *a homomorphism* $f_* : H_n(X,A) \to H_n(Y,B)$ *for each map of pairs* $f : (X,A) \to (Y,B)$;

E3) *a homomorphism* $\partial : H_n(X,A) \to H_{n-1}(A)$ *for each integer* n *(where* $H_n(A)$ *is an abbreviation for* $H_n(A,\emptyset)$ *),*

such that:

A1) $1_* = 1$;

A2) $(gf)_* = g_* f_*$;

A3) ∂ *is natural. That is, given* $f : (X,A) \to (Y,B)$, *the diagram*

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_n(Y,B) \arrow[d, "\partial"] \\ H_{n-1}(A) \arrow[r, "(f|_A)_*"] & H_{n-1}(B) \end{tikzcd}$$

commutes;

A4) *Exactness:*

$$\ldots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \longrightarrow \ldots$$

is exact for every pair (X,A) *in* $\mathcal{A}$, *where* $H_*(A) \to H_*(X)$ *and* $H_*(X) \to H_*(X,A)$ *are induced by the inclusion maps* $(A,\emptyset) \to (X,\emptyset)$ *and* $(X,\emptyset) \to (X,A)$;

A5) *Homotopy:* $f \simeq g \Rightarrow f_* = g_*$.

A6) *Excision: If* (X,A) *is in* $\mathcal{A}$ *and* U *is an open subset of* X *such that* $\overline{U} \subset \mathring{A}$ *and* $(X \setminus U, A \setminus U)$ *is in* $\mathcal{A}$ *then the inclusion map* $(X \setminus U, A \setminus U) \to (X,A)$ *induces an isomorphism* $H_n(X \setminus U, A \setminus U) \xrightarrow{\cong} H_n(X,A)$ *for all* n;

A7) *Dimension:* $H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases}$

Many homology theories also satisfy the following "Compactness Axiom".

A8) For each $\alpha \in H_n(X, A)$ there exists a pair of compact subspaces (X_0, A_0) in $\mathcal{A}$ such that $\alpha \in \operatorname{Im} j_*$, where $j : (X_0, A_0) \to (X, A)$ is the inclusion map.

Remark 14.1.2

1. *Some people include the 8th axiom (which is not on Eilenberg-Steenrod's list) in their definition, but many people would call anything satisfying the 1st 7 axioms a homology theory.*

2. *A1 and A2 simply say that $H_n(\ )$ is a functor for each n.*

Remark 14.1.3 *Under the presence of the other axiom, the excision is equivalent to the Mayer-Vietoris property, stated below as Theorem 14.2.34 and to the Suspension property, stated below as Theorem 15.0.41.*

14.2 Singular Homology Theory

Definition 14.2.1 *A set of points $\{a_0, a_1, \ldots, a_n\} \in \mathbb{R}^N$ is called* geometrically independent *if the set*

$$\{a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0\}$$

is linearly independent.

Proposition 14.2.2 *$a_0, \ldots, a_n$ geometrically independent if and only if the following statement holds: $\sum_{t=0}^{n} t_i a_i = 0$ and $\sum_{t=0}^{n} t_i = 0$ implies $t_i = 0$ for all i.*

Proof: Exercise $\square$

Definition 14.2.3 *Let $\{a_0, \ldots, a_n\}$ be geometrically independent. The* n-simplex *σ spanned by $\{a_0, \ldots, a_n\}$ is the convex hull of $\{a_0, \ldots, a_n\}$. Explicitly*
$\sigma = \{x \in \mathbb{R}^n \mid x = \sum_{i=0}^{n} t_i a_i$ *where* $t_i \geq 0$ *and* $\sum t_i = 1\}$.

For a given n-simplex σ, each $x \in \sigma$ has a unique expression $x = \sum_{i=0}^{n} t_i a_i$ with $t_i \geq 0$ and $\sum t_i = 1$. The t_i's are called the *barycentric coordinates* of x (with respect to $a_0, \ldots, a_n$). The *barycentre* of the n-simplex is the point all of whose barycentric coordinates are $1/(n+1)$.

$a_0, \ldots, a_n$ are called the *vertices* of σ.

n is called the *dimension* of σ.

Any simplex formed by a subset of $\{a_0, \ldots, a_n\}$ is called a *face* of σ.

Special case:

$a_0 = \epsilon_0 := (0, 0, \ldots, 0)$, $a_1 = \epsilon_1 := (1, 0, \ldots, 0)$, $a_2 = \epsilon_2 := (0, 1, 0, \ldots, 0)$, $a_n = \epsilon_n := (0, 0, \ldots, 0, 1)$ in $\mathbb{R}^n$ gives what is known as the *standard n-simplex*, denoted Δ^n.

Definition 14.2.4 *Suppose $A \subset \mathbb{R}^m$ is convex. A function $f : A \to \mathbb{R}^k$ is called* affine *if $f\big(ta + (1-t)b\big) = tf(a) + (1-t)f(b)\ \forall a, b \in \mathbb{R}^m$ and $0 \le t \le 1 \in \mathbb{R}$.*

Let σ be an n-simplex with vertices $v_0, \ldots, v_n$. Given $(n+1)$ points $p_0, \ldots, p_n$ in $\mathbb{R}^k$, $\exists!$ affine map f taking v_j to p_j.
Note: $p_0, \ldots, p_n$ need not be geometrically independent.

Notation: Given $a_0, \ldots, a_n \in \mathbb{R}^N$, let $l(a_0, \ldots, a_n)$ denote the unique affine map taking e_j to a_j.
Explictly, $l(a_0, \ldots, a_n)(x_1, \ldots, x_n) = a_0 + \sum_{i=1}^{n}(a_i - a_0)x_i$

Note: $l(\epsilon_0, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_n)$ is the inclusion of the (ith face of Δ^n) into Δ^n.

Definition 14.2.5 *Given a topological space X, a continuous function $f : \Delta^p \to X$ is called a* singular p-simplex *of X.*

Let $S_p(X) :=$ free abelian group on $\{$singular p-simplices of $X\}$.
Wish to define a boundary map making $S_p(X)$ into a chain complex.
Given a singular p-simplex T, can define $(p-1)$-simplices by the compositions

$$\Delta^{p-1} \xrightarrow{l(\epsilon_0, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_n)} \Delta^p \xrightarrow{T} X.$$

A homomorphism from a free group is uniquely determined by its effect on generators.
Define homomorphism
$\partial : S_p(X) \to S_{p-1}(X)$ by $\partial(T) := \sum_{i=0}^{p}(-1)^i T \circ l(\epsilon_0, \ldots, \hat{\epsilon_i}, \ldots, \epsilon_n)$.

Given $g : X \to Y$, define homomorphism $g_* : S_p(X) \to S_p(Y)$ by defining it on generators by $g_*(T) := g \circ T$. $\quad \Delta^p \xrightarrow{T} X \xrightarrow{g} Y$

Lemma 14.2.6 $g_* \partial = \partial g_*$

(Thus after we show $S_p(X)$, $S_p(Y)$ are chain complexes, we will know that g_* is a chain map.)

Proof: Sufficient to check $g_*\partial(T) = \partial g_*(T)\ \forall T$. (Exercise: Essentially, left multiplication commutes with right multiplication.) $\square$

Lemma 14.2.7 *$S_*(X)$ is a chain complex. (i.e. $\partial^2 = 0$)*

Proof:
Special Case: $X = \sigma$ spanned by $a_0, \ldots, a_p$ and $T = l(a_0, \ldots, a_p)$.

Then

$$\begin{aligned}\partial T &= \partial l(a_0, \ldots, a_p)\\ &= \textstyle\sum_{j=0}^{p}(-1)^j l(a_0, \ldots, a_p) \circ l(\epsilon_0, \ldots, \hat{\epsilon_j}, \ldots, \epsilon_p)\\ &= \textstyle\sum_{j=0}^{p}(-1)^j l(a_0, \ldots, \hat{a}_j, \ldots, a_p)\end{aligned}$$

Therefore

$$\begin{aligned}\partial^2 T &= \textstyle\sum_{j=0}^{p}(-1)^j \partial l(a_0, \ldots, \hat{a}_j, \ldots a_p)\\ &= \textstyle\sum_{j=0}^{p}(-1)^j \Big(\sum_{i<j}(-1)^i l(a_0, \ldots, \hat{a}_i \ldots, \hat{a}_j, \ldots, a_p)\\ &\qquad\qquad + \textstyle\sum_{i>j}(-1)^{i-1} l(a_0, \ldots, \hat{a}_i \ldots, \hat{a}_j, \ldots, a_p)\Big)\\ &\quad \text{(Note: removal of } a_j \text{ moves } a_i \text{ to } (i-1)st \text{ position)}\\ &= 0\end{aligned}$$

since each term appears twice (once with $i < j$ and once with $j < i$) with opposite signs so they cancel.

General Case: $f : \Delta^p \to X$. Let $I = 1_{\Delta^p} = l(\epsilon_0, \ldots, \epsilon_p) \in S_p(\Delta^p)$. Then $f = f_*(I) \in S_p(X)$. So $\partial^2 f = f_*(\partial^2 I) \xlongequal{\text{(special case)}} f_*(0) = 0$. □

Corollary 14.2.8 (Corollary of previous Lemma)
$g : X \to Y$ implies $g_* : S_*(X) \to S_*(Y)$ is a chain map. □

Definition 14.2.9 *$H_*\big(S_*(X), \partial\big)$ is denoted $H_*(X)$ and called the* singular homology *of the space X.*

Proposition 14.2.10 *Singular homology is a functor from the category of topological spaces to the category of abelian groups.*

Proof: Requirements are $1_* = 1$ and $(gf)_* = g_* f_*$. Both are trivial. □

Corollary 14.2.11 *If $f : X \to Y$ is a homeomorphism then f_* is an isomorphism.* □

Let A be a subspace of X with inclusion map $j : A \hookrightarrow X$. Then $j_* : S_*(A) \to S_*(X)$ is an inclusion ($S_*(X)$ is the free abelian group on a larger set — in general strictly larger since not all functions into X factor through A) so can form the quotient complex $S_*(X)/S_*(A)$ (strictly speaking the denominator is $j_*\big(S_*(A)\big)$).

Definition 14.2.12 *$H_*\big(S_*(X)/S_*(A)\big)$ is written $H_*(X, A)$ and is called the relative homology of the pair (X, A).*

Notice, if $A = \emptyset$ then $S_*(A) =$ Free-Abelian-Group$(\emptyset) = 0$ so $H_*(X, \emptyset) = H_*(X)$.

14.2.1 Verification that Singular Homology is a Homology Theory

A pair (X, A) gives rise to a short exact sequence of chain complexes:

$$0 \to S_*(A) \to S_*(X) \to S_*(X)/S_*(A) \to 0$$

in such a way that a map of pairs $(X, A) \to (Y, B)$ gives a commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r] \arrow[d] & S_*(X) \arrow[r] \arrow[d] & S_*(X)/S_*(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(B) \arrow[r] & S_*(Y) \arrow[r] & S_*(Y)/S_*(B) \arrow[r] & 0
\end{tikzcd}$$

It follows from the homological algebra section that there are induced long exact homology sequences

$$\begin{tikzcd}
\ldots \arrow[r, "\partial"] & H_p(A) \arrow[r] \arrow[d] & H_p(X) \arrow[r] \arrow[d] & H_p(X,A) \arrow[r, "\partial"] \arrow[d] & H_{p-1}(A) \arrow[r] \arrow[d] & H_{p-1}(X) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r, "\partial"] & H_p(A) \arrow[r] & H_p(X) \arrow[r] & H_p(X,A) \arrow[r, "\partial"] & H_{p-1}(A) \arrow[r] & H_{p-1}(X) \arrow[r] & \ldots
\end{tikzcd}$$

making the squares commute.

This in the definition of a homology theory we immediately have the following: E1, E2, E3, A1, A2, A3, A4.

Proposition 14.2.13 *A7 is satisfied.*

Proof: By definition, if $p \geq 0$,

$S_p(*) = \text{Free-Abelian-Group}\big(\{\text{maps from } \Delta^p \text{ to } *\}\big) = \mathbb{Z}$,

generated by T_p where T_p is the unique continuous map from Δ^p to $*$.

$\partial T_p = \sum_{i=0}^{p} (-1)^i T_p \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)$.

For $p > 0$, $T_p \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p) = T_{p-1}\ \forall i$, so $\partial T_p = \begin{cases} T_{p-1} & p \text{ even;} \\ 0 & p \text{ odd.} \end{cases}$ □

Proposition 14.2.14 *A8 is satisfied.*

Proof: Let $\alpha \in H_p(X, A)$. So α is represented by a cycle of $S_p(X)/S_p(A)$ for which we choose a representative $c = \sum_{i-1}^{k} n_i T_i \in S_p(X)$. Thus $\partial c = \sum_{i=1}^{r} m_i V_i \in S_p(A)$.

Let $X_0 = \left(\cup_{i=1}^{k} \operatorname{Im} T_i\right) \cup \left(\cup_{i=1}^{r} \operatorname{Im} V_i\right)$ and let $A_0 = \left(\cup_{i=1}^{r} \operatorname{Im} V_i\right)$.

Since $T_i : \Delta^p \to X$ and $V_i : \Delta^{p-1} \to A \hookrightarrow X$, each of X_0 and A_0 are a finite union of compact sets and thus compact. It is immediate from the definitions that $\alpha \in \operatorname{Im} j_* : H_*(X_0, A_0) \to H_*(X, A)$ where $j : (X_0, A_0 \hookrightarrow (X, A)$ is the inclusion map, since the chain representing α exists back in $S_*(X_0)/S_*(A_0)$. $\square$

Theorem 14.2.15 *$H_0(X) \cong F_{\text{ab}}(\{\text{path components of } X\})$.*

Proof: $S_0(X) = F_{\text{ab}}(\{\text{singular 0-simplices of } X\})$.

$S_1(X)$ is generated by maps $f : I = \Delta^1 \to X$.

$\partial f = f(1) - f(0)$. Hence $\operatorname{Im} \partial = \{f(1) - f(0) \mid f : I \to X\}$.

Therefore

$$\begin{aligned} H_0(X) &= \ker \partial_0 / \operatorname{Im} \partial_0 = S_0(X)/\operatorname{Im} \partial_1 \\ &= F_{\text{ab}}(\text{points of } X)/\sim \qquad \text{where } f(1) - f(0) \sim 0 \ \forall f : I \to X \\ &\cong F_{\text{ab}}(\{\text{path components of } X\}). \end{aligned}$$

$\square$

14.2.2 Reduced Singular Homology

Define the "augmentation map" $\epsilon : S_0(X) \to \mathbb{Z}$ by $\epsilon(\sum_{i \in I} n_i x_i) = \sum_{i \in I} n_i$.

If f is a generator of $S_1(X)$ with $f(0) = x$ and $f(1) = y$ then $\partial f = y - x$ so $\epsilon \partial f = 0$.

$$\begin{tikzcd} {} \arrow[r] & S_p(X) \arrow[r, "\partial"] \arrow[d] & S_{p-1}(X) \arrow[r] \arrow[d] & \ldots \arrow[r] & S_1(X) \arrow[r, "\partial"] \arrow[d] & S_0(X) \arrow[r] \arrow[d, "\epsilon"] & 0 \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & {} \end{tikzcd}$$

commutes.

The chain complex formed by taking termwise kernels of this chain map is denoted $\tilde{S}_*(X)$ and its homology, denote $\tilde{H}_*(X)$, is called the *reduced homology* of X.

The short exact sequence of chain complexes defining $\tilde{S}_*(X)$ yields a long exact sequence

$$0 \to \tilde{H}_p(X) \to H_p(X) \to 0 \to \ldots \to 0 \to \tilde{H}_1(X) \to H_1(X) \to 0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Therefore $H_n(X) \cong \begin{cases} \tilde{H}_n(X) & n > 0; \\ \tilde{H}_0(X) \oplus \mathbb{Z} & n = 0. \end{cases}$

Consider the special case $X = *$.

$$\begin{tikzcd} S_*(*) & \to \mathbb{Z} \arrow[r, "\cong"] \arrow[d] & Z \to \ldots \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r, "\cong"] \arrow[d] & \mathbb{Z} \to 0 \arrow[d, "\epsilon"] \\ & \to 0 \arrow[r] & 0 \to \ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \to 0 \end{tikzcd}$$

In this case ϵ becomes the identity map so that $\epsilon_* : H_0(*) \to \mathbb{Z}$ is an isomorphism. (We already knew $H_*(X) \cong \mathbb{Z}$; just want to check that ϵ_* gives the isomorphism.)

Theorem 14.2.16 $\tilde{H}_*(X) \cong H_*(X, *)$.

Proof: We have a long exact sequence

$$\begin{tikzcd} 0 \arrow[d, equal] & & & 0 \arrow[d, equal] & & 0 \arrow[d, equal] & \\ H_n(*) \arrow[r] & H_n(X) \arrow[r] & H_n(X,*) \arrow[r] & H_{n-1}(*) \arrow[r] & \ldots \arrow[r] & H_1(*) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & H_1(X) \arrow[r] & H_1(X,*) \arrow[r, "\partial"] & H_0(*) \arrow[r, "i"] & H_0(X) \arrow[r] & H_n(X,*) \arrow[r] & 0 \end{tikzcd}$$

Let $f : X \to *$.

$$\begin{tikzcd} H_0(*) \arrow[r, "i"] \arrow[dr, "1"] & H_0(X) \arrow[d, "f_*"] \arrow[dr, "\epsilon_*"] & \\ & H_0(*) \arrow[r, "\epsilon_*"] & \mathbb{Z} \end{tikzcd}$$

Therefore $\epsilon_* i = \epsilon_*$ is an isomorphism so i is an injection. It follows algebraically that $\partial = 0$ and that the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_0(*) \arrow[r] \arrow[dr, "\cong"] & H_0(X) \arrow[r] \arrow[d, "\epsilon"] & H_0(X,*) \arrow[r] & 0 \\ & & \mathbb{Z} & & \end{tikzcd}$$

splits and $\tilde{H}_0(X) = \ker \epsilon \cong \operatorname{coker} i \cong H_0(X*)$. $\square$

Theorem 14.2.17 *Let $X \subset \mathbb{R}^N$ be convex. Then $\tilde{H}_*(X) = 0$.*

Proof: Let $w \in X$ be any point. Define a homomorphism $S_p(X) \to S_{p-1}(X)$ by defining it on generators as follows.

Let $T : \Delta^p \to X$ be a generator of $S_p(X)$.

To define $\phi(T) \in S_{p+1}(X)$: Let $\phi(T) : \Delta^{p+1} \to X$ be the generator of $S_{p+1}(X)$ defined as follows: Given $y \in \Delta^{p+1}$ we can write $y = t\epsilon_p + (1-t)z$ for some $z \in \Delta^p$, $t \in [0,1]$ (where $\epsilon_p = (0, \ldots, 0, 1)$). Let $\phi(T)(y) = tw + (1-t)T(z)$.

Lemma 14.2.18 *Let $c \in S_p(X)$. Then* $\partial\big(\phi(c)\big) = \begin{cases} \phi(\partial c) + (-1)^{p+1}c & p > 0 \\ \epsilon(c)T_w - c & p = 0 \end{cases}$
where $T_w : \Delta^0 \to X$ by $T_x(*) = w$.

Proof: It suffices to check this when c is a generator. Let $T : \Delta^p \to X$ be a generator of $S_p(X)$.

If $p = 0$:

$\phi(T)$ is a line joining $T(*)$ to w so $\partial\big(\phi(T)\big) = T_w - T = \epsilon(T)T_w - T$ as required.

If $p > 0$:

$\partial\big(\phi(T)\big) = \sum_{i=0}^{p+1}(-1)^i\phi(T) \circ l_i$ where l_i is short for $l(\epsilon_0, \ldots \hat{\epsilon}_i, \ldots, \epsilon_p)$.

If $i = p+1$, l_i is the inclusion of Δ^p into Δ^{p+1} so $\phi(T) \circ l_p = \phi \circ T\big|_{\Delta^p} = T$.

If $i \leq p$, $\phi(T) \circ l_i = \phi\big(T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\big)$, extended by sending the last vertex to w.

Therefore

$$\begin{aligned} \partial\big(\phi(T)\big) &= \textstyle\sum_{i=0}^{p}(-1)^i\phi\big(T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\big) + (-1)^{p+1}T \\ &= \phi\big(\textstyle\sum_{i=0}^{p}(-1)^i T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\big) + (-1)^{p+1}T \\ &= \phi(\partial T) + (-1)^{p+1}T \end{aligned}$$

Proof of Theorem (cont.)

$p = 0$:

Suppose $c \in \tilde{S}_0(X)$. So $\epsilon(c) = 0$.

$\partial\big(\phi(c)\big) = 0 - c$ so $[c] = 0 \in \tilde{H}_0(X)$.

$p > 0$:

Let $c \in Z_p(X)$.

$\partial\big(\phi(c)\big) = \phi(\partial c) + (-1)^{p+1}c = \phi(0) + (-1)^{p+1}c = (-1)^{p+1}c$.

Therefore $[c] = 0$ in $H_p(X) = \tilde{H}_p(X)$. $\square$

Corollary 14.2.19 $\tilde{H}_p(\Delta^n) = 0\ \forall p.$ $\square$

14.2.3 Proof that A5 is satisfied: Acyclic Models

Let $f, g : X \to Y$ s.t. $f \overset{H}{\simeq} g$.

$X \overset{i}{\underset{j}{\rightrightarrows}} X \times I \xrightarrow{H} Y$ where $i(x) = (x, 0)$, $j(x) = (x, 1)$.

Then $H \circ = f$ and $H \circ j = g$. Therefore $f_* = H_* \circ i_*$ and $g_* = H_* \circ j_*$. Show to show $f_* = g_*$ it suffices to show that $i_* = j_*$.

We show this by showing that at the chain level $i_* \simeq j_* : S_*(X) \to S_*(X \times I)$.

We will show that $i_* \simeq j_*$ by "acyclic models".

Intuitively, acyclic models is a method of inductively constructing chain homotopies which makes use of the fact that in an acyclic space equations of the form $\partial x = y$ can always be "solved" for x provided $\partial y = 0$. (In general there will be many choices for the solution x.) The method does not give an explicit formula for the chain homotopy but merely proves that one exists. In fact, the final result is non-canonical and depends upon the choices of the solutions. In the case of chain homotopy $i_* \simeq j_*$ which we are considering at present, it would be possible to directly write down a chain homotopy and check that it works without using acyclic models. However we will need the method in other places where it would not be so easy to simply write down the formula so we introduce it here.

The acyclic spaces ("models") used in this particular application of the method are the spaces Δ^n. Intuitively we make used of the fact that equations can be solved in Δ^n to solve the same equations in $S_*(X)$ using that elements in $S_*(X)$ are formed from maps $\Delta^n \to X$.

Lemma 14.2.20 *$\exists$ a natural chain homotopy $D_X : i \simeq j : S_*(X) \to S_*(X \times I)$.*

In more detail:

1. *$\forall x$ and $\forall p$, $\exists D_X : S_p(X) \to S_{p+1}(X \times I)$ s.t. $\forall c \in S_p(X)$,*
   $\partial D_X c + D_X \partial c = j_*(c) - i_*(c)$.
2. *$\forall f : X \to Y$,*

$$\begin{array}{ccc} S_p(X) & \xrightarrow{D_X} & S_{p+1}(X \times I) \\ \Big\downarrow f_* & & \Big\downarrow (f \times 1)_* \\ S_p(Y) & \xrightarrow{D_Y} & S_{p+1}(Y \times I) \end{array}$$

commutes.

Proof: Since $S_p(X)$ is a free abelian group it suffices to define D_X on generators and check its properties on them.

If $p < 0$, $S_p(X) = 0$ so $D_X = 0$-map.

Continue constructing D_X inductively. The induction assumptions are for <u>all</u> spaces. More precisely:

Induction Hypothesis: $\exists$ integer p such that for all $k < p$ and $\forall X$ we have constructed homomorphisms $D_X : S_k(X) \to S_{k+1}(X \times I)$ s.t. $\forall c \in S_k(C)$

1. $\forall x$ and $\forall p$, $\exists D_X : S_p(X) \to S_{p+1}(X \times I)$ s.t. $\forall c \in S_p(X)$,
   $\partial D_X c + D_X \partial c = j_X * (c) - i_X * (c)$.
2. $\forall f : X \to Y$,

$$\begin{array}{ccc} S_k(X) & \xrightarrow{D_X} & S_{k+1}(X \times I) \\ \Big\downarrow f_* & & \Big\downarrow (f \times 1)_* \\ S_k(Y) & \xrightarrow{D_Y} & S_{k+1}(Y \times I) \end{array}$$

   commutes.

(We have this initially for $p = 0$.)

To construct $D_X : S_p(X) \to S_{p+1}(X \times I)$ for any X, consider first the special case ("model case"):

Let $X = \Delta^p$ and let $\iota_p = 1_{\Delta^p} \in S_p(\Delta^p)$.

$i, j : \Delta^p \to \Delta^p \times I$.

Want to define $D_{\Delta^p}(\iota_p)$ so that $\partial D_{\Delta^p}(\iota_p) = j_*(\iota_p) - i_*(\iota_p) - D_{\Delta^p}(\partial \iota_p)$.

That is, solve the equation $\partial x = j_*(\iota_p) - i_*(\iota_p) - D_{\Delta^p}(\partial \iota_p)$ for x and set $D_{\Delta^p(\iota_p)} :=$ solution.

Since $\Delta^p \times I$ is acyclic, solving the equation is equivalent (except when $p = 0$: see below) to checking $\partial(RHS) = 0$.

$$\begin{aligned}
\partial(\text{RHS}) &= \partial j_*(\iota_p) - i_*(\iota_p) - D_{\Delta^p}(\partial \iota_p) \\
&\overset{\text{(chain maps)}}{=\!=\!=\!=\!=} j_*(\partial \iota_p) - i_*(\partial \iota_p) - \partial D_{\Delta^p}(\partial \iota_p) \\
&\overset{\text{(induction)}}{=\!=\!=\!=\!=} \partial J_*(\iota_p) - \partial i_*(\iota_p) - (j_* \partial \iota_p - i_* i_* \partial \iota_p - D_{\Delta^p} \partial \partial \iota_p) \\
&= 0.
\end{aligned}$$

Hence $\exists$ solution. Choose any solution and define $D_{\Delta^p}(*\iota_p) =$ solution.

Must do the case $p=0$ separately, since $H_0(\Delta^0 \times I) \neq 0$. For the generator $1_{\Delta^0} : \Delta^0 = * \to *$, set $D_{\Delta^0}(x) := 1_I \in S_1(I = \Delta^0 \times I) = \mathrm{Hom}(\Delta^1, I) = \mathrm{Hom}(I, I)$. Then $\partial D_{\Delta^0}(x) := \partial 1_I = j_*(*) - i_*(*)$ as desired.

Note: We could have avoided doing $p=0$ separately by writing our argument using reduced homology.

Now to define $S_p(X) \to S_{p+1}(X)$ in general:

Let $T : \Delta^p \to X$ be a generator of $S_p(X)$. Define $D_X(T)$ in the only possible such that (2) is satisfied. That is, want

$$\begin{tikzcd} S_p(\Delta^p) \arrow[r, "D_{\Delta^p}"] \arrow[d, "T_*"] & S_{p+1}(\Delta^p \times I) \arrow[d, "(T\times 1)_*"] \\ S_p(X) \arrow[r, "D_X"] & S_{p+1}(X \times I) \end{tikzcd}$$

Observe that $T = T_*(\iota_p) \in S_p(X)$ so we are forced to define $D_X(T)$ by $D_X(T) := (T \times 1)_* D_{\Delta^p} \iota_0$.

Check that this works:

$$\begin{tikzcd} \Delta^p \arrow[r, "\iota_p"] \arrow[dr, "T"'] & \Delta^p \arrow[r, "j_{\Delta^p}"] \arrow[d, "T"] & \Delta^p \times I \arrow[d, "T \times 1"] \\ & X \arrow[r, "j"] & X \times I \end{tikzcd} \qquad \begin{tikzcd} \Delta^p \arrow[r, "\iota_p"] \arrow[dr, "T"'] & \Delta^p \arrow[r, "i_{\Delta^p}"] \arrow[d, "T"] & \Delta^p \times I \arrow[d, "T \times 1"] \\ & X \arrow[r, "i"] & X \times I \end{tikzcd}$$

$$\begin{aligned} \partial D_X T &= \partial (T\times 1)_* D_{\Delta^p}\iota_p \\ &= (T \times 1)_* \partial D_{\Delta^p} \iota_p \\ &= (T\times 1)_*(j_*\iota_p - i_*\iota_p - D_{\Delta^p}\partial\iota_p) \\ &= (T \times 1 \circ j)_*\iota_p - (T\times 1 \circ i)_*\iota_p - (T\times 1)_* D_{\Delta^p}\partial\iota_p \\ &= j_*(T) - i_*(T) - (T\times 1)_* D_{\Delta^p}\partial\iota_p \\ &\overset{\text{((2) of induction hypothesis)}}{=\!=\!=} j_*(T) - i_*(T) - D_X T_*(\partial \iota_p) \\ &\overset{(T_* \text{ is a chain map})}{=\!=\!=} j_*(T) - i_*(T) - D_X \partial T_* \iota_p \\ &= j_*(T) - i_*(T) - D_X \partial T \end{aligned}$$

Also, if $f : X \to Y$ then

$$(f\times 1)_* D_X(T) \overset{\text{(defn)}}{=} (f\times 1)_*(T\times 1)_* D_{\Delta^p}\iota_p = \big((f\circ T)\times 1\big)_* D_{\Delta^p}\iota_p \overset{\text{(defn)}}{=} D_Y(f\circ T) = D_Y(f_*T).$$

This competes the induction step and proves the lemma. $\square$

Theorem 14.2.21 *Singular homology satisfies A5.*

Proof: Let $f, g : (X, A) \to (Y, B)$ s.t. $f \simeq g$.

Then $\exists F : X \times I \to Y$ s.t. $F : f \simeq g$ and $F\big|_{A\times I} : f\big|_A \to g\big|_A$. That is, $(X, A) \underset{j}{\overset{i}{\rightrightarrows}} (X \times I, A \times I) \xrightarrow{F} (Y, B)$ where $i(x) = (x, 0)$, $j(x) = (x, 1)$, $F \circ i = f$, $F \circ j = g$. Therefore, to show $f_* = g_*$ it suffices to show $i_* = j_*$.

By (2) of the lemma, the restriction of D_X to A equals D_A. (since the diagram commutes and $S_*(A) \to S_*(X)$ is a monomorphism. Thus there is an induced homomorphism on the relative chain groups:

$$\begin{tikzcd}
0 \arrow[r] & S_p(A) \arrow[r] \arrow[d, "D_A"] & S_p(X) \arrow[r] \arrow[d, "D_X"] & S_p(X, A) \arrow[r] \arrow[d, dashed, "D_{X,A}"] & 0 \\
0 \arrow[r] & S_{p+1}(A) \arrow[r] & S_{p+1}(X) \arrow[r] & S_{p+1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

with $D_{X,A}$ a chain homotopy between i_* and j_*. Hence $i_* = j_*$ and so $f_* = g_*$. $\square$

14.2.4 Barycentric Subdivision

(to prepare for excision:)

Definition 14.2.22 *Let σ be a (geometric) p-simplex spanned by $p + 1$ geometrically independent points $v_0,\ldots,v_p$. The* barycenter *of σ, denoted $\hat{\sigma}$ is defined by $\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{p+1} v_i$.*

(This is, the unique point all of whose barycentric coordinates are equal)

$\hat{\sigma}$ = centroid of σ.

Define the *barycentric subdivision* $\operatorname{sd}\sigma$ of a simplex as follows.

Join $\hat{\sigma}$ to the barycenter of each face of σ to get $\operatorname{sd}\sigma$. (This includes joining $\hat{\sigma}$ to each vertex since vertices are faces and are their own barycenters.)

$\operatorname{sd}\sigma$ writes σ as a union of p-simplices.

Can then perform barycentric subdivision on each of these to get $\operatorname{sd}^2\sigma$ and so on.
Notation: $\tau \prec \sigma$ shall mean: τ is a face of σ.

Lemma 14.2.23 *Every p-simplex of $\operatorname{sd}\sigma$ is spanned by vertices $\hat{\sigma_0}$, $\hat{\sigma_1}$, $\ldots$, $\hat{\sigma_p}$ where $\sigma_0 \prec \sigma_1 \prec \cdots \sigma_p$.*

Proof: By induction on $\dim \sigma$.

True if $\dim \sigma = 0$.

Observe: $\operatorname{sd} \sigma$ is formed by forming $\operatorname{sd}(\text{Boundary}\,\sigma)$ and then joining $\hat{\sigma}$ to each vertex in $\operatorname{sd}(\text{Boundary}\,\sigma)$. Thus, of the $(p+1)$ vertices spanning a simplex τ in $\operatorname{sd} \sigma$, p of them span a simplex τ' in Boundary σ and the last is $\hat{\sigma}$. By induction, τ' is spanned by $\hat{\sigma_0}$, $\hat{\sigma_1}$, ..., $\widehat{\sigma_{p-1}}$ where $\sigma_0 \prec \sigma_1 \prec \cdots \sigma_{p-1}$ and so τ has the desired form with $\sigma_p = \hat{\sigma}$.

Lemma 14.2.24 *Let σ be a p-simplex and let d be any metric on σ which gives it the standard topology. Then $\forall \epsilon > 0$, $\exists N$ s.t. the diameter or each simplex of $\operatorname{sd}^N \sigma$ is less than ϵ.*

Proof:

Step 0: If true for one metric than true for any metric.

Proof:

Let d_1, d_2 be metrics on σ each giving the correct topology. Then $1 : \sigma \to \sigma$ is a homeomorphism so continuous and thus uniformly continuous by compactness of σ. Therefore, given ϵ, $\exists \delta > 0$ s.t. any set with d_1-diameter less than δ has d_2-diameter less then ϵ. Thus if the theorem holds for d_1 then it holds for d_2 also.

For the rest of the proof use the metric on $\mathbb{R}$ given by $d(x, y) = \max_{i=1,\ldots,N} |x_i - y_i|$, which yields the same topology as the standard one. Notice that in this metric:

1. $d(x, y) = d(x - a, y - a)$
2. $d(0, nx) = nd(0, x)$
3. $d(0, x + y) \leq d(0, x) + d(x, x + y) = d(0, x) + d(0, y)$
4. For a p-simplex τ spanned by v_0, ..., v_p, $\operatorname{diam}(\tau) = \max\{d(v_i, v_j)\}$

Step 1: If $\dim \sigma = p$ then $\forall z \in \sigma$, $d(z, \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$.

Proof:

First consider the special case $z = v_0$.

$$
\begin{aligned}
d(v_0, \hat{\sigma}) &= d\left(v_0, \sum_{i=0}^{p} \frac{v_i}{p+1}\right) \\
&= d\left(0, \sum_{i=0}^{p} \frac{v_i - v_0}{p+1}\right) \\
&= \frac{1}{p+1}\, d\left(0, \sum_{i=0}^{p} (v_i - v_0)\right) \\
&= \frac{1}{p+1}\, d\left(0, \sum_{i=1}^{p} (v_i - v_0)\right) \\
&\leq \sum_{i=1}^{p} \frac{1}{p+1}\, d(0, v_i - v_0) \\
&= \sum_{i=1}^{p} \frac{1}{p+1}\, d(v_0, v_i) \\
&\leq \sum_{i=1}^{p} \frac{1}{p+1} \operatorname{diam} \sigma \\
&= \frac{p}{p+1} \operatorname{diam} \sigma.
\end{aligned}
$$

Similarly $d(v_j \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$ $\forall$ vertices of σ. Therefore the closed ball $B_{\frac{p}{p+1} \operatorname{diam} \sigma}[\hat{\sigma}]$ contains all vertices of σ so, being convex it contains all of σ. Hence $d(z, \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$ $\forall z \in \sigma$.

Step 2: For any simplex τ of $\operatorname{sd} \sigma$, $\operatorname{diam} \tau \leq \frac{p}{p+1} \operatorname{diam} \sigma$.

Proof: By induction on $p = \dim \sigma$.

Trivial if $p = 0$. Suppose true in dimensions less than p.

Write $\tau = \hat{\sigma_0} \ldots \hat{\sigma_p}$ where $\sigma_p = \sigma$.

Then $\operatorname{diam} \tau = \max\{d(\hat{\sigma_i}, \hat{\sigma_j})\}$. Suppose $i < j$.

If $j < p$ then by induction: $d(\hat{\sigma_i}, \hat{\sigma_j}) \leq \frac{j}{j+1} \operatorname{diam} \sigma_j \leq \frac{p}{p+1} \operatorname{diam} \sigma_j \leq \frac{p}{p+1} \operatorname{diam} \sigma$ since $j < p$ and $\sigma_j \subset \sigma$.

If $j = p$ then $d(\hat{\sigma_i}, \hat{\sigma_p}) = d(\hat{\sigma_i}, \hat{\sigma}) \leq \frac{p}{p+1} \operatorname{diam} \sigma$ by Step 1.

Hence $\operatorname{diam} \tau \leq \frac{p}{p+1} \operatorname{diam} \sigma$. □

Definition 14.2.25 *Let X be a topological space. Define the* barycentric subdivision operator*,* $\operatorname{sd}_X : S_p(X) \to S_p(X)$ *inductively as follows:*

$\operatorname{sd}_X : S_0(X) \to S_0(X)$ *is defined as the identity map.*

Suppose sd_X *defined in degrees less than p for all spaces.*

Recall: Given convex $Y \subset \mathbb{R}^N$ and $y \in Y$, in the proof of Theorem 14.2.17 we defined a homomorphism $S_q(Y) \to S_{q+1}(Y)$, which we will denote $T \mapsto [T, y]$, by

$$[T,y](v) := ty + (1-t)T(z)$$

where $v = t\epsilon_{p+1} + (1-t)z$ with $z \in \Delta^p$. Recall that $\partial[c,y] = \begin{cases} [\partial c, y] + (-1)^{q+1}c & q > 0; \\ \epsilon(c)T_y - c & q = 0, \end{cases}$
where $T_y : \Delta^0 \to Y$ by $T_y(*) = y$.

We will apply this with $Y = \Delta^p$, $y = \hat{\sigma} =$ barycenter of Δ^p.

To define $S_p(X) \xrightarrow{\mathrm{sd}_X} S_p(X)$, first consider $\iota_p :=$ identity map $: \Delta^p \to \Delta^p \in S_p(\Delta^p)$.

Define $\mathrm{sd}_{\Delta^p}\, \iota_p) := (-1)^p[\mathrm{sd}_{\Delta^p}(\partial\iota_p), \hat{\sigma}] \in S_{p+1}(\Delta^p)$.

Then given generator $T : \Delta^p \to X \in S_p(X)$ for arbitrary X, define
$\mathrm{sd}_X(T) := T_*\big(\mathrm{sd}_{\Delta^p}(\iota_p)\big) = (-1)^p\big[T_*\big(\mathrm{sd}_{\Delta^p}(\partial\iota_p)\big), T(\hat{\sigma})\big]$.

Letting SD denote geometric barycentric subdivision, by construction, $\mathrm{sd}_{\Delta^p}(\iota_p) = \sum \pm\sigma_i$ where $\mathrm{SD}(\Delta^p) = \cup_i \tau_i$ and $\sigma \in S_p(\Delta^p)$ is the affine map sending ϵ_j to $\hat{\tau}_j$ where $\hat{\tau}_0, \ldots, \hat{\tau}_p$ are the vertices of $\hat{\tau}_i$.

Lemma 14.2.26 sd_X *is a natural augmentation-preserving chain map.*

Note: Natural means
$$\begin{tikzcd} S_p(X) \arrow[r, "\mathrm{sd}_X"] \arrow[d, "f_*"] & S_p(Y) \arrow[d, "f_*"] \\ S_p(Y) \arrow[r, "\mathrm{sd}_Y"] & S_p(Y) \end{tikzcd}$$
commutes.

Proof:

Let $\epsilon : S_0(X) \to \mathbb{Z}$ be the augmentation. If $c \in S_0(X)$ then $\operatorname{sd}_X(c) = c$ so $\epsilon\big(\operatorname{sd}(c)\big) = \epsilon(c)$. Hence sd_X is augmentation preserving.

To show naturality:

$f_X \operatorname{sd}_X T = f_* T_* \operatorname{sd}_{\Delta^p} \iota_p = (f \circ T)_* \operatorname{sd}_{\Delta^p} \iota_p = \operatorname{sd}_Y (f \circ T)_* \iota_p = \operatorname{sd}_Y f_* T$.

We show that sd_X is a chain map by induction on p. Suppose we know, for all spaces, that $\partial \operatorname{sd}_X = \operatorname{sd}_X \partial$ in degrees less than p. Then in Δ^p we have

$$\begin{aligned}
\partial \operatorname{sd} \iota_p &= (-1)^p \partial[\operatorname{sd} \partial \iota_p, \hat{\sigma}] \\
&= \begin{cases} (-1)^p[\partial \operatorname{sd} \partial \iota_p, \hat{\sigma}] + (-1)^p(-1)^p \operatorname{sd} \partial \iota_p & p > 1 \\ -\epsilon(\operatorname{sd} \partial \iota_1) T_{\hat{\sigma}} + \operatorname{sd} \partial \iota_1 & p = 1 \end{cases} \\
&= \begin{cases} (-1)^p[\operatorname{sd} \partial\partial \iota_p, \hat{\sigma}] + \operatorname{sd} \partial \iota_p & p > 1 \\ -\epsilon \partial \iota_1 T_{\hat{\sigma}} + \operatorname{sd} \partial \iota_1 & p = 1 \end{cases} \\
&= \begin{cases} 0 + \operatorname{sd} \partial \iota_p & p > 1 \\ 0 + \operatorname{sd} \partial \iota_1 & p = 1 \end{cases} \\
&= \operatorname{sd} \partial \iota_p.
\end{aligned}$$

Now for arbitrary $T \in S_p(X)$,

$\partial \operatorname{sd} T = \partial T_*(\operatorname{sd} \iota_p) = T_*(\partial \operatorname{sd} \iota_p) \overset{\text{(naturality of sd)}}{=} \operatorname{sd} T_* \partial \iota_p = \operatorname{sd} \partial T_* \iota_p = \operatorname{sd} \partial T.$ □

Theorem 14.2.27 *Let $\mathcal{A}$ be a collection of subset of X whose interiors cover X. Let $T : \Delta^p \to X$ be a generator of $S_p(X)$. Then $\exists N$ s.t. $\operatorname{sd}^N T = \sum_i n_i T_i$ with $\operatorname{Im} T_i$ contained in some set in $\mathcal{A}$ for each i. (Need not be the same set of $\mathcal{A}$ for different i.)*

Proof: Since $\{\operatorname{Int} A\}_{A \in \mathcal{A}}$ covers X, $\{T^{-1}(\operatorname{Int} A)\}_{A \in \mathcal{A}}$ covers Δ^p which is compact. Let λ be a Lebesgue number for the covering $\{T^{-1}(\operatorname{Int} A)\}_{A \in \mathcal{A}}$ of Δ^p. Choose N s.t. for each simplex σ of $\operatorname{SD}^N \Delta^p$, $\operatorname{diam} \sigma < \lambda$ (where SD denotes geometric barycentric subdivision).

Thus writing $\operatorname{sd}^N \sigma = \sum n_i \sigma_i$, for each i $\exists A \in \mathcal{A}$ s.t. $\operatorname{Im} \sigma_i \subset T^{-1}(\operatorname{Int} A)$. (Each n_i is ± 1, but we don't need this.)

By naturality $\operatorname{sd}^N T = \sum n_i T(\sigma_i)$ and so $\forall i$ $\exists A \in \mathcal{A}$ s.t. $\operatorname{Im} T\sigma_i \subset A$ □

Theorem 14.2.28 *For each m, $\exists$ natural chain homotopy $D_X : 1 \simeq \operatorname{sd}^m : S_*(X) \to S_*(X)$.*

That is,

1. $\forall p$ $\exists D_X : S_p(X) \to S_{p+1}(X)$ *s.t.* $\partial D_X c + D_X \partial c = \operatorname{sd}^m c - c$ $\forall c \in S_p(X)$

2. Given $f : X \to Y$,

$$\begin{tikzcd} S_p(X) \arrow[r, "D_X"] \arrow[d, "f_*"] & S_{p+1}(X) \arrow[d, "f_*"] \\ S_p(Y) \arrow[r, "D_Y"] & S_{p+1}(Y) \end{tikzcd}$$ *commutes.*

Proof: By "acyclic models". i.e. D_X is defined on all spaces by induction on p.
For $p = 0$, define $D_X = 0 : S_*(X) \to S_1(X)$:

Since for $c \in S_0(X)$, $\operatorname{sd}^m(c) = c$, so $\partial D_X c + D_X \partial c = \partial 0 + D_X 0 = 0 = \operatorname{sd}^m c - c$ is satisfied.

Now suppose by induction that for all $k < p$ and for all spaces X, $D_X : S_k(X) \to S_{k+1}(X)$ has been defined satisfying (1) and (2) above.

Define $D_X T$ first in the special case $X = \Delta^p$, $T = \iota_p : \Delta^p \to \Delta^p \in S_p(\Delta^p)$.

To define $D_X \iota_p$ need to "solve" equation $\partial c = \operatorname{sd}^m \iota_p - \iota_p - D_{\Delta^p}(\partial \iota_p)$ for c and define $D_X \iota_p$ to be a solution.

Since Δ^p is acyclic, it suffices to check that $\partial(RHS) = 0$.

$\partial \operatorname{sd}^m \iota_p - \partial \iota_p - \partial D_{\Delta^p}(\partial \iota_p) = \partial \operatorname{sd}^m \iota_p - \partial \iota_p - \big(\operatorname{sd}^m \partial \iota_p - \partial \iota_p - D_{\Delta^p}(\partial\partial \iota_p)\big) = 0$. Therefore can define $D_X \iota_p$ s.t. (1) is satisfied.

Given $T : \Delta^p \to X \in S_p(X)$, define $D_X T := T_*\big(D_{\partial^p}(\iota_p)\big)$. Then

$$\begin{aligned} \partial D_X T &= \partial T_*(D_{\partial^p} \iota_p) \\ &= T_* \partial (D_{\partial^p} \iota_p) \\ &\overset{\text{(induction)}}{=} \operatorname{sd}^m T_* \iota_p - T_* \iota_p - D_{\Delta^p} T_* \partial \iota_p \\ &= \operatorname{sd}^m T - T - D_{\Delta^p} \partial T \iota_p \\ &= \operatorname{sd}^m T - T - D_{\Delta^p} \partial T \end{aligned}$$

Also $f_X D_X(T) = f_* T_*(D_{\Delta^p} \iota_p) = (f \circ T)_*(D_{\Delta^p} \iota_p) = D_Y(f \circ T) = D_Y f_*(T)$. $\square$

Let A be a subspace of X. Since sd_A is the same as sd_X restricted to A, $\exists$ induced $\operatorname{sd}_{X,A} : S_*(X, A) \to S_*(X, A)$. By property (2) of D_X, restrcion of D_X to A equals D_A so $\exists$ an induced homomorphism

$$\begin{tikzcd} 0 \arrow[r] & S_p(A) \arrow[r] \arrow[d, "D_A"'] & S_p(X) \arrow[r] \arrow[d, "D_X"] & S_p(X, A) \arrow[r] \arrow[d, dashed, "D_{X,A}"] & 0 \\ 0 \arrow[r] & S_{p+1}(A) \arrow[r] & S_{p+1}(X) \arrow[r] & S_{p+1}(X, A) \arrow[r] & 0 \end{tikzcd}$$

with $D_{X,A} : 1 \simeq \mathrm{sd}^m_{X,A} : S_*(X,a) \to S_*(X,A)$.

Notation: Let $\mathcal{A}$ be a collection of sets which cover X.

Set $S^{\mathcal{A}}_p(X) :=$ free abelian group$\{T : \Delta^p \to X \mid \operatorname{Im} T \subset A$ for some $A \in \mathcal{A}\}$.

$S^{\mathcal{A}}_p(X)$ is a subgroup of $S_p(X)$.

Notice that if $\operatorname{Im} T \subset A$ then writing $\partial T = \sum n_i T_i$, for each i $\operatorname{Im} T_i \subset \operatorname{Im} T \subset A$ so $\partial T \in S^{\mathcal{A}}_{p-1}(X)$. Thus the restriction of ∂ to $S^{\mathcal{A}}_p(X)$ turns $S^{\mathcal{A}}_p(X)$ into a chain complex and the inclusion map becomes a chain map.

Notice also that if T is a generator of $S^{\mathcal{A}}_p(X)$ then $D_X T \in S^{\mathcal{A}}_{p+1}(X)$ because:

if $D_{\Delta^p}(\iota_p) = \sum n_i S_i$ then $D_X T = T_*(D_{\Delta^p}\iota_p = \sum n_i T_* S_i = \sum n_i (T \circ S_i)$. But $\operatorname{Im} T \subset A$ for some $A \in \mathcal{A}$ and $\operatorname{Im} T \circ S_i \subset \operatorname{Im} T$.

Theorem 14.2.29 *Let $\mathcal{A}$ be a collection of subsets of X whose interiors cover X. Then $H_*\big(S^{\mathcal{A}}_*(X), \partial\big) \to H_*\big(S_*(X), \partial\big)$ is an ismorphism.*

Remark 14.2.30 *The even stronger statement $i_* : S^{\mathcal{A}}_*(X) \to S_*(X)$ is a chain homotopy equivalence is true, but we will not show this.*

Proof: The short exact sequence of chain complexes

$0 \to S^{\mathcal{A}}_*(X) \xrightarrow{i} S_*(X) \xrightarrow{q} S_*(X)/S^{\mathcal{A}}_*(X) \to 0$

induces a long exact homology sequence. Showing that i_* is an isomorphism on homology for all p is equivalent to showing that $H_p\left(S_*(X)/S^{\mathcal{A}}_*(X)\right) = 0\ \forall p$.

Let $qc \in S_*(X)/S^{\mathcal{A}}_*(X)$ be a cycle representing an element of $H_p\left(S_*(X)/S^{\mathcal{A}}_*(X)\right)$, where $c \in S_p(X)$. That is, $\partial qc = 0$ or equivalently $\partial c \in S^{\mathcal{A}}_{p-1}(X)$.

We wish to show that there exists $d \in S_{p+1}(X)$ s.t. $\partial qd = qc$ or equivalently $c - \partial d \in S^{\mathcal{A}}_p(X)$.

Since c is a finite sum of generators $c = \sum n_j T_j$, find N s.t. we can write $\mathrm{sd}^N T_j = \sum n_{ij} T_{ij}$ where $\forall i, j\ \exists A \in \mathcal{A}$ (depending upon i and j) with $\operatorname{Im} T_{ij} \subset A$. Let D_X be the chain homotopy $D_X : 1 \simeq \mathrm{sd}^N$ for this N. Show $c + \partial D_X c \in S^{\mathcal{A}}_p(X)$ and then let $d = -D_X c$.

$\partial D_X c + D_X \partial c = \mathrm{sd}^N c - c$ so $c + \partial D_X c = \mathrm{sd}^N c - D_X \partial c$.

By definition of N, $\mathrm{sd}^N c \in S^{\mathcal{A}}_p(X)$. Also $\partial c \in S^{\mathcal{A}}_{p-1}(X)$ as noted earlier and so $D_X \partial x \in S^{\mathcal{A}}_p(X)$. Thus the requred d exists. Hence ∂c represents the zero homology class in $H_p\left(S_*(X)/S^{\mathcal{A}}_*(X)\right)$.

$\square$

Let X, $\mathcal{A}$ be as in the preceding theorem, and let B be a subspace of X. Let $\mathcal{A} \cap B$ denote the covering of B obtained by intersecting the sets in$\mathcal{A}$ with B. Write $S^{\mathcal{A}}_*(X,B)$ for $S^{\mathcal{A}}_*(X)/S^{\mathcal{A}\cap B}_*(B)$.

Corollary 14.2.31 *$S^{\mathcal{A}}_*(X,B)$ to $S_*(X,B)$ induces an isomorphism on homology.*

Proof:

$$\begin{tikzcd}
0 \arrow[r] & S_*^{\mathcal{A}\cap B}(B) \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X) \arrow[r] \arrow[d] & S_*^{\mathcal{A}}(X,B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_*(B) \arrow[r] & S_*^{(}X) \arrow[r] & S_*(X,B) \arrow[r] & 0
\end{tikzcd}$$

induces

$$\begin{tikzcd}
\to H_{p+1}^{\mathcal{A}}(X,B) \arrow[r] \arrow[d] & H_p^{\mathcal{A}\cap B}(B) \arrow[r] \arrow[d, "\cong"] & H_p^{\mathcal{A}}(X) \arrow[r] \arrow[d, "\cong"] & H_p^{\mathcal{A}}(X,B) \arrow[r] \arrow[d] & H_{p-1}^{\mathcal{A}\cap B}(B) \arrow[r] \arrow[d, "\cong"] & H_{p-1}^{\mathcal{A}}(X) \to \arrow[d, "\cong"] \\
\to H_{p+1}(X,B) \arrow[r] & H_p(B) \arrow[r] & H_p(X) \arrow[r] & H_p(X,B) \arrow[r] & H_{p-1}(B) \arrow[r] & H_{p-1}(X) \to
\end{tikzcd}$$

Since the marked maps are isomorphisms from the theorem, the remaining vertical maps are also, by the 5-lemma. $\square$

Theorem 14.2.32 *(Excision)*

Let A be a subspace of X and suppose that U is a subspace of A s.t. $\overline{U} \subset \operatorname{Int} A$. Then $j : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism on singular homology.

Remark 14.2.33 *Note that this is slightly stronger than axiom A5 which requires that U be open in X.*

Proof: Let $\mathcal{A}$ denote the collection $\{X - U, A\}$ in 2^X.

$\operatorname{Int}(X \setminus U) = X \setminus \overline{U}$. Since $\overline{U} \subset \operatorname{Int} A$, the interiors of $X - U$ and A cover X. Hence $S_*^{\mathcal{A}}(X, A) \to S_*(X, A)$ induces an isomorphism on homology. To conclude the proof we show that $S_*(X \setminus U, A \setminus U) \cong S_*^{\mathcal{A}}(X, A)$ as chain complexes.

Define $\phi : S_p(X \setminus U) \to S_p^{\mathcal{A}}(X)/S_p^{\mathcal{A}\cap A}(A)$ by $T \mapsto [T]$, which makes sense since $\operatorname{Im} T \subset X - U$ which belongs to $\mathcal{A}$.

Every element of $S_p^{\mathcal{A}}(X)$ can be written $c = \sum m_i S_i + \sum n_j T_j$ where $\operatorname{Im} S_i \subset A\ \forall i$ and $\operatorname{Im} T_j \subset X \setminus U\ \forall j$. Since $\sum m_i S_i \in S_p^{\mathcal{A}\cap A}(A)$, in $S_p^{\mathcal{A}}(A)/S_p^{\mathcal{A}\cap A}(A)$, $[c] = \left[\sum n_j T_j\right] = \phi(\sum_j T_j)$. Therefore ϕ is onto.

$\ker \phi = S_p(X - U) \cap S_p^{\mathcal{A}\cap A}(A)$.

Notice that $\mathcal{A} \cap A = \{(X \setminus U) \cap A, A \cap A\} = \{A - U, A\}$ and since this colleciton includes A itself, $S_p^{\mathcal{A}\cap A}(A) = S_p(A)$.

In general $S_p(A) \cap S_p(B) = S_p(A \cap B)$ since a simplex has image in A and B if and only if its image lies in $A\cap$. Hence $\ker\phi = S_p(X \setminus U) \cap S_p^{\mathcal{A}\cap A}(A) = S_p\big((X \setminus U) \cap A\big) = S_p(A \setminus U)$.

Thus $S_p(X \setminus U, A \setminus U) \cong S_p(X - U)/S_p(A \setminus U) \overset{\phi}{\cong} S_p^{\mathcal{A}}(X)/S + p^{\mathcal{A}\cap A}(A) = S_p^{\mathcal{A}}(X, A)$. □

Let X_1, X_2 be subspaces of Y, let $A = X_1 \cap X_2$ and let $X = X_1 \cup X_2$. Notice that $X_2 \setminus A = X \setminus X_1$. Call this U. Thus $X_2 \setminus U = A$; $X \setminus U = X_1$.

Theorem 14.2.34 *(Mayer-Vietoris): Suppose that* $(X_1, A) \xrightarrow{j} (X, X_2)$ *induces an isomorphism on homology. (e.g. if* $\overline{U} \subset \operatorname{Int} X_2$. *) Then there is a long exact homology sequence*

$$\ldots \to H_{n+1}(X) \xrightarrow{\Delta} H_n(A) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \xrightarrow{\Delta} H_{n-1}(A) \to \ldots$$

Remark 14.2.35 *The hypothesis is satisfied of* X_1 *and* X_2 *are open since that* $\overline{U} = U$ *and* $\operatorname{Int} X_2 = X_2$.

Proof: Follows by algebraic Mayer-Vietoris from:

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}(X_1, A) \arrow[r] \arrow[d, "\cong"] & H_n(A) \arrow[r] \arrow[d] & H_n(X_1) \arrow[r] \arrow[d] & H_n(X_1, A) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}(A) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_{n+1}(X, X_2) \arrow[r] & H_n(X_2) \arrow[r] & H_n(X) \arrow[r] & H_n(X, X_2) \arrow[r, "\partial"] & H_{n-1}(X_2) \arrow[r] & {}
\end{tikzcd}$$

□

14.2.5 Exact Sequences for Triples

Suppose $A \hookrightarrow B \hookrightarrow C$.

$0 \to S_*(B)/S_*(A) \to S_*X/S_*(A) \to S_*(X)/S_*(B) \to 0$ is a short exact sequence of chain complexes. Therefore we have a long exact sequence

$$\ldots \to H_{n+1}(X, B) \xrightarrow{\partial} H_n(B, A) \to H_n(X, A) \to H_n(X, B) \xrightarrow{\partial} H_{n-1}(X, A) \to \ldots$$

called the long exact homology sequence of the triple. From

$$\begin{tikzcd}
0 \arrow[r] & S_*(B) \arrow[r] \arrow[d] & S_*(X) \arrow[r] \arrow[d] & S_*(X)/S_*(B) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S_*(B)/S_*(A) \arrow[r] & S_*(X)/S_*(A) \arrow[r] & S) * (X)/S_*(B) \arrow[r] & 0
\end{tikzcd}$$

we get

$$
\begin{tikzcd}
{} \arrow[r] & H_n(X) \arrow[r] \arrow[d] & H_n(X,B) \arrow[r, "\partial"] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, "j"'] & {} \\
{} \arrow[r] & H_n(A) \arrow[r] & H_n(X,B) \arrow[r, "\tilde{\partial}"] & H_{n-1}(B,A) \arrow[r] & {}
\end{tikzcd}
$$

so $\tilde{\partial} = j\partial$ which relates the boundary homomorphism of the triple to ones we have seen before.

Chapter 15

Applications of Homology

First we need some calculations.

Theorem 15.0.36 *Suppose* $n > 0$*. Then* $H_q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & \text{otherwise.} \end{cases}$

Proof: By induction on n using Mayer-Vietoris. $\square$

Corollary 15.0.37 S^n *is not homotopy equivalent (and in particular not homeomorphic) to* S^m *for* $n \neq m$*.* $\square$

Corollary 15.0.38 $\mathbb{R}^n$ *is not homeomorphic to* $\mathbb{R}^m$ *for* $n \neq m$*.*

Proof: If $\mathbb{R}^n$ were homotopy equivalent to $\mathbb{R}^m$ then $\mathbb{R}^n \setminus \{*\}$ would be homeomorphic to $\mathbb{R}^m \setminus \{*\}$. But $S^{n-1} \simeq \mathbb{R}^n \setminus \{*\}$ and $S^{m-1} \simeq \mathbb{R}^m \setminus \{*\}$. $\square$

Theorem 15.0.39 $\nexists f : D^n \to S^{n-1}$ *s.t.*

$$\begin{tikzcd} S^{n-1} \arrow[rr] \arrow[dr, "1_{S^{n-1}}"'] & & D^n \arrow[dl] \\ & S^{n-1} & \end{tikzcd}$$

commutes. $\square$

Corollary 15.0.40 *(Brouwer Fixed Point Theorem) Let $g : D^n \to D^n$. Then $\exists x \in D^n$ s.t. $g(x) = x$.*

Proof: Same as proof in case $n = 2$. $\square$

Definition and Notation:

Let X be a topological space. Define the (unreduced) *cone* on X, denoted CX by $CX := \frac{X \times I}{X \times \{0\}}$.

CX is contractible $\forall X$. ($H : CX \times I \to CX$ by $H\big((x,s),t\big) := (x, st)$.)

Define the (unreduced) suspension of X, denoted SX, by $SX := \frac{X \times I}{X \times \{0\} \cup X \times \{1\}}$. SS^n is homeomorphic to S^{n+1}

C and S are functors from Topological Spaces to Topological Spaces. e.g. Given $f : X \to Y$, $\exists$ induced $S(f) : SX \to SY$ given by $(x,t) \mapsto \big(f(x), t\big)$ satisfying $S(1) = 1$ and $S(g \circ f) = S(g) \circ S(f)$.

Theorem 15.0.41 *(Suspension) $\exists$ a natural isomorphism $\tilde{H}_q(X) \cong \tilde{H}_{q+1}(SX)$ $\forall q$ and $\forall X$.*

Note: Natural means, $\forall f : X \to Y$,
$$\begin{tikzcd} \tilde{H}_q(X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & \tilde{H}_{q+1}(SX) \arrow[d, "Sf_*"] \\ \tilde{H}_q(X) \arrow[r, "\cong"] & \tilde{H}_{q+1}(SX) \end{tikzcd}$$
commutes.

Proof: Let C^+X and C^-X denote the upper and lower cones on X, within SX. Enlarge them slightly to open sets. i.e. Replace them by

$$C^+X := \frac{X \times (\frac{1}{2} - \epsilon, 1)}{X \times \{1\}}, \qquad C^-X := \frac{X \times (0, \frac{1}{2} + \epsilon)}{X \times \{0\}}.$$

Then we have Mayer-Vietoris sequences for C^+X, C^-X, where $C^+X \cup C^-X = SX$ and $C^+X \cap C^-X \simeq X$

$$\begin{tikzcd} 0 \arrow[d, equal] & & & 0 \arrow[d, equal] \\ \tilde{H}_{q+1}(C^+X) \oplus \tilde{H}_{q+1}(C^-X) \arrow[r] \arrow[d] & \tilde{H}_{q+1}(SX) \arrow[r, "\Delta", "\cong"'] \arrow[d, "(Sf)_*"'] & \tilde{H}_q(X) \arrow[r] \arrow[d, "f_*"'] & \tilde{H}_q(C^+X) \oplus \tilde{H}_{q+1}(C^-X) \arrow[d] \\ \tilde{H}_{q+1}(C^+Y) \oplus \tilde{H}_{q+1}(C^-Y) \arrow[r] \arrow[d, equal] & \tilde{H}_{q+1}(SY) \arrow[r, "\Delta", "\cong"'] & \tilde{H}_q(Y) \arrow[r] & \tilde{H}_q(C^+Y) \oplus \tilde{H}_{q+1}(C^-Y) \arrow[d, equal] \\ 0 & & & 0 \end{tikzcd}$$

□

Remark 15.0.42 *Under the presence of the other axioms, Suspension⇔Mayer-Vietoris⇔Excision.*

Theorem 15.0.43 *Let $f : S^n \to S^n$ be the reflection $(x_0, \ldots, x_n) \mapsto (-x_0, \ldots, x_n)$. Then $r_* : \mathbb{Z} \cong \tilde{H}_n(S^n) \to \tilde{H}_n(S^n) \cong \mathbb{Z}$ is multiplication by -1.*

Proof: Notice that if we denote $r : S^n \to S^n$ by r_n then $r_n = Sr_{n-1}$. Therefore by naturality of suspension it suffices to prove the theorem in the case $n = 0$ when it is trivial. □

Corollary 15.0.44 *Let : $S^n \to S^n$ be the antipodal map $x \mapsto -x$. Then $a_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ is multiplication by $(-1)^{n+1}$.*

Proof: : Write a as the composition of the $n+1$ reflections $r_j : S^n \to S^n$ given by $r_j(x_0, \ldots, x_n) := (x_0, \ldots, -x_j, \ldots, x_n)$. □

Definition 15.0.45 *Let $f : S^n \to S^n$. Then $f_* : \mathbb{Z} \cong \tilde{H}_n(S^n) \to \tilde{H}_n(S^n) \cong \mathbb{Z}$ is multiplication by k for some integer k. k is called the* degree *of f.*

Theorem 15.0.46 *Let $f : S^n \to S^n$. Suppose $\deg f \neq (-1)^{n+1}$. Then f has a fixed point.*

Proof: If f has no fixed point then the great circle joining $f(x)$ to $-x$ has a well defined shorter and longer segment. Contruct a homotopy $H : f \simeq a$ by moving $f(x)$ towards $-x$ along the shorter seqment. Explicitly, $H(x,t) = \frac{(1-t)f(x)+t(-x)}{||(1-t)f(x)+t(-x)||}$. (The only way the denominator can be zero is if $(1-t)f(x) = tx$ which is doesn't hold for $t = 0$ or 1 and would otherwise require that $f(x) = tx/(1-t)$ which doesn't hold since $f(x)$ is never a multiple of x.) Hence $\deg f = \deg a = (-1)^{n+1}$, which is a contradiction. □

Theorem 15.0.47 *Let $f : S^n \to S^n$. If $\deg f \neq 1$, then $f(x) = -x$ for some x.*

Proof: Since $\deg f \neq 1$, $\deg af \neq (-1)^{n+1}$, so af has a fixed point x. i.e. $x = af(x) = -f(x)$. Hence $f(x) = -x$. □

Theorem 15.0.48 *$\exists$ continuous nowhere vanishing "vector field" on S^n if and only if n is odd. That is, if $T(S^n)$ denotes the tangent bundle to S^n then ($\exists$ continuous $v : S^n \to T(S^n)$ s.t. $v(x) \neq 0$ $\forall x \in S^n$) if and only if n is odd.*

Proof:
$\longleftarrow$ If n is odd, then $v(x_0, x_1, \ldots, x_{2n+1}) := (-x_1, x_0, \ldots, -x_{2n+1}, x_{2n}$ is a nowhere vanishing vector field on S^n.
$\longrightarrow$ Suppose $\exists$ such a v. Define $w : S^n \to S^n$ by $w(x) := v(x)/||v(x)\}||$. Then $x \perp w(x)$ $\forall x \in S^n$. In particular, $w(x) \neq x$ $\forall x$ and $w(x) \neq -x$ $\forall x$. Thus w has no fixed point and hence $\deg w = (-1)^{n+1}$. But since $\nexists x$ s.t. $w(x) = -x$ we also have $\deg w = 1$. Hence $1 = (-1)^{n+1}$, so n is odd.

An alternate more direct argument (not using the two preceding theorems) is as follows:

To get the conclusion $1 = (-1)^{n+1}$ is suffices to show that both $w \simeq 1_{S^n}$ and $w \simeq a$ hold. Define $F : S^n \times I \to S^n$ by $F(x,t) := x\cos(t\pi) + w(x)\sin(t\pi)$. Then $F_0 = 1$, $F_{1/2} = w$ and $F_1 = a$ so F provides a homotopy from 1 to a. Therefore by the homotopy axiom $1 = (-1)^{n+1}$.
$\square$

15.1 Jordan-Brouwer Separation Theorems

Definition 15.1.1 *Suppose $A \subset X$. We say that A* separates X *if $X \smallsetminus A$ is disconnected (i.e, not path connected), or equivalently if $\tilde{H}_*(X \smallsetminus A) \neq 0$.*

Terminology: If B is homeomorphic to D^k then B is called a *k-cell*.

Theorem 15.1.2 *Let $B \subset S^n$ be a k-cell. Then $S^n \smallsetminus B$ is acyclic. (i.e. $\tilde{H}(S^n \smallsetminus B) = 0 \ \forall\ q$.) In particular, B does not separate S^n.*

Remark 15.1.3 *$B \simeq *$ and $S^n \smallsetminus \{*\} = \mathbb{R}^n$, but in general $A \simeq B$ does not imply that $X \smallsetminus A \simeq X \smallsetminus B$.*

Proof: By induction on k.
$k = 0$ is trivial since then $B = *$ and $S^n \smallsetminus \{*\} = \mathbb{R}^n$.
Suppose that the theorem is true for $(k-1)$-cells.
Let $h : I^k \to B$ be a homeomorphism.
Write $B = B_1 \cup B_2$ where $B_1 := h(I^{k-1} \times [0, 1/2])$ and $B_2 := h(I^{k-1} \times [1/2, 1])$.
Let $C = B_1 \cap B_2$; a $(k-1)$-cell.
Let $i : (S^n \smallsetminus B) \to S^n \smallsetminus B_1$, $j : (S^n \smallsetminus B) \to (S^n \smallsetminus B_2)$.
Suppose $0 \neq \alpha \in \tilde{H}_p(S^n \smallsetminus B)$.

Lemma 15.1.4 *Either $i_*(\alpha) \neq 0$ or $j_*(\alpha) \neq 0$.*

Proof: $S^n \smallsetminus B_1$ and $S^n \smallsetminus B_2$ are open so they have a Mayer-Vietoris sequence.
$(S^n \smallsetminus B_1) \cap (S^n \smallsetminus B_2) = S^n \smallsetminus B$ $\quad (S^n \smallsetminus B_1) \cup (S^n \smallsetminus B_2) = S^n \smallsetminus (B_1 \cap B_2) = S^n \smallsetminus C$.

$$\begin{tikzcd}
\tilde{H}_{p+1}(S^n \smallsetminus C) \arrow[r, "\Delta"] \arrow[d, equal, "\text{(by hypothesis)}"'] & \tilde{H}_p(S^n \smallsetminus B) \arrow[r, tail, "{(i_*, j_*)}"] & \tilde{H}_p(S^n \smallsetminus B_1) \oplus \tilde{H}_p(S^n \smallsetminus B_2) \\
0 & &
\end{tikzcd}$$

so either $i_*(\alpha) \neq 0$ or $j_*(\alpha) \neq 0$. □

Proof of Theorem (cont.): By the lemma, continuing to subdivide we obtain a nested decreasing sequence of closed intervals I_n s.t. if we let $j_m : (S^n \smallsetminus B) \hookrightarrow (S^n \smallsetminus Q_m)$, where $Q_m := h(I^{k-1} \times I_m)$, then $j_{m*}\alpha \neq 0$.
By the Cantor Intersection Theorem, $\cap_m I_m =$ a single point $\{e\}$.

$$
\begin{array}{c}
H_p(S^n \smallsetminus B) \longrightarrow \dots \longrightarrow H_p(S^n \smallsetminus Q_m) \longrightarrow \dots \longrightarrow H_p\big(S^n \smallsetminus h(I^{k-1} \times \{e\})\big) \\
\hfill (\text{induction})\Big\| \\
\hfill 0
\end{array}
$$

where we have used that $E := h(I^{k-1} \times \{e\})$ is a $(k-1)$-cell. Since $S^n \smallsetminus Q_m$ is open and nested and $S^n \smallsetminus E = \cup_{m=1}^{\infty}(S^n \smallsetminus Q_m)$, $H_*(S^n \smallsetminus E) = \varinjlim H_*(S^n \smallsetminus Q_m)$.

Therefore $\alpha \mapsto 0$ in $H_*(S^n \smallsetminus E)$ implies that $j_{m*}(\alpha) = 0$ in $H_*(S^n \smallsetminus Q_m)$ for some m, which is a contradiction. Hence $\nexists$ nonzero $\alpha \in H_p(S^n \smallsetminus B)$. □

Theorem 15.1.5 *Suppose* $h : S^k \hookrightarrow S^n$. *Then* $\tilde{H}_i\big(S^n \smallsetminus h(S^k)\big) = \begin{cases} \mathbb{Z} & i = n-k-1; \\ 0 & \text{otherwise.} \end{cases}$

Proof: By induction on k.

If $k = 0$, $\tilde{H}_p\big(S^n \smallsetminus h(S^0)\big) = \tilde{H}_p\big(S^n \smallsetminus \{2 \text{ points}\}\big) = \tilde{H}_p\big(\mathbb{R}^n \smallsetminus \{\text{point}\}\big) = \tilde{H}_p(S^{n-1})$.

$\surd$

Suppose that the theorem is true for $k-1$.

Let E_+^k, E_-^k be the upper and lower hemispheres of S^k. Notice that by compactness, h is a homeomorphism onto its image, so $h(E_+^k)$ and $h(E_-^k)$ are k-cells.

Also $S^n \smallsetminus h(E_+^k)$, $S^n \smallsetminus h(E_-^k)$ are open so Mayer-Vietoris applies.

$\big(S^n \smallsetminus h(E_+^k)\big) \cup \big(S^n \smallsetminus h(E_-^k)\big) = \big(S^n \smallsetminus h(E_+^k \cap E_-^k)\big) = \big(S^n \smallsetminus h(S^{k-1})\big)$

$\big(S^n \smallsetminus h(E_+^k)\big) \cap \big(S^n \smallsetminus h(E_-^k)\big) = \big(S^n \smallsetminus h(E_+^k \cup E_-^k)\big) = \big(S^n \smallsetminus h(S^k)\big)$

$$
\begin{array}{l}
\qquad\qquad\quad 0 \\
\qquad\qquad\quad \| \\
\tilde{H}_p\big(S^n \smallsetminus h(E_+^k)\big) \oplus \tilde{H}_p\big(S^n \smallsetminus h(E_-^k)\big) \to \tilde{H}_p\big(S^n \smallsetminus h(S^{k-1})\big) \xrightarrow[\cong]{\Delta} \tilde{H}_{p-1}\big(S^n \smallsetminus h(S^k)\big) \\
\hfill \to \tilde{H}_{p-1}\big(S^n \smallsetminus h(E_+^k)\big) \oplus \tilde{H}_{p-1}\big(S^n \smallsetminus h(E_-^k)\big) \\
\hfill \| \qquad\qquad \\
\hfill 0 \qquad\qquad
\end{array}
$$

□

Theorem 15.1.6 *(Jordan Curve Theorem) Suppose $n > 0$. Let C be a subset of S^n which is homeomorphic to S^{n-1}. Then $S^n \smallsetminus C$ has precisely two path components and C is their common boundary. (Furthermore, the components are open in S^n.)*

Proof: By the preceding theorem, $\tilde{H}_0(S^n \setminus C) \cong \mathbb{Z}$, so $S^n \setminus C$ has two path components. Denote these components W_1 and W_2.

C is closed in S^n so $S^n \setminus C$ is open. Hence by local path connectedness of S^n, its components W_1 and W_2 are open. Thus $\overline{W_1} \subset W_2^c$.

If $x \in \partial W_1 = \overline{W_1} \setminus W_1$, then $x \notin W_2$ (since $x \in \overline{W_1} = W_2^c$) and $x \notin W_1$. So $x \in (W_1 \cup W_2)^c = C$. Hence $\partial W_1 \subset C$.

Conversely let $x \in C$.

Let U be an open neighbourhood of x. Show $U \cap \overline{W_1} \neq \emptyset$. Since U arbitrary, it will follow that x is an accumulation point of $\overline{W_1}$ so that $x \in \overline{W_1}$. But $x \in C$ so $x \notin W_1$, resulting in $x \in \overline{W_1} \setminus W_1 = \partial W_1$.
To show $U \cap \overline{W_1} \neq \emptyset$:

$U \cap C$ is homeomorphic to an open subset of S^{n-1} (since $C \cong S^{n-1}$ by hypothesis) so it contains the closure of an $(n-1)$-sphere. Let C_1 be this closure. Under the homeomorphism $C \cong S^{n-1}$, $C_1 \cong N_r[x]$ for some r and x. Thus $C_1 \subset C$ is an $(n-1)$-cell. Let $C_2 = \overline{C \setminus C_1}$. Then C_2 is also an $(n-1)$-cell (up to homeomorphism it is the closure of the complement of $N_r[x]$ in S^{n-1}) and $C_1 \cup C_2 = C$ which is closed. By Theorem 15.1.2, C_2 does not separate S^n so $\exists$ path α in $S^n \setminus C_2$ joining $p \in W_1$ to $q \in W_2$. $(\operatorname{Im}\alpha) \cap (\overline{W_1} \setminus W_1) = \alpha\big(\alpha^{-1}(\overline{W_1}) \setminus \alpha^{-1}(W_1)\big)$. If this is empty then $\alpha^{-1}(\overline{W_1}) = \alpha^{-1}(W_1)$. However the equality of these open and closed subsets of I means that either $\alpha(W_1) = \emptyset$ or $\alpha^{-1}(W_1) = I$. We know $\alpha^{-1}(W_1) \neq \emptyset$ since $0 \in \alpha^{-1}(W_1)$ (since $p = \alpha(0) \in W_1$). And $1 \notin \alpha^{-1}(W_1)$ since $q \notin W_1$. Therefore $(\operatorname{Im}\alpha) \cap (\overline{W_1} \setminus W_1) \neq \emptyset$. Thus $\exists y \in (\operatorname{Im}\alpha) \cap (\overline{W_1} \setminus W_1) \subset \partial W_1 \subset C = C_1 \cup C_2$. Since $\operatorname{Im}\alpha \subset S^n \setminus C_2$, $y \notin C_2$ so $y \in C_1 \subset U$. Hence $y \in U \cap \overline{W_1}$. $\surd$

So $\partial W_1 = C$. Similarly $\partial W_2 = C$, as desired. □

Corollary 15.1.7 *(Jordan Curve Theorem - standard version): Supppose $n > 1$. Let C be asubspace of $\mathbb{R}^n$ which is homeomorphic to S^{n-1}. Then $\mathbb{R}^n \setminus C$ has precisely two components (one bounded, one unbounded — known as the "inside of C" and "outside of C" respectively) and C is their common boundary.*

Proof: Include $\mathbb{R}^n$ into S^n, writing $\mathbb{R}^n = S^n = \{P\}$. Then $S^n \setminus C$ is the union of two components W_1, W_2 whose common boundary is C. One of the components, say W_1 contains P so $W_1 \setminus \{P\}$, W_2 are the components of $\mathbb{R}^n \setminus C$ and their common boundary is C. □

Theorem 15.1.8 *(Invariance of Domain): Let V be open in $\mathbb{R}^n$ and let $f : V \to \mathbb{R}^n$ be continuous and injective. Then $f(V)$ is open in $\mathbb{R}^n$ and $f : V \to f(V)$ is a homeomorphism.*

Remark 15.1.9 *Compare the inverse function theorem which asserts this under the stronger hypothesis that f is continuously differentiable with non-singular Jacobian, but also asserts differentiability of the inverse map.*

Proof:

Include $\mathbb{R}^n$ into S^n. Let U be an open subset of V. Let $y \in f(U)$. We show that $f(U)$ contains an open neighbourhood of y.

Write $y = f(x)$, Find ϵ s.t. $N_\epsilon[x] \subset U$. Set $A := N_\epsilon[x] \smallsetminus N_\epsilon(x)$. So A is homeomorphic to S^{n-1}. Since $f\big|_{N_\epsilon[x]} \subset U$ is a homeomorphism (an injective map from a compact set to a Hausdorff space), $f(A)$ is homeomorphic to S^{n-1}. Therefore $f(A)$ separates S^n into two components W_1 and W_2 which are open in S^n.

$N_\epsilon(x)$ is connected and disjoint from A, so $f\big(N_\epsilon(x)\big)$ is connected and disjoint from $f(A)$. Thus $f\big(N_\epsilon(x)\big)$ is contained entirely within either W_1 or W_2. Say $f\big(N_\epsilon(x)\big) \subset W_1$.

$$S^n \smallsetminus f(A) \smallsetminus f\big(N_\epsilon(x)\big) = S^n \smallsetminus f\big(A \cup N_\epsilon(x)\big) = S^n \smallsetminus f\big(N_\epsilon[x]\big)$$

(which the later argument will show is equal to $S^n \smallsetminus W_2^c = W_2$). Since $f\big(N_\epsilon[x]\big)$ is an n-cell, it does not disconnect S^n, i.e. $S^n \smallsetminus f\big(N_\epsilon[x]\big)$ is connected. Because $f\big(N_\epsilon[x]\big) \subset \overline{W_1} \subset W_2^c$ which is equivalent to $W_2 \subset S^n \smallsetminus f\big(N_\epsilon[x]\big)$, we get $W_2 = S^n \smallsetminus f\big(N_\epsilon[x]\big)$ (as remarked earlier), since W_2 is a path component of S^n. Hence $f\big(N_\epsilon[x]\big) = W_2^c = \overline{W_1}$. Thus $f\big(N_\epsilon(x)\big) = W_1$. (i.e. If $z \in W_1 \smallsetminus f\big(N_\epsilon(x)\big)$ then $z \in S^n \smallsetminus f(A) \smallsetminus f\big(N_\epsilon(x)\big) = S^n \smallsetminus f\big(N_\epsilon[x]\big) = S^n \smallsetminus W^c = W_2$, which contradicts $W_1 \cap W_2 = \emptyset$.)

Therefore we have shown that $\exists$ an open set W_1 s.t. $y \in W_1 \subset f(U)$ and thus $f(U)$ is open. Applying the above argument with $U := V$ gives that $f(V)$ is open. It also shows that $f : V \to f(V)$ is an open map, so it is a homeomorphism. $\square$

Chapter 16

Homology of CW-complexes

Let X be a CW-complex.

If $T : \Delta_n \to X \in S_n(X)$ is a generator, then $\operatorname{Im} T$ is compact so $\operatorname{Im} T \subset X^{(p)}$ for some p. Therefore $S_*(X) = \cup_p S_*(X^{(p)})$.

How does this tell us $H_*(X)$ in terms of the $H_*(X^{(p)})$'s?

16.1 Direct Limits

Definition 16.1.1 *A partially ordered set J is called a* directed set *if $\forall i, j \in J$ $\exists k$ s.t. $i \leq k$ and $j \leq k$.*

Definition 16.1.2 *Given a directed set J, a* directed system *of abelian groups indexed by J consists of:*

1. *An abelian group G_j for each $j \in J$;*
2. *For each pair $i, j \in J$ a group homomorphism $\phi_{j,i} : G_i \to G_j$ s.t. $\phi_{j,j} = 1_{G_j}$ and $\phi_{k,j} \circ \phi_{j,i} = \phi_{k,i}$.*

Examples

1. $J = \mathbb{Z}^+$; $\quad G_n = M_n(\mathbf{k})$ $\quad$ ($n \times n$ matrices over a field $\mathbf{k}$)

   $\phi_{ij} : M_i(\mathbf{k}) \to M_j(\mathbf{k})$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.

2. $J = \{$finite subcomplexes of a CW complex X, ordered by inclusion$\}$

   $G_Y = H_p(Y)$ $\quad$ (where Y is a finite subcomplex of X)

3. X topological space; $\quad J = \{\text{open subsets of } X \text{ ordered by inclusion}\}$
   $G_U = H_p(U)$.
4. $J = \mathbb{Z}^+$; $\quad G_n = \mathbb{Z}$; $\quad \phi_{j,i} : \mathbb{Z} \to \mathbb{Z}$ by $1 \mapsto p^{j-i}$

Definition 16.1.3 *The* direct limit *of the direct system* $\{G_j\}_{j \in J}$ *consists of an abelian group* G *and homomorphisms* $\phi_j : G_j \to G$ *s.t.*

1. $$\begin{tikzcd} G_i \arrow[rr, "\phi_{j,i}"] \arrow[dr, "\phi_i"'] & & G_j \arrow[dl, "\phi_j"] \\ & G & \end{tikzcd}$$ commutes $\forall i, j$

2. G is univesal w.r.t. property (1). i.e., given H and homomorphisms $\psi_j : G_j \to H$ s.t. $\psi_i \circ \phi_{j,i} = \psi_j$, $\exists!\, \theta : G \to H$ s.t. $\forall i, j$

$$\begin{tikzcd} G_i \arrow[rr, "\phi_{j,i}"] \arrow[dr, "\phi_i"] \arrow[ddr, "\psi_i"'] & & G_j \arrow[dl, "\phi_j"'] \arrow[ddl, "\psi_j"] \\ & G \arrow[d, dashed, "\theta"] & \\ & H & \end{tikzcd}$$

We write $G = \varinjlim_J \{G_j\}$.
Note: By the usual categorical argument, a direct system has at most one direct limit up to isomorphism. As we shall see, every direct system of abelian groups has a direct limit.

Observe that if $\phi_{j,i}$ is an inclusion map $\forall i, j$ then $G = \cup_{j \in J} G_j$ is the direct limit of the system.

Theorem 16.1.4 *Every direct direct system of abelian groups has a direct limit.*

Proof: Let $H = \oplus_{j \in J} G_j$ with $\alpha_j : G_j \to H$ the canonical inclusion.

Let $G = H/\!\sim$ where $\alpha_i(g) \sim \alpha_j(g)$ $\forall i, j$ and $\forall g \in G_i$. More precisely, $G = H/H'$ where H' is the subgroup of H generated by $\{\alpha_i(g) - \alpha_j \phi_{j,i}(g)\}$.

Let $\pi : H \to G$ be the quotient map.

Define ϕ_j to be the composite $G_j \xrightarrow{\alpha_j} H \xrightarrow{\phi} G$.

Then $\forall i, j$ and $\forall g \in G$, $\phi_j \phi_{j,i}(g) = \pi \alpha_j \phi_{j,i}(g) = \pi \alpha_i(g) = \phi_i(g)$.

Also, given k and maps $\psi_j : G_j \to K$ s.t. $\psi_i \circ \phi_{j,i} = \psi_j$: The maps ψ_j induce a unique map $\theta : H \to K$ (by the universal property of direct sum). Furthermore, since $\psi_i \circ \phi_{j,i} = \psi_j$, $\theta\big|_{H'}$ is the trivial map so by the universal property of quotient

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\pi"] & K \\ G \arrow[ur, dotted, "\theta"'] & \end{tikzcd}$$

□

Remark 16.1.5 *The definitions make sense and this proof still works even if the poset J is not a direct system. There is a more general notion called* colimit *when the poset J is not directed.*

From now on we will omit the inclusion maps α_j.

Notice: Any element of G has a representative of the form $\phi_k(g)$ for some $g \in G_k$.
Proof: Let $X = (g_j)_{j \in J}$ represent an element of G. Since x has only finitely many nonzero components, the definition of direct system implies that $\exists k \in J$ s.t. $j \le k$ $\forall j$ s.t. $g_j \ne 0$. Then adding $\phi_{k,j}(g_j) - g_j$ to x for all j s.t. $g_j \ne 0$ gives a new representative for x with only one nonzero component. (i.e. for some k, $x = \phi_k(g)$ with $g \in G_k$.)

Lemma 16.1.6 *If $g \in G_k$ s.t. $\phi)k(g) = 0$ then $\phi_{m,k}(g) = 0$ for some m.*

Proof:
Notation: For "homogeneous" elements of $\oplus_{\alpha \in J} G_\alpha$ (i.e. elements with just 1 nonzero component) write $|h| = \alpha$ to mean that $h \in G_\alpha$, or more precisely that the only nonzero component of h lies in G_α.

$\phi_k(g) = 0 \Rightarrow g \in H' \Rightarrow$

$$g = \sum_{t=1}^{n} \phi_{j_t, i_t} g_t - g_t \qquad \text{where } g_t \in G_{i_t} \tag{16.1}$$

Find m s.t. $k \leq m$ and $i_r \leq m$ and $j_r \leq m$ $\forall r$. Set $g' = \phi_{m,k}g$.

Adding $g' - g = \phi_{m.k}g - g$ to equation 16.1 gives

$$g' = \sum_{t+0}^{n} \phi_{j_t,i_t} g_t - g_t \qquad \text{where } g_0 = g \tag{16.2}$$

Note that for any $\alpha < m$, collecting terms on RHS in G_α gives 0, since LHS is 0 in degree α.

Among $S := \{i_0, \ldots, i_n, j_0, \ldots, j_n, m\}$ find α which is minimal. (i.e. each other index occuring is either greater or not comparable) Since j_t i_t, α is one of the i's so this means $J - t \neq \alpha$ for any t.

For each t with $|g_t| = \alpha$, add $g_t - \phi_{m,|g_t|}g_t$ to both sides of equation 16.2.

As noted above, $\sum_{\{t||g_t|=\alpha\}} g_t = 0$ so $\sum_{\{t||g_t|=\alpha\}} \phi_{m,|g_t|}g_t$ is also 0 and so we are actually adding 0 to the equation. However we can rewrite it using:

$\phi_{j_t,i_t}g_t - g_t + g_t - \phi_{m,|g_t|}g_t = \phi_{j_t,i_t}g_t - \phi_{m,|g_t|}g_t \overset{(|g_t|=i_t)}{=\!=\!=} \phi_{j_t,i_t}g_t - \phi_{m,j_t}\phi_{j_t,i_t}g_t = \phi_{m,j_t}\tilde{g}_t$ where $\tilde{g}_t = -\phi_{j_t,i_t}g_t$. Therefore we now have a new expression of the form $g' = \sum \phi_{j_t,i_t}g)t - g_t$; however the new2 set S is smaller than before since it no longer contains α (and no new index was added).

Repeat this process until the set S consists of just $\{m\}$. Then no i's are left in S (since $i_t < m$ $\forall t$) which means that there are no terms left in the sum. That is, Equation 16.1 reads $g' = 0$, as required. □

Notice that from the construction: If J is totally ordered and $\exists N$ s.t. $\phi_{n,k}$ is an isomorphism $\forall k, n \geq N$ (in which case we say the system *stabilizes*) then the direct limit is isomorphism to the "stable" group G_N.

Remark 16.1.7 *Above can be dualized by turning the arrows around: That is, define*

Inversely directed system = poset J s.t. $\forall k, n \in J$ $\exists j \in J$ s.t. $j \leq k$, $j \leq n$.

Define an inverse system of abelian groups to be a collection of abelian groups G_j and "compatible" group homomorphisms $\phi_{k,j}$ indexed by the inverse system. The inverse limit, $\varprojlim_J G_j$, of the inverse system is defined as an abelian group which has the property that there exists a "compatible" collection of homomorphisms $\phi_k : \varprojlim_J G_j \to G_k$ and such that given any group H with the same properties $\exists!$ $\theta : H \varprojlim_J G_j$ making the diagrams commute. The construction of a group satisfying this definition is given by $\varprojlim_J G_j = \{(x_j) \in \prod_{j\in J} G_j \mid \phi_{k,j}x_j = x_k\}$.

Theorem 16.1.8 *("Homology commutes with direct limits")*

Let $C = \varinjlim (C_j)_*$. Then $H(C) = \varinjlim_J H_*(C_j)$.

Remark 16.1.9 *Even if $\varinjlim_J C_j$ is just a union, $\{H_*(C_j)\}$ may be a non-trivial direct system. (Homology need not preserve monomorphisms.)*

Proof: Let $\psi_{j,i} : (C_i)_* \to (C_j)_*$ be the maps in the direct system $\varinjlim_J C_j$. Definition of maps $\phi_{j,i} : H(C_{i*}) \to H(C_{j*})$ is $\phi_{j,i} = (\psi_{j,i})_*$.

$$\begin{tikzcd}
H(C_{i*}) \arrow[rr, "\phi_{j,i} = (\psi_{j,i})_*"] \arrow[dr, "\phi_i"] \arrow[ddr, "(\psi_i)_*"'] & & H(C_{j*}) \arrow[dl, "\phi_j"'] \arrow[ddl, "(\psi_j)_*"] \\
& \varinjlim_J H(C_j) \arrow[d, dashed, "\theta"] & \\
& H(C) &
\end{tikzcd}$$

Claim θ is onto:

Given $[x] \in H(C)$, where $x \in C$, find a representative $x_k \in C_{k*}$ for x. (That is, $x = \psi_k x_k$).

Since x represents a homology class, $\partial x = 0$. Hence $\psi_k \partial x_k = \partial \psi_k x_k = \partial x = 0$. Replacing x_k by $x_m = \phi_{m,k} x_k$ for some m, get a new representative for x s.t. $\partial x_m = 0$. Therefore x_m represents a homology class $[x_m] \in H(C_{m*})$ and

$$\begin{tikzcd}
{[x_m]} \arrow[dr] & H(C_{m*}) \arrow[rr] \arrow[dr] & & \varinjlim_J H(C_j) \arrow[dl, "\theta"] \\
& {[x]} & H(C) &
\end{tikzcd}$$

shows $\in \operatorname{Im} \theta$.

Claim θ is $1-1$:

Let $y \in \lim_J H(C_j)$ s.t. $\theta(y) = 0$.

Find a representative $[x_k] \in H(C_{k*})$ for y, where $x_k \in X_{k*}$. (That is, $y = \phi_k(x_k)$.)

$$\begin{tikzcd}
{[x_k]} \arrow[rr] & & {[y]} \\
H(C_{k*}) \arrow[rr] \arrow[dr, "\psi_{k*}"'] & & \varinjlim_J H(C_j) \arrow[dl, "\theta"] \\
& H(C) &
\end{tikzcd}$$

Since $\theta y = 0$, $[\psi_k x_k] = 0$ in $H(C)$. That is, $\exists v \in C$ s.t. $\partial v = \psi_k x_k$.

May choose l s.t. $v = \psi_{l*}(w_l)$.

Find m s.t. $k, l \leq m$. Then replacing x_k, w_l by their images in $(C_m)_*$ we get that $x - \partial w_m$ stabilizes to 0 so that $\exists m' \geq m$ s.t. $[x_{m'}] = [\partial w_{m'}] = 0$. Hence $y = 0$. $\square$

Theorem 16.1.10 $H_*(X) = \varinjlim_p H_*(X^{(p)})$

Proof: Every compact subset of X is contained in $X^{(N)}$ for some N, so by $A8$, $S_*(X) = \cup_p S(X^{(p)}) = \varinjlim_p S_*(X^{(p)})$. Therefore $H_*(X) = \varinjlim_p H_*(X^{(p)})$. $\square$

Theorem 16.1.11 *If $X = \cup_{n=1}^{\infty} V_n$ where V_n open in X and $V_n \subset V_{n+1}$ then $H_*(X) = \varinjlim_n H_*(V_n)$.*

Proof: Sufficient to show that $S_*(X) \cup_{n=1}^{\infty} S_*(V_n)$.

If $T \in S_*(X)$ is a generator then $\operatorname{Im} T$ is compact.

$\{V_n\}$ covers X so $\operatorname{Im} T \subset V_n$ for some n (since V_n's nested).

Hence $T \in S_*(V_n)$ for that n. $\square$

16.2 Cellular Homology

Let X be a CW-complex.

By convention $X^{(p)} = \emptyset$ if $p < 0$.

Let $D_p(X) = H_p(X^{(p)}, X^{(p-1)})$.

Define $\partial_D : D_p(X) \to D_{p-1}(X)$ to be the connecting homorphism from the exact sequence of the triple $(X^{(p)}, X^{(p-1)}, X^{(p-2)})$. Therefore ∂_D factors as

$H_p(X^{(p)}, X^{(p-1)}) \xrightarrow{\partial} H_{p-1}(X^{(p-1)}) \xrightarrow{j_*} H_p(X^{(p-1)}, X^{(p-2)})$.

Hence $\partial_D^2 = 0$ since

$$H_p(X^{(p)}, X^{(p-1)}) \xrightarrow{\partial} H_{p-1}(X^{(p-1)}) \xrightarrow{j_*} H_{p-1}(X^{(p-1)}, X^{(p-2)}) \xrightarrow{\partial} H_{p-2}(X^{(p-2)}) \xrightarrow{j_*} H_{p-2}(X^{(p-2)}, X^{(p-3)}$$

contains the consecutive maps $H_{p-1}(X^{(p-1)}) \xrightarrow{j_*} H_{p-1}(X^{(p-1)}, X^{(p-2)}) \xrightarrow{\partial} H_{p-2}(X^{(p-2)})$ which is 0 from the exact sequence of the pair $(X^{(p-1)}, X^{(p-2)})$.

Therefore $\big(D_*(X), \partial_D\big)$ forms a chain complex called the *cellular chain complex* of X. Its homology is called the *cellular homology* of X, written $H_*^{\text{cell}}(X)$.

Lemma 16.2.1 $H_q(X^{(p)}, X^{(p-1)}) \cong \begin{cases} \mathrm{F}_{\text{ab}}\{p-\text{cells of } X\} & q = p \\ 0 & \text{otherwise} \end{cases}$

Proof: In each p-cell of X, select a point x_j.

Notice that $X^{(p-1)} \cup (e_j^p - x_j) \simeq X^{(p-1)}$. That is, $X^{(p-1)} \cup (e_j^p - x_j)$ is the subspace of $X^{(p)}$ formed by attaching D^p to $X^{(p-1)}$ along ∂D^p. $X^{(p-1)} \cup (e_j^p - x_j)$ is formed by attaching $D^p - \{*\}$ to $X^{(p-1)}$ along ∂D^p. But using the homotopy equivalence $D^p - \{*\} \simeq \partial D^p$ can construct a continuous deformation of $X^{(p-1)} \cup (e_j^p - x_j)$ back to $X^{(p-1)}$. (i.e. gradually enlarge the hole.)

$$X^{(p-1)} \simeq X^{(p-1)} \cup \Bigg(\bigcup_{p-\text{cells of } X} (e_j^p \setminus \{x_j\}) \Bigg)$$

Note: If $A \overset{j}{\hookrightarrow} B \subset X$ where j is a homotopy equivalence then $H_*(X, A) \xrightarrow{\cong} H_*(X, B)$ using

$$\begin{tikzcd}
\arrow[r] & H_q(A) \arrow[r] \arrow[d, "\cong"] & H_q(X) \arrow[r] \arrow[d, equal] & H_q(X,A) \arrow[r] \arrow[d] & H_{q-1}(A) \arrow[r] \arrow[d, "\cong"] & H_{q-1}(X) \arrow[r] \arrow[d, equal] & {} \\
\arrow[r] & H_q(B) \arrow[r] & H_q(X) \arrow[r] & H_q(X,B) \arrow[r] & H_{q-1}(B) \arrow[r] & H_{q-1}(X) \arrow[r] & {}
\end{tikzcd}$$

and the 5-lemma. (This avoids using the homotopy axiom directly, which would require a homotopy equivalence of pairs.)

Therefore

$$H_*(X^{(p)}, X^{(p-1)}) \cong H_*\left(X^{(p)}, X^{(p-1)} \cup \left(\bigcup_{p-\text{cells of } X} (e_j^p \setminus \{x_j\})\right)\right)$$

Notice that $X^{(p-1)} \cup \left(\bigcup_{p-\text{cells of } X} (e_j^p \setminus x_j)\right) = X^{(p)} \setminus (\cup\{x_j\})$ which is open.

By excision

$$\begin{aligned} H_*\left(X^{(p)}, X^{(p-1)} \cup \left(\bigcup_{p-\text{cells of } X} (e_j^p \setminus \{x_j\})\right)\right) &\cong H_*\left(\left(\bigcup_{p-\text{cells of } X} (e_j^p)\right), \left(\bigcup_{p-\text{cells of } X} (e_j^p \setminus \{x_j\})\right)\right) \\ &\cong \bigoplus_{p-\text{cells of } X} H_*\big(e_j^p, e_j^p \setminus \{x_j\}\big) \end{aligned}$$

where we have excised the closed set $X^{(p-1)}$ from the open set $X^{(p)} \setminus \big(\cup\{x_j\}\big)$.

Up to homeomorphism, $e_j^p = \overset{\circ}{D}{}^p$ and $H_q\left(\overset{\circ}{D}{}^p, \overset{\circ}{D}{}^p \setminus \{*\}\right) = \begin{cases} \mathbb{Z} & q = p \\ 0 & \text{otherwise} \end{cases}$ since

$$\begin{array}{ccccccccc} H_q(\overset{\circ}{D}{}^p \setminus \{*\}) & \longrightarrow & H_q(\overset{\circ}{D}{}^p) & \longrightarrow & H_q(\overset{\circ}{D}{}^p, \overset{\circ}{D}{}^p \setminus \{*\}) & \xrightarrow{\cong} & H_{q-1}(\overset{\circ}{D}{}^p - \{*\}) & \longrightarrow & H_{q-1}(\overset{\circ}{D}{}^p) \longrightarrow \\ & & \| & & & & \| & & \\ & & 0 & & & & H_{q-1}(S^{p-1}) & & \end{array}$$

Hence

$$H_q(X^{(p)}, X^{(p-1)}) = \begin{cases} \displaystyle\bigoplus_{p-\text{cells of } X} \mathbb{Z} & \text{if } q = p\,; \\ 0 & \text{otherwise} \end{cases} \cong \begin{cases} F_{\text{ab}}\{p-\text{cells of } X\} & \text{if } q = p\,; \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 16.2.2 $H_q(X^{(n)}) = \begin{cases} H_q(X) & q < n\,; \\ 0 & q > n. \end{cases}$

Proof:

If $q > n$:

$$\begin{array}{ccccccccc}
H_{q+1}(X^{(n)}) & \longrightarrow & H_{q+1}(X^{(n)}, X^{(n-1)}) & \longrightarrow & H_q(X^{(n-1)}) & \xrightarrow{\cong} & H_q(X^{(n)}) & \longrightarrow & H_q(X^{(n)}, X^{(n-1)}) \\
 & & \| & & & & & & \| \\
 & & 0 & & & & & & 0
\end{array}$$

So $H_q(X^{(n)}) \cong H_q(X^{(n-1)}) \cong \ldots \cong H_q(X^{(-1)}) \cong H_q(\emptyset) \cong 0$.

Similarly if $q < n$:

$$\begin{array}{ccccccc}
H_{q+1}(X^{(n+1)}, X^{(n)}) & \longrightarrow & H_{q+1}(X^{(n)}) & \xrightarrow{\cong} & H_q(X^{(n+1)}) & \longrightarrow & H_q(X^{(n+1)}, X^{(n)}) \\
\| & & & & & & \| \\
0 & & & & & & 0
\end{array}$$

So $H_q(X^{(n)}) \cong H_q(X^{(n+1)}) \cong \ldots \cong H_q(X^{(p)})\ \forall p$.

Therefore $H_q(X) \cong \varinjlim H_q(X^{(p)}) = H_q(X^{(n)})$. $\square$

Theorem 16.2.3 $H\big(D_*(X)\big) = H_*(X)$.

Proof: From the triples $(X^{(n+1)}, X^{(n)}, X^{(n-2)})$ and $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$ we have

$$\begin{tikzcd}
 & H_n(X^{(n-1)}, X^{(n-2)}) \arrow[r, equal] \arrow[d] & 0 & 0 \arrow[d, equal] \\
H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\Delta"] \arrow[d, "\partial"'] & H_n(X^{(n)}, X^{(n-2)}) \arrow[r] \arrow[d, "j_*"'] & H_n(X^{(n+1)}, X^{(n-2)}) \arrow[r] & H_n(X^{(n+1)}, X^{(n)}) \\
H_n(X^{(n)} \arrow[ur, "i_*"] & H_n(X^{(n)}, X^{(n-1)}) \arrow[r, equal] \arrow[d, "\partial_D"'] & D_n & \\
 & H_{n-1}(X^{(n-2)}, X^{(n-1)}) \arrow[r, equal] & D_{n-1} &
\end{tikzcd}$$

$j_* i_*$ is induced by the canonical map of pairs $\big(X^{(n)}, \emptyset\big) \to \big(X^{(n)}, X^{(n-1)}\big)$ so $j_*\Delta = j_* i_* \partial = \partial_D$.

The diagram shows that $\ker(\partial_D)_n \cong H_n(X^{(n)}, X^{(n-1)})$.

Therefore $H_n(D_*) = \ker(\partial D)_n / \operatorname{Im}(\partial D)_n \cong H_n(X^{(n)}, X^{(n-1)}) / \operatorname{Im}\Delta \cong H_n(X^{(n+1)}, X^{(n-2)})$.

$$\begin{array}{ccccccc} H_n(X^{(n-2)}) & \longrightarrow & H_n(X^{(n+1)}) & \xrightarrow{\cong} & H_n(X^{(n+1)}, X^{(n-2)}) & \longrightarrow & H_{n-1}(X^{(n-2)}) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

Thus $H_n(D_*) \cong H_n(X^{(n+1)}, X^{(n-2)}) \cong H_n(X^{(n+1)}) \cong H_n(X)$ $\square$

16.2.1 Application: Calculation of $H_*(\mathbb{R}P^n)$ (for $1 \le n \le \infty$)

$p : S^n \to \mathbb{R}P^n$ $\quad$ $p =$ quotient map (covering projection)

Want to find "compatible" CW-complex structures on S^n and $\mathbb{R}P^n$ (i.e. such that p is a "cellular" map).

$S^n = e_0^+ \cup e_0^- \cup e_1^+ \cup e_1^- \cup \ldots \cup e_n^+ \cup e_n^-$ where $e_j^+ = \{(x_0, \ldots, x_j) \in S^j \mid x_j > 0\}$.

Let $e_j = p(e_j^+) \subset \mathbb{R}P^n$.

$p\big|_{e_j^+}$ is a homeomorphism. In fact, $e_j = p(e_j^+) = p(e_j^-)$ is an evenly covered open set in $\mathbb{R}P^n$ with $p^{-1}(e_j) = e_j^+ \cup e_j^-$. So e_j is an open j-cell and $\mathbb{R}P^n = e_0 \cup e_1 \cup \ldots \cup e_n$ is a CW-complex structure on $\mathbb{R}P^n$ (and p is a cellular map).

We define $\mathbb{R}P^\infty := \cup_n \mathbb{R}P^n = e_0 \cup e_1 \cup \ldots \cup e_n \cup \ldots$ and topologize it by declaring that $A \subset \mathbb{R}P^n$ shall be closed if and only if $A \cup \overline{e_n}$ is closed in $\overline{e_n}$ for all n. Thus by construction $\mathbb{R}P^\infty$ is also a CW-complex.

p induces a map of cellular chain complexes $p_* : D_*(S^n) \to D_*(\mathbb{R}P^n)$.

$D_j(S^n) \cong \mathrm{F}_{\mathrm{ab}}\{j\text{-cells of} S^n\} \cong \mathbb{Z} \oplus \mathbb{Z}$ $\qquad$ $D_j(\mathbb{R}P^n) \cong \mathbb{Z}$

$$\begin{array}{ccccccccccccc} & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & & & \mathbb{Z}\oplus\mathbb{Z} & & & & \mathbb{Z}\oplus\mathbb{Z} & & \\ & & \| & & \| & & & & \| & & & & \| & & \\ 0 & \longrightarrow & D_n(S^n) & \xrightarrow{\partial} & D_{n-1}(S^n) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & D_j(S^n) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & D_0(S^n) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle p_*} & & \\ 0 & \longrightarrow & D_n(\mathbb{R}P^n) & \xrightarrow{\partial} & D_{n-1}(\mathbb{R}P^n) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & D_j(\mathbb{R}P^n) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & D_0(\mathbb{R}P^n) & \longrightarrow & 0 \\ & & \| & & \| & & & & \| & & & & \| & & \\ & & \mathbb{Z} & & \mathbb{Z} & & & & \mathbb{Z} & & & & \mathbb{Z} & & \end{array}$$

To determine $\partial : D_j(\mathbb{R}P^n) \to D_{j-1}(\mathbb{R}P^n)$ first determine $\partial : D_j(S^n) \to D_{j-1}(S^{n-1})$.

Let $a : S^n \to S^n$ denote the antipodal map $a(x) = x$.

a respects the cellular structure of S^n: $\quad a(e_j^+) = e_j^- \qquad a(e_j^-) = e_j^+$
so it induces a chain map $a_* : D_*(S^n) \to D_*(S^n)$.

We pick generators for $D_*(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ as follows.

In the summand $\mathbb{Z} \subset D_0(S^n)$ corresponding to e_0^+ pick one of the two generators and call it f_0^+. Then $a_*(f_0^+)$ will be a generator for the other $\mathbb{Z}$ summand in $D_0(S^n)$ so set $f_0^- := a_* f_0^+$.

Lemma 16.2.4 *$f_0^+ - f_0^-$ generates* $\operatorname{Im} \partial$.

Proof: a induces the identity on $H_0(S^n)$ (any self-map of a connected space does), so $[f_0^-] = a_*[f_0^-] = a_*[f_0^+] = [f_0^+]$. Hence $[f_0^+] - [f_0^-]$ is the zero homology class so $f_0^+ - f_0^- \in \operatorname{Im} \partial$.

Since $D_*(S^n)$ is a complex whose homology gives $H_*(S^n)$ and we know $H_0(S^n) \cong \mathbb{Z}$, we conclude that $f_0^+ - f_0^-$ generates $\operatorname{Im} \partial$. $\square$

Pick a generator of the $\mathbb{Z}$ summand of $D_1(S^n)$ corresponding to e_1^+ and call it f_1^+. So $\partial f_1^+ = m(f_0^+ - f_0^-)$ for some m. Replacing f_1^+ by $-f_1^+$ if necessary, we may assume that $m \geq 0$. Let $f_1^- = a f_1^+$. Then $\partial f_1^- = m(a f_0^+ - a f_0^-) = m(f_0^- - a^2 f_0^+) = m(f_0^- - f_0^+) = -m(f_0^+ - f_0^-)$. Since $\partial\big(D_1(S^n)\big)$ is generated by ∂f_1^+ and ∂f_1^-, the only way it can be generated by $f_0^+ - f_0^-$ is if $m = 1$.

$\partial f_1^+ = f_0^+ - f_0^- \qquad \partial f_1^- = -(f_0^+ - f_0^-)$

Therefore $\ker \partial_1 : D_1(S^n) \to D_0(S^n)$ is generated by $f_1^+ + f_1^-$. But since $H_1(S^n) = 0$, $\ker \partial_1 = \operatorname{Im} \partial_2$.

Pick a generator $f_2^+ \in D_2(S^n)$ corresponding to e_2^+. Then $\partial f_2^+ = m(f_1^+ - f_1^-)$ for some m, and as above we may assume $m \geq 0$. Let $f_2^- = a_* f_2^+$. Then $\partial f_2^- = m(f_1^- + f_1^+)$ and so as above we conclude that $m = 1$.

$\partial f_2^+ = f_1^+ + f_1^- \qquad \partial f_2^- = f_1^+ + f_1^-$ Therefore $\ker \partial_2$ is generated by $f_2^+ - f_2^-$. As above, pick f_3^+ and f_3^- s.t. $f_3^- = a_* f_3^+$, $\partial f_3^+ = f_2^+ - f_2^-$ and $\partial f_2^- = -(f_2^+ - f_2^-)$.

Continuing, get f_j^+ and f_j^- for $j = 0, \ldots, n$ s.t. $f_j^- = a_* f_j^+$ and $\partial f_j^+ = \partial f_j^- = f_{j-1}^+ - f_{j-1}^-$ when j is even, while $\partial f_j^+ = f_{j-1}^+ - f_{j-1}^-$ and $\partial f_j^- = -(f_{j-1}^+ - f_{j-1}^-)$ when j is odd.

For each j, $f_j := p_*(f_j) = p_*(f_j^-) \in D_j(\mathbb{R}P^n)$ since $p_* a_* = p_*$.

Therefore $\partial f_j = \begin{cases} f_{j-1} + f_{j-1} = 2 f_{j-1} & j \text{ even;} \\ f_{j-1} - f_{j-1} = 0 & j \text{ odd.} \end{cases}$

$$D_*(\mathbb{R}P^n) \qquad \longrightarrow \mathbb{Z} \longrightarrow \ldots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

n even:

$$H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ odd, } q < n \\ 0 & q \text{ even or } q > n \end{cases}$$

n odd:

$$H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ odd, } q < n \\ 0 & q \text{ even or } q > n. \end{cases}$$

That is, $H_q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & n \text{ (if } n \text{ odd)} \\ \vdots & \\ 0 & 4 \\ \mathbb{Z}/(2\mathbb{Z}) & 3 \\ 0 & 2 \\ \mathbb{Z}/(2\mathbb{Z}) & 1 \\ \mathbb{Z} & 0. \end{cases}$

16.2.2 Complex Projective Space

Regard S^{2n+1} as the unit sphere of $\mathbb{C}^{n+1}$.

An action $S^1 \times S^{2n+1}$ of S^1 on ${}^{2n+1}$ is given by $\big(\lambda, (z_0, \ldots, z_n)\big) \mapsto (\lambda z_0, \ldots, \lambda z_n)$. Note that $|\lambda z_0|^2 + \ldots + |\lambda z_n|^2 = \lambda|(|z_0|^2 + \ldots + |z_n|^2) = 1 \cdot 1 = 1$ so $(\lambda z_0, \ldots, \lambda z_n) \in S^{2n+1}$.

Define as the orbit space $\mathbb{C}P^n := S^{2n+1}/S^1$.

The inclusions $\mathbb{C}^n \overset{i}{\hookrightarrow} \mathbb{C}^{n+1}$, $(z_0, \ldots, z_{n-1}) \mapsto (z_0, \ldots, z_{n-1}, 0)$ respects the S^1 action so i induces $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$.

Proposition 16.2.5 *$\mathbb{C}P^n$ has a CW-structure: $e^0 \cup e^2 \cup \ldots \cup e^{2n}$*

Proof: Suppose by induction that we have given $\mathbb{C}P^{n-1}$ a CW-structure with one cell in each even degree up to $2n-2$: $\mathbb{C}P^{n-1} = e_0 \cup^2 \cup \ldots \cup e^{2n-1}$.

Let $z = (z_0, \ldots, z_n)$ represent a point in $\mathbb{C}P^n$. Then z lies in $\mathbb{C}P^{n-1}$ if and only if $z_n = 0$. By multiplying by a suitable $\lambda \in S^1$ we may choose to new representative for z in which z_n is real and $z_n \geq 0$. Unless $z_n = 0$, z will have a unique representativve of this form. Writing $z_j = x_j + x_j + iy_j$ (with $y_n = 0$) we have $z = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n, 0)$ with $x_n \geq 0$.

Let $E^{2n}_+ = \{(w_0, \ldots, w_{2n}) \in S^{2n} \mid w_{2n} \geq 0\}$. E^{2n}_+ is a $2n$-cell.

Define f^{2n} to be the composite $E^{2n}_+ \hookrightarrow S^{2n} \hookrightarrow S^{2n+1} \xrightarrow{\text{quotient}} \mathbb{C}P^n$. (That is, $(w_0, \ldots, w_{2n}) \mapsto [(w_0 + iw_1, w_2 + iw_3, \ldots, w_{2n-2} + iw_{2n-1}, w_{2n})]$.)

$e^{2n} = \{w_0, \ldots, w_{2k} \in S^{2k} \mid w_{2n} > 0\}$. By the above, the restriction of f_{2n} to e^{2n} is a bijection. It is also an open map (by definition of quotient topology a set map is open if an donly if its inverse image is open and the inverse image of $f^{2n}(U)$ is $\cup_{\lambda \in S^1} \lambda \cdot U$) so it is a homeomorphism. Therefore $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup e^{2n} = e^0 \cup e^2 \cup \ldots \cup e^{2n}$ is a CW-complex.

(Note: By compactness, the $3rd$ condition is automatic when there are only finitely many cells.) □

Can define a CW-complex $\mathbb{C}P^\infty$ by $\mathbb{C}P^\infty := \cup_n \mathbb{C}p^n = e^0 \cup e^2 \cup \ldots \cup e^{2n} \cup \ldots$ topologized by $A \subset \mathbb{C}P^\infty$ is closed if and only if $A \cap \overline{e^2 n}$ is closed in $\overline{e_n}$ for all n.

Theorem 16.2.6 $H_q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & q \text{ even, } q \leq 2n \\ 0 & q \text{ odd, } q > 2n. \end{cases}$

Proof:

$$\begin{array}{ccccccccccccc} 0 \longrightarrow & D_{2n}(\mathbb{C}P^n) & \longrightarrow & D_{2n-1}(\mathbb{C}P^n) & \longrightarrow & D_{2n-1}(\mathbb{C}P^n) & \longrightarrow & \dots & \longrightarrow & D_1(\mathbb{C}P^n) & \longrightarrow & D_0(\mathbb{C}P^n) & \longrightarrow 0 \\ & \| & & \| & & \| & & & & \| & & \| & \\ & \mathbb{Z} & & 0 & & \mathbb{Z} & & & & 0 & & \mathbb{Z} & \end{array}$$

Every $2nd$ group is 0 so the boundary maps are all 0. Therefore $H_*(\mathbb{C}P^n)$ is as stated. □

Remark 16.2.7 *Using the same ideas as above, one can define quaternionic projective space $\mathbb{H}P^n$ by $\mathbb{H}P^n := S^{4n+3}/S^3$ where we think of S^3 as the unit sphere of the quaternions $\mathbb{H}$ and S^{4n+3} as the unit sphere in $\mathbb{H}P^{n+1}$ with quaternionic multiplication as the action. n this case we get that $\mathbb{H}P^n$ is a CW-complex of the form $\mathbb{H}P^n = e^0 \cup e^4 \cup \dots e^{4n}$. We can also define $\mathbb{H}P^\infty = \cup_n \mathbb{H}P^n = e^0 \cup e^4 \cup \dots e^{4n} \dots$. As above we get* $H_q(\mathbb{H}P^n) = \begin{cases} \mathbb{Z} & q \equiv 0(4), q \leq 4n; \\ 0 & q \not\equiv 0(4), \text{ or } q > 4n. \end{cases}$

(Details left as an exercise.)

Chapter 17

Cohomology

Definition 17.0.8 *A* cochain complex (C, d) *of abelian groups consists of an abelian group* C^p *for each integer* p *together with a morphism* $d^p : C^p \to C^{p-1}$ *for each* p *such that* $d^{p+1} \circ d^p = 0$. *The maps* d^p *are called* coboundary operators *or* differentials.

Aside from the fact that we have chosen to number the groups differently, the concept of cochain complex is identical to that of chain complex. (Given a cochain complex (C, d) we could make it into a chain complex by renumbering the groups, letting $C_p := C^{-p}$, and vice versa.) So we can make all the same homological definitions and get the same homological theorems. A summary follows:

$\ker d^{p+1} : C^p \to C^{p+1}$ is denoted $Z^p(C)$. Its elements are called *cocycles*.

$\operatorname{Im} d^p : C^{p-1} \to C^p$ is denoted $B^p(C)$. Its elements are called *coboundaries*.

$H^p(C) := Z^p(C)/B^p(C)$ called the *pth cohomology group* of C.

A *cochain map* $f : C \to D$ consists of a group homomorphism f^p for each p s.t.

$$\begin{tikzcd} C^p \arrow[r, "d^{p+1}"] \arrow[d, "f^p"] & C^{p+1} \arrow[d, "f^{p+1}"] \\ D^p \arrow[r, "d^{p+1}"] & D^{p+1} \end{tikzcd} \quad \text{commutes.}$$

Proposition 17.0.9

A cochain map f induces a homomorphism denoted $f^* : H^*(C) \to H^*(D)$. □

Theorem 17.0.10 *Let* $0 \to P \to Q \to R \to)$ *be a short exact sequence of chain complexes. Then there is an induced natural (long) exact cohomology sequence*

$$\ldots \to H^n(P) \to H^n(Q) \to H^n(R) \xrightarrow{\delta} H^{n+1}(P) \to H^{n+1}(Q) \to \ldots$$

Let (C, ∂) be a chain complex. Form a cochain complex (Q, δ) as follows.

$Q^p := \text{Hom}(C_p, \mathbb{Z})$.

Notation: for $c \in C_p$, $f \in Q^p = \text{Hom}(C, \mathbb{Z})$ write $\langle f, c\rangle$ for $f(c)$.

Define $\delta : Q^p \to Q^{p+1}$ bu $\langle \delta f, c\rangle := (-1)^{p+1}\langle f, \partial c\rangle$ where $c \in C_{p+1}$.

$\partial^2 = 0$ implies $\delta^2 = 0$.

Remark 17.0.11 *Changing one or more boundary maps by minus signs has no affect on kernels or images so it does not affect homology. The sign convention $(-1)^{p+1}$ chosen above makes the signs come out better in some of the later formulas. This is the convention used in Dold, Milnor, Mac Lane, and Selick. An explanation of the intuition behind it can be found in Dold (page 173) or Selick (page 30). Notice Dold's convention on page 167 chosen so that when $n = 0$, $\partial f = 0$ implies f is a chain map. There are also other sign conventions ($(-1)^p$ or no sign at all) in the literature (e.g. Greenberg-Harper, Eilenberg-Steenrod, Munkres, Spanier, Whitehead) but they lead to less aesthetic formulas in several places and/or diagrams which only commute up to sign.*

Let $[c]$ and $[f]$ be homology and cohomology classes in C_*, Q_* respectively. Then $\langle [f], [c]\rangle$ has a well-defined meaning since if c' is another representative for c then for some d, $\langle f, c' - c\rangle = \langle f, \partial d\rangle = \pm\langle \delta f, d\rangle \pm \langle 0, d\rangle = 0$ and similarly if $f - f' = \delta g$ for some g then $\langle f - f', c\rangle = \langle \delta g, c\rangle = \pm\langle g, \partial c\rangle = 0$

$\langle\ ,\ \rangle$ is often called the *Kronecker product* or *Kronecker pairing.*

Any chain map $\phi : C \to D$ induces, by duality, a cochain map $\phi^* : \text{Hom}(D, \mathbb{Z}) \to \text{Hom}(C, \mathbb{Z})$. $\langle \phi^p(g), c\rangle := \langle g, \phi_p c\rangle$.

If C is a free chain complex (i.e. C_p is a free abelian group $\forall p$) then there is a formula, called the "Universal Coefficient Theorem" giving $H^*\big(\text{Hom}(C, \mathbb{Z})\big)$ in terms of $H_*C()$. An immediate corollary of the Universal Coefficient Theorem is that if C, D are free chain complexes and $\phi : C \to D$ s.t. $\phi_* : H_p(C) \to H_p(D)$ is an isomorphism $\forall p$, then $\phi^* : H^p\big(\text{Hom}(D, \mathbb{Z})\big) \to H^p\big(\text{Hom}(C, \mathbb{Z})\big)$ is an isomorphism $\forall p$. We will not get to the Universal Coefficient Theorem in this course but we will give a direct proof of this corollary now.

From algebra recall:

Theorem 17.0.12 *If R is a PID and M is a free R-module than any R-submodule of M is a free R-module. In particular: letting $R = \mathbb{Z}$: A subgroup of a free abelian group is a free abelian group.* □

Proposition 17.0.13 *Let C be a free chain complex s.t. $H_q(C) = 0\ \forall\ q$. Then $H^q\big(\text{Hom}(C, \mathbb{Z})\big) = 0\ \forall\ q$.*

Proof: $C_p/\ker\partial_p \cong \operatorname{Im}\partial_p = B_{p-1}$.

Since $H_*(C) = 0$, $\ker\partial_p = \operatorname{Im}\partial_{p+1} = B_p$. That is, $0 \to B_p \to C_p \xrightarrow{\partial_p} B_{p-1} \to 0$ is a short exact sequence. Since $B_{p-1} \subset C_{p-1}$ is a free abelian group, the sequence splits: $0 \to B_p \to C_p \underset{s}{\overset{\partial_p}{\rightleftarrows}} B_{p-1} \to 0$. i.e. $\exists$ a subgroup $U_p := \operatorname{Im} s$ of C_p s.t. $\partial U_p \cong B_{p-1}$ and $C_p \cong B_p \oplus U_p$ with $\partial(b,u) = (\partial u, 0)$.

C

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, bend left, "\partial"] \arrow[r] & (B_{p+1} & \oplus U_{p+1}) \arrow[r] \arrow[r, bend left, "\partial"] & (B_p & \oplus\ U_p) \arrow[r] \arrow[r, bend left, "\partial"] & (B_{p-1} & \oplus U_{p-1}) \arrow[r] \arrow[r, bend left, "\partial"] & {}
\end{tikzcd}$$

so dualizing gives a similar picture in $\operatorname{Hom}(C,\mathbb{Z})$. That is, letting $U^p := \operatorname{Hom}(U^p,\mathbb{Z})$ and $V^p := \operatorname{Hom}(B^p,\mathbb{Z})$:

$\operatorname{Hom}(C,\mathbb{Z})$

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, bend left] \arrow[r] & (U^{p-1} & \oplus V_{p-1}) \arrow[r] \arrow[r, bend left] & (U^p & \oplus\ V^p) \arrow[r] \arrow[r, bend left] & (U^{p+1} & \oplus V_{p+1}) \arrow[r] \arrow[r, bend left] & {}
\end{tikzcd}$$

So $H^*\big(\operatorname{Hom}(C,\mathbb{Z}\big) = 0$. □

Corollary 17.0.14

Let $0 \to C \xrightarrow{\phi} D \xrightarrow{\alpha} E \to 0$ be a short exact sequence of chain complexes. Suppose that E is a free chain complex. If $\phi_* : H_q(C) \to H_q(D)$ is an isomorphism $\forall q$ then so is $\phi^* : H^*\big(\operatorname{Hom}(D,\mathbb{Z}\big) \to H^*\big(\operatorname{Hom}(C,\mathbb{Z}\big) = 0.$.

Proof: Since E_p is free $\forall p$, $D_p \cong C_p \oplus E_p$ and thus

$\operatorname{Hom}(D_p,\mathbb{Z}) \cong \operatorname{Hom}(C_p,\mathbb{Z}) \oplus \operatorname{Hom}(E_p,\mathbb{Z})$. Thus in particular,

$0 \to \operatorname{Hom}(E,\mathbb{Z}) \xrightarrow{\alpha^*} \operatorname{Hom}(D,\mathbb{Z}) \xrightarrow{\phi^*} \operatorname{Hom}(C,\mathbb{Z}) \to 0$ is again exact (a short exact sequence of cochain complexes). To show that ϕ^* is an isomorphism on cohomology, by the long exact sequence it suffices to show that $\operatorname{Hom}(E,\mathbb{Z}) \cong 0\ \forall q$. But $H_q(E) = 0\ \forall q$ by the long exact homology sequence of $0 \to C \xrightarrow{\phi} D \xrightarrow{\alpha} E \to 0$ so the corollary follows from the previous proposition. □

Note: The hypothesis that E be free is really needed. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/(2\mathbb{Z}) \to 0$ is short exact but

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(\mathbb{Z}/(2\mathbb{Z}) & \longrightarrow & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) & \longrightarrow & \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & 0 & & \mathbb{Z} & & \mathbb{Z} & &
\end{array}$$

is not.

Theorem 17.0.15 *(Algebraic Mapping Cylinder) Let C, D be free chain complexes and let $\phi : C \to D$. Then $\exists$ an injective chain homotopy equivalence $j : D \underset{k}{\overset{\simeq}{\rightleftarrows}} \tilde{D}$ (with chain* *homotopy inverse k) and an injection $i : C \to \tilde{D}$ s.t. $\phi = k \circ i$, $i \simeq j \circ \phi$, and $\tilde{D}/\operatorname{Im} j$ is free, and $\tilde{D}/\operatorname{Im} i$ is free.*

Corollary 17.0.16 *Let C, D be free chain complexes. Suppose $\phi^* C \to D$ such that $\phi_q : H_q(C) \to H_q(D)$ is an isomorphism $\forall q$. Then $\phi^* : H^q\big(\operatorname{Hom}(D, \mathbb{Z})\big) \to H^q\big(\operatorname{Hom}(C, \mathbb{Z})\big)$ is an isomorphism $\forall q$.*

Warning: To use this theorem to conclude that ϕ^p is an isomorphism for some particular p, we must know that ϕ_q is an isomorphism $\forall q$, not just for $q = p$. However it will follow from the Universal Coefficient Theorem that it is sufficient to know that ϕ_p and ϕ_{p-1} are isomorphisms to conclude that ϕ^p is an isomorphism.

Proof of Corollary (given Theorem.):

Previous lemma applied to $0 \to D \xrightarrow{j} \tilde{D} \to (\tilde{D}/\operatorname{Im} j) \to 0$ shows j^q is an isomorphism $\forall q$, which implies that $(\phi \circ j)_*$ is an isomorphism, which implies that i_* is an isomorphism. (Exercise: $f \simeq g \Rightarrow f^* \simeq g^*$.) Applying the lemma to $0 \to C \xrightarrow{i} \tilde{D} \to (\tilde{D}/\operatorname{Im} i) \to 0$ shows that i^q is an isomorphism $\forall q$. Therefore ϕ^q is an isomorphism $\forall q$. $\square$

17.0.3 Digression: Mapping Cylinders

Let $f : X \to Y$. If f is an injection then $\exists$ relative homology groups $H_*(Y, X)$ which "measure the difference" between $H_*(X)$ and $H_*(Y)$ and this is often convenient. What if f is not an injection? Then we can replace Y by a homotopy equivalent but "larger" space $\tilde{Y}$, called the mapping cylinder of f, such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \arrow[d, "j", "\simeq"'] \\ & Y \end{tikzcd}$$

homotopy commutes ($j \circ f \simeq i$) with i an injection. The construction is as follows: $\tilde{Y} := (X \times I) \cup_{f'} Y$ where $f' : X \times \{0\} \to Y$ by $(a, 0) \mapsto f(x)$.

$X \hookrightarrow \tilde{Y}$ by $x \mapsto (x, 1)$. $\tilde{Y}$ can be "homotoped" to Y by squashing the cylinder.

Proof of Theorem 17.0.15:

17.1 Cohomology of Spaces

For a simplicial complex K we define the simplicial cochain complex of K by $C^*(K) := \mathrm{Hom}\big(C_*(K), \mathbb{Z}\big)$. Its cohomology is written $H^*(K)$ and called the simplicial cohomology of K.

For a topological space X we define its singular cohomology by $H^*(X) := H^*\big(S^*(X)\big)$ where $S^*X := \mathrm{Hom}\big(S_*(X), \mathbb{Z}\big)$.

And for a CW-comples, its cellular cohomology is defined as $H^*\big(D^*(X)\big)$ where $D^*X := \mathrm{Hom}\big(D_*(X), \mathbb{Z}\big)$.

From the isomorphisms on homology we get immediately $H^*(X) = H^*(|K|)$ and $H^*\big(D^*(X)\big) \cong H^*(X)$.

Can similarly define relative and reduced cohomology groups. e.g.

$H^*(X, A) := H^*\big(S^*(X, A)\big)$ where $S^*(X, A) := \mathrm{Hom}\big(S_*(X, A), \mathbb{Z}\big)$

Definition 17.1.1 (Eilenberg-Steenrod) *Let $\mathcal{A}$ be a class of topological pairs such that:*

1) (X, A) in $\mathcal{A} \Rightarrow (X, X)$, $(X, \emptyset)$, (A, A), $(A, \emptyset)$, and $(X \times I, A \times I)$ are in $\mathcal{A}$;

2) $(*, \emptyset)$ is in $\mathcal{A}$

A cohomology theory *on $\mathcal{A}$ consists of:*

E1) an abelian group $H^n(X, A)$ for each pair (X, A) in $\mathcal{A}$ and each integer n;

E2) a homomorphism $f^* : H^n(Y, B) \to H^n(X, A)$ for each map of pairs $f : (X, A) \to (Y, B)$;

E3) a homomorphism $\delta : H^n(X, A) \to H^{n+1}(A)$ for each integer n

such that:

A1) $1_* = 1$;

A2) $(gf)^* = f^* g^*$;

A3) δ is natural. That is, given $f : (X, A) \to (Y, B)$, the diagram

$$\begin{tikzcd} H^n(B) \arrow[r, "(f \mid_A)^*"] \arrow[d, "\delta"] & H^n(A) \arrow[d, "\delta"] \\ H_{n+1}(Y, B) \arrow[r, "f^*"] & H_{n+1}(X, A) \end{tikzcd}$$

commutes;

A4) Exactness:

$$\longrightarrow H^{n-1}(A) \longrightarrow H^n(X,A) \longrightarrow H^n(X) \longrightarrow H^n(A) \longrightarrow H^{n+1}(X,A) \longrightarrow$$

is exact for every pair (X,A) in $\mathcal{A}$

A5) Homotopy: $f \simeq g \Rightarrow f^* = g^*$.

A6) Excision: If (X,A) is in $\mathcal{A}$ and U is an open subset of X such that $\overline{U} \subset \overset{\circ}{A}$ and $(X-U, A-U)$ is in $\mathcal{A}$ then the inclusion map $(X \smallsetminus U, A \smallsetminus U) \to (X,A)$ induces an isomorphism $H^n(X,A) \xrightarrow{\cong} H^n(X \smallsetminus U, A \smallsetminus U)$ for all n;

A7) Dimension: $H^n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases}$

Theorem 17.1.2 *Singular cohomology is a cohomology theory.*

Proof: For exactness, observe that because all the complexes are free, the fact that $0 \to S_*(A) \to S_*(X) \to S_*(X,A) \to 0$ is exact (and thus $S_*(X) \cong S_*(A) \oplus S_*(X,A)$) implies that $0 \to S^*(X,A) \to S^*(X) \to S^*(A) \to 0$ is exact. Everything else is immediate from the previous theorem and the corresponding statment for homology (and, of course, we get the slightly stronger version of excision, not requiring that U be open, since singular homology satisfies that).

The following theorems also follow easily from the homological counterparts:

Theorem 17.1.3 *(Mayer-Vietoris): Suppose that $(X_1, A) \xrightarrow{j} (X, X_2)$ induces an isomorphism on cohomology. (e.g. if X_1 and X_2 are open. Then there is a long exact cohomology sequence*

$$\ldots \to H_{n-1}(A) \xrightarrow{\Delta} H^n(X) \to H^n(X_1) \oplus H^n(X_2) \to H^n(A) \xrightarrow{\Delta} H^{n+1}(X) \to \ldots$$ □

Theorem 17.1.4

$$H^n(X) \cong \begin{cases} \tilde{H}^n(X) & n > 0; \\ \tilde{H}^0(X) \oplus \mathbb{Z} & n = 0. \end{cases}$$

Also $\tilde{H}^q(X) \cong H^q(X,*)$ □

Theorem 17.1.5 $H^q(S^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ 0 & q \neq 0, n \end{cases}$

Proof: Use cellular cohomology. Write $S^{=}e^0 \cup e^n$.

$$D_*(S^n) \qquad \underset{}{0} \longrightarrow \underset{n th\ \text{pos.}}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow \underset{0th\ \text{pos.}}{\mathbb{Z}} \longrightarrow 0$$

$$D^*(S^n) \qquad \underset{0th\ \text{pos.}}{0 \longrightarrow \mathbb{Z}} \longrightarrow 0 \longrightarrow \ldots \longrightarrow \underset{nth\ \text{pos.}}{0 \longrightarrow \mathbb{Z}} \longrightarrow 0$$

□

Theorem 17.1.6

n even:

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ even, } q < n \\ 0 & q \text{ odd or } q > n \end{cases}$$

n odd:

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ \mathbb{Z}/(2\mathbb{Z}) & q \text{ even, } q < n \\ 0 & q \text{ odd or } q > n. \end{cases}$$

□

Theorem 17.1.7

$$H^q(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & q \text{ even, } q \leq 2n \\ 0 & q \text{ odd, } q > 2n. \end{cases}$$

$$H^q(\mathbb{H}P^n) = \begin{cases} \mathbb{Z} & q \equiv 0(4); \\ 0 & q \not\equiv 0(4). \end{cases}$$

□

Proof: Write $\mathbb{C}P^n = e^0 \cup e^2 \cup \ldots e^{2n}$. Write $\mathbb{H}P^n = e^0 \cup e^4 \cup \ldots e^{4n}$. □

17.2 Cup Products

From the last section (and the Universal Coefficient Theorem), we know that $H^*(X)$ is completely determined by $H_*(X)$, so why bother with cohomology at all? In any potential applicaiton, why not just use homology instead? One answer is that there is a natural way to put a multiplication called the "cup product" on $H^*(X)$ so that $H^*(X)$ becomes a ring. This might be used, for example, in a case where the $H^*(X)$ and $H^*(Y)$ to show that $X \not\simeq Y$ if it should turn out that the multiplications on $H^*(X$ and $H^*(Y)$ were different.

Let $f \in S^p(X)$ and $S^q(X)$. Define $f \cup g \in S^{p+1}(X)$ as follows.
For a generator $T : \Delta^{p+q} \to X$ of $S_{p+q}(X)$ we define
$\langle f \cup g, T\rangle := (-1)^{pq}\big\langle f, T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q})\big\rangle \in \mathbb{Z}$
where $\Delta^p \xhookrightarrow{l(\epsilon_0, \ldots, \epsilon_p)} \Delta^{p+q} \xhookrightarrow{\quad T \quad} X$.

(Since g has moved $T \circ l(\epsilon_0, \ldots, \epsilon_p)$, the sign convention is in keeping with the convention of introducing a sign of $(-1)^{pq}$ whenever interchanging symbols of degree p and q.)

Notation: Let $1 \in S^0(X)$ be the element defined by $\langle 1, T\rangle = 1$ for all generators $T \in S_0(X)$. (Thus as a function in $\mathrm{Hom}\big(S_0(X), \mathbb{Z}\big) \cong \mathbb{Z}$, $1 = \epsilon =$ a generator.)

The following properties follow immediately from the definitions:

1. $f \cup (g + h) = (f \cup g) + (f \cup h)$
2. $(f + g) \cup h = (f \cup g) + (h \cup g)$
3. $(f \cup g) \cup h = f \cup (g \cup h)$
4. $1 \cup g = g \cup 1 = g$

So $\cup$ turns $S^*(X)$ into a ring (with unit). It is called a graded ring with $S^p(X)$ being the p gradation where:

Definition 17.2.1 *A ring R is called a graded ring if $\exists$ subgroups R_p s.t. $R = \oplus_p R_p$ and the multiplication satisfies $R_p \cdot R_q \subset R_{p+q}$.*

Lemma 17.2.2 *Let $f \in S^p(X)$ and $g \in S^q(X)$. Then $\delta(f \cup g) = \delta f \cup g + (-1)^p f \cup \delta g$.*

Proof: Let $T : \Delta^{p+q+1} \to X$ be a generator of $S_{p+q+1}(X)$.
$\langle \delta(f \cup g), T\rangle$
$= (-1)^{(p+1)q}\big\langle \delta f, T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q+1})\big\rangle$
$= (-1)^{(pq+q}(-1)^{p+1}\big\langle f, \partial T \circ l(\epsilon_0, \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q+1})\big\rangle$
$= (-1)^{pq+p+q+1} \sum_{i=0}^{p+1}(-1)^i\big\langle f, T \circ l(\epsilon_0, \ldots, \hat{\epsilon_i} \ldots, \epsilon_p)\big\rangle\big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q+1})\big\rangle.$

Similarly
$(-1)^p f \cup \delta g$
$= (-1)^p(-1)^{pq+p+q+1} \sum_{i=p}^{p+q+1}(-1)^{i-p}\big\langle f, T\circ l(\epsilon_0,\ldots,\epsilon_p)\big\rangle\big\langle g, T\circ l(\epsilon_p,\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+q+1})\big\rangle$
$= (-1)^{pq+p+q+1} \sum_{i=p}^{p+q+1}(-1)^{i}\big\langle f, T\circ l(\epsilon_0,\ldots,\epsilon_p)\big\rangle\big\langle g, T\circ l(\epsilon_p,\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+q+1})\big\rangle.$

Notice that the term of $\langle \delta f \cup g, T\rangle$ corresponding to $i = p+1$ equals that of $(-1)^p\langle\delta(f \cup \delta g), T\rangle$ corresponding to $i = p$ except that the signs are opposite so they cancel when we form $\langle \delta f \cup g, T\rangle + (-1)^p\langle\delta(f\cup\delta g), T\rangle$. On the other hand,
$\langle\delta(f\cup g), T\rangle$
$= (-1)^{p+q+1}\langle f\cup\delta g, \partial T\rangle$
$$= (-1)^{p+q+1}\sum_{i=0}^{p+q+1}(-1)^i\big\langle f\cup g, T\circ l(\epsilon_0,\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+q+1})\big\rangle$$
$= (-1)^{p+q+1}(-1)^{pq}\sum_{i=0}^{p}(-1)^i$
$\big\langle f, T\circ l(\epsilon_0,\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+1})\big\rangle\big\langle g, T\circ l(\epsilon_{p+1},\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+q+1})\big\rangle$
$$= (-1)^{pq+p+q+1}\sum_{i=0}^{p}(-1)^i$$
$\big\langle f, T\circ l(\epsilon_0,\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+1})\big\rangle\big\langle g, T\circ l(\epsilon_{p+1},\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+q+1})\big\rangle$
$$+ (-1)^{pq+p+q+1}\sum_{i=p+1}^{p+q+1}(-1)^i$$
$\big\langle f, T\circ l(\epsilon_0,\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+1})\big\rangle\big\langle g, T\circ l(\epsilon_{p+1},\ldots,\hat{\epsilon}_i\ldots,\epsilon_{p+q+1})\big\rangle$
$= \langle\delta f\cup g + (-1)^p f\cup\delta g, T\rangle.$ □

Corollary 17.2.3 *If $[f] \in H^p(X)$ and $[g] \in H^q(X)$ then $[f]\cup[g]$ is a well defined element of $H^{p+q}(X)$.*

Proof:
If $\delta f = 0$ and $\delta g = 0$ then $\delta(f\cup g) = 0$ by the lemma.
Also, if $f - f' = \delta h$ then $\delta(h\cup g) = \delta h\cup g + (-1)^{p+1}h\cup\delta g = (f-f')\cup g + 0 = f\cup g - f'\cup g$. Hence $[f\cup g] = [f'\cup g]$.
Similarly if $[g] = [g'] = \delta h$ then $[f\cup g] = [f\cup g']$. □

Proposition 17.2.4 $\delta 1 = 0$

Proof:
Let $T : I = \Delta_1 \to X$ be a generator of $S_1(X)$.
$\langle\delta 1, T\rangle = -\langle 1, \partial T\rangle = -\langle 1, T(1) - T(0)\rangle = -(1-1) = 0$ □

Corollary 17.2.5 *$H^*(X)$ is a graded ring with $[1]$ as unit.* $\square$

From now on we will write 1 for $[1] \in H^0(X)$.

Proposition 17.2.6 *Let $\phi : X \to Y$. Then $\phi^* : S^*(Y) \to S^*(X)$ and $\phi^* : H^*(Y) \to H^*(X)$ are ring homomorphisms.*

Proof:
$\langle \phi^*(f \cup g), T\rangle = \langle (f \cup g), \phi_* T\rangle = (-1)^{pq}\big\langle f, \phi_* T \circ l(\epsilon_0 \ldots, \epsilon_p)\big\rangle\big\langle g, \phi_* T \circ l(\epsilon_p \ldots, \epsilon_{p+q})\big\rangle = (-1)^{pq}\big\langle \phi^* f, T \circ l(\epsilon_0 \ldots, \epsilon_p)\big\rangle\big\langle \phi^* g, T \circ l(\epsilon_p \ldots, \epsilon_{p+q})\big\rangle\langle \phi^*(f) \cup \phi^*(g), T\rangle$ $\square$

Definition 17.2.7 *A graded ring $R = \oplus_p R_p$ is called* graded commutative *if for $a \in R_p$, $b \in R_q$, $ab = (-1)^{pq} ba$.*

Theorem 17.2.8 *$H^*(X)$ is graded commutative.*

Remark 17.2.9 *It is note true that $S^*(X)$ is grade commutative. Instead, $ab - (-1)^{pq} ba = \delta$(something).*

Proof:
Define $\theta : S_*(X) \to S_*(X)$ as follows. For a generator $T : \Delta^p \to X \in S_p(X)$ define $\theta(T) = (-1)^{\frac{1}{2}p(p+1)} T \circ l(\epsilon_p, \epsilon_{p-1}, \ldots, \epsilon_1, \epsilon_0) \in S_p(X)$.
Write $\lambda_p := (-1)^{\frac{1}{2}p(p+1)}$.

Lemma 17.2.10 *θ is a chain map.*

(The factor λ_p was included so that this would be true.)
Proof: For a generator $T \in S_p(X)$,
$\partial\theta(T) = \lambda_p \partial T \circ l(\epsilon_p, \ldots, \epsilon_0) = \lambda_p \sum_{i=0}^{p} (-1)^{p-i} T \circ l(\epsilon_p, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_0)$.
$\theta\partial(T) = \theta\left(\sum_{i=0}^{p} (-1)^i T \circ l(\epsilon_0, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_p)\right) = \lambda_{p-1} \sum_{i=0}^{p} (-1)^i T \circ l(\epsilon_p, \ldots, \hat{\epsilon}_i, \ldots, \epsilon_0)$.
However $\lambda_p (-1)^{p-i} = \lambda_p (-1)^{i-p} = (-1)^{\frac{1}{2}p(p+1)+i-p} = (-1)^{\frac{1}{2}(p^2-p)+i} = (-1)^i \lambda_p$. $\square$

Lemma 17.2.11 *$\theta \simeq i$*

Proof: Acyclic models.
If you examine the proof that sd $\simeq 1$ you discover that the only properties of sd use are:

1. $\forall f : X \to Y$, $f \circ \mathrm{sd}_X = \mathrm{sd}_Y \circ f$

2. $\mathrm{sd}_0 = 1 : S_0(X) \to S_0(X)$.

Since θ satisfies these also, the proof can be repeated, word for word, with θ replacing sd.

Proof of Theorem (cont.):

Since $\theta = \mathrm{id} : H_*(X) \to H_*(X,\ \theta^* = \mathrm{id} : H^*(X) \to H^*(X)$.

Let $[f] \in H^p(X)$, $[g] \in H^q(X)$. For a generator $T \in S_{p+q}(X)$:

$$\begin{aligned}
\langle \theta^*(f \cup g), T\rangle &= \langle (f \cup g), \theta T\rangle \\
&= \lambda_{p+q}\big\langle (f \cup g), \theta T \circ l(\epsilon_{p+q}, \ldots, \epsilon_0)\big\rangle \\
&= \lambda_{p+q}(-1)^{pq}\big\langle f, T \circ l(\epsilon_{p+q}, \ldots, \epsilon_q)\big\rangle\big\langle g, T \circ l(\epsilon_q, \ldots, \epsilon_0)\big\rangle \\
&= \lambda_{p+q}(-1)^{pq}\big\langle f, \lambda_p \theta T \circ l(\epsilon_q, \ldots, \epsilon_{p+q})\big\rangle\big\langle g, \lambda_q \theta T \circ l(\epsilon_0, \ldots, \epsilon_q)\big\rangle \\
&= \lambda_{p+q}\lambda_p\lambda_q(-1)^{pq}\big\langle \theta^* f, \theta T \circ l(\epsilon_q, \ldots, \epsilon_{p+q})\big\rangle\big\langle \theta^* g, T \circ l(\epsilon_0, \ldots, \epsilon_q)\big\rangle \\
&= \lambda_{p+q}\lambda_p\lambda_q\langle \theta^* f \cup \theta^* g, T\rangle
\end{aligned}$$

So $\theta^*(f \cup g) = \lambda_{p+q}\lambda_p\lambda_q \theta^* g \cup \theta^* f$.

Hence $[f \cup g] = [\theta^*(f \cup g)] = \lambda_{p+q}\lambda_p\lambda_q[\theta^* g] \cup [\theta^* f] = \lambda_{p+q}\lambda_q\lambda_q[g] \cup [f]$.

However

$$\begin{aligned}
\lambda_{p+q}\lambda_p\lambda_q &= (-1)^{\frac{1}{2}(p+q)(p+q+1)+\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)} \\
&= (-1)^{\frac{1}{2}(p^2+2pq+q^2+p+q+p^2+p+q^2+q)} \\
&= (-1)^{\frac{1}{2}(2p^2+2pq+2q^2+2p+2q)} \\
&= (-1)^{p^2+pq+q^2+p+q} \\
&= (-1)^{pq}(-1)^{p(p+1)}(-1)^{q(q+1)} = (-1)^{pq}.
\end{aligned}$$

□

This is a "real" sign: does not depend upon the sign conventions.

17.2.1 Relative Cup Products

Let $j : A \hookrightarrow X$.

$0 \to S_*(A) \xhookrightarrow{j_*} S_*(X) \xrightarrow{c_*} S_*(X, A) \to 0.$

$0 \to S^*(X, A) \xhookrightarrow{c^*} S^*(X) \xrightarrow{j^*} S^*(A) \to 0.$

Let $f \in S^p(X)$ and let $g \in S^q(X, A)$.

j^* is a ring homomorphism, so $S^*(X, A)$ is an ideal in $S^*(X)$. i.e. $f \cup c^* g \in S^{p+q}(X, A)$.

Write $f \cup g$ for $f \cup c^* g) \in S^{p+q}(X, A) \subset S^*(X)$. That is, $c^*(f \cup g) := f \cup c^* g$. (Explicitly, observe that $j^*(f \cup c^* g) = j^f \cup j^* c^* g = j^* f \cup 0 = 0$ so $f \cup c^* g \in \operatorname{Im} c^*$ and therefore it defines an element of S^{p+q} which we are writing as $f \cup g$.) In computer science language, we are "overloading" the symbol $\cup$, meaning that its interpretation depends upon its arguments.

Similarly if $f \in S^p(X,A)$ and '$g \in S^q(X)$ we can define an element of $S^{p+q}(X,A)$ denoted again $f \cup g$ by $c^*(f \cup g) := f \cup c^* g$.

If $\delta f = 0$ and $\delta g = 0$ then $c^*\delta(f \cup g) = \delta c^*(f \cup g) = \delta(f \cup c^g) = 0$, and so $\delta(f \cup g) = 0$ since c^* is a monomorphism. Therefore $[f] \cup [g] \in H^{p+q}(X,A)$.
Check that it this is well defined:

If $f - f' = \delta h$ then $c^*\delta(h \cup g) = \delta(h \cup c^* g) = \delta h \cup c^* g = f \cup c^* g - f' \cup c^* g = c*(f \cup g - f' \cup g)$. Therefore $\delta(h \cup g) = f \cup g - f' \cup g$ so $[f \cup g] = [f' \cup g]$. Also if $g - g' = \delta k$ then $c^*\delta(f \cup k) = \delta(f \cup c^* k) = \pm\big(f \cup c^*(g - g')\big) = \pm c^*(f \cup g - f \cup g')$. Hence $\delta(f \cup k) = \pm(f \cup g - f \cup g')$ so $[f \cup g] = [f \cup g']$ in $H'(X,A)$. Therefore $f \cup g$ is well defined.

Lemma 17.2.12 *Let $\phi : (X,A) \to (Y,B)$ be a map of pairs. Let $f \in S^p(Y)$ and let $g \in S^q(Y,B)$. Then $\phi^*(f \cup g) = (\phi^* f \cup \phi^* g) \in S^q(X,A)$.*

$$\begin{tikzcd}
0 \arrow[r] & S^*(Y,B) \arrow[r, "c_B^*"] \arrow[d, "\phi^*"] & S^*(Y) \arrow[r] \arrow[d, "\phi^*"] & S^*(B) \arrow[r] \arrow[d, "\phi^*"] & 0 \\
0 \arrow[r] & S^*(X,A) \arrow[r, "c_A^*"] & S^*(X) \arrow[r] & S^*(A) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{aligned}
c_A^*\phi^*(f \cup g) &= \phi^* c_B^*(f \cup g) \\
&\overset{\text{(definition of rel. cup)}}{=} \phi^*(f \cup c_B^* g) \\
&\overset{(\phi^*\text{ring homom.})}{=} \phi^* f \cup \phi^* g \\
&= \phi^* f \cup c_A^* \phi^* g \\
&\overset{\text{(definition of rel. cup)}}{=} c_A^*(\phi^* f \cup \phi^* g)
\end{aligned}$$

Since c_A^* is a monomorphism. $\phi^*(f \cup g) = \phi^* f \cup \phi^* g)$. □

17.3 Cap Products

Given $g \in S^q(X)$ and $x \in S_{p+q}(X)$ define $g \cap x \in S_p(X)$ by $\langle f, g \cap x >:= \langle f \cup g, x \rangle$ for all $f \in S^p(X)$.
Note: This uniquely defines $g \cap x$ (if it defines it all; i.e. $\exists$ at most one element satisfying this definition) since:

Given an abelian group G, write G^* $\mathrm{Hom}(G, \mathbb{Z})$, If G is free abelian then the canonical map $G \to G^{**}$ is a monomorphism.

Proof: The corresponding statement for vector spaces is standard. Since G is free abelian, can choose a basis and repeat the vector space proof, or:

Let $V = G \otimes \mathbb{Q}$. Since G is free abelian the map $G \to V$ given by $g \mapsto g \otimes 1$ is a monomorphism so
$$\begin{tikzcd} G \arrow[r, tail] \arrow[d] & V \arrow[d, tail] \\ G^{**} \arrow[r] & V^{**} \end{tikzcd}$$
shows $G \to G^{**}$ is a monomorphism.

Remark 17.3.1 *Even in the vector space case, $V \to V^{**}$ is not an isomorphism unless V is finite dimensional.*

Explicitly, for a generator $T : \Delta^{p+q} \to X$ of $S_{p+q}(X)$, the above "definition" for $g \cap x$ is becomes $g \cup T = (-1)^{pq} \big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q}) \big\rangle T \circ l(\epsilon_0, \ldots, \epsilon_p)$

(This formula shows that there does indeed exist an element satisfying the above definition.)

Proof: $\forall f \in S^p(X)$,

$$(-1)^{pq} \Big\langle f, \big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q}) \big\rangle, T \circ l(\epsilon_0, \ldots, \epsilon_p) \Big\rangle$$
$$= (-1)^{pq} \big\langle f, T \circ l(\epsilon_0, \ldots, \epsilon_p) \big\rangle \big\langle g, T \circ l(\epsilon_p, \ldots, \epsilon_{p+q}) \big\rangle = \langle f \cup g, T \rangle$$

Lemma 17.3.2 *If $g \in S^q(X)$, $x \in S_{p+q}(X)$ then $\partial(g \cap x) = \delta g \cap x + (-1)^q (g \cap \partial x)$.*

Proof: Given $f \in S^{p-1}(X)$,

$$\begin{aligned} \langle f, g \cap \partial x \rangle &= \langle f \cup g, \partial x \rangle \\ &= (-1)^{p+q} \langle \delta(f \cup g), x \rangle \\ &= (-1)^{p+q} \langle \delta f \cup g + (-1)^{p-1} f \cup \delta g), x \rangle \\ &= (-1)^{p+q} \langle \delta f \cup g \rangle + (-1)^{q-1} \langle f \cup \delta g), x \rangle \\ &= (-1)^{p+q} \langle \delta f, g \cap x \rangle + (-1)^{q-1} \langle f, \delta g \cap x \rangle \\ &= (-1)^{p+q} (-1)^p \langle f, \partial(g \cap x) \rangle + (-1)^{q-1} \langle f, \delta g \cap x. \rangle \end{aligned}$$

Therefore $g \cap \partial x = (-1)^{-q} \partial(g \cap x) \rangle + (-1)^{q-1} \delta g \cap x \rangle$ or equivalently $\partial(g \cap x) = \delta g \cap x + (-1)^q (g \cap \partial x)$. □

It follows that if $[g] \in H^q(X)$, $[x] \in H_{p+q}(X)$, then $[g] \cap [x]$ is an element of $H_p(X)$. (Proof that it is well defined left as an exercise.)

There are also two versions of a relative cap product:

Let $j : A \hookrightarrow X$.

$0 \to S_*(A) \xhookrightarrow{j_*} S_*(X) \xrightarrow{c_*} S_*(X, A) \to 0.$

$0 \to S^*(X,A) \xhookrightarrow{c^*} S^*(X) \xrightarrow{j^*} S^*(A) \to 0.$

Let $g \in S^q(X)$ and let $x \in S^{p+q}(X,A)$.

Define $g \cap x \in S_p(X,A)$ by $\langle f, g \cap x\rangle = f \cup g, x >$ for $f \in S^p(X,A)$ (where $f \cup g$ is the relative cup product).

Or: If $g \in S^q(X,A)$, $x \in S_{p+q}(X,A)$ can define $g \cap x \in S_p(X)$ by $\langle f, g \cap x\rangle = f \cup g, x >$ for $f \in S^p(X)$ (where again $f \cup g$ is the relative cup product).

In each case, whenever g and x represent homology classes, $[g] \cap [x]$ is a well defined homology class of $H_p(X,A)$ or $H_p(X)$ respectively. (Exercise)

Lemma 17.3.3 *Let $\phi : (X,A) \to (Y,B)$. Let $g \in S^q(Y,B)$ and let $x \in S_{p+q}(X,A)$. Then $\phi_*(\phi^* g \cap x) = g \cap \phi_* x$ in $S_p(Y)$.*

Proof: Let $\in S^p(Y)$. Then

$$\begin{aligned}
&\langle f, \phi_*(\phi^* g \cap x)\rangle \\
&\quad = \langle \phi^* f, \phi^* g \cap x\rangle \\
&\quad = \langle \phi^* f \cup \phi^* g, x\rangle \qquad \text{(where } \cup \text{ is the relatively cup product)} \\
&\quad \overset{\text{(lemma 17.2.12)}}{=} \langle \phi^*(f \cup g), x\rangle \\
&\quad = \langle f \cup g, \phi_* x\rangle \\
&\quad = \langle f, g \cap \phi_* x\rangle
\end{aligned}$$

so $\phi_*(\phi^* g \cap x) = g \cap \phi_* x$. □

Lemma 17.3.4 *Suppose $Y \subset X$. Suppose $Y = Y_1 \cup Y_2$ and $X = X_1 \cup X_2$ where Y_ϵ and X_ϵ are open in X. Let $A = X_1 \cap X_2$, $B = Y_1 \cap Y_2$. Suppose also that $X_\epsilon \cup Y_\epsilon = X$ for $\epsilon = 1, 2$. Let $[v] \in H_n(X,B)$. Then the following diagram commutes $\forall q \le n$:*

$$\begin{tikzcd}
H^{q-1}(X,B) \arrow[r, "\Delta^*"] \arrow[dd, "\cap[v]"] & H^q(X,Y) \arrow[d, "\cong \text{ (excision)}"] \\
 & H^q(A, A \cap Y) \arrow[d, "\cap[v']"] \\
H_{n-q+1}(X) \arrow[r, "\Delta_*"] & H_{n-q}(A)
\end{tikzcd}$$

where:

$$\begin{tikzcd} {[v]} & & {[v']} \\ H_n(X,B) \arrow[r] & H_n(X,Y) & H_n(A,A\cap Y) \arrow[l, "\cong"', "\text{(excision)}"] \end{tikzcd}$$

defines [v'] and Δ_* and Δ^* are the connectiing homomorphisms from the Mayer-Vietoris sequences

$$\ldots H_{n-q+1}(X) \xrightarrow{\Delta_*} H_{n-q}(A) \to H_{n-q}(X_1) \oplus H_{n-q}(X_2) \to H_{n-q}(X) \xrightarrow{\Delta_*} \ldots$$
$$\ldots H^{q-1}(X,B) \xrightarrow{\Delta^*} H^q(X,Y) \to H^q(X,Y_1) \oplus H^q(X,Y_2) \to H^q(X,B) \xrightarrow{\Delta_*} \ldots$$

Proof: By definition of Δ_* and Δ^* they factor as show below:

$$\begin{tikzcd}
H^{q-1}(X,B) \arrow[r] \arrow[rrr, bend left, "\Delta^*"] \arrow[dd, "\cap[v]"] & H^{q-1}(Y_1,B) & H^{q-1}(Y,Y_1) \arrow[l, "\cong"', "\text{(excision)}"] \arrow[r, "\delta^*"] & H^q(X,Y) \arrow[d] \\
 & \text{commutes?} & & H^q(A,A\cap Y) \arrow[d, "\cap[v']"] \\
H_{n-q+1}(X) \arrow[rrr, bend right, "\Delta_*"'] & H_{n-q+1}(X,X_1) \arrow[l] & H_{n-q+1}(X_1,A) \arrow[l, "\cong"', "\text{(excision)}"] \arrow[r, "\partial_*"] & H_{n-q}(A)
\end{tikzcd}$$

where ∂_* and ∂^* are connecting maps from long exact sequences.
The open sets $\{X_1 \cap Y_2, X_2 \cap Y_1, A\}$ cover X because:

$$\begin{aligned}
(X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup A &= (X_1 \cap Y_2) \cup (X_2 \cap Y_1) \cup (X_1 \cap X_2) \\
&= (X_1 \cap Y_2) \cup \big(X_2 \cap (Y_1 \cup X_1)\big) \\
&= (X_1 \cap Y_2) \cup X_2 \\
&= (X_1 \cup X_2) \cap (Y_2 \cup X_2) = X \cap X = X
\end{aligned}$$

Therefore by corollary 14.2.31 (used in the proof of excision,) $[v]$ has a representative $u \in S_n(X)$ where $u = u_1 + u_2 + u'$ with $u_1 \in S_n(X_1 \cap Y_2)$, $u_2 \in S_n(X_2 \cap Y_1)$, $u' \in S_n(A)$, and $\partial u \in S_{n-1}(B)$.

That is, by corollary 14.2.31, $S^{\mathcal{A}}_*(X,B) \to S_*(X,B)$ induces an isomorphism on homology, where $\mathcal{A} = \{X_1 \cap Y_2, X_2 \cap Y_1, A\}$. Therefore $\exists$ a representative $\tilde{u}$ of $[v]$ lying in $S^{\mathcal{A}}_n(X,B)$ which means that if we take a preimage u of $\tilde{u}$ back in $S^{\mathcal{A}}_n(X)$ then $u = u_1 + u_2 + u'$ as above with $\partial y \in S_{n-1}(B)$.

Notice that since $u_1, u_2 \in S_n(Y)$, then their images in $S_n(X,Y)$ vanish so that the image of $[v]$ under $H_n(X,B) \to H_n(X,Y)$ is represented by the reduction of $u' \mod S_n(Y)$. Hence $[v'] = [u'] \mod S_n(Y)$.

Left-bottom image of $[f] \in H^{q-1}(X,B)$ is

$$\Delta_*(f \cap u) = \Delta_*(f \cap u_1] + \Delta_*(f \cap u_2] + \Delta_*(f \cap u'].$$

However $U_2 \in S_n(X_2 \cup Y_1) \subset S_n(X_2)$ and $u' \in S_n(A) \subset S_n(X_2)$.

Therefore $f \cap u_2 \in S_{n-q+1}(X_2)$ and $f \cup u' \in S_{n-q+1}(X_2)$. (More precisely, if $j_2 : X_2 \hookrightarrow X$ then $j_{2*}(j_2^* f \cap u_2) = f \cap j_{2*} u_2 = f \cap u_2$, identifying u_2 with its image under the monomorphism j_{2*}. So $f \cap u_2 \in \operatorname{Im} j_{2*}$.)

Hence $f \cap u_2$ and $f \cap u'$ die under the map $S_{n-1+1}(X) \to S_{n-q+1}(X, X_2)$, (which is part of Δ_*) and thus $\Delta_*[f \cap u] = \Delta_*[f \cap u_1]$.

Notice that $\Delta_*[f \cap u_1] = \partial[f \cap u_1]$ because as above $f \cap u_1 \in S_*(X_1)$ and so its reduction mod $S_*(A)$ gives the image under the excision isomorphism and thus *it* serves as a suitable pre-image of the reduction to be used when computing the connecting homomorphism ∂.

Finally, $\partial[f \cap u_1] = [\partial f \cap u_1] + (-1)^{q-1}[f \cap \partial u_1] = (-1)^{q-1}[f \cap \partial u_1]$, since f is a cocycle.

To summarize, the left-bottom image of $[f]$ is $(-1)^{q-1}[f \cap \partial u_1]$

To compute the other way around the figure:

The image of $[f]$ under $H^{q-1}(X,B) \to H^{q-1}(Y_2,B)$ is represented by the restriction of f to $S_{q-1}(Y_2)$. The image under the excision isomorphism is represented by a cocycle $f' \in S^{q-1}(Y,Y_1)$ whose restriction to Y_2 is homologous to $f\big|_{S_{q-1}(Y_2)}$ within $S^{q-1}(Y_2,B)$. That is, $\exists \in S^{q-2}(Y_2,B)$ s.t. $f'\big|_{S_{q-1}}(Y_2) = f\big|_{S_{q-1}}(Y_2) + \delta g$.

We modify f' so as to eliminate δg as follows:

$g \in S^{q-2}(Y_2,B)$ is defined on $S_{q-2}(Y_2)$. Extend it to a g' defined on $S_{q-2}(Y_2)$ by defining it to be zero on all generators of $S_{q-2}(Y)$ lying outside $S_{q-2}(Y_2)$. (We are using, in effect, that $S_{q-2}(Y_2) \hookrightarrow S_{q-2}(Y)$ splits.) Let $f'' = f' - \delta g' \in S^{q-1}(Y)$. Then f'' is still a cocycle, $[f''] = [f']$ and $f''\big|_{S_{q-1}}(Y_2) = f\big|_{S_{q-1}}(Y_2)$. Extend f'' to an element $\tilde{f} \in S^{q-1}(X)$ (for example, by setting it to be zero on generators outside $S_{q-1}(Y)$. Note: $\tilde{f}$ need no longer be a cocycle.) $\tilde{f}$ is thus a pre-image of f'' under the surjection $S^{q-1}(X,Y_1) \twoheadrightarrow S^{q-1}(Y,Y_2)$ and so is a suitable element for computing $\delta^*[f'']$. That is $\delta^*[f''] = [\delta\tilde{f}]$. (It needn't be the 0 homology class because $\tilde{f} \notin S^q(X,Y)$: it isn't zero on $S_*(Y)$.) So $\Delta^*[f] = [\delta\tilde{f}]$.

Thus the top-right image of $[f]$ is $[\delta\tilde{f}] \cap [v'] = [\delta\tilde{f} \cap u']$ (where, more precisely, we should write the restriction of $\delta\tilde{f}$ to $S_*(A)$ rather than $\delta\tilde{f}$.)

Since $u' \in S_*(A)$, $\tilde{f} \cap u' \in S_*(A)$, so $[\partial(\tilde{f} \cap u')] = 0$ in $S_{n-q}(A)$.

$\partial(\tilde{f} \cap u') = \delta\tilde{f} \cap u' + (-1)^{q-1}\tilde{f} \cap \partial u'$ so $[\partial(\tilde{f} \cap u')] = -(-1)^{q-1}[\tilde{f} \cap \partial u']$.

Therefore it remains to show that $[\tilde{f} \cap \partial u'] = -[f \cap \partial u_1]$.

However $\tilde{f} \cap \partial u' = \tilde{f} \cap \partial u - \tilde{f} \cap \partial u_1 - \tilde{f} \cap \partial u_2$.

$\partial u \in S_{n-1}(B) \subset S_{n-1}(Y_1)$ and $u_2 \in S_n(X_2 \cap Y_1) \subset S_{n-1}(Y_1)$ and so $\partial u_1 \in S_{n-1}(Y_2)$.

Similarly $\partial U_1 \in S_{n-1}(Y_2)$.

But $\tilde{f}\big|_{S_*(Y)} = f''\big|_{S_*(Y)}$ and $\tilde{f}\big|_{S_*(Y_2)} = f''\big|_{S_*(Y_2)} = f\big|_{S_*(Y_2)}$. Hence $\tilde{f} \cap \partial u = f'' \cap \partial u$, $\tilde{f} \cap \partial u_2 = f'' \cap \partial u_2$, $\tilde{f} \cap \partial u_1 = f'' \cap \partial u_2$.

The first two terms are zero, since $f''\big|_{Y_1} = 0$. Thus $[\tilde{f} \cap \partial u'] = -[f \cap \partial u_1]$, as desired. $\square$

Chapter 18

Homology and Cohomology with Coefficients

18.1 Tensor Product

Let R be a commutative ring and let M and N be R-modules.

The *tensor product* $M \otimes_R N$ is the R-module with the universal property

$$\begin{tikzcd} M \times N \arrow[rr, "R \text{ bilinear}"] \arrow[dr] & & X \\ & M \otimes_R N \arrow[ur, dotted, "\exists!"] & \end{tikzcd}$$

Explicity, $M \otimes_R N = F_{ab}(M \times N)/\sim$ where

$(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$

$(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$

$(mr, n) \sim (m, rn)$

with the R-modules structure $f(m, n) := (rm, n) = (m, rn)$,

$[(m, n)]$ in $M \otimes_R N$ is written $m \otimes n$.

Thus elements of $M \otimes_R N$ are of the form $\sum_{i=1^k} m_i \otimes n_i$.

18.2 (Co)Homology with Coefficients

Let C be a chain complex and let G be an abelian group. Define a chain complex denoted $C \otimes G$ by $C \times G)_p := C_p \otimes G$ with boundary operator defined to be $d \times 1_G : C_p \otimes G \to C_{p-1} \otimes G$, where d is the boundary operator on C. Similarly if C is a cochain complex, can define a cochain complex $C \otimes G$ by $C \otimes G)^p := C^p \otimes G$ with boundary operator $d \otimes 1_G$.

There is a version of the Universal Coefficient Theorem which gives the homology (resp. cohomology) of $C \otimes G$ in terms of the homology (resp. cohomology) of C whenever C is either free abelian or G is free abelian. However we will now give a direct proof that if C, D are free chain complexes and $\phi : C \to D$ s.t. $\phi_* : H_*(C) \to H_*(D)$ is an isomorphism then $\phi_* \otimes G : H_*(C \otimes G) \to H_*(D \otimes G)$ is an isomorphism.

Proposition 18.2.1 *Let C be a free chain complex s.t. $H_q(C) = 0\ \forall q$. Then $H_q(C \otimes G) = 0\ \forall q$.*

Proof: As in the proof that $H^q(\operatorname{Hom} C, \mathbb{Z}) = 0$, we can describe C as follows:

C

$$\xrightarrow{\partial} (B_{p+1} \oplus U_{p+1}) \xrightarrow{\partial} (B_p \oplus U_p) \xrightarrow{\partial} (B_{p-1} \oplus U_{p-1}) \xrightarrow{\partial}$$

where $C_p \cong B_p \oplus U_p$ with $\partial_p : U_p \cong B_{p-1}$.

Therefore

$C \otimes G$

$$\longrightarrow (B_{p+1} \otimes G \oplus U_{p+1} \otimes G) \xrightarrow{\partial \otimes 1_G} (B_p \otimes G \oplus U_p \otimes G) \xrightarrow{\partial \otimes 1_G} (B_{p-1} \otimes G \oplus U_{p-1} \otimes G) \longrightarrow$$

so $H_p(C) = 0\ \forall p$. □

Proposition 18.2.2 *Let $0 \to C \xrightarrow{\phi} D \to E \to)$ be a short exact sequence of chain complexes s.t. E is a free chain complex. If $\phi_* : H_q(C) \xrightarrow{\cong} H_q(D)\ \forall q$ then $\phi_* \otimes G : H_q(C \otimes G) \to H_q(D \otimes G)$ is an isomorphism $\forall q$.*

Proof: Since E_p is free $\forall\ p$, $D_p \cong C_p \oplus E_p$ and thus $D_p \otimes G \cong C_p \otimes G \oplus E_p \otimes G$.

Hence $0 \to C \otimes G \xrightarrow{\phi \otimes G} D \otimes G \longrightarrow E \otimes G \to 0$ is again a short exact sequence so $H_q(E) = 0\ \forall q \Rightarrow H_q(E \otimes G) = 0 \forall q \Rightarrow \phi_q \otimes G$ is an isomorphism $\forall q$. □

Without the freeness condition, $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/(2\mathbb{Z}) \to 0$ is exact but tensoring with $G = \mathbb{Z}/(2\mathbb{Z})$ gives $0 \to \mathbb{Z}/(2/\mathbb{Z}) \xrightarrow{2} \mathbb{Z}/(2/ZZ) \to \mathbb{Z}/(2\mathbb{Z}) \to 0$ which is not exact.

Theorem 18.2.3 *Let C, D be free chain complexes such that ϕ_* is an isomorphism on (co)homology $\forall q$. Then $\phi_* \otimes G$ is an isomorphism on (co)homology $\forall$ q.*

Proof: The homology case follows from the preceding propositions, given the earlier theorem on existence of algebraic mapping cones. This also proves the cohomology statement, since a cochain complex is merely a chain complex with the groups renumbered.

For a simplicial complex K, we define the simplicial homology of K with coefficients in G, denoted $H_*(K;G)$ by $H_*(K;G) := H_*\big(C_*(K) \otimes G\big)$. Similarly if X is a topological space, its singular homology with coefficients in G is defined by $H_*(X;G) := H_*\big(S_*(X) \otimes G\big)$ and if X is a CW-comples, its cellular homology with coefficients in G is $H_*\big(D_*(X) \otimes G\big)$. Can likewise define $H^*(K;G) := H_*\big(C^*(K) \otimes G\big)$. $H^*(X;G) := H_*\big(S^*(X) \otimes G\big)$ and cellular cohomology of a CW-complex X as $H^*\big(S^*(X) \otimes G\big)$. We can also define relative and reduced homology and cohomology groups with coefficients in G.

From the preceding theorem we get $H_*(K;G) := H_*\big(|K|;G\big)$ and $H^*(K;G) := H^*\big(|K|;G\big)$ and $H_*(D(X);G) := H_*(X;G$ and $H^*(D(X);G) := H^*(X;G$. It is also immediate that $H_*(X;G)$ and $H^*(X;G)$ satisfy all the axioms for a homology (resp. cohomology) theory except for $A7$ which has to be replaced by

$$H_p(*;G) = \begin{cases} 0 & p \neq 0; \\ G & p = 0; \end{cases} \qquad H^p(*;G) = \begin{cases} 0; & p \neq 0 \\ G & p = 0. \end{cases}$$

Similarly Mayer-Vietoris works, Also $\tilde{H}_*(X;G)$ satisfies

$$H_n(X;G) = \begin{cases} \tilde{H}(X;G) & n > 0; \\ H_0(G) \oplus G & n = 0; \end{cases}$$

and $\tilde{H}_n(X;G) \cong H_n\big((X,*);G\big)$. The cohomology versions work also.

If $G \to H$ is a homomorphism of abelian groups, then it induces a (co)chain map $C \otimes G \to C \otimes H$ for any (co)chain complex C and thus induces $H_*(X;G) \to H_*(X;H)$ and $H^*(X;G) \to H^*(X;H)$ (notice that the direction of latter arrow does not get reversed).

If G happens to have an R-module structure for some commutative ring R (with 1) then for any abelian group A, $A \otimes G$ becomes an R-module by defining on generators $r(a \otimes g) := a \otimes rg$. In this case, for $c \in C_p$¡ $g \in G$:

$r\big(\partial(c \otimes g)\big) = r(\partial c \otimes g) = \partial c \otimes rg = \partial(c \otimes rg) = \partial\big(r(c \otimes g)\big)$. That is, the boundary operator on $C \otimes G$ becomes an R-module homomorphism, so $\ker \partial$ and $\operatorname{Im} \partial$ are R-modules and so their quotient, $H_*(C \otimes G)$ inherits an R-module structure.

Suppose now that G is a ring R (commutative, with 1) and C is a free chain complex. The Kronecker product induces a bilinear pairing between $C \otimes R$ and $\operatorname{Hom}(C, \mathbb{Z}) \otimes R$ with values in R, which is again called the Kronecker product. Explicitly, given generators $f \otimes r$ of $\mathrm{Hom}(C, \mathbb{Z}) \otimes R$ and $c \otimes r'$ of $C \otimes R$, $\langle f \otimes r, c \otimes r' \rangle := rr\langle f, c\rangle$, where the multiplication takes place in R after taking the image of the integer-valued Kronecker prduct $\langle f, c\rangle$ under the unique ring homomorphism $\mathbb{Z} \to R$ (sending $1 \in \mathbb{Z}$ to $1 \in R$). This results in a bilinear R-module pairing (also called the Kronecker product) between the homology and cohomology groups as well.

We can also define cup products on cohomology with coefficients in R. Namely, for generators $f \otimes f \in S^p(X; R)$ and $g \otimes f' \in S^q(X; R)$ define $(f \otimes f) \cup (g \otimes r') \in S^{p+q}(X; R)$ by $(f \otimes f) \cup (g \otimes r') := (f \cup g) \otimes rr;$. Thus $S^*(X; R)$ and $H^*(X; R)$ become graded rings (with 1) and $H^*(X; R)$ is graded commutative. If $A \to R$ is a ring homomorphism then it follows immediately from the definitions that $S^*(X; A) \to S^*(X; R)$ and $H^*(X; A) \to H^*(X; R)$ are ring homomorphisms. (Note the special case were $A = \mathbb{Z} \to R$ given by $1 \mapsto 1$).

Given generators $\otimes r \in S^q(X; R)$ and $x \otimes r' \in S_{p+q}(X)$, can define cap product by $(g \otimes r) \cap (x \otimes r') := (g \cap x) \otimes rr$. Similarly one can define the relative cup and cap products.

Remark 18.2.4 *In practice, there are sometimes advantages to having a field as coefficients. Thus, besides $\mathbb{Z}$, the most common coefficients are $\mathbb{Z}/(p\mathbb{Z})$ and $\mathbb{Q}$. Sometimes $R = \mathbb{Z}_{(p)}$, $\mathbb{R}$, or $\mathbb{C}$ are also useful.*

Theorem 18.2.5

$$H_q(S^n; R) = \begin{cases} R & q = 0, n; \\ 0 & q \neq 0, n; \end{cases} \qquad H^q(S^n; R) = \begin{cases} R & q = 0, n; \\ 0 & q \neq 0, n; \end{cases}$$

$$H_q(\mathbb{C}P^n; R) = \begin{cases} R & q \text{ even, } q \leq 2n; \\ 0 & q \text{ odd or } q > 2n; \end{cases} \qquad H^q(\mathbb{C}P^n; R) = \begin{cases} R & q \text{ even, } q \leq 2n; \\ 0 & q \text{ odd or } q > 2n; \end{cases}$$

Proof: Use cellular (co)homology. e.g.
$D_*(\mathbb{C}P^n \otimes R)$ $\qquad R \to 0 \to R \to 0 \to R \to \ldots \to R \to 0 \to R \to 0$ $\square$

Theorem 18.2.6

$$H_q(\mathbb{R}P^n; \mathbb{Z}/(2\mathbb{Z})) = \begin{cases} \mathbb{Z}/(2\mathbb{Z}) & q \leq n; \\ 0 & q > n; \end{cases}$$

$$H^q(\mathbb{R}P^n; \mathbb{Z}/(2\mathbb{Z})) = \begin{cases} \mathbb{Z}/(2\mathbb{Z}) & q \leq n; \\ 0 & q > n; \end{cases}$$

$$H_q(\mathbb{R}P^n; \mathbb{Q}) = H^q(\mathbb{R}P^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & q = n \text{ when } n \text{ is even, or } q = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Use cellular (co)homology.

$D_*(\mathbb{R}P^n)$ $\qquad \mathbb{Z} \to \mathbb{Z} \to \dots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

Therefore

$D_*(\mathbb{R}P^n \otimes \mathbb{Z}/(2\mathbb{Z}))$

$$\mathbb{Z}/(2\mathbb{Z}) \to \mathbb{Z}/(2\mathbb{Z}) \to \dots \xrightarrow{0} \mathbb{Z}/(2\mathbb{Z}) \xrightarrow{2=0} \mathbb{Z}/(2\mathbb{Z}) \xrightarrow{0} \mathbb{Z}/(2\mathbb{Z}) \to 0$$

Thus $H_q(\mathbb{R}P^n; \mathbb{Z}/(2\mathbb{Z})) = \begin{cases} \mathbb{Z}/(2\mathbb{Z}) & q \leq n; \\ 0 & q > n, \end{cases}$

$D_*(\mathbb{R}P^n \otimes \mathbb{Q})$ $\qquad \mathbb{Q} \to \mathbb{Q} \to \dots \xrightarrow{0} \mathbb{Q} \xrightarrow[\cong]{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0$

Since $2 : \mathbb{Q} \to \mathbb{Q}$ is an isomorphism (with mult. by $1/2$ as inverse), $H_*(\mathbb{R}P^n; \mathbb{Q})$ is as stated. Similarly one gets the cohomology results.

Remark 18.2.7 *If R is a field, then it follows from the Universal Coefficient Theorem that $H^*(X; R) \cong \mathrm{Hom}_{R-\mathrm{mods}}\big(H_*(X, R), R\big)$.*

Chapter 19

Orientation for Manifolds

Recall

Definition 19.0.8 *A (paracompact) Hausdorff space M is called an n-dimensional* manifold *of for each $x \in M$ $\exists$ open neighbourhood U of x s.t. U is homeomorphic to $\mathbb{R}^n$.*

U is called an open coordinate neighbourhood. (If the neighbourhoods are diffeomorphic to $\mathbb{R}^n$ then M is a called a differentiable manifold. Similarly can define C^∞ manifolds, etc.)

Let M denote an n-dimensional manifold. Given open coordinate neighbourhood V of x, can choose smaller open neighbourhood U of x s.t. the homeomorphism of V to $\mathbb{R}^n$ restricts to a homeomorphism of U with an open ball of radius 1. Thus U is also homeomorphic to $\mathbb{R}^n$. From now on whenever we pick a coordinate neighbourhood U of x we shall always assume that we have chosen one which is contained in a larger coordinate neighbourhood V as above so that $\overline{U} \subset V$ and $V \smallsetminus U \simeq S^{n-1}$.

Proposition 19.0.9 *$\forall x \in M$,*

$$H_q(M, M \smallsetminus \{x\}) \cong \begin{cases} \mathbb{Z} & q = n \\ 0; & q \neq n. \end{cases}$$

Proof: Let U be an open coordinate neighbourhood of x. Then $\overline{M \setminus U} = M \setminus U \subset M - \{x\} = \mathrm{Int}(M \setminus \{x\})$ so

$$\begin{aligned} H_q(M, M \setminus \{x\}) & \overset{\text{(excision)}}{\cong} H_q(U, U \setminus \{x\}) \\ & \cong H_q(\mathbb{R}, \mathbb{R} \setminus \{x\}) \\ & \overset{\text{(long exact sequence)}}{\cong} \tilde{H}_{q-1}(\mathbb{R} \setminus \{x\}) \\ & \cong \tilde{H}_{q-1}(S^{n-1}) \\ & \cong \begin{cases} \mathbb{Z} & q = n \\ 0 & q \neq n \end{cases} \end{aligned}$$

□

Definition 19.0.10 *A choice of one of the two generators for $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ is called a* local orientation *for M and x.*

Notation: Given $x \in A \subset K \subset M$, let $j_x^A : (M, M \setminus A) \to (M, M - \setminus \{x\})$ denote the map of pairs induced by inclusions. If $A = K = M$, just write j_x for j_x^A.

Lemma 19.0.11 *1. Given open neighbourhood W of x, $\exists$ open neighbourhood U of x s.t. $U \subset W$ and $j^U_{y\,*} : H_*(M, M \setminus U) \to H_*(M, M \setminus \{y\})$ is an isomorphism $\forall y \in U$.*

2. Let $\zeta \in H_n(M, M \setminus W)$. Let U be any open neighbourhood of x satisfying part (1) (i.e. $j^U_{y\,*}$ iso. $\forall y \in U$.) If $\alpha \in H_n(M, M \setminus U)$ s.t. $j^U_{y\,*}(\alpha) = j^W_{y\,*}(\zeta)$ for some $y \in U$ then $j^U_{y\,*}(\alpha) = j^W_{y\,*}(\zeta)$ $\forall y \in U$.

Proof: Within W find a pair $U \subset V$ of open coordinate neighbourhoods of x (as outlined earlier) s.t. $V \setminus U \simeq S^{n-1}$. Then $\forall y \in U$

$$\begin{tikzcd} \zeta \arrow[d] & H_*(M, M \setminus W) \arrow[dl, "j_*"'] \arrow[dr, "j^W_{y\,*}"] & \\ \alpha \quad H_*(M, M \setminus U) \arrow[rr, "j^U_{y\,*}"] \arrow[d, "\cong"', "\text{(excision)}"] & & H_*(M, M \setminus \{y\}) \arrow[d, "\cong"', "\text{(excision)}"] \\ H_*(V, V \setminus U) \arrow[rr, "\cong", "\text{(homotopy)}"'] & & H_*(V, V \setminus \{y\}) \end{tikzcd}$$

Therefore $j^U_{y\,*}$ is an isomorphism as required in (1). If $y_0 \in U$ s.t. $j^U_{y_0}(\alpha) = j^W_{y_0}(\zeta)$ then the diagram with $y = y_0$ shows that $j_*(\zeta) = \alpha$. Hence the diagram with arbitrary $y \in U$ gives $j^U_y(\alpha) = j^W_y(\zeta)$. $\square$

Theorem 19.0.12 *Let K be compact, $K \subset M$. Then*

1. $H_q(M, M \smallsetminus K) = 0 \qquad q > n$
2. *For $\zeta \in H_n(M, M \smallsetminus K)$ if $j^K_{x\,(}\zeta) = 0$, then $\zeta = 0$.*

Proof:
Case 1: $M = \mathbb{R}^n$, K compact convex subset.
Then for $x \in K$, $\mathbb{R}^n \smallsetminus K \cong \mathbb{R}^n \smallsetminus \{x\}$, so (1) and (2) are immediate. $\surd$
Case 2: $K = K_1 \cup K_2$ when theorem is known for K_1, K_2, and $K_1 \cap K_2$.
Apply (relative) Mayer-Vietoris to open sets $M \smallsetminus K_1$, $M \smallsetminus K_2$.
$(M \smallsetminus K_1) \cap (M \smallsetminus K_2) = M \smallsetminus (K_1 \cup K_2) = M \smallsetminus K$
$(M \smallsetminus K_1) \cup (M \smallsetminus K_2) = M \smallsetminus (K_1 \cap K_2)$

$$
\begin{array}{l}
\qquad\qquad\qquad\qquad 0 \\
\qquad\qquad\qquad\qquad \| \\
\longrightarrow H_{n+1}\big(M, M \smallsetminus (K_1 \cap K_2)\big) \xrightarrow{\Delta} H_n(M, M \smallsetminus K) \xrightarrow{(j_{K_1 *}, j_{K_2 *})} \\
\qquad H_n(M, M \smallsetminus K_2) \oplus H_n(M, M \smallsetminus K_2) \longrightarrow H_n\big(M, M \smallsetminus (K_1 \cap K_2)\big)
\end{array}
$$

(1) follows immediately. For (2):
$\forall x \in K_1$

$$
\begin{tikzcd}
H_n(M, M \smallsetminus K) \arrow[rr, "j_{K_1}"] \arrow[dr, "j^K_{x\,*}"'] & & H_n(M, M \smallsetminus K_1) \arrow[dl, "j^{K_1}_{x\,*}"] \\
& H_n(M, M \smallsetminus \{x\}) &
\end{tikzcd}
$$

Hence $j^K_{x\,X}\big(j_{K_1}(\zeta)\big) = j^K_{x\,*}(\zeta) = 0$. So (since true $\forall x \in K_1$, by the theorem applied to K_1 gives $j_{K_1}(\zeta) = 0$. Similarly $j_{K_1}(\zeta) = 0$.
But by exactness, $\ker(j_{K_1}, j_{K_2}) = 0$ so $\zeta = 0$. $\surd$
Case 3: $M = \mathbb{R}^n$, $K = K_1 \cup \ldots \cup K_r$ where K_i is compact and convex.
Follows by induction on r from Cases 1 and 2.
Note: Intersection of convex sets is convex. To prove the theorem for, say, $K_1 \cup K_2 \cup K_3$ will have to know it already for $(K_1 \cup K_2) \cap K_3$. This will be done by a subsidiary induction. It can

best be phrased by taking as the induction hypothesis that the theorem holds for *any* union of $r-1$ compact convex subsets).

$$\surd.$$

Case 4: $M = \mathbb{R}^n$, K arbitrary compact set.
(This is the heart of the proof of the theorem.)

$$H_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K) \overset{\text{(exactness)}}{\cong} H_{q-1}(\mathbb{R}^n \smallsetminus K).$$

Given $z \in H_{q-1}(\mathbb{R}^n - K)$, by axiom $A8$, $\exists$ compact set (depending on z) $L_z \overset{j}{\hookrightarrow} \mathbb{R}^n \smallsetminus K$ s.t. $z = \iota_*(z')$ for some $z' \in H_{q-1}(L_z)$.

Given A s.t. $K \subset A \subset (L_z)^c$,

$$\begin{tikzcd}
z' \arrow[d] & H_{q-1}(L_z) \arrow[d] \arrow[dr, "i_*"] & \\
a_z & H_{q-1}(\mathbb{R}^n \smallsetminus A) \arrow[r, "i'_*"] & H_{q-1}(\mathbb{R}^n \smallsetminus K)
\end{tikzcd}$$

shows $z = i'_*(a_z)$ for some $a_z \in H_{q-1}(\mathbb{R} - A)$.

Will also use a_z and z to denote their isomorphic images under $H_q(\mathbb{R}^n, \mathbb{R} \smallsetminus A) \cong H_{q-1}(\mathbb{R}^n \smallsetminus A)$, etc.

Wish to select A_z s.t. A_z is a finite union of compact convex sets and $K \subset A_z \subset (L_z)^c$.

Cover K by open balls whose closures are disjoint from L_z (using normality). By compactness can choose a finite subcover and let A_z be the union of their closures. By Case 3, the theorem holds for A_z.

If $q > n$, by (1) of the theorem applied to A_Z, $A_z = 0$ so $z = 0$. Hence (1) holds for K.
To prove (2):

Suppose $z = \zeta$ where $j^K_{x*}(\zeta) = 0$ $\forall x \in K$. It suffices to show that ${j_x^{A_\zeta}}_*(a_\zeta) = 0$ $\forall x \in A_\zeta$ since we can apply (2) of the theorem for A_ζ to conclude that $a_\zeta = 0$ so that $\zeta = 0$. (It is immediate that ${j_x^{A_\zeta}}_*(a_\zeta) = 0$ if $x \in K \subset A_\zeta$.)

Write $A_\zeta = B_1 \cup \ldots B_r$ where B_i is a closed n-ball s.t. $B_i \cap K \neq \emptyset$ (using defn. of A_ζ).

Given $x \in A_\zeta$, suppose $x \in B_i$ and find $y \in B_i \cap K$.

$$
\begin{tikzcd}
 & a_z \\
 & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A_\zeta) \arrow[d, "\gamma_*"] \arrow[ddl, "j^{A_\zeta}_{x*}"'] \arrow[drr, "i'_*"] & & \\
 & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_i) \arrow[dl, "j^{B_i}_{z*}"', "\cong"] \arrow[dr, "j^{B_i}_{y*}", "\cong"'] & & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus K) \arrow[dl, "j^K_{z*}"] \\
H_n\big(\mathbb{R}, \mathbb{R} \setminus \{x\}\big) & \leftarrow (B_i \text{ convex}) \rightarrow & H_n\big(\mathbb{R}, \mathbb{R} \setminus \{y\}\big) &
\end{tikzcd}
$$

Since $j^K_{y\;*}(\zeta) = 0$ by hypothesis, $j^{B_i}_{y\;*}\gamma_*(a_\zeta) = 0$ so $\gamma_*(a_\zeta) = 0$ so that $j^{A_\zeta}_{x\;*}(a_\zeta) = j^{B_i}_{x\;*}(a_\zeta) = 0$. Thus $j^{A_\zeta}_{x\;*}(a_\zeta) = 0$, as desired. $\surd$

Case 5: $K \subset U \subset M$, where U is an open coordinate neighbourhood.

Follows immediate from Case 4 since $H_*(M, M \setminus K) \overset{\text{(excision)}}{\cong} H_*(U, U \setminus K)$. $\surd$

Case 6: General Case

By covering K with coordinate neighbourhoods whose closures are contained in larger coordinate neighbourhoods, write $K = K_1 \cup \ldots K_r$ where for each i, $K_i \subset U_i$ with U_i is an open coordinate neighbourhood. Then use Case 5, Case 2, and induction on r. $\square$

Theorem 19.0.13 *For each $x \in M$, let α_x be a generator of $H_n(M, M \setminus \{x\})$. Suppose that these generators are compatible in the sense that $\forall x\ \exists$ open coordinate neighbourhood U_x of x and $\exists \alpha_{U_x} \in H_n(M, M \setminus U_x)$ s.t. $j^{U_x}_y = \alpha_y\ \forall y \in U_x$. Then given $K \subset M$, $\exists! \alpha_K \in H_n(M, M \setminus K)$ s.t. $j^K_{y\;*}(\alpha_K) = \alpha_y\ \forall y \in K$.*

Proof: Unique is immediate from the previous theorem. To prove existence:

Case 1: $K \subset U_x$ for some x

Use $\alpha_K = j_*(\alpha_{U_x})$ where $j_* : H_n(M, M \setminus U_x) \to H_n(M.M \setminus K)$.

Case 2: $K = K_1 \cup K_2$ where α_{K_1}, α_{K_2} exist.

$$H_{n+1}\big(M, M \setminus (K_1 \cap K_2)\big) \to H_n(M, M \setminus K) \xrightarrow{(j_{K_1}, j_{K_2})}$$
$$H_n(M, M \setminus K_1) \oplus H_n(M, M \setminus K_2) \xrightarrow{j'_* - j''_*} H_n\big(M, M \setminus (K_1 \cap K_2)\big) \to$$

For any $x \in K_1 \cap K_2$, $j^{K_1 \cap K_2}_{x*}(j'_* - j''_*)(\alpha_{K_1}, \alpha_{K_2}) = j^{K_1}_{x*}(\alpha_{K_1}) - j^{K_2}_{x*}(\alpha_{K_2}) = \alpha_x - \alpha_x = 0$ Therefore by the previous theorem applied to $K_1 \cup K_2$, $(j'_* - j''_*)(\alpha_{K_1}, \alpha_{K_2}) = 0$ so from the

exact sequence $\exists \alpha_K \in H_n(M, M \smallsetminus K)$ s.t. $j_{K_1}(alpha_K) = \alpha_{K_1}$ and $j_{K_2}(alpha_K) = \alpha_{K_2}$. Then α_K satisfies the conditions of the theorem. (To check it from y, find $\epsilon \in K_\epsilon$ and use naturality.)
Case 3: General case

Write $K = K_1 \cup \ldots \cup K_r$ with each $K_i \subset U_x$ for some x by covering K with open sets each having its closure in some U_x. Now use Cases 1, 2 and induction on r. $\square$

Remember: j_x means j_x^M.

Definition 19.0.14 *Suppose M is a compact n-dimensional manifold. If $\exists \zeta \in H_n(M)$ s.t. $j_{x_*}(\zeta)$ is a local orientation for M at x for each $x \in M$ then M is called* orientable *and ζ is called a (global)* orientation *for M.*

If M is not compact than such a global orientation class will not exist. (Consider, for example, $M = \mathbb{R}^n$). More generally we define:

Definition 19.0.15 *An* orientation *for M consists of a family of elements $\{\zeta_K\}_{K \subset M}$ with $\zeta_K \in H_n(M, M \smallsetminus K)$ such that $J^K_{x_*}(\zeta_K)$ is a local orientation for M at x $\forall x \in K$, K compact and furthermore if $x \in K_1 \cap K_2$ then $j^{K_1}_{x_*}(\zeta_{K_1}) = j^{K_2}_{x_*}(\zeta_{K_2})$.*

Of course, this second definition works equally well in the compact case, since a global class can be restricted.

The preceding theorem says that if M has a "compatible" collection of local orientations at each point then M is orientable.

Corollary 19.0.16 *Let M be orientable and connected. Then any two orientations of M which induce the same local orientation at any point are equal.*

Proof: Let $\{\alpha_y\}_{y \in M}$ and $\{\beta_y\}_{y \in M}$ be the sets of local orientations induced by the two orientations $\{\zeta_{K \subset M}$ and $\{\zeta'_{K \subset M}$.

By earlier lemma, if the orientations agree at x then they agree on an open neighbourhood of x ($\exists U$ s.t. $J^U_{y_*} : H_*(M.M \smallsetminus U) \to H_*(M, M \smallsetminus \{y\})$ is iso. $\forall y \in U$) so $A = \{x \mid \alpha_x = \beta_x\}$ is open.

On the other hand, if $\alpha_x \neq \beta_x$, then $\alpha_x = -\beta_x$ (there are only 2 generators of $\mathbb{Z}$ and they are related in this way) so by the same lemma $\exists$ open set U containing x s.t. $\alpha_y = -\beta_y$ $\forall y \in U$. Hence $B = \{x \mid \alpha_x \neq \beta_x\}$ is also open.

Since $A \cup B = M$ and $A \cap B = \emptyset$, by connectivity of M one of A, B is $\emptyset$. By hypothesis $A \neq \emptyset$ so $B = \emptyset$ and $A = M$. Hence $\alpha_x = \beta_x$ $\forall x \in M$, which by earlier theorem says that $\zeta_K = \zeta'_K$ $\forall$ K. $\square$

Corollary 19.0.17 *If M is connected and orientable then it has precisely 2 orientations and a choice of orientations at one point uniquely determines one of the orientations.* $\square$

Theorem 19.0.18 *Let X be a connected nonorientable (compact) manifold. Then there is a 2-fold covering space $p : E \to X$ s.t. E is a connected orientable (compact) manifold.*

Proof: Let $E := \{(x, \alpha_x) \mid x \in X$ and α_x is a local orientation for X at $x\}$. Set $p(x, \alpha_x) := x$.

Topologize E as follows.

Given open set $U \subset X$ and element $\alpha_U \in H_n(X, X \smallsetminus U)$ s.t. ${j^U_x}_*(\alpha_U)$ is a generator of $H_n(X, X \smallsetminus \{x\})$ for all $x \in U$, let $\langle U, \alpha_U \rangle = \{(x, {j^U_x}_*(\alpha_U))\} \subset E$.

To show that these sets form a base for a topology:

Suppose $\langle U, \alpha_U \rangle \cap \langle U, \alpha_U \rangle \neq \emptyset$. Let $(x, \alpha_x) \in \langle U, \alpha_U \rangle \cap \langle U, \alpha_U \rangle$. By earlier lemma $\exists$ open nbhd U of x, $U \subset U' \cap U$" s.t. ${j^U_x}_*$ is an isomorphism $\forall y \in U$. Let $\alpha_U = (j^U_x)^{-1}_*(\alpha_x)$. Show $\langle U, \alpha_U \rangle \subset \langle U, \rangle \alpha_{U'} \cap \langle U", \alpha_U" \rangle$.

Let $\big(w, j^U_{w*}(\alpha_U)\big) \in \langle U, \alpha_U \rangle$. To show $\big(w, j^U_{w*}(\alpha_U)\big) \in \langle U', \alpha'_U \rangle$ we must show $j^U_{w*}(\alpha_U) = {j^{U'}_w}_*(\alpha_{U'})$. However ${j^U_x}_*(\alpha_U)\big) = {j^{U'}_x}_*(\alpha_{U'})\big)$, so by part 2 of the lemma that produced U, we have ${j^U_y}_*(\alpha_U) = {j^{U'}_y}_*(\alpha_{U'})$ for all $y \in U$ and in particular for $y = w$. Therefore $\big(w, j^U_{w*}(\alpha_U)\big) \in \langle U', \alpha'_U \rangle$ and similarly $\big(w, j^U_{w*}(\alpha_U)\big) \in \langle U", \alpha_U" \rangle$.

So $\{\langle U, \alpha_U \rangle\}$ forms a base for a topology.

By (1) of the Lemma, X can be covered by open sets U s.t. j^U_{y*} is an isomorphism for all $y \in U$.

For such sets

$$p^{-1}(U) = \langle U, \rho \rangle \amalg \langle U, -\rho \rangle$$

where

$$\zeta, -\zeta \in H_n(X, X \smallsetminus U) \cong H_n(X, X \smallsetminus \{y\}) \cong \mathbb{Z}$$

are the two generators and the restrictions $\rho : \langle U, \zeta \rangle \to U$ and $\rho : \langle U, -\zeta \rangle \to U$ are homeomorphisms. So ρ is a 2-fold covering projection.

Therefore E is a manifold.

X is compact, so E is compact since a finite cover of a compact Hausdorff space is compact.

(Proof: Cover the base with evenly covered open sets. By normality, we can find another open cover in which the closures of the sets are contained in evenly covered open sets. Take a finite subcover. Then the inverse images of the closures of these sets under the covering projections write the total space as a finite union of compact sets.)

To show E is orientable:

Given $(x, \alpha_x) \in E$, by (1) of the Lemma, there is an open neighbourhood U_x of x s.t. ${j^{U_x}_y}_*$ is an isomorphism for all $y \in U_x$.

Let $\alpha_{U_x} = (j^{U_x}_{x*})^{-1}(\alpha_x)$. So $< U_{x,\alpha_{U_x}} >$ is an open neighbourhood of (x, α_x) s.t. the restriction of p to $< U_x, \alpha_{U_x} >$ is a homeomorphism.

So

$$H_n\bigl(E, E \smallsetminus \{(x, \alpha_x)\}\bigr) \cong H_n\bigl(\langle U_x, \alpha_{U_x}\rangle, \langle U_x, \alpha_{U_x}\rangle \smallsetminus \{(x, \alpha_x)\}\bigr) \cong H_n(U_x, U_x \smallsetminus \{x\}) \cong H_n(X, X \smallsetminus \{x\}) \cong \mathbb{Z}.$$

Let $\beta_{(x,\alpha_x)} \in H_n\bigl(E, E \smallsetminus \{(x, \alpha_x)\}\bigr)$ correspond to α_x under this isomorphism.

By (2) of the Lemma (and naturality of the above isomorphism), we see that these local orientations β_x are "compatible" in the sense of the earlier Theorem. The required open neighbourhood is $\langle U_x, \alpha_{U_x}\rangle$. Note that $j^{<U_x, \alpha_{U_x}>}_{(e,\alpha_e)*}$ is an isomorphism for all $(e, \alpha_e) \in \langle U_x, \alpha_{U_x}\rangle$ to get the required homology class.

So by that Theorem, the classes $\beta_{(x,\alpha_x)}$ determine an orientation so that E is orientable.

Finally, to show E is connected:

If E had two components (as a 2-fold cover of a connected space, it can have at most 2), each would be a covering space of X (a component of a covering space of a connected space is a covering space). So each would be a 1-fold cover and thus a homeomorphism.

But then each component would be nonorientable (since X is) which would mean that E is nonorientable. This is a contradiction. So E is connected.

Corollary 19.0.19 *If M is simply connected, then M is orientable. (More generally, if $\pi_1(M)$ does not have a subgroup of index 2 then M is orientable.)*

Proof: M has no 2-fold covering space.

19.1 Orientability with Coefficients

Let R be a commutative ring with 1/ We can make the same definitions of orientability using homology with R-coefficients (e.g., a local orientation is a generator of $H_n(M, M \smallsetminus \{x\}) \cong R$) although the theorems might not all work. In practice, besides $\mathbb{Z}$ the only useful coefficient ring for the purpose of orientations is $R = \mathbb{Z}/(2\mathbb{Z})$. In that case there is only one generator so all compatiblity conditions are automatic. This means that every manifold is $(\mathbb{Z}/(2\mathbb{Z})$-orientable. Sometimes theorems which hold (using $\mathbb{Z}$-coefficients) only for orientable manifolds *can* be extended to non-orientable manifolds if $(\mathbb{Z}/(2\mathbb{Z})$-coeffiecients are used.

Example 19.1.1 *Consider $\mathbb{R}P^2$. It is a 2-dimensional manifold.*

$$H_q(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/(2\mathbb{Z}) & q = 1 \\ 0 & q = 2 \end{cases} \qquad H_q\bigl(\mathbb{R}P^2; \mathbb{Z}/(2\mathbb{Z})\bigr) = \begin{cases} \mathbb{Z}/(2\mathbb{Z}) & q = 0 \\ \mathbb{Z}/(2\mathbb{Z}) & q = 1 \\ \mathbb{Z}/(2\mathbb{Z}) & q = 2 \end{cases}$$

Examining the $\mathbb{Z}$-coefficients, since $H_2(\mathbb{R}P^2) = 0$ there can be no global orientation class, so $\mathbb{R}P^2$ is non-orientable. Notice that there *is* a candidate for a global $\mathbb{Z}/(2\mathbb{Z})$-orientation calss, and since every manifold is $\mathbb{Z}/(2\mathbb{Z})$-orientable it must indeed *be* a $\mathbb{Z}/(2\mathbb{Z})$-orientation class.

Chapter 20

Poincaré Duality

Let M be an oriented n-dimensional manifold and let $\{\zeta_K\}_{\left(\substack{K\subset M \\ L \text{ compact}}\right)}$ be its chosen orientation, where $\zeta_K \in H_n)M, M \smallsetminus K)$. If M is compact, let $\zeta = \zeta_M$.

(The following also works in M is non-orientable provided $\mathbb{Z}/(2/ZZ)$-coefficients are used.)

Consider first the case where M is compact.

Let $D : H^i(M) \to H_{n-i}(M)$ by $D(z) = z \cap \zeta$.

Theorem 20.0.2 *(Poincaré Duality) $D : H^i(M) \to H_{n-i}(M)$ is an isomorphism $\forall i$.*

In the case where M is not compact:

For each compact $K \subset M$, define $D_K : H^i(M, M \smallsetminus K) \to H_{n-i}(M)$ by $D_K(z) = z \cap \zeta_K$.

If $K \subset L \subset M$, K, L compact, then by theorem 19.0.12 $j^L_{K_*}(\zeta_L) = \zeta_K$ where $j^L_K : (M.M \smallsetminus L) \to (M, M \smallsetminus K)$.

Therefore

$$\begin{tikzcd}
H^i(M, M \smallsetminus K) \arrow[dr, "D_K"] \arrow[dd, "j_K^{L^*}"'] & \\
& H_{n-i}(M) \\
H^i(M, M \smallsetminus L) \arrow[ur, "D_L"] &
\end{tikzcd}$$

commutes since $D_K(z) = z \cap \zeta_K = z \cap j^L_{K_*}(\zeta_L) \overset{\text{lemma 17.3.3}}{=} j^{L^*}_K z \cap \zeta_L = D_L j^{L^*}_K(z)$. Thus the various maps D_K induce (by universal property) a unique map

$$D : \varinjlim_{\substack{K \subset M \\ K \text{ compact}}} H^i(M, M \smallsetminus K) \to H_{n-i}(M)$$

where the partial ordering is induced by inclusion.

Notation: Write $H^i_c(M) = \varinjlim_{\substack{K \subset M \\ K \text{ compact}}} H^i(M, M \smallsetminus K)$.

$H^*_c(M)$ is called the cohomology of M with compact support. An element of $H^*_c(M)$ is represented by a singular cochain which vanishes outside of some compact set. Of course, if M is already compact then each element in the direct system maps into $H^i(M)$ so that $H^i_c(M) = H^i(M)$ in this case.

Theorem 20.0.3 *(Poincaré Duality) $D : H^i_c(M) \to H_{n-i}(M)$ is an isomorphism $\forall i$.*

Proof:
Case 1: $M = \mathbb{R}$

Lemma 20.0.4 *Let $B \subset \mathbb{R}^n$ be a closed ball. Then $D_B : H^i(\mathbb{R}, \mathbb{R} \smallsetminus B) \to H_{n-i}(\mathbb{R}^n)$ is an isomorphism $\forall i$.*

Proof: $H_q(\mathbb{R}, \mathbb{R} \smallsetminus B) \cong H_q(\mathbb{R}, \mathbb{R} \smallsetminus \{*\} \cong \tilde{H}_{q-1}(\mathbb{R}^n \smallsetminus \{*\}) \cong \tilde{H}_{q-1}(S^{n-1})$. Similarly $H^q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus B) \cong \tilde{H}^{q-1}(S^{n-1})$. Thus if $i \neq n$ the lemma is trivial since both groups are 0.
For i=n:

The groups are isomorphic (both are $\mathbb{Z}$). Must show that D_B is an isomorphism.

ζ_B is a generator of $H_n(\mathbb{R}, \mathbb{R} \smallsetminus B) \cong \mathbb{Z}$. Find generator $f \in H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus B)$ s.t. $\langle f, \zeta_B \rangle = 1$. To see that one of the two generators of $H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus B)$ must have this property, examine the Kronecker pairing of $\tilde{H}_{n-1}(S^{n-1})$ with $\tilde{H}^{n-1}(S^{n-1})$. Using the cellular chain complex $0 \to \mathbb{Z} \to 0 \ldots \to 0$ makes it obvious that the Kronecker pariting gives an ismorphism $\tilde{H}_{n-1}(S^{n-1}) \cong \operatorname{Hom}\big(\tilde{H}_{n-1}(S^{n-1}), \mathbb{Z}\big) \cong \mathbb{Z}$ and that the ring identity $1 \in H^0(\mathbb{R}^n)$ is a generator. Thus

$\langle 1, D_B(f) \rangle = \langle 1, f \cap \zeta_B \rangle = \langle 1 \cup f, \zeta_B \rangle = \langle f, \zeta_B \rangle = 1$

so that $D_B(f)$ must be a generator of $H_0(\mathbb{R}^n)$. Hence D_B is an isomorphism.

Proof of theorem in case 1: Let $\alpha \in H^i_c(\mathbb{R}^n) = \varinjlim_{\substack{K \subset \mathbb{R}^n \\ K \text{ compact}}} H^i(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K)$. Pick a representative $f \in H^i(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K)$ of α for some compact $K \subset \mathbb{R}^n$. Let B be a closed ball containing K.

Replacing f by $j_K^{B^*}(f)$ gives a new representative for α lying in $H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B)$, and by definition of D, $D(\alpha) - D_B(f)$. Since D_B is an isomorphism by the lemma, if $D(\alpha) = 0$ then $f = 0$ and so $\alpha = 0$. Hence D is $1-1$. Conversely, given $x \in H_{n-i}(\mathbb{R}^n)$, $\exists f \in H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ s.t. $D_B(f) = x$ and so the element α of $H_c^i(\mathbb{R}^n)$ represented by f satisfies $D(\alpha) = D_B(f) = x$. Hence D is onto.

(In effect, there is a cofinal subsystem which has stabilized. Therefore the direct limit map is the same as the map induced by this stabilized subsystem.) $\surd$

Case 2: $M = U \cap V$ where U, V are open subsets of M (thus submanifolds) s.t. the theorem is known for U, V, and $W := U \cap V$

Proof: Let K, L be compact subsets of U, V respectively. Let $A = K \cap L$, $N = K \cup L$. Then we have a Mayer-Vietoris sequence

$$\begin{tikzcd}
H^q(M, M \setminus A) \arrow[r] \arrow[d, "\cong"', "\text{(excision)}"] & H^q(M, M \setminus K) \oplus H^q(M, M \setminus L) \arrow[r] \arrow[d, "\cong"', "\text{(excision)}"] & H^q(M, M \setminus N) \arrow[r] \arrow[d, equal] & H^{q+1}(M, M \setminus A) \arrow[d, "\cong"'] \\
H^q(W, W \setminus A) & H^q(U, U \setminus K) \oplus H^q(V, V \setminus L) \arrow[r] & H^q(M, M \setminus N) \arrow[r] & H^{q+1}(W, W \setminus A)
\end{tikzcd}$$

Lemma 20.0.5

$$\begin{tikzcd}
H^{q-1}(M, M \setminus N) \arrow[r] \arrow[d, "D_N"'] \arrow[dr, phantom, "①"] & H^q(W, W \setminus A) \arrow[r] \arrow[d, "D_A"'] \arrow[dr, phantom, "②"] & H^q(U, U \setminus K) \oplus H^q(V, V \setminus L) \arrow[r] \arrow[d, "D_K \oplus D_K"'] \arrow[dr, phantom, "③"] & H^{q+1}(M, M \setminus N) \arrow[d, "D_N"'] \\
H_{n-q+1}(M) \arrow[r, "\Delta_*"] & H_{n-q}(W) \arrow[r] & H_{n-q}(U) \oplus H_{n-q}(V) \arrow[r] & H_{n-q}(M)
\end{tikzcd}$$

commutes.

Proof:

For square ②: Let $j_W^U : (W, W \setminus A) \to (U, U \setminus A)$ denote the inclusion map of pairs. (It induces an excision isomorphism.)

$$\begin{array}{ccc}
\tilde{f} \in H^q(U, U \setminus A) & \xrightarrow{j_A^{K*}} & H^q(U, U \setminus K) \\
{\scriptstyle j_W^{U*}} \downarrow \cong & & \| \\
f \in H^q(W, W \setminus A) & \xrightarrow{j_A^{K*}} & H^q(U, U \setminus K) \\
\downarrow D_A & ? & \downarrow D_K \\
H_{n-q}(W) & \xrightarrow{j_W^{U*}} & H_{n-q}(U)
\end{array}$$

Let $f \in H^q(W, W \setminus A)$.

By the excision isomorphism, $\exists \tilde{f} \in H^q(U, U \setminus A)$ s.t. $j_W^{U\,*}(\tilde{f}) = f$.

Let $\zeta_A^U \in H_n(U, U - A)$ be the restriction of ζ_k to A. i.e. $\zeta_A^U := j^K_{A\,*}\zeta_K$. By compatibility of orientations, $j^U_{W*}(\zeta_A) = \zeta_A^U$ (where ζ_A means ζ_A^W).

$$\begin{aligned}
j^U_{W*} D_A f &= j^U_{W*}(f \cap \zeta_A) \\
&= j^U_{W*}\big(j_W^{U\,*}(\tilde{f}) \cap \zeta_A\big) \\
&\overset{\text{(lemma 17.3.3)}}{=} \tilde{f} \cap j^U_{W*}\zeta_A \\
&= \tilde{f} \cap \zeta_A^U \\
&= \tilde{f} \cap j^K_{A\,*}\zeta_K \\
&\quad \text{(map of pairs is } (U, U \setminus K) \to (U, U \setminus A) \text{ whose restriction to } U \text{ is } 1) \\
&\overset{\text{(lemma 17.3.3)}}{=} j_A^{K*}\tilde{f} \cap \zeta_K \\
&= D_K j_A^{K*}\tilde{f}
\end{aligned}$$

so the diagram commutes. Get the same diagram with V replacing U, so square 2 commutes.

Similarly, doing the same arguments with the pairs (M, U) replacing (U, W) and then (M, V) replacing U, W), we get that the third square commutes.

For square 1:

$$
\begin{tikzcd}
H^{q-1}(M, M \smallsetminus N) \arrow[r, "\Delta^*"] \arrow[dd, "D_N"] \arrow[dr, phantom, "?"] & H^q(M, M \smallsetminus A) \arrow[d, "\cong"] \\
 & H^q(W, W \smallsetminus A) \arrow[d, "D_A"'] \\
H_{n-q+1}(M) \arrow[r, "\Delta_*"] & H_{n-q}(W)
\end{tikzcd}
$$

apply lemma 17.3.4

$$
\begin{tikzcd}
H^{q-1}(X, B) \arrow[r, "\Delta^*"] \arrow[dd, "\cap[v]"] & H^q(X, Y) \arrow[d, "\cong \text{ (excision)}"] \\
 & H^q(A, A \cap Y) \arrow[d, "\cap[v']"] \\
H_{n-q+1}(X) \arrow[r, "\Delta_*"] & H_{n-q}(A)
\end{tikzcd}
$$

in the case:

$X := M$; $X_1 := U$; $X_2 := V$; $Y := M \smallsetminus A$; $Y_1 := M \smallsetminus K$; $[v] := \zeta_N$.

(Thus $A = U \cap V = W$ and $B = Y_1 \cap Y_2 = M \smallsetminus (K \cup L) = M \smallsetminus N$. Note: $X_1 \cap Y_1 = U \cap (M \smallsetminus K) = M$ since $K \subset U$.) $\surd$

Proof of Case 2 (cont.): Passing to the limit gives a commutative diagram with exact rows (recall the homology commutes with direct limits so exactness is preserved)

$$\begin{tikzcd}
H_c^q(W) \arrow[r] \arrow[d, "D"', "\cong"] & H_c^q(U)\oplus H_c^q(V) \arrow[r, "\Delta^*"] \arrow[d, "(D\oplus D)"', "\cong"] & H_c^{q+1}(M) \arrow[r] \arrow[d, "D"'] & H_c^{q+1}(W) \arrow[r] \arrow[d, "D"', "\cong"] & H_c^{q+1}(U)\oplus H_c^{q+1}(V) \arrow[d, "\cong"] \\
H_{n-q}(W) \arrow[r] & H_{n-q}(U)\oplus H_{n-q}(V) \arrow[r, "\Delta^*"] & H_{n-q}(M) \arrow[r] & H_{n-q-1}(W) \arrow[r] & H_{n-q-1}(U)\oplus H_{n-q-1}(V)
\end{tikzcd}$$

so by the 5-lemma, $D : H_c^q(M) \to H_{n-q}(M)$ is an isomorphism. $\checkmark$

Case 3: M is the union of a nested family of open sets U_α where the duality theorem is known for each U_α.

Since $M = \cup_\alpha U_\alpha$ and U_α is open, $S_*(M) = \cup_\alpha S_*(U_\alpha)$ so $H_*(M) = \varinjlim_\alpha H_*(U_\alpha)$.

Similarly each generator of $S_c^*(M)$ vanishes outside some compact K, where $S_c^*(M) := \varinjlim_{\substack{K\subset M\\ K \text{ compact}}} S^*(M, M\smallsetminus K)$. Since homology commutes with direct limits, $H_c^*(M) = H\big(S_*^c(M)\big)$.

Find U_{α_0} s.t. $K \subset U_{\alpha_0}$ s.t. $K \subset U_{\alpha_0}$. Then $f \in \operatorname{Im} S_c^*(U_{\alpha_0})$. Thus again $S_c^*(M) = \cup_\alpha s_c^*(U_\alpha)$ and so $H_c^*(M) = \varinjlim_\alpha H_c^*(U_\alpha)$. $\checkmark$

Case 4: M is an open subset of $\mathbb{R}^n$

If V is a convex open subset of M, then the theorem holds for V by Case 1. (i.e. V is homemorphic $\mathbb{R}^n$.)

If V, W are converx open then so is $V \cap W$ so the theorem holds for $V \cup W$ by Case 2.

Hence if $V = V_1 \cup \ldots \cup V_k$ where V_i is convex open, then the theorem holds for V.

Write $M = \cup_{i=1}^\infty V_i$ by letting $\{V_i\}$ be
$\{N_r(x) \mid N_r(x) \subset M, r \text{ rational}, x \text{ has rational coordinates}\}$ (which is countable).

Let $W_l = \cup_{i=1}^k V_i$. Then by the above, the theorem holds for W_k $\forall k$, $\{W_k\}$ are nested, and $M = \cup_{k=1}^\infty W_k$. Therefore the theorem holds for M by Case 3. $\checkmark$

Case 5: General Case

By Zorn's Lemma $\exists$ a maximal open subset U of M s.t.the theorem holds for U. If $U \neq M$, find $x \in M \smallsetminus U$ and find an open coordinate neighbourhood C of x. Then by Case 4, the theorem holds for V and $U \cap V$ so by Case 2 the theorem holds for $U \cup V$.$\Rightarrow\Leftarrow$.

Therefore $U = M$. $\square$

20.1 Cohomology Ring Calculations

S^n

degree	H_*	H^*
n	$\mathbb{Z}$	$\mathbb{Z}$
$n-1$	0	0
	$\vdots$	$\vdots$
1	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$S^1 \times S^1$

degree	H_*	H^*
2	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{C}P^n$

degree	H_*	H^*
$2n$	$\mathbb{Z}$	$\mathbb{Z}$
$2n-1$	0	0
$2n-2$	$\mathbb{Z}$	$\mathbb{Z}$
	$\vdots$	$\vdots$
1	0	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{R}P^{2n+1}$

degree	H_*	H^*
$2n+1$	$\mathbb{Z}$	$\mathbb{Z}$
$2n$	0	$\mathbb{Z}/(2\mathbb{Z})$
$2n-1$	$\mathbb{Z}/(2\mathbb{Z})$	0
	$\vdots$	$\vdots$
3	$\mathbb{Z}/(2\mathbb{Z})$	0
2	0	$\mathbb{Z}/(2\mathbb{Z})$
1	$\mathbb{Z}/(2\mathbb{Z})$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{R}P^2$ (nonorientable)

degree	H_*	H^*
2	0	$\mathbb{Z}/(2\mathbb{Z})$
1	$\mathbb{Z}/(2\mathbb{Z})$	0
0	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{R}P^2$

degree	$H_*(\ ;\mathbb{Z}/(2\mathbb{Z}))$	$H^*(\ ;\mathbb{Z}/(2\mathbb{Z}))$
2	$\mathbb{Z}/(2\mathbb{Z})$	$\mathbb{Z}/(2\mathbb{Z})$
1	$\mathbb{Z}/(2\mathbb{Z})$	$\mathbb{Z}/(2\mathbb{Z})$
0	$\mathbb{Z}/(2\mathbb{Z})$	$\mathbb{Z}/(2\mathbb{Z})$

Cup Products:

$H^*(S^n)$:
Group generators: $1 \in H^0(S^n)$, $x \in H^n(S^n)$.
No choices: $1 \cup 1 = 1 \qquad 1 \cup x = x \cup 1 = x \qquad x \cup x = 0$ $\checkmark$

Before proceding to the other spaces we need a lemma.

Let X be a connected compact oriented manifold s.t. all the boundary maps in some cellular chain complex for X are trivial. (e.g. $X = S^n$; $S^1 \times S^1$; $\mathbb{C}P^n$. Also $X = \mathbb{R}P^n$ if we use $\mathbb{Z}/(2\mathbb{Z})$

coefficients.)

$H^n(X) \cong H_0(X) \cong \mathbb{Z}$ (in the cases with $\mathbb{Z}$-coefficients). Let μ be a generator of $H^n(X)$. Replacing μ by $-\mu$ is necessary, we may assume that $\langle \mu, \zeta \rangle = 1$, where $\zeta \in H_n(X)$ the chosen orientation. Let $g \in H^q(X)$ be a basis element. (Note: The boundary maps equal to 0 implies that $H^q(X) \cong \operatorname{Hom}\big(D_q(X), \mathbb{Z}\big)$ is a free abelian group.)

Lemma 20.1.1 *$\exists f \in H^{n-q}(X)$ s.t. $f \cup g = \mu$*

Proof: Being a basis element, g is not divisible by p for any p so neither is $D(g) \in H_{n-q}(X)$ (since D is an isomorphism). Therefore by the hypothesis on the cellular chain complex for X, $\exists f \in H^{n-q}(X)$ s.t. $\langle \mu, \zeta \rangle = 1 = \langle f, D(g) \rangle \langle f, g \cap \zeta \rangle = \langle f \cup g, \zeta \rangle$ Hence $f \cup g$ is a generator of $H^n(X)$ and $f \cup g = \pm\mu$. $\square$

$H^*(S^1 \times S^1)$.

Group generators: $1 \in H^0(\;)$, $y, z \in H^1(\;)$, $\mu \in H^2(\;)$.

$S^{\times} S^1 \xrightarrow{\pi_1} (S^1) \qquad \pi_1^*(x) = y$, $\pi_2^*(x) = z$.

Since $x^2 = 0$ in $H^*(S^1)$, $y^2 = (\pi_1^* x)^2 = 0$ (ring homomorphism). Similarly $z^2 = 0$.

By the lemma, $y \cup f = \mu$ for some f so $f = \pm z$.

Reversing the roles of y and z if necessary, $y \cup z = \mu$ and $z \cup y = (-1)^{1 \cdot 1} y \cup z = -\mu$.

Aside from the multiplications by the identity and the multiplications which must be 0 for degree reasons, this describes all of the cup products in $H^*(S^1 \times S^1)$. $\surd$

Lemma 20.1.2 *Let $X = Y \vee Z$ so that $\tilde{H}^*(X) \cong \tilde{H}^*(Y) \oplus \tilde{H}^*(Z)$ If $f \in H^p(X)$ and $g \in H^q(Z)$ then $f \cup g = 0$ in $H^{p+q}(X)$.*

Proof: Let $i : Y \to Y \vee Z$ by $y \mapsto (y, *)$ and $j : Z \to Y \vee Z$ by $z \mapsto (*, z)$ denote the injections.

$i^* : \tilde{H}^*(Y) \oplus \tilde{H}^*(Z) \to \tilde{H}^*(Y)$ is the first projection and j^* is the second projection. Thus for $x \in \tilde{H}^*(Y) \oplus \tilde{H}^*(Z)$, $x = 0$ is equivalent to $i^* x = 0$ and $j^* x = 0$.

$i^*(f \cup g) = i^* f \cup i^* g = f \cup 0$ since $g = (0, g) \in H^*(Z)$ has no $H^*(Y)$ component. Thus $i^*(f \cup g) = 0$. Similarly $j^*(f \cup g) = 0$. Thus $f \cup g = 0$. $\square$

Corollary 20.1.3 *$S^1 \times S^1 \not\simeq S^1 \vee S^1 \vee S^2$ (although they have the same homology groups).*

$H^*(\mathbb{C}P^n)$:

Let $x_j \in H^{2j}(\mathbb{C}P^n)$ be a generator, choosing $x_0 = 1$ and x_μ. Set $x := x_1$.

$n = 2$: Basis is 1, $x = x_1$, $\mu = x_2$.

By the lemma, $\exists g$ s.t. $x \cup g = \mu$, and so g must be $\pm x$. Replacing μ by $-\mu$ if necessary, we may assume $x \cup x = \mu$. Aside from the multiplications by the identity and those that must be 0 for degree reasons, this describes all of the multiplications in $H^*(\mathbb{C}P^2)$.

$n = 3$:

Consider $i : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$. It is clear from the cellular chain complex that $i^*(x_j) = x^j$ for $j \leq n-1$ (and $i^* x_n = 0$ for degree reasons). So in $H^*(\mathbb{C}P^3)$, $x \cup x = x_2$ (else applying i^* gives a contradiction to the above calculations in $H^*(\mathbb{C}P^2)$). Now by the lemma, $x \cup (x \cup x)$ must be a generator of $H^6(\mathbb{C}P^3)$, so $x \cup x \cup x = \mu$ (or at least we can choose μ so that this is true). This describe all the non-obvious multiplications in $H^*(\mathbb{C}P^3)$.

For general n: Using induction on n and the same argument as in the previous cases, $x_j = x \cup x \cup \cdots x$ (j times). In other words, as a graded ring $H*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with degree $x = 2$. Passing to the limit gives $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$. $\checkmark$

If we use $\mathbb{Z}/(2\mathbb{Z})$ coefficients, the same method shows that $H*(\mathbb{R}P^n); \mathbb{Z}/(2\mathbb{Z}) \cong \mathbb{Z}/(2\mathbb{Z})[x]/(x^{n+1})$ with degree $x = 1$ and $H*(\mathbb{R}P^\infty); \mathbb{Z}/(2\mathbb{Z}) \cong \mathbb{Z}/(2\mathbb{Z})[x]$.

Chapter 21

Classification of Surfaces

Definition 21.0.4 *A* surface *is a 2-dimensional manifold.*

Definition 21.0.5 *Let* S_1 *and* S_2 *be two manifolds of dimension* n*. The* connected sum $S_1 \# S_2$ *is the manifold obtained by removing a disk* D^n *from* S_1 *and* S_2 *and gluing the resulting manifold with boundary* $S^1 \amalg S^1$ *to the cylinder* $S^1 \times [0, 1]$*.*

Theorem 21.0.6 *(a) Any compact orientable surface is homeomorphic to a sphere, or to the connected sum*

$$T^2 \# \ldots \# T^2$$

.

(b) Any compact nonorientable surface is homeomorphic to the connected sum

$$P \# \ldots P \#$$

where P *is the projective plane* $\mathbb{R}P^2$*.*

Alternative version of part (b) of Theorem 21.0.6:

Theorem 21.0.7 *Any compact orientable surface is homeomorphic to the connected sum of an orientable surface with either one copy of the projective plane* P *or one copy of the Klein bottle* K*.*

Proof of Theorem 21.0.6:

Definition 21.0.8 Euler Characteristic

The Euler characteristic of a topological space M *is the alternating sum of the dimensions of the homology groups (with rational coefficients):*

$$\chi(M) = h_0(M) - h_1(M) + \dots$$

where $h_j(M) = \dim H_j(M; \mathbb{Q})$.

For a manifold of dimension 2 equipped with a triangulation, the Euler characteristic is given by

$$\chi(M) = V - E + F$$

where V is the number of vertices, E the number of edges and F the number of faces. The Euler characteristic is independent of the choice of triangulation.

Proposition 21.0.9 *The Euler characteristic of a connected sum of surfaces S_1 and S_2 is given by*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

(This is proved by counting the number of vertices, edges and faces in a natural triangulation of the connected sum.)

Lemma 21.0.10 *The Euler characteristics of surfaces are as follows:*

genus $= 0$

$$\chi(S^2) = 2$$

genus $= g$

$$\chi(T^2 \# \dots T^2) = 2 - 2g$$

(the genus is the number of copies of T^2)

(connected sum of n copies of the projective plane)

$$\chi(P \# \dots \# P) = 2 - n$$

(connected sum of K with genus g orientable surface)

$$\chi(K \# T^2 \# \dots \# T^2) = -2g$$

(connected sum of P with genus g orientable surface)

$$\chi(P \# T^2 \# \dots \# T^2) = 1 - 2g$$

Lemma 21.0.11 *Surfaces are classified by:*
(i) whether they are orientable or nonorientable
(ii) their Euler characteristic

Proof of Theorem 21.0.6:

1. Take a triangulation of the surface S. Glue together some (not all) of the edges to form a surface D which is a closed disk. (This comes from a Lemma which asserts that if we glue together two disks along a common segment of their boundaries, the result is again a disk.) The edges along the boundary of D form a word where each edge is designated by a letter x_1 or x_2, with the same letter used to designate edges that are glued.

2. We now have a polygon D whose edges must be identified in pairs to obtain S. We subdivide the edges as follows.

(i) Edges *of the first kind* are those for which the letter designating the edge appears with both exponents $+1$ and -1.

(ii) Edges *of the second kind* are those for which the letter designating the edge appears with only one exponent ($+1$ or -1)

Adjacent edges of the first kind can be eliminated if there are at least four edges. (See Figure 1.17, p. 22, figure #2.)

3. Identify all vertices to a single vertex. If there are at least 2 different equivalence classes, then the polygon must have an adjacent pair of vertices which are not equivalent, call them P and Q.

Cut along the edge c from Q to the other vertex of a. Then glue together the two edges labelled a. The new polygon has one less vertex in the equivalence class of P. (See Figure 1.18, p. 23, figure #3.)

Perform step 2 again if possible (eliminate adjacent edges). Then perform step 3 again, reducing the number of vertices in the equivalence class of P. If more than one equivalence class of vertices remains, repeat the procedure to reduce the number of equivalence classes of vertices to 1, in other words we reduce to a polygon where all vertices are to be identified to a single vertex.

4. Make all pairs of edges of the second king adjacent. (See Figure 1.19, p. 24, #4.) Thus if there are no pairs of edges of the first kind, the symbol becomes

$$x_1x_1x_2x_2 \dots x_nx_n$$

In this case the surface is

$$S = P\# \dots \#P$$

(the connected sum of n copies of P).

Otherwise there is at least one pair of edges of the first kind (label these c) One can argue that there is a second pair of edges of the first kind interspersed (label these d. It is possible to transform these so they are consecutive, so the symbol includes

$$cdc^{-1}d^{-1}$$

This corresponds to the connected sum of one copy of T^2 with a surface with fewer edges in its triangulation. (See Figure 1.21, p. 25, #5.) $\square$

Lemma 21.0.12

$$T^2\#P \cong P\#P\#P$$

Remark 21.0.13 $P\#P \cong K$ *This is because we can carve up the diagram representing the Klein bottle, a square with two parallel edges identified in the same direction, and the two remaining parallel edges identified in opposite directions. (See Figure 1.5, p. 10, #1) This is the union of two copies of the Möbius strip along their boundary, using the fact that a Möbius strip is the same as the complement of a disk in the real projective plane.*

This reduces the proof of Lemma 21.0.12 to proving

Lemma 21.0.14 $P\#K \cong P\#T$

This is proved by decomposing a torus and a Klein bottle as the union of two rectangles. We excise a disk from one of the rectangles, and glue a Möbius strip to the boundary of the excised disk (to form the connected sum of P with the torus or Klein bottle). The text (Massey, see handout, Lemma 1.7.1) argues that the resulting objects are homeomorphic. Indeed, we can regard this as taking the connected sum of a Möbius strip with a torus or Klein bottle, and then gluing a disk to the boundary of the Möbius strip. The first step (connected sum of Möbius strip with torus or Klein bottle) yields two spaces that are manifestly homeomorphic. So they remain homeomorphic after gluing a disk to the boundary of the Möbius strip. See Figure 1.23, p. 27, #6. $\square$

References: 1. William S. Massey, *Algebraic Topology: An Introduction* (Harcourt Brace and World, 1967), Chapter 1.

(All figures are taken from Chapter 1 of Massey's book.)

2. James R. Munkres, *Topology* (Second Edition), Chapter 12.

Chapter 22

Group Structures on Homotopy Classes of Maps

For basepointed spaces X, Y, recall that $[X, Y]$ denotes the based homotopy classes of based maps from X to Y. In general $[X, Y]$ has no canonical group structure, but we define concepts of H-group and co-H-group such that $[X, Y]$ has a natural group structure provided either Y is an H-group or X is a co-H-group.

It is easy to check that if G is a topological group (regarded as a pointed space with the identity as basepoint) then $[X, G]$ has a group structure defined by $[f][g] = [h]$, where $h(x)$ is the product $f(x)g(x)$ in G. But a topological group is more than we need: all we need is a group "up to homotopy". We generalize topological group to H-group as follows:

A pointed space (H, e) is called an *H-space* if $\exists$ a (continuous pointed) map $m : H \times H \to H$ such that

$$\begin{tikzcd} H \arrow[rr, "i_1"] \arrow[dr, "1_H"'] & & H \times H \arrow[dl, "m"] \\ & H & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H \arrow[rr, "i_1"] \arrow[dr, "1_H"'] & & H \times H \arrow[dl, "m"] \\ & H & \end{tikzcd}$$

are homotopy commutative, where $i_1(x) := (x, e)$ and $i_2(X) := (e, x)$.

An H-space is called *homotopy associative* if

$$\begin{tikzcd} H \times H \times H \arrow[r, "1_H \times m"] \arrow[d, "m \times H"] & H \times 1_H \arrow[d, "m"] \\ H \times H \arrow[r, "m"] & H \end{tikzcd}$$

If H is an H-space, a map $c : H \to H$ is called a *homotopy inverse* for H if

$$\begin{tikzcd} H \arrow[rr, "(1_H, *)"] \arrow[dr, "*"'] & & H \times H \arrow[dl, "m"] \\ & H & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H \arrow[rr, "(*, 1_H)"] \arrow[dr, "m"'] & & H \times H \arrow[dl, "*"] \\ & H & \end{tikzcd}$$

are homotopy commutative.

A homotopy associative H-space with a homotopy inverse is called an H-group.

An H-space is called homotopy abelian if

$$\begin{tikzcd} H \arrow[rr, "T"] \arrow[dr, "m"'] & & H \times H \arrow[dl, "m"] \\ & H & \end{tikzcd}$$

is homotopy commutative, where T is the swap map $T(x, y) = (y, x)$.

Proposition 22.0.15 *Let H be an H-group. Then $\forall X$, $[X, H]$ has a natural group structure given by $[f][g] = [m \circ (f, g)]$. If H is homotopy abelian then the group is abelian.*

Remark 22.0.16 *"Natural" means that any map $q : W \to X$ induces a group homomorphism denoted $q^{\#} : [X, H] \to [W, H]$ defined by $q^{\#}([f]) = [f \circ q]$. The assignment $X \mapsto [X, H]$ is thus a contravariant functor.*

Proof:
Associative:

$m \circ (m \times 1_H) \circ (f, g, h) = m(\circ 1_H) \times m \circ (f, g, h)$ so $([f][g])[h] = [f]([g][h])$.
Identity:

$m \circ (*, f) = m \circ i_1 \circ f = 1_H \circ f = f$ so $[*][f] = [f]$ and similarly $[f][*] = [f]$, and thus $[*]$ forms a 2-sided for $[X, H]$.
Inverse:

Given $[f]$, define $[f]^{-1}$ to be the class represented by $c \circ f$. $m \circ (f, f^{-1}) = m \circ (1_H, c) \circ (f \times f) = 1_H \circ f = (f \times) f \circ * = *$ so $[f][f^{-1}] = [*]$ and similarly $[f^{-1}][f] = [*]$. Thus $[f^{-1}]$ forms a 2-sided inverse for f.

Finally, if H is homotopy abelian then $[f][g] = m \circ (f, g) = m \circ T \circ (f, g) = m \circ (g, f) = [g][f]$, so that $[X, H]$ is abelian. $\square$

Two H-space structures m, m' on X are called *equivalent* if $m \simeq m'$ (rel $*$) as maps from $X \times X$ to X. It is clear that equivalent H-space structures on X result in the same group structure on $[W, X]$.

A basepoint-preserving map $f : X \to Y$ between H-spaces is called an *H-map* if

$$\begin{tikzcd} X \times X \arrow[r, "f \times f"] \arrow[d, "m_X"] & Y \times Y \arrow[d, "m_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

homotopy commutes.

An H-map $f : X \to Y$ induces, for any space A, a group homomorphism $f_{\#} : [A, X] \to [A, Y]$ given by $f_{\#}([g]) = [f \circ g]$.

Remark 22.0.17 *The collection of H-spaces forms a category with H-maps as morphisms.*

Examples

1. A topological group is clearly an H-group.

2. $\mathbb{R}^8$ has a continous (non-associative) multiplication as the "Cayley Numbers", also called "octonians" $\mathbb{O}$.

3. Loop space on X:

   Given pointed spaces W and X, we define the function space X^W, also denoted $\mathrm{Map}_*(W, X)$. Set $X^W := \{\text{continuous } f : W \to X\}$. Topologize X^W as follows: For each pair (K, U)

where $K \subset W$ is compact and $U \subset X$ is open, let $V_{(K,U} = \{f \in X^W \mid f(K) \subset U\}$. Take the set of all such sets $V_{(K,U)}$ as the basis for the topology on X^W.

X^{S^1} is called the "loop space" of X and denote ΩX. Define a multiplication on ΩX which resembles the multiplication in the group $\pi_1(X)$ by $m(f,g) := f \cdot g$.

To show that m is continuous:

Let $V_{(K,U)}$ be a subbasic open set in ΩX. Write $K = K' \cup K''$ where $K' = K \cap [0,1/2]$ and $K'' = K \cap [1/2,1]$. Then $m^{-1}(V_{(K,U)}) = V_{(L',U)} \times V_{(L'',U)}$ where L' is the image of K' under the homeomorphism $[0,1/2] \to [0,1]$ given by $t \mapsto 2t$ and L'' is the image of K'' under the homeomorphism $[1/2,1] \to [0,1]$ given by $t \mapsto 2t-1$. Therefore m is continuous.

The facts that ΩX is homotopy associative, that the constant map c_{x_0} is a homotopy identity, and that $f \to f^{-1}$ (where $f^{-1}(t) = f(1-t)$) is a homotopy inverse follow, immediately from the facts used in the proof that $\pi_1(X, x_0)$ is a group.

ΩX is an example of an H-group which is not a group. By definition a path from f to g in ΩX is the same as a homotopy $H : f \simeq g \ \mathrm{rel}(0,1)$. The group $[S^0, \Omega X]$ defined using the H-space structure on ΩX is clearly the same as $\pi_1(X, x_0)$.

Remark 22.0.18 *Given continous $f : X \to Y$, it is easy to see that there is a continous induced map $\Omega f : \Omega X \to \Omega Y$ given by $(\Omega f)(\alpha) := f \circ \alpha$. Thus the correspondence $X \mapsto \Omega X$ defines a functor from the category of topological spaces to the category of H-spaces.*

The preceding can be generalized as follows.

Observe that a pointed map $A \vee B \to Y$ is equivalent to a pair of pointed map $A \to Y$, $B \to Y$. (In other words, $A \vee B$ is the coproduct of A and B in the category of pointed topological spaces.) We write $f \perp g : A \vee B \to Y$ for the map corresponding to f and g.

A pointed space X is called a *co-H-space* if $\exists$ a (continuous pointed) map $\psi : X \to X \vee X$ such that

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "1_X"'] & & X \vee X \arrow[dl, "1_X \perp *"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "1_X"'] & & X \vee X \arrow[dl, "* \perp 1_X"] \\ & X & \end{tikzcd}$$

are homotopy commutative.

A co-H-space is called *homotopy coassociative* if

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\psi"] & X \vee X \arrow[d, "1_X \perp \psi"] \\ X \vee X \arrow[r, "\psi \perp 1_X"] & X \vee X \vee X \end{tikzcd}$$

If X is a co-H-space, a map $c : X \to X$ is called a *homotopy inverse* for X if

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "*"'] & & X \vee X \arrow[dl, "1_X \perp c"] \\ & X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "*"'] & & X \vee X \arrow[dl, "c \perp 1_X"] \\ & X & \end{tikzcd}$$

are homotopy commutative.

A homotopy coassociative co-H-space with a homotopy inverse is called a co-H-group.

A co-H-space is called homotopy coabelian if

$$\begin{tikzcd} & X \arrow[dl, "\psi"] \arrow[dr, "\psi"] & \\ X \vee X \arrow[rr, "T"] & & X \vee X \end{tikzcd}$$

is homotopy commutative, where T is the swap map.

Proposition 22.0.19 *Let X be a co-H-group. Then for any pointed space Y, $[X, Y]$ has a natural group structure. If X coabelian then $[X, Y]$ is abelian.*

Proof: The group structure is given by $[f][g] = [(f \perp g) \circ \psi]$ where $f, g : X \to Y$. The proof is essentially the same as the dual proof for H-groups with arrows reversed. Further, as before, a map $q : Y \to Z$ induces a group homomorphism $q_\# : [X, Y] \to [X, Z]$ defined by $q_\#([f]) = [q \circ f]$. (The association $Y \mapsto [X, Y]$, $q \mapsto q_\#$ is a functor from topological spaces to groups.) $\square$

Example of a co-H-group:

S^n is a co-H-space for $n \geq 1$. The map $\psi : S^n \to S^n \vee S^n$ is given by "pinching" the equator to a point.

Thus for any pointed space X, $\pi_n(X) := [S^n, X]$ has a natural group structure for $n \geq 1$, called the nth *homotopy group* of X. Looking at the case $n = 1$, the group structure that we get on $[S^1, X]$ is the same as that of the fundamental group.
More generally:

Let X be a topological space. Define a space denoted SX, called the (reduced) *suspension* of X, by $SX := (X \times I)/\big((X \times \{0\}) \cup (X \times \{1\}) \cup (* \times I)\big)$. For any X, SX becomes a co-H-group by pinching the equator, $X \times \{1/2\}$, to a point. That is, $\psi : SX \to SX \vee SX$ by

$$\psi(x,t) = \begin{cases} (x, 2t) \text{ in the first copy of } SX & \text{if } t \leq 1/2; \\ (x, 2t-1) \text{ in the second copy of } SX & \text{if } t \geq 1/2. \end{cases}$$

When $t = 1/2$ the definitions agree since each gives the common point at which the two copies of SX are joined.

This generalizes the preceding example since:

Lemma 22.0.20 *SS^n is homeomorphic to S^{n+1}.*

Proof: Intuitively, think of S^{n+1} as the one point compactification of $\mathbb{R}^{n+1}$ and notice that after removal of the point at which the identifications have been made, SS^n opens up to become an open $(n+1)$-disk. For a formal proof, write S^k as $I^k/\partial(I^k)$ and notice that both SS^n and S^{n+1} becomes quotients of I^{n+1} with exactly the same identifications. $\square$

Remark 22.0.21 *As in the case of Ω, given $f : X \to Y$ there is an induced map $Sf : SX \to SY$ defined by $Sf(x,t) := \big(f(x), t\big)$ and so S defines a functor from the category of pointed spaces to itself.*

Theorem 22.0.22 *For each pair of pointed spaces X and Y there is a natural bijection between the sets $\mathrm{Map}_*(SX, Y)$ and $\mathrm{Map}_*(X, \Omega Y)$. This bijection takes homotopic maps to homotopy maps and thus induces a bijection $[SX, Y] \to [X, \Omega Y]$. Furthermore, the group structure on $[SX, Y]$ coming from the co-H-space structure on SX coincides under this bijection with that coming from the H-space structure on ΩY.*

Proof: Define $\phi : \mathrm{Map}_*(SX, Y) \to \mathrm{Map}_*(X, \Omega Y)$ by $\phi(f) = g$ where $g(x)(t) = f(x,t)$. Notice that $g(x)(0) = f(x, 0) = f(*) = y_0$ and $g(x)(1) = f(x, 1) = f(*) = y_0$ since the identified subspace $\big((X \times \{0\}) \cup (X \times \{1\}) \cup (* \times I)\big)$ is used as the basepoint of SX. Thus $g(x)$ is an element of ΩY.

Must show that g is continuous.

Let $q : X \times I \to SX$ denote the quotient map. Let $V_{(K,U)}$ be a subbasic open set in ΩY. Then $g^{-1}(V_{(K,U)}) = \{x \in X \mid f(x, k) \in U \ \forall k \in K\}$. Pick $x \in g^{-1}(V_{(K,U)})$. By continuity of q and f, for each $k \in K$ find basic open set $A_k \times W_k \subset X \times I$ s.t. $x \in A_k$, $k \in W_k$ and $A_k \times W_k \subset (q \circ f)^{-1}(U)$. $\{W_k\}_{k\in K}$ covers I so choose a finite subcover $W_{k_1}, \dots, W_{k_n}$ and let $A = A_{k_1} \cap \cdots \cap A_{k_n}$. Then $x \in A$ and $A \subset g^{-1}(V_{(K,U)})$ so x is an interior point of $g^{-1}(V_{(K,U)})$ and since this is true for arbitrary x, $g^{-1}(V_{(K,U)})$ is open. Therefore g is continuous.

Show ϕ is $1-1$:

Clearly if $\phi(f) = \phi(f')$ then $f(x,t) = \big(\phi(f)(x)\big)(t) = \big(\phi(f')(x)\big)(t) = f'(x,t)$ for all x, t so $f = f'$.

Show ϕ is onto:

Given $g : X \to \Omega Y$, define : $SX \to Y$ by $f(x,t) = \big(g(x)\big)(t)$. For all x, $f(x,0) = \big(g(x)\big)(0) = y_0$ and $f(x,1) = \big(g(x)\big)(1) = y_0$ and for all t, $f(x_0,t) = \big(g(x_0)\big)(t) = c_{y_0}(t) = y_0$ and thus f is well defined.

Must show that f is continuous. Given open $U \subset Y$,

$$f^{-1}(U) = \{(x,t) \in SX \mid \big(g(x)\big)(t) \in U\}.$$

By the universal property of the quotient map, showing that $f^{-1}(U)$ is open is equivalent to showing that $(f \circ q)^{-1}(U)$ is open in $X \times I$.

For a pair $(x,t) \in X \times I$:

Since g is continuous $A := g^{-1}(V_{I,U}) \subset X$ is open. Thus $A \times I$ is an open subset of $X \times I$ which contains (x,t), and if $(a,t') \in A \times I$ then $f \circ q(a,t') = \big(g(a)\big)(t) \in U$ since $g(a)$ takes all of I to U. Thus $A \times I \subset (f \circ q)^{-1}(U)$ and thus (x,t) is an interior point of $A \times I$, and since this is true for arbibrary (x,t), f is continuous. Therefore f lies in $\mathrm{Map}_*(SX,Y)$ and clearly $\phi(f) = g$, so ϕ is onto.

It is easy to see that $f \simeq f' \Leftrightarrow \phi(f) = \phi(f')$. (e.g., if $H : f \simeq f'$ define $\big(g_s(x)\big)(t) := H_s(x,t)$.)

To show that the group structures coincide:

$$(ff')(x,t) = \begin{cases} f(x,2t) & \text{if } t \le 1/2; \\ f'(x,2t-1) & \text{if } t \ge 1/2. \end{cases}$$

so $\big(\phi(ff')\big)(x) = \Big(\big(\phi(f)\big)(x) \cdot \big(\phi(f')\big)(x)\Big)$ by the definition of multiplication of paths. Therefore $\phi(ff') = \phi(f)\phi(f')$. Thus ϕ is an isomorphism, or equivalently, the group structures coincide.

□

Corollary 22.0.23 $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ □

According to the previous theorem, there is a natural bijection between $\mathrm{Map}_*(SX, Y)$ and $\mathrm{Map}_*(X, \Omega Y)$ where natural means that for any map $j : A \to X$.

$$\begin{array}{ccc}
\mathrm{Map}_*(SX, Y) & \xrightarrow{\phi} & \mathrm{Map}(X, \Omega Y) \\
\Big\uparrow (Sj)^{\#} & & \Big\uparrow j^{\#} \\
\mathrm{Map}_*(SA, Y) & \xrightarrow{\phi} & \mathrm{Map}(A, \Omega Y)
\end{array}$$

commutes, and similarly for any $k : Y \to Z$

$$\begin{array}{ccc}
\mathrm{Map}_*(SX, Y) & \xrightarrow{\phi} & \mathrm{Map}(X, \Omega Y) \\
\Big\downarrow (Sk)_{\#} & & \Big\downarrow k_{\#} \\
\mathrm{Map}_*(SX, Z) & \xrightarrow{\phi} & \mathrm{Map}(X, \Omega Z)
\end{array}$$

For this reason, S and Ω are called adjoint functors. More generally:

Definition 22.0.24 *Functors $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ are called adjoint functors if there is a natural set bijection $\phi : \mathrm{Hom}_{\underline{\underline{C}}}(F\overline{\overline{X}}, Y) \to \mathrm{Hom}_{\underline{\underline{D}}}(\overline{\overline{X}}, G\overline{\overline{Y}})$ for all X in $\mathrm{Obj}\,\underline{\underline{C}}$ and Y in $\mathrm{Obj}\,\underline{\underline{D}}$. F is called the left adjoint or co-adjoint and G is called the right adjoint or simply adjoint.*

Another example: Let T : Vector Spaces/$\mathbf{k} \to$ Algebras/$\mathbf{k}$ by sending V to the tensor algebra on V, and let J : Algebras/$\mathbf{k} \to$ Vector Spaces/$\mathbf{k}$ be the forgetful functor. Then $\mathrm{Hom}_{\mathrm{Alg}}(TV, W) = \mathrm{Hom}_{VS}(V, JW)$ for any vector space V and algebra W over $\mathbf{k}$.

Let X be an H-space. Then ΩX has a second H-space structure (in addition to the one coming from the loop-space structure) given by $m' : \Omega X \times \Omega X \to \Omega X$ with m' is defined by $m'(\alpha, \beta) = \gamma$ where $\gamma(t) = \alpha(t)\beta(t)$ (where $\alpha(t)\beta(t)$ denotes the product $m_X\big(\alpha(t), \beta(t)\big)$ in the H-space structure on X.

Theorem 22.0.25 *Let X be an H-space. Then the H-space structure on ΩX induced from that on X as above is equivalent to the one coming form the loop-space multiplication. Furthermore, this common H-space structure is homotopy abelian.*

Proof: In one H-space structure $(\alpha\beta)(s) = \alpha(s)\beta(s)$, while in the other the product is

$$(\alpha \cdot \beta)(s) := \begin{cases} \alpha(2s) & \text{if } s \le 1/2; \\ \beta(2s-1) & \text{if } s \ge 1/2. \end{cases}$$

We construct a homotopy by homotoping α until it becomes $\alpha \cdot c_{x_0}$ and β until it comes $c_{x_0} \cdot \beta$ while at all times "multiplying" the paths in X using the H-space structure on X. Explicitly $H : \Omega X \times \Omega X \times I \to \Omega X$ by

$$H(\alpha, \beta, t)(s) = \begin{cases} \alpha\big(2s/(t+1)\big)\beta(0) & \text{if } 2s \le 1-t; \\ \alpha\big(2s/(t+1)\big)\beta\big((2s+t-1)/(t+1)\big) & \text{if } 1-t \le 2s \le 1+t; \\ \alpha(0)\beta\big((2s+t-1)/(t+1)\big) & \text{if } 2s \ge 1+t; \end{cases}$$

The definitions agree on the overlaps do the function is well defined and is continuous. Check that H is a homotopy rel $*$:

The basepoint of $\Omega X \times \Omega X$ is (c_{x_0}, c_{x_0}).

$H\big((c_{x_0}, c_{x_0}, t)\big)(s) = x_0 x_0 = x_0\ \forall s, t$. Hence $H(c_{x_0}, c_{x_0}, t) = c_{x_0}\ \forall t$ so H is a homotopy rel $*$. Note: Although for arbitrary x, $x_0 x$ and $x x_0$ need not equal x, since multiplication by x_0 is only required to be homotopic to the identity rather that equal to the identity, it is nevertheless true that $x_0 x_0 = x_0$ since multiplication is a basepoint-preserving map.

$H(\alpha, \beta, 1)(s) = \alpha(s)\beta(s)\ \forall s$ which is the product of α and β in the H-space structure induced from that on X.

Since $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = x_0$,

$$H(\alpha, \beta, 0)(s) = \begin{cases} \alpha(2s)\beta(0) & \text{if } 2s \le 1; \\ \alpha(1)\beta(2s-1) & \text{if } 2s \ge 1, \end{cases} = \begin{cases} \alpha(2s)x_0 & \text{if } 2s \le 1; \\ x_0\beta(2s-1) & \text{if } 2s \ge 1, \end{cases} = \tilde{\alpha} \cdot \tilde{\tilde{\beta}}$$

where $\tilde{\alpha}(s) = \alpha(s)x_0$ and $\tilde{\tilde{\beta}}(s) = x_0\beta(s)$. Since multiplication by x_0 is homotopy to the identity (rel $*$), $\tilde{\alpha} \simeq \alpha$ (rel $*$) and similarly $\tilde{\tilde{\beta}} \simeq \beta$ (rel $*$). Thus the multiplications maps are homotopic and so the two H-space structures are equivalent.

To show that this structure is homotopy abelian, observe there is a homotopy analogous to H given by

$$J(\alpha, \beta, t)(s) = \begin{cases} \alpha(0)\beta\big(2s/(t+1)\big) & \text{if } 2s \le 1-t; \\ \alpha\big((2s+t-1)/(t+1)\big)\beta\big(2s/(t+1)\big) & \text{if } 1-t \le 2s \le 1+t; \\ \alpha\big((2s+t-1)/(t+1)\big)\beta(1) & \text{if } 2s \ge 1+t. \end{cases}$$

As before $J_1(s) = \alpha(s)\beta(s)$ but

$$J_0(s) = \begin{cases} x_0\beta(2s) & \text{if } 2s \le 1; \\ \alpha(2s-1)x_0 & \text{if } 2s \ge 1. \end{cases}$$

Since $J_0 \simeq \beta \cdot \alpha$, we get that the H-space structure is homotopy abelian. $\square$

Corollary 22.0.26 *Suppose Y is an H-space. Then for any space X the group structure on $[SX, Y]$ coming from the co-H-space structure on SX agrees with that coming from the H-space structure on Y. Furthermore this common group structure is abelian.*

Proof: By Theorem 22.0.22 there is a bijection from $[SX, Y] \cong [X, \Omega Y]$ which is a group isomorphism from $[SX, Y]$ with the group structure coming from the suspension structure on $[SX]$, to $[X, \Omega Y]$ with the group structure coming from the loop space H-space structure on ΩY. It is easy to check that the group space structure on $[SX, Y]$ coming from the H-space structure on Y corresponds under this bijection with that on $[X, \Omega Y]$ coming from the H-space structure on Y. Since these H-space structures agree and are homotopy abelian, the result follows. $\square$

Corollary 22.0.27 *If Y is an H-space, $\pi_1(Y)$ is abelian.*

$\square$

Corollary 22.0.28 *For any spaces X and Y, $[S^2X, Y]$ is abelian.*

Proof: $[S^2X, Y] \cong [SX, \Omega Y]$. $\square$

Corollary 22.0.29 *$\pi_n(Y)$ is abelian for all Y when $n \geq 2$.*

$\square$

22.1 Hurewicz Homomorphism

Suppose $n \geq 1$ and let ι_n be a generator of $H_n(S^n)$. Define $h : \pi_n(X) \to H_n(X)$ by $h([f]) := f_*(\iota_n)$ for a representative $f : S^n \to X$. This is well defined by the homotopy axiom.

Check that h is a group homomorphism:

$[fg] = [(f \perp g) \circ \psi]$ where $\psi : S^n \to S^n \vee S^n$ pinches the equator to a point.

$H_n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $e_1 := j_1(\iota)$, $e_2 := j_1(\iota)$, where $j_1, j_2 : S^n \to S^n \vee S^n \subset S^n \times S^n$ by $j_1(x) = (x, *)$ and $j_2(x) = (*, x)$. $\psi_*(\iota) = e_1 + e_2$. To determine $(f \perp g)_*(e_1)$ use the commutative diagram

$$\begin{tikzcd} S^n \arrow[r, "j_1"] \arrow[dr, "f \simeq f \cdot *"'] & S^n \vee S^n \arrow[d, "f \perp g"] \\ & X \end{tikzcd}$$

to obtain $(f \perp g)_*(e_1) = f_*(\iota)$. Similary $(f \perp g)_*(e_2) = g_*(\iota)$. Therefore $h([fg]) = (fg)_*(\iota) = (f \perp g)_*(e_1 + e_2) = f_*(\iota) + g_*(\iota) = h[f] + h[g]$ and so h is a homomorphism.

We now specialize to the case $n = 1$.

As before, let $S_*(X)$ denote the singular chain complex of X. Let exp be the generator of $S_1(S^1)$ defined by $\exp : \Delta^1 = I \to S^1$ where $\exp(t) = e^{2\pi i t}$. In S^1, set $v = 1 = \exp(0)$ and $w = -1 = \exp(1)$. Clearly $\partial(\exp) = v - v = 0$ so $[\exp]$ is a cycle and thus represents a homology class in $H_1(S^1)$.

Lemma 22.1.1 $[\exp]$ *is a generator of* $H_1(S^1)$.

Proof: Set $D := [-1, 1]$, $D^+ := [0, 1]$ and $D^- := [-1, 1]$. Let f be the composite $\Delta^1 = I \cong D^+ \xrightarrow{\tilde{f}} S^1$ where $\tilde{f}(t) = e^{\pi i t}$ and let g be the composite $\Delta^1 = I \cong D^- \xrightarrow{\tilde{g}} S^1$ where $\tilde{g}(t) = e^{\pi i (t+1)}$. We have isomorphisms

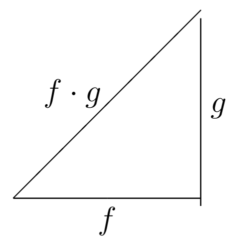
$$\begin{tikzcd} H_1(D^+, S^0) \arrow[r, "\cong", "\text{excision}"'] \arrow[d, "\cong"', "\partial"] & H_1(S^1, D^-) \\ \mathbb{Z} \cong \tilde{H}_0(S^0) & H_1(S^1) \arrow[u, "\cong"'] \end{tikzcd}$$

$w - v$ is a generator of $\tilde{H}_0(S^0)$ so its image under the isomorphisms is a generator of $H_1(S^1)$. $f \in S_1(D^+)$ has the property that $\partial f = w - v$ so it represents the generator of $H_1(D^+, S^0)$

which hits $w - v$ under the isomorphism ∂, and thus its image in $S_1(S^1)/S_1(D^-)$ represents a generator of $H_1(S^1, D^-)$. $f + g \in S_1(S^1)$ projects to f in $S_1(S^1)/S_1(D^-)$, and $f + g$ is a cycle so the homology class $[f + g]$ is a generator of $H_1(S^1)$. Since $\exp = f \cdot g$, we conclude the proof by the following Lemma which shows that $[\exp] = [f + g]$.

Lemma 22.1.2 *Let $f, g : I \to X$ such that $g(0) = f(1)$. Then as elements of $S_1(X)$, $f \cdot g$ is homologous to $f + g$.*

Proof: Define $T : \Delta^2 \to X$ by extending the map shown around the boundary:



This is possible since the map around the boundary is null homotopic.

$\partial T = f - f \cdot g + g$, so $f \cdot g$ is homologous to $f + g$. □

We will use $[\exp]$ for ι_1.

Theorem 22.1.3 *(Baby Hurewicz Theorem)*

Suppose X is connected. Then $h : \pi_1(X) \to H_1(X)$ is onto and its kernel is the commutator subgroup of $\pi_1(X)$. ie. $H_1(X) \cong \pi_1(X)/(\text{commutator subgroup}) = \text{abelianization of } \pi_1(X)$.

Proof: Let x_0 be the basepoint of X.
Show that h is onto:

Let $z = \sum n_i T_i$ represent a homology class in $H_1(X)$. Thus $0 = \partial z = \sum n_i \big(T_i(1) - T_i(0)\big)$.

Let γ_{i0} and γ_{i1} be paths joining x_0 to $T_i(0)$ and $T_i(1)$ respectively.

Let $S_i = \gamma_{i0} + T_i - \gamma_{i1} \in S_1(X)$ Thus $z = \sum n_i S_i$ since the γ's cancel out, using $\partial z = 0$. (Each γ_i appears equally often with $\epsilon = 0$ as with $\epsilon = 1$.)

Set $f_i := \gamma_{i_0} \cdot T_i \cdot \gamma_i^{-1} \in \pi_1(X)$.

Let $\bar{f}_i$ denote the composite $I \twoheadrightarrow I/\sim = S^1 \xrightarrow{f_i} X \in \pi_1(X)$.

By the preceding Lemma, $\bar{f}_i$ is homologous to $\gamma_{i0} + T_i - \gamma_{i1} = S_i \in S_1(X)$. Therefore $h(\prod f_i^{n_i}) = (\prod f_i^{n_i})_*(\iota_1) = [\sum n_i \bar{f}_i] = [\sum n_i S_i] = [z]$.

Show $\ker h =$ commutator subgroup:

$H_1(X)$ is abelian so (commutator subgroup) $\subset \ker h$.

Conversely, suppose $f \in \ker h$. Then, regarded as a generator of $S_1(X)$, $f = \partial z$ for some $z \in S_2(X)$.

Write $f = \partial(\sum n_i T_i) = \sum n_i \partial T_i$. Let $\partial T_i = \alpha_{i0} - \alpha_{i1} + \alpha_{i2}$ and for $j = 0, 1, 2$ choose paths γ_{ij} joining x_0 to the endpoints of α_{ij} as shown, making sure to always choose the same path γ_{ij} if a given point occurs as an endpoint more than once.

Set

$g_{i0} := \gamma_{i1}\alpha_{i0}\gamma_{i2}^{-1}$

$g_{i1} := \gamma_{i0}\alpha_{i1}\gamma_{i2}^{-1}$

$g_{i0} := \gamma_{i0}\alpha_{i2}\gamma_{i1}^{-1}$

Set $g_i = g_{i0}g_{i1}^{-1}g_{i2} = \gamma_{i1}\alpha_{i0}\alpha_{i1}^{-1}\alpha_{i2}\gamma_{i1}^{-1}$.

Since $\alpha_{i0}\alpha_{i1}^{-1}\alpha_{i2}$ can be extended to a map on the interior (namely T_i,) it is null homotopic, so $g_i \simeq *$. Therefore $\prod_i (g_i)^{n_i} = 1 \in \pi_1(X)$. But $f = \sum n_i \partial T_i = \sum n_i(\alpha_{i0} - \alpha_{i1}^{-1} + \alpha_{i2})$ in the free abelian group $S_1(X)$. This means that when terms are collected on the right, f remains with coefficient 1 and all other terms cancel. Thus modulo the commutator subgroup the product $\pi_i(g_i)^{n_i}$ can be reordered to give f with the $\gamma' s$ cancelling out. Therefore, modulo commutators, $f = 1$ so that $f \in$ (commutator subgroup). $\square$

Chapter 23

Universal Coefficient Theorem

Theorem 23.0.4 *Universal Coefficient Theorem – homology*
Let G be an abelian group. Then

$$H_q(X,A;G) \cong H_q(X,A) \otimes G \oplus Tor(H_{q-1}(X,A),G)$$

More precisely there is a short exact sequence

$$0 \to H_q(X,A) \otimes G \to H_q(X,A;G) \to \mathrm{Tor}(H_{q-1}(X,A),G) \to 0$$

This sequence splits (implying the preceding statement) but not canonically (the splitting requires some choices).

Theorem 23.0.5 *Universal Coefficient Theorem – cohomology*
Let G be an abelian group. Then

$$H^q(X,A;G) \cong \mathrm{Hom}(H_q(X,A),G) \oplus \mathrm{Ext}(H_{q-1}(X,A),G)$$

More precisely there is a short exact sequence

$$0 \to Ext(H_{q-1}(X,A),G) \to H^q(X,A;G) \to Hom(H_q(X,A),G) \to 0$$

This sequence splits (implying the preceding statement) but not canonically (the splitting requires some choices).

Definition 23.0.6 *Let $R = \mathbb{Z}$ and let M be a left $\mathbb{Z}$-module. A free resolution of M is a sequence of left $\mathbb{Z}$-modules and an exact sequence*

$$\to C_q \xrightarrow{d} C_{q-1} \xrightarrow{d} C_{q-2} \xrightarrow{d} \dots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\epsilon} M \to 0 \tag{23.1}$$

where all the C_j are free.

Free resolutions exist. To construct one, we choose ϵ mapping a free module C_0 onto M, then choose d mapping a free module C_1 onto $\mathrm{Ker}(\epsilon)$, etc.

Definition 23.0.7 *To form* Tor*, we tensor the sequence (23.1) by G on the left, forming*

$$G \otimes C_q \xrightarrow{d^*} G \otimes C_{q-1} \xrightarrow{d^*} \dots$$

The resulting sequence is not exact. We define

$$\mathrm{Tor}_q(G, M) = \frac{\mathrm{Ker}(d_* : G \otimes C_q) \to G \otimes C_{q-1})}{\mathrm{Im}(d_* : G \otimes C_{q+1} \to G \otimes C_q)}$$

Similarly we define

$$\mathrm{Ext}_q(G, M) = \frac{\mathrm{Ker}(d^* : Hom(C_q, G) \to Hom(C_{q+1}, G)}{\mathrm{Im}(d^* : Hom(C_{q-1}, G) \to Hom(C_q, G)}$$

We use $q = 1$ for Ext and Tor. For $q \geq 2$, we can arrange that Ext $=$ Tor $= 0$.

Remark: if $G = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ (a field of characteristic zero) we have $H_n(X; G) = H_n(X) \otimes G$ and $H^n(X; G) = \mathrm{Hom}(H^n(X), G)$.

Remark: If H_n and H_{n-1} are finitely generated, then $H_n(X; \mathbb{Z}_p)$ has

- a $\mathbb{Z}_p$ summand for every $\mathbb{Z}$ summand of H_n
- a $\mathbb{Z}_p$ summand for every $\mathbb{Z}_{p^k}$ summand of H_n (for $k \geq 1$)
- a $\mathbb{Z}_p$ summand for every $\mathbb{Z}_{p^k}$ summand of H_{n-1} (for $k \geq 1$)

Remark: If A or B is free or torsion free, then $\mathrm{Tor}(A, B) = 0$

If H is free then $\mathrm{Ext}(H, G) = 0$.

$$\mathrm{Ext}(\mathbb{Z}_n, G) = G/nG.$$

Remark: If $H_n(X)$ and $H_{n-1}(X)$ are finitely generated with torsion subgroups T_n resp. T_{n-1}, then $H^n(X) \cong H_n/T_n \oplus T_{n-1}$.

Example:

$H_j(\mathbb{R}P^n; \mathbb{Z}_2) =$

$\mathbb{Z}_2$ when $H_j = \mathbb{Z}$ or $\mathbb{Z}_2$,

$\mathbb{Z}_2$ when $H_{j-1} = \mathbb{Z}_2$.

Example: orientable 2-manifolds of genus g

degree	H_*	H^*
2	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$
0	$\mathbb{Z}$	$\mathbb{Z}$

Example: nonorientable 2-manifolds

degree	H_*	H^*	$H^*(-,\mathbb{Z}_2)$
2	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$\mathbb{Z}^n \oplus \mathbb{Z}_2$	$\mathbb{Z}^n$	$\mathbb{Z}^{n+1}$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$

Chapter 24

Hodge Star Operator

Let M be a compact oriented manifold of dimension n.

Definition 24.0.8 *The Hodge star operator is a linear map*

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

which satisfies

- $$* \circ * = (-1)^{k(n-k)}$$

- $$\alpha \wedge *\alpha = |\alpha|^2 vol$$

  where vol is the standard volume form and $|\alpha|^2$ is the usual norm on $\alpha(x)$ viewed as an element of $\Lambda^k T_x^* M$.

The definition of the Hodge star operator requires the choice of a Riemannian metric on the tangent bundle to M.

Let d be the exterior differential. Then $d^* := *d*$ is the formal adjoint of d, in the sense that $(d^*a, b) = (a, db)$. This is because $(*a, *b) = (a, b)$ for any $a, b \in \Omega^k M$, so

$$(da, b) = \int da^* b = (-1)^k \int a^d * b$$

(by Stokes' theorem)

$$= (-1)^{k(n-k)}(-1)^k(a, *d * b)$$

Definition 24.0.9 *A k-form α on M is* harmonic *if $d\alpha = d^*\alpha = 0$.*

Theorem 24.0.10 *The set of harmonic k-forms is isomorphic to $H^k(M; \mathbb{R})$.*

Theorem 24.0.11 *If α is a harmonic k-form on M, its Poincare dual is represented by $*\alpha$. The pairing between an element α and its Poincare dual is nondegenerate, i.e. for any α $\int_M \alpha \wedge *\alpha = 0 \longrightarrow \alpha = 0$.*

For the definition of the Hodge star operator, see J. Roe, *Elliptic Operators, Topology and Asymptotic Methods* (Pitman, 1988). I have reproduced two pages from this book (p. 18-19) which give the definition. See the link on this website.