

THE FUNDAMENTAL GROUP OF REDUCTIVE BOREL-SERRE AND SATAKE COMPACTIFICATIONS

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ABSTRACT. Let \mathbf{G} be an almost simple, simply connected algebraic group defined over a number field k , and let S be a finite set of places of k including all infinite places. Let X be the product over $v \in S$ of the symmetric spaces associated to $\mathbf{G}(k_v)$, when v is an infinite place, and the Bruhat-Tits buildings associated to $\mathbf{G}(k_v)$, when v is a finite place. The main result of this paper is to compute explicitly the fundamental group of the reductive Borel-Serre compactification of $\Gamma \backslash X$, where Γ is an S -arithmetic subgroup of \mathbf{G} . In the case that Γ is neat, we show that this fundamental group is isomorphic to Γ/ET , where ET is the subgroup generated by the elements of Γ belonging to unipotent radicals of k -parabolic subgroups. Analogous computations of the fundamental group of the Satake compactifications are made. It is noteworthy that calculations of the congruence subgroup kernel $C(S, \mathbf{G})$ yield similar results.

1. INTRODUCTION

Let X be a symmetric space of noncompact type, and let $G = \text{Isom}(X)^0$. Pick a basepoint $x_0 \in X$, with isotropy group $K \subset G$. Then $X \cong G/K$. Suppose that $G = \mathbf{G}(\mathbb{R})^0$, where \mathbf{G} is a connected almost simple algebraic group defined over \mathbb{Q} , and that Γ is an arithmetic subgroup of G . The associated locally symmetric space $\Gamma \backslash X$ is typically not compact. Noncompact arithmetic locally symmetric spaces $\Gamma \backslash X$ admit several different compactifications such as the Satake compactifications, the Borel-Serre compactification and the reductive Borel-Serre compactification. If $\Gamma \backslash X$ is a Hermitian locally symmetric space, then one of the Satake compactifications, called the Baily-Borel or Baily-Borel Satake compactification, is a projective variety.

The cohomology and homology groups of locally symmetric spaces $\Gamma \backslash X$ and of their compactifications have been intensively studied because of their relation to the cohomology of Γ and to automorphic forms. Our interest here is in the fundamental group. There are a number of results on the fundamental group of the Baily-Borel Satake compactification of particular Hermitian locally symmetric spaces (see [15, 17–20, 24, 32]). In this paper we deal with arithmetic locally symmetric spaces in general (not necessarily Hermitian). A natural compactification of $\Gamma \backslash X$ to consider in this context is the reductive Borel-Serre compactification, which more and more is playing a central role (see, for example [2, 23, 35]). We show that its fundamental group can be described in terms of “elementary matrices”. We also determine the fundamental group of arbitrary Satake compactifications of a locally symmetric space and actually treat the more general case of Γ an S -arithmetic group.

In the remainder of this section we state our results more precisely. In sections 2 and 3 we recall results about compactifications of locally symmetric spaces and Bruhat-Tits buildings,

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which we will need in our computations of fundamental groups. Of particular importance are descriptions of the topology of these spaces and of the stabilizers of points under the built-in group action. In section 4 we define the reductive Borel-Serre and Satake compactifications of $\Gamma \backslash X$ for a general S -arithmetic group Γ . In sections 5 and 6 we prove our results.

1.1. S -arithmetic groups and elementary matrices. Let k be a number field and let S be a finite set of places of k which contains the infinite places S_∞ . Set $S_f = S \setminus S_\infty$. For $v \in S$ let k_v denote the completion of k with respect to a norm associated to v . Denote by \mathcal{O} the ring of S -integers

$$\mathcal{O} = \{ x \in k \mid \text{ord}_v(x) \geq 0 \text{ for all } v \notin S \} .$$

The corresponding group of units \mathcal{O}^\times is finite if and only if $|S| = 1$.

Let \mathbf{G} be an algebraic group defined over k and fix a faithful representation

$$\rho: \mathbf{G} \longrightarrow \mathbf{GL}_N$$

defined over k . Set

$$\mathbf{G}(\mathcal{O}) = \rho^{-1}(\mathbf{GL}_N(\mathcal{O})) \subset \mathbf{G}(k).$$

Note that $\mathbf{G}(\mathcal{O})$ depends on the representation ρ .

A subgroup $\Gamma \subset \mathbf{G}(k)$ is an S -arithmetic subgroup if it is commensurable with $\mathbf{G}(\mathcal{O})$; an S_∞ -arithmetic subgroup is simply called an *arithmetic subgroup*. This definition is independent of the choice of ρ . Note that if $S_1 \subset S_2$, then an S_1 -arithmetic group is not necessarily an S_2 -arithmetic group. (For example, $\mathbf{SL}_2(\mathbb{Z})$ is of infinite index in $\mathbf{SL}_2(\mathbb{Z}[1/p])$ for any prime p and in particular they are not commensurable.) However if Γ is an S_2 -arithmetic subgroup and K_v is a compact open subgroup of $\mathbf{G}(k_v)$ for each $v \in S_2 \setminus S_1$, then $\Gamma \cap \bigcap_{v \in S_2 \setminus S_1} K_v$ is an S_1 -arithmetic subgroup.

For any S -arithmetic subgroup Γ let

$$E\Gamma \subset \Gamma$$

be the subgroup generated by the elements of Γ belonging to the unipotent radical of any parabolic k -subgroup of \mathbf{G} (the subgroup of ‘‘elementary matrices’’). Let

$$S\text{-rank } \mathbf{G} = \sum_{v \in S} k_v\text{-rank } \mathbf{G} .$$

If $k\text{-rank } \mathbf{G} > 0$ and $S\text{-rank } \mathbf{G} \geq 2$, then $E\Gamma$ is S -arithmetic [27; 29, Theorem A, Corollary 1].

1.2. Fundamental groups. Now let \mathbf{G} be connected, absolutely almost simple, and simply connected. Let \mathbf{H} denote the restriction of scalars $\text{Res}_{k/\mathbb{Q}} \mathbf{G}$ of \mathbf{G} ; this is a group defined over \mathbb{Q} with $\mathbb{Q}\text{-rank } \mathbf{H} = k\text{-rank } \mathbf{G}$. Let $X_\infty = \mathbf{H}(\mathbb{R})/K$ be the symmetric space associated to \mathbf{H} , where K is a maximal compact subgroup of $\mathbf{H}(\mathbb{R})$, and for $v \in S_f$, let X_v be the Bruhat-Tits building of $\mathbf{G}(k_v)$.

Consider $X = X_\infty \times \prod_{v \in S \setminus S_\infty} X_v$. By extending the work of Borel and Serre [7, 8] and of Zucker [42], we define in §§2.4, 4.3 the reductive Borel-Serre bordification \overline{X}^{RBS} of X . For an S -arithmetic subgroup Γ of $\mathbf{G}(k)$, the action of Γ on X by left translation extends to \overline{X}^{RBS} and the quotient $\Gamma \backslash \overline{X}^{RBS}$ is a compact Hausdorff topological space, called the *reductive Borel-Serre compactification* of $\Gamma \backslash X$. Our main result (Theorem 5.1) is the computation of

the fundamental group of $\Gamma \backslash \overline{X}^{RBS}$. Under the mild condition that Γ is a neat S -arithmetic group, we show (Corollary 5.3) that

$$(1) \quad \pi_1(\Gamma \backslash \overline{X}^{RBS}) \cong \Gamma / E\Gamma .$$

If k -rank $\mathbf{G} > 0$ and S -rank $\mathbf{G} \geq 2$ this is finite. The Satake compactifications of the locally symmetric space $\Gamma \backslash X_\infty$ are important as well, as mentioned at the beginning of this introduction. In §4.4 we define compactifications $\Gamma \backslash_{\mathbb{Q}} \overline{X}^T$ of $\Gamma \backslash X$ which generalize the Satake compactifications of $\Gamma \backslash X_\infty$ and in §5 we calculate that their fundamental groups are a certain quotient of $\pi_1(\Gamma \backslash \overline{X}^{RBS})$.

Remark 1.1. There is an intriguing similarity between our results on the fundamental group and computations of the congruence subgroup kernel $C(S, \mathbf{G})$ (see [38] and [28]). For more details, see the Appendix.

2. THE REDUCTIVE BOREL-SERRE AND SATAKE COMPACTIFICATIONS: THE ARITHMETIC CASE

In order to establish notation and set the framework for later proofs, we recall in §§2.3–2.5 several natural compactifications of the locally symmetric space $\Gamma \backslash X_\infty$ associated to an arithmetic group Γ ; in each case a bordification of X_∞ is described on which $\mathbf{G}(k)$ acts. We also examine the stabilizer subgroups of points in these bordifications. The case of general S -arithmetic groups will be treated in §4. Throughout the paper, \mathbf{G} will denote a connected, absolutely almost simple, simply connected algebraic group defined over a number field k .

2.1. Proper and discontinuous actions. Recall [10, III, §4.4, Prop. 7] that a discrete group Γ acts *properly* on a Hausdorff space Y if and only if for all $y, y' \in Y$, there exist neighborhoods V of y and V' of y' such that $\gamma V \cap V' \neq \emptyset$ for only finitely many $\gamma \in \Gamma$. We will also need the following weaker condition on the group action:

Definition 2.1 ([17, Definition 1]). The action of a discrete group Γ on a topological space Y is *discontinuous* if

- (i) for all $y, y' \in Y$ with $y' \notin \Gamma y$ there exists neighborhoods V of y and V' of y' such that $\gamma V \cap V' = \emptyset$ for all $\gamma \in \Gamma$, and
- (ii) for all $y \in Y$ there exists a neighborhood V of y such that $\gamma V \cap V = \emptyset$ for $\gamma \notin \Gamma_y$ and $\gamma V = V$ for $\gamma \in \Gamma_y$.

It is easy to check that a group action is proper if and only if it is discontinuous and the stabilizer subgroup Γ_y is finite for all $y \in Y$.

Definition 2.2. The *stabilizer* Γ^X of a subset $X \subset Y$ is the subgroup

$$\Gamma^X = \{ \gamma \mid \gamma X = X \} .$$

The *fixing* group (*fixateur*) Γ_X is

$$\Gamma_X = \{ \gamma \mid \gamma x = x, \text{ for all } x \in X \} .$$

Thus for $y \in Y$, $\Gamma^y = \Gamma_y$. Lastly, let Γ_{fix} be the subgroup generated by the stabilizer subgroups Γ_y for all $y \in Y$. (This subgroup is obviously normal.)

2.2. The locally symmetric space associated to an arithmetic subgroup. Let S_∞ be the set of all infinite places of k . For each $v \in S_\infty$, let k_v be the corresponding completion of k with respect to a norm associated with v ; thus either $k_v \cong \mathbb{R}$ or $k_v \cong \mathbb{C}$. For each $v \in S_\infty$, $\mathbf{G}(k_v)$ is a (real) Lie group.

Define $G_\infty = \prod_{v \in S_\infty} \mathbf{G}(k_v)$, a semisimple Lie group with finitely many connected components. Fix a maximal compact subgroup K of G_∞ . When endowed with a G -invariant metric, $X_\infty = G_\infty/K$ is a Riemannian symmetric space of noncompact type and is thus contractible. Embed $\mathbf{G}(k)$ into G_∞ diagonally. Then any arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$ is a discrete subgroup of G_∞ and acts properly on X_∞ . It is known that the quotient $\Gamma \backslash X_\infty$ is compact if and only if the k -rank of \mathbf{G} is equal to 0. In the following, we assume that the k -rank of \mathbf{G} is positive so that $\Gamma \backslash X_\infty$ is noncompact.

Since the theory of compactifications of locally symmetric spaces is usually expressed in terms of algebraic groups defined over \mathbb{Q} , let $\mathbf{H} = \text{Res}_{k/\mathbb{Q}} \mathbf{G}$ be the algebraic group defined over \mathbb{Q} obtained by restriction of scalars; it satisfies

$$(2) \quad \mathbf{H}(\mathbb{Q}) = \mathbf{G}(k) \quad \text{and} \quad \mathbf{H}(\mathbb{R}) = G_\infty .$$

The space X_∞ can be identified with the symmetric space of maximal compact subgroups of $\mathbf{H}(\mathbb{R})$, $X_\infty = \mathbf{H}(\mathbb{R})/K$, and the arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$ corresponds to an arithmetic subgroup $\Gamma \subset \mathbf{H}(\mathbb{Q})$. Restriction of scalars yields a one-to-one correspondence between parabolic k -subgroups of \mathbf{G} and parabolic \mathbb{Q} -subgroups of \mathbf{H} so that the analogue of (2) is satisfied.

2.3. The Borel-Serre compactification. (For details see the original paper [7], as well as [6].) For each parabolic \mathbb{Q} -subgroup \mathbf{P} of \mathbf{H} , consider the Levi quotient $\mathbf{L}_\mathbf{P} = \mathbf{P}/\mathbf{N}_\mathbf{P}$ where $\mathbf{N}_\mathbf{P}$ is the unipotent radical of \mathbf{P} . This is a reductive group defined over \mathbb{Q} . There is an almost direct product $\mathbf{L}_\mathbf{P} = \mathbf{S}_\mathbf{P} \cdot \mathbf{M}_\mathbf{P}$, where $\mathbf{S}_\mathbf{P}$ is the maximal \mathbb{Q} -split torus in the center of $\mathbf{L}_\mathbf{P}$ and $\mathbf{M}_\mathbf{P}$ is the intersection of the kernels of the squares of all characters of $\mathbf{L}_\mathbf{P}$ defined over \mathbb{Q} . The real locus $L_P = \mathbf{L}_\mathbf{P}(\mathbb{R})$ has a direct product decomposition $A_P \cdot M_P$, where $A_P = \mathbf{S}_\mathbf{P}(\mathbb{R})^0$ and $M_P = \mathbf{M}_\mathbf{P}(\mathbb{R})$. The dimension of A_P is called the *parabolic \mathbb{Q} -rank* of \mathbf{P} .

The real locus $P = \mathbf{P}(\mathbb{R})$ has a Langlands decomposition

$$(3) \quad P = N_P \times (\tilde{A}_P \cdot \tilde{M}_P),$$

where $N_P = \mathbf{N}_\mathbf{P}(\mathbb{R})$ and $\tilde{A}_P \cdot \tilde{M}_P$ is the lift of $A_P \cdot M_P$ to the unique Levi subgroup of P which is stable under the Cartan involution θ associated with K .

Since P acts transitively on X_∞ , the Langlands decomposition induces a horospherical decomposition

$$(4) \quad X_\infty \cong A_P \times N_P \times X_P, \quad u\tilde{a}\tilde{m}K \mapsto (\tilde{a}, u, \tilde{m}(K \cap \tilde{M}_P)),$$

where

$$X_P = \tilde{M}_P / (K \cap \tilde{M}_P) \cong L_P / (A_P \cdot K_P)$$

is a symmetric space (which might contain an Euclidean factor) and is called the *boundary symmetric space associated with \mathbf{P}* . The second expression for X_P is preferred since $\mathbf{L}_\mathbf{P}$ is defined over \mathbb{Q} ; here $K_P \subseteq \mathbf{M}_\mathbf{P}(\mathbb{R})$ corresponds to $K \cap \tilde{M}_P$.

For each parabolic \mathbb{Q} -subgroup \mathbf{P} of \mathbf{H} , define the Borel-Serre boundary component

$$e(P) = N_P \times X_P$$

which we view as the quotient of X_∞ obtained by collapsing the first factor in (4). The action of P on X_∞ descends to an action on $e(P) = N_P \times X_P$ given by

$$(5) \quad p \cdot (u, y) = (pu\tilde{m}_p^{-1}\tilde{a}_p^{-1}, \tilde{a}_p\tilde{m}_py), \quad \text{for } p = u_p\tilde{a}_p\tilde{m}_p \in P.$$

Define the Borel-Serre partial compactification \overline{X}_∞^{BS} (as a *set*) by

$$(6) \quad \overline{X}_\infty^{BS} = X_\infty \cup \coprod_{\mathbf{P} \subset \mathbf{H}} e(P).$$

Let Δ_P be the simple ‘‘roots’’ of the adjoint action of A_P on the Lie algebra of N_P and identify A_P with $(\mathbb{R}^{>0})^{\Delta_P}$ by $a \mapsto (a^{-\alpha})_{\alpha \in \Delta_P}$. Enlarge A_P to the topological semigroup $\overline{A}_P \cong (\mathbb{R}^{\geq 0})^{\Delta_P}$ by allowing a^α to attain infinity and define

$$\overline{A}_P(s) = \{a \in \overline{A}_P \mid a^{-\alpha} < s^{-1} \text{ for all } \alpha \in \Delta_P\} \cong [0, s^{-1})^{\Delta_P}, \quad \text{for } s > 0.$$

Similarly enlarge the Lie algebra $\mathfrak{a}_P \subset \overline{\mathfrak{a}}_P$. The inverse isomorphisms $\exp: \mathfrak{a}_P \rightarrow A_P$ and $\log: A_P \rightarrow \mathfrak{a}_P$ extend to isomorphisms

$$\overline{A}_P \xrightarrow{\log} \overline{\mathfrak{a}}_P \quad \text{and} \quad \overline{\mathfrak{a}}_P \xrightarrow{\exp} \overline{A}_P.$$

To every parabolic \mathbb{Q} -subgroup $\mathbf{Q} \supseteq \mathbf{P}$ there corresponds a subset $\Delta_P^{\mathbf{Q}} \subseteq \Delta_P$ and we let $o_{\mathbf{Q}} \in \overline{A}_P$ be the point with coordinates $o_{\mathbf{Q}}^{-\alpha} = 1$ for $\alpha \in \Delta_P^{\mathbf{Q}}$ and $o_{\mathbf{Q}}^{-\alpha} = 0$ for $\alpha \notin \Delta_P^{\mathbf{Q}}$. Then $\overline{A}_P = \coprod_{\mathbf{Q} \supseteq \mathbf{P}} A_P \cdot o_{\mathbf{Q}}$ is the decomposition into A_P -orbits.

Define the *corner associated to \mathbf{P}* to be

$$(7) \quad X_\infty(P) = \overline{A}_P \times e(P) = \overline{A}_P \times N_P \times X_P.$$

We identify $e(Q)$ with the subset $(A_P \cdot o_Q) \times N_P \times X_P$. In particular, $e(P)$ is identified with the subset $\{o_P\} \times N_P \times X_P$ and X_∞ is identified with the open subset $A_P \times N_P \times X_P \subset X_\infty(P)$ (compare (4)). Thus we have a bijection

$$(8) \quad X_\infty(P) \cong X_\infty \cup \coprod_{\mathbf{P} \subset \mathbf{Q} \subset \mathbf{H}} e(Q).$$

Now give \overline{X}_∞^{BS} the finest topology so that for all parabolic \mathbb{Q} -subgroups \mathbf{P} of \mathbf{H} the inclusion of (8) into (6) is a continuous inclusion of an open subset. Under this topology, a sequence $x_n \in X$ converges in \overline{X}_∞^{BS} if and only if there exists a parabolic \mathbb{Q} -subgroup \mathbf{P} such that if we write $x_n = (a_n, u_n, y_n)$ according to the decomposition of (4), then (u_n, y_n) converges to a point in $e(P)$ and $a_n^\alpha \rightarrow \infty$ for all $\alpha \in \Delta_P$. The space \overline{X}_∞^{BS} is a manifold with corners. It has the same homotopy type as X_∞ and is thus contractible [7].

The action of $\mathbf{H}(\mathbb{Q})$ on X_∞ extends to a continuous action on \overline{X}_∞^{BS} which permutes the boundary components: $g \cdot e(P) = e(gPg^{-1})$ for $g \in \mathbf{H}(\mathbb{Q})$. The normalizer of $e(P)$ is $\mathbf{P}(\mathbb{Q})$ which acts according to (5).

It is shown in [7] that the action of Γ on \overline{X}_∞^{BS} is proper and the quotient $\Gamma \backslash \overline{X}_\infty^{BS}$, the *Borel-Serre compactification*, is a compact Hausdorff space. It is a manifold with corners if Γ is torsion-free.

2.4. The reductive Borel-Serre compactification. This compactification was first constructed by Zucker [42, §4] (see also [16]). For each parabolic \mathbb{Q} -subgroup \mathbf{P} of \mathbf{H} , define its reductive Borel-Serre boundary component $\hat{e}(P)$ by

$$\hat{e}(P) = X_P$$

and set

$$\overline{X}_\infty^{RBS} = X_\infty \cup \coprod_{\mathbf{P}} \hat{e}(P).$$

The projections $p_P: e(P) = N_P \times X_P \rightarrow \hat{e}(P) = X_P$ induce a surjection $p: \overline{X}_\infty^{BS} \rightarrow \overline{X}_\infty^{RBS}$ and we give $\overline{X}_\infty^{RBS}$ the quotient topology. Its topology can also be described in terms of convergence of interior points to the boundary points via the horospherical decomposition in equation (4). Note that $\overline{X}_\infty^{RBS}$ is not locally compact, although it is compactly generated (being a Hausdorff quotient of the locally compact space \overline{X}_∞^{BS}). The action of $\mathbf{H}(\mathbb{Q})$ on \overline{X}_∞^{BS} descends to a continuous action on $\overline{X}_\infty^{RBS}$.

Lemma 2.3. *Let \mathbf{P} be a parabolic \mathbb{Q} -subgroup of \mathbf{H} . The stabilizer $\mathbf{H}(\mathbb{Q})_z = \mathbf{G}(k)_z$ of $z \in X_P$ under the action of $\mathbf{H}(\mathbb{Q})$ on $\overline{X}_\infty^{RBS}$ satisfies a short exact sequence*

$$1 \rightarrow \mathbf{N}_{\mathbf{P}}(\mathbb{Q}) \rightarrow \mathbf{H}(\mathbb{Q})_z \rightarrow \mathbf{L}_{\mathbf{P}}(\mathbb{Q})_z \rightarrow 1$$

where $\mathbf{L}_{\mathbf{P}}(\mathbb{Q})_z$ is the stabilizer of z under the action of $\mathbf{L}_{\mathbf{P}}(\mathbb{Q})$ on X_P .

Proof. The normalizer of X_P under the action of $\mathbf{H}(\mathbb{Q})$ is $\mathbf{P}(\mathbb{Q})$ which acts via its quotient $\mathbf{L}_{\mathbf{P}}(\mathbb{Q})$. \square

By the lemma, the action of Γ on $\overline{X}_\infty^{RBS}$ is not proper since the stabilizer of a boundary point in X_P contains the infinite group $\Gamma_{N_P} = \Gamma \cap N_P$. Nonetheless

Lemma 2.4. *The action of an arithmetic subgroup Γ on $\overline{X}_\infty^{RBS}$ is discontinuous and the arithmetic quotient $\Gamma \backslash \overline{X}_\infty^{RBS}$ is a compact Hausdorff space.*

Proof. We begin by verifying Definition 2.1(ii). Let $x \in X_P \subseteq \overline{X}_\infty^{RBS}$. Set $\Gamma_P = \Gamma \cap P$ and $\Gamma_{L_P} = \Gamma_P / \Gamma_{N_P}$. Since Γ_{L_P} acts properly on X_P there exists a neighborhood O_x of x in X_P such that $\bar{\gamma}O_x \cap O_x \neq \emptyset$ if and only if $\bar{\gamma} \in \Gamma_{L_P, x}$, in which case $\bar{\gamma}O_x = O_x$. We can assume O_x is relatively compact. Set $V = p(\overline{A}_P(s) \times N_P \times O_x)$, where we chose s sufficiently large so that that only identifications induced by Γ on V already arise from Γ_P [44, (1.5)]. Thus $\gamma V \cap V \neq \emptyset$ if and only if $\gamma \in \Gamma_P$ and $\gamma \Gamma_{N_P} \in \Gamma_{L_P, x}$; by Lemma 2.3 this occurs if and only if $\gamma \in \Gamma_x$ as desired.

To verify Definition 2.1(i) we will show the equivalent condition that $\Gamma \backslash \overline{X}_\infty^{RBS}$ is Hausdorff (compare [42, (4.2)]). Compactness will follow since it is the image of a compact space under the induced projection $p': \Gamma \backslash \overline{X}_\infty^{BS} \rightarrow \Gamma \backslash \overline{X}_\infty^{RBS}$. Observe that p' is a quotient map and that its fibers, each being homeomorphic to $\Gamma_{N_P} \backslash N_P$ for some \mathbf{P} , are compact. For $y \in \Gamma \backslash \overline{X}_\infty^{RBS}$ and W a neighborhood of $p'^{-1}(y)$, we claim there exists $U \ni y$ open such that $p'^{-1}(U) \subseteq W$. This suffices to establish Hausdorff, for if $y_1 \neq y_2 \in \Gamma \backslash \overline{X}_\infty^{RBS}$ and W_1 and W_2 are disjoint neighborhoods of the compact fibers $p'^{-1}(y_1)$ and $p'^{-1}(y_2)$, there must exist U_1 and U_2 , neighborhoods of y_1 and y_2 , such that $p'^{-1}(U_i) \subseteq W_i$ and hence $U_1 \cap U_2 = \emptyset$.

To prove the claim, choose $x \in X_P$ such that $y = \Gamma x$. Let $q: \overline{X}_\infty^{BS} \rightarrow \Gamma \backslash \overline{X}_\infty^{BS}$ be the quotient map. The compact fiber $p'^{-1}(y)$ may be covered by finitely many open subsets

$q(\overline{A}_P(s_\mu) \times C_{P,\mu} \times O_{P,\mu}) \subseteq W$ where $C_{P,\mu} \subseteq N_P$ and $x \in O_{P,\mu} \subseteq X_P$. Define a neighborhood V of the fiber by

$$p'^{-1}(y) \subset V = q(\overline{A}_P(s) \times C_P \times O_P) \subseteq W$$

where $s = \max s_\mu$, $O_P = \bigcap O_{P,\mu}$, and $C_P = \bigcup C_{P,\mu}$. Since $\Gamma_{N_P} C_P = N_P$, we see $V = p'^{-1}(U)$ for some $U \ni y$ as desired. \square

2.5. Satake compactifications. For arithmetic quotients of X_∞ , the Satake compactifications $\Gamma \backslash_{\mathbb{Q}} \overline{X}_\infty^\tau$ form an important family of compactifications. When X_∞ is Hermitian, one example is the Baily-Borel Satake compactification. The construction has three steps.

- (i) Begin¹ with a representation (τ, V) of \mathbf{H} which has a nonzero K -fixed vector $v \in V$ (a *spherical representation*) and which is irreducible and nontrivial on each noncompact \mathbb{R} -simple factor of \mathbf{H} . Define the Satake compactification \overline{X}_∞^τ of X to be the closure of the image of the embedding $X_\infty \hookrightarrow \mathbb{P}(V)$, $gK \mapsto [\tau(g)v]$. The action of G_∞ extends to a continuous action on \overline{X}_∞^τ and the set of points fixed by N_P , where \mathbf{P} is any parabolic \mathbb{R} -subgroup, is called a *real boundary component*. The compactification \overline{X}_∞^τ is the disjoint union of its real boundary components.
- (ii) Define a partial compactification ${}_{\mathbb{Q}}\overline{X}_\infty^\tau \subseteq \overline{X}_\infty^\tau$ by taking the union of X_∞ and those real boundary components that meet the closure of a Siegel set. Under the condition that \overline{X}_∞^τ is *geometrically rational* [14], this is equivalent to considering those real boundary components whose normalizers are parabolic \mathbb{Q} -subgroups; call these the *rational boundary components*. Instead of the subspace topology induced from \overline{X}_∞^τ , give ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ the Satake topology [37].
- (iii) Still under the condition that \overline{X}_∞^τ is geometrically rational, one may show that the arithmetic subgroup Γ acts continuously on ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ with a compact Hausdorff quotient, $\Gamma \backslash_{\mathbb{Q}} \overline{X}_\infty^\tau$. This is the *Satake compactification* of $\Gamma \backslash X_\infty$.

The geometric rationality condition above always holds if the representation (τ, V) is rational over \mathbb{Q} [34]. It also holds for the Baily-Borel Satake compactification [3], as well as most equal-rank Satake compactifications including all those where \mathbb{Q} -rank $\mathbf{H} > 2$.

We will now describe an alternate construction of ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ due to Zucker [43]. Instead of the Satake topology, Zucker gives ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ the quotient topology under a certain surjection $\overline{X}_\infty^{RBS} \rightarrow {}_{\mathbb{Q}}\overline{X}_\infty^\tau$ described below. It is this topology we will use in this paper. Zucker proves that the resulting two topologies on $\Gamma \backslash_{\mathbb{Q}} \overline{X}_\infty^\tau$ coincide.

Let (τ, V) be a spherical representation as above. We assume that \overline{X}_∞^τ is geometrically rational. For any parabolic \mathbb{Q} -subgroup \mathbf{P} of \mathbf{H} , let $X_{P,\tau} \subseteq \overline{X}_\infty^\tau$ be the real boundary component fixed pointwise by N_P ; geometric rationality implies that $X_{P,\tau}$ is actually a rational boundary component. The transitive action of P on $X_{P,\tau}$ descends to an action of $L_P = P/N_P$. The geometric rationality condition ensures that there exists a normal \mathbb{Q} -subgroup $\mathbf{L}_{P,\tau} \subseteq \mathbf{L}_P$ with the property that $L_{P,\tau} = \mathbf{L}_{P,\tau}(\mathbb{R})$ is contained in the centralizer $\text{Cent}(X_{P,\tau})$ of $X_{P,\tau}$ and $\text{Cent}(X_{P,\tau})/L_{P,\tau}$ is compact. Then $X_{P,\tau}$ is the symmetric space associated to the \mathbb{Q} -group $\mathbf{H}_{P,\tau} = \mathbf{L}_P/\mathbf{L}_{P,\tau}$. There is an almost direct product decomposition

$$(9) \quad \mathbf{L}_P = \widetilde{\mathbf{H}}_{P,\tau} \cdot \mathbf{L}_{P,\tau} ,$$

¹Here we follow [14] in beginning with a spherical representation. Satake's original construction [36] started with a non-spherical representation but then constructed a spherical representation by letting G_∞ act on the space of self-adjoint endomorphisms of V with respect to an admissible inner product. See [34] for the relation of the two constructions.

where $\tilde{\mathbf{H}}_{\mathbf{P},\tau}$ is a lift of $\mathbf{H}_{\mathbf{P},\tau}$; the root systems of these factors may be described using the highest weight of τ . We obtain a decomposition of symmetric spaces

$$(10) \quad X_P = X_{P,\tau} \times W_{P,\tau} .$$

Different parabolic \mathbb{Q} -subgroups can yield the same rational boundary component $X_{P,\tau}$; if \mathbf{P}^\dagger is the maximal such parabolic \mathbb{Q} -subgroup, then

$$(11) \quad P^\dagger = \mathbf{P}^\dagger(\mathbb{R})$$

is the normalizer of $X_{P,\tau}$. The parabolic \mathbb{Q} -subgroups that arise as the normalizers of rational boundary components are called τ -saturated. For example, all parabolic \mathbb{Q} -subgroups are saturated for the maximal Satake compactification, while only the maximal parabolic \mathbb{Q} -subgroups are saturated for the Baily-Borel Satake compactification when \mathbf{H} is \mathbb{Q} -simple. In general, the class of τ -saturated parabolic \mathbb{Q} -subgroups can be described in terms of the highest weight of τ .

Define

$$\mathbb{Q}\overline{X}_\infty^\tau = X_\infty \cup \coprod_{\mathbf{Q} \text{ } \tau\text{-saturated}} X_{\mathbf{Q},\tau} .$$

A surjection $p: \overline{X}_\infty^{RBS} \rightarrow \mathbb{Q}\overline{X}_\infty^\tau$ is obtained by mapping X_P to $X_{P,\tau} = X_{P^\dagger,\tau}$ via the projection on the first factor in (10). Give $\mathbb{Q}\overline{X}_\infty^\tau$ the resulting quotient topology; the action of $\mathbf{H}(\mathbb{Q})$ on $\overline{X}_\infty^{RBS}$ descends to a continuous action on $\mathbb{Q}\overline{X}_\infty^\tau$.

Let \mathbf{P}_τ be the inverse image of $\mathbf{L}_{\mathbf{P},\tau}$ under the projection $\mathbf{P} \rightarrow \mathbf{P}/\mathbf{N}_{\mathbf{P}}$.

Lemma 2.5. *Let \mathbf{P} be a τ -saturated parabolic \mathbb{Q} -subgroup of \mathbf{H} . The stabilizer $\mathbf{H}(\mathbb{Q})_z = \mathbf{G}(k)_z$ of $z \in X_{P,\tau}$ under the action of $\mathbf{H}(\mathbb{Q})$ on $\mathbb{Q}\overline{X}_\infty^\tau$ satisfies a short exact sequence*

$$1 \rightarrow \mathbf{P}_\tau(\mathbb{Q}) \rightarrow \mathbf{H}(\mathbb{Q})_z \rightarrow \mathbf{H}_{\mathbf{P},\tau}(\mathbb{Q})_z \rightarrow 1,$$

where $\mathbf{H}_{\mathbf{P},\tau}(\mathbb{Q})_z$ is the stabilizer of z under the action of $\mathbf{H}_{\mathbf{P},\tau}(\mathbb{Q})$ on $X_{P,\tau}$.

Proof. As in the proof of Lemma 2.3, the normalizer of $X_{P,\tau}$ is $\mathbf{P}(\mathbb{Q})$ which acts via its quotient $\mathbf{P}(\mathbb{Q})/\mathbf{P}_\tau(\mathbb{Q}) = \mathbf{H}_{\mathbf{P},\tau}(\mathbb{Q})$. \square

Similarly to \overline{X}^{RBS} , the space $\mathbb{Q}\overline{X}_\infty^\tau$ is not locally compact and Γ does not act properly. Nonetheless one has the

Lemma 2.6. *The action of an arithmetic subgroup Γ on $\mathbb{Q}\overline{X}_\infty^\tau$ is discontinuous and the arithmetic quotient $\Gamma \backslash \mathbb{Q}\overline{X}_\infty^\tau$ is a compact Hausdorff space.*

The proof is similar to Lemma 2.4 since the fibers of p' are again compact, being reductive Borel-Serre compactifications of the $W_{P^\dagger,\tau}$. The Satake compactification of $\Gamma \backslash X_\infty$ associated to τ is $\Gamma \backslash \mathbb{Q}\overline{X}_\infty^\tau$.

In the case when the representation τ is generic one obtains the maximal Satake compactification $\overline{X}_\infty^{\max}$. This is always geometrically rational and the associated $\mathbb{Q}\overline{X}_\infty^{\max}$ is very similar to $\overline{X}_\infty^{RBS}$. Indeed in this case $X_P = X_{P,\tau} \times (\mathbb{R}A_P/A_P)$, where $\mathbb{R}A_P$ is defined like A_P but using a maximal \mathbb{R} -split torus instead of a maximal \mathbb{Q} -split torus, and the quotient map simply collapses the Euclidean factor $\mathbb{R}A_P/A_P$ to a point. In particular, if \mathbb{Q} -rank $\mathbf{H} = \mathbb{R}$ -rank \mathbf{H} , then $\Gamma \backslash \mathbb{Q}\overline{X}_\infty^{\max} \cong \Gamma \backslash \overline{X}_\infty^{RBS}$.

3. THE BRUHAT-TITS BUILDINGS

For a finite place v , let k_v be the completion of k with respect to a norm associated with v . Bruhat and Tits [12, 13] constructed a building X_v which reflects the structure of $\mathbf{G}(k_v)$. The building X_v is made up of subcomplexes called *apartments* corresponding to the maximal k_v -split tori in \mathbf{G} and which are glued together by the action of $\mathbf{G}(k_v)$. We give an outline of the construction here together with the properties of X_v which are needed in the sections below; in addition to the original papers, we benefited greatly from [21, §3.2; 25; 40].

In this section we fix a finite place v and a corresponding discrete valuation ω .

3.1. The apartment. Let \mathbf{S} be a maximal k_v -split torus in \mathbf{G} and let $X^*(\mathbf{S}) = \text{Hom}_{k_v}(\mathbf{S}, \mathbf{G}_m)$ and $X_*(\mathbf{S}) = \text{Hom}_{k_v}(\mathbf{G}_m, \mathbf{S})$ denote the k_v -rational characters and cocharacters of \mathbf{S} respectively. Denote by $\Phi \subset X^*(\mathbf{S})$ the set of k_v -roots of \mathbf{G} with respect to \mathbf{S} . Let \mathbf{N} and \mathbf{Z} denote the normalizer and the centralizer, respectively, of \mathbf{S} ; set $N = \mathbf{N}(k_v)$, $Z = \mathbf{Z}(k_v)$. The Weyl group $W = N/Z$ of Φ acts on the real vector space

$$V = X_*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} = \text{Hom}_{\mathbb{Z}}(X^*(\mathbf{S}), \mathbb{R})$$

by linear transformations; for $\alpha \in \Phi$, let r_α denote the corresponding reflection of V .

Let A be the affine space underlying V and let $\text{Aff}(A)$ denote the group of invertible affine transformations. We identify V with the translation subgroup of $\text{Aff}(A)$. There is an action of Z on A via translations, $\nu: Z \rightarrow V \subset \text{Aff}(A)$, determined by

$$\chi(\nu(t)) = -\omega(\chi(t)) \ , \quad t \in Z, \ \chi \in X^*(\mathbf{Z}) \ ;$$

note that $V = \text{Hom}_{\mathbb{Z}}(X^*(\mathbf{Z}), \mathbb{R})$ since $X^*(\mathbf{Z}) \subseteq X^*(\mathbf{S})$ is a finite index subgroup.

We now extend ν to an action of N by affine transformations. Let $H = \ker \nu$, which is the maximal compact subgroup of Z . Then Z/H is a free abelian group with rank = $\dim_{\mathbb{R}} V = k_v$ -rank \mathbf{G} . The group $W' = N/H$ is an extension of W by Z/H and there exists an affine action of W' on A which makes the following diagram commute [25, 1.6]:

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z/H & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \text{Aff}(A) & \longrightarrow & \text{GL}(V) & \longrightarrow & 1 \ . \end{array}$$

The action of W' lifts to the desired extension $\nu: N \rightarrow \text{Aff}(A)$.

For each $\alpha \in \Phi$, let U_α be the k_v -rational points of the connected unipotent subgroup of \mathbf{G} which has Lie algebra spanned by the root spaces \mathfrak{g}_α and (if 2α is a root) $\mathfrak{g}_{2\alpha}$. For $u \in U_\alpha \setminus \{1\}$, let $m(u)$ be the unique element of $N \cap U_{-\alpha} u U_{-\alpha}$ [25, 0.19]; in \mathbf{SL}_2 , for example, $m\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}$. The element $m(u) \in N$ acts on A by an affine reflection $\nu(m(u))$ whose associated linear transformation is r_α . The hyperplanes fixed by these affine reflections for all α and u are the *walls* of A . The connected components of the complement of the union of the walls are called the *chambers* of A ; since we assume \mathbf{G} is almost simple, these are (open) simplices. A *face* of A is an open face of a chamber. The affine space A is thus a simplicial complex (with the open simplices being faces) and the action of N is simplicial.

For convenience we identify A with V by choosing a “zero” point $o \in A$. For $\alpha \in \Phi$, define $\phi_\alpha: U_\alpha \rightarrow \mathbb{R} \cup \{\infty\}$ by setting $\phi_\alpha(1) = \infty$ and requiring for $u \neq 1$ that the function $x \mapsto \alpha(x) + \phi_\alpha(u)$ vanishes on the wall fixed by $\nu(m(u))$. For $\ell \in \mathbb{R}$, let

$$U_{\alpha, \ell} = \{ u \in U_\alpha \mid \phi_\alpha(u) \geq \ell \} \ .$$

These are compact open subgroups and define a decreasing exhaustive and separated filtration of U_α which has “jumps” only for ℓ in the discrete set $\phi_\alpha(U_\alpha \setminus \{1\})$. The affine function $\alpha + \ell$ is called an *affine root* if for some $u \in U_\alpha \setminus \{1\}$, $\ell = \phi_\alpha(u)$ and (if 2α is a root) $\phi_\alpha(u) = \sup \phi_\alpha(uU_{2\alpha})$; let $r_{\alpha,\ell} = \nu(m(u))$ be the corresponding affine reflection. Note that the zero set of an affine root is a wall of A and every wall of A arises in this fashion.

Denote the set of affine roots by Φ_{af} ; it is an *affine root system* in the sense of [26]. The Weyl group W_{af} of the affine root system Φ_{af} is the group generated by $r_{\alpha,\ell}$ for $\alpha + \ell \in \Phi_{\text{af}}$; it is an affine Weyl group in the sense of [9, Ch. VI, §2] associated to a reduced root system (not necessarily Φ). Since we assume \mathbf{G} is simply connected, $W_{\text{af}} = \nu(N) \cong W'$.

The *apartment* associated to \mathbf{S} consists of the affine simplicial space A together with the action of N , the affine root system Φ_{af} , and the filtration of the root groups, $(U_{\alpha,\ell})_{\substack{\alpha \in \Phi \\ \ell \in \mathbb{R}}}$.

3.2. The building. For $x \in A$, let U_x be the group generated by $U_{\alpha,\ell}$ for all $\alpha + \ell \in \Phi_{\text{af}}$ such that $(\alpha + \ell)(x) \geq 0$. The *building* of \mathbf{G} over k_v is defined [12, (7.4.2)] to be

$$X_v = (G \times A) / \sim ,$$

where $(gnp, x) \sim (g, \nu(n)x)$ for all $n \in N$ and $p \in HU_x$. We identify A with the subset of X_v induced by $\{1\} \times A$.

The building X_v has an action of $\mathbf{G}(k_v)$ induced by left multiplication on the first factor of $G \times A$. Under this action, N acts on $A \subset X_v$ via ν and $U_{\alpha,\ell}$ fixes the points in the half-space of A defined by $\alpha + \ell \geq 0$. The simplicial structure on A induces one on X_v and the action of $\mathbf{G}(k_v)$ is simplicial. The subcomplex $gA \subset X_v$ may be identified canonically with the apartment corresponding to the maximal split torus $g\mathbf{S}g^{-1}$.

Choose an inner product on V which is invariant under the Weyl group W ; the resulting metric on A may be transferred to any apartment by using the action of $\mathbf{G}(k_v)$. These metrics fit together to give a well-defined metric on X_v which is invariant under $\mathbf{G}(k_v)$ [12, (7.4.20)] and complete [12, (2.5.12)]. Given two points $x, y \in X_v$, there exists an apartment gA of X_v containing them [12, (7.4.18)]. Since gA is an affine space we can connect x and y with a line segment, $t \mapsto tx + (1-t)y$, $t \in [0, 1]$; this segment is independent of the choice of apartment containing the two points and in fact is the unique geodesic joining x and y .

Proposition 3.1 ([12, (7.4.20)]). *The mapping $t \mapsto tx + (1-t)y$ of $[0, 1] \times X_v \times X_v \rightarrow X_v$ is continuous and thus X_v is contractible.*

In fact it follows from [12, (3.2.1)] that X_v is a CAT(0)-space².

3.3. Stabilizers. For $\Omega \subset X_v$, let $\mathbf{G}(k_v)_\Omega$ be the fixing subgroup of Ω (see 2.1). Suppose now that $\Omega \subseteq A$ and set

$$U_\Omega = \langle U_{\alpha,\ell} \mid (\alpha + \ell)(\Omega) \geq 0, \alpha + \ell \in \Phi_{\text{af}} \rangle .$$

Since \mathbf{G} is simply connected and the valuation ω is discrete, $\mathbf{G}(k_v)_\Omega = HU_\Omega$ (see [12, (7.1.10), (7.4.4)]). In particular, the stabilizer of $x \in A$ is the compact open subgroup $\mathbf{G}(k_v)_x = HU_x$.

If F is a face of A and $x \in F$, then the set of affine roots which are nonnegative at x is independent of the choice of $x \in F$. Thus $\mathbf{G}(k_v)_F = \mathbf{G}(k_v)_x$. Note that an element of $\mathbf{G}(k_v)$

²Recall that a CAT(0)-space is a metric space where the distance between any two points is realized by a geodesic and every geodesic triangle is thinner than the corresponding triangle of the same side lengths in the Euclidean plane; see [11] for a comprehensive discussion of CAT(0)-spaces. Besides affine buildings such as X_v , simply connected, non-positively curved Riemannian manifolds such as X_∞ are CAT(0)-spaces.

which stabilizes F also fixes the barycenter x_F of F ; thus $\mathbf{G}(k_v)_{x_F}$ is the stabilizer subgroup $\mathbf{G}(k_v)^F$ of F .

Remark 3.2. The stabilizer subgroups for the building of \mathbf{SL}_2 (a tree) are calculated in [39, II, 1.3].

Let \mathbf{P} be a parabolic k_v -subgroup which without loss of generality we may assume contains the centralizer of \mathbf{S} ; let $\mathbf{N}_{\mathbf{P}}$ be its unipotent radical. Let $\Phi_P = \{ \alpha \in \Phi \mid U_\alpha \subseteq \mathbf{N}_{\mathbf{P}}(k_v) \}$ and set $E_P = \{ v \in V \mid \alpha(v) \geq 0, \alpha \in \Phi_P \}$; note that Φ_P is contained in a positive system of roots and hence E_P is a cone with nonempty interior.

Lemma 3.3. *For each $u \in \mathbf{N}_{\mathbf{P}}(k_v)$ there exists $x \in A$ such that $x + E_P$ is fixed pointwise by u . In particular, u belongs to a compact open subgroup.*

Proof. Since $\mathbf{N}_{\mathbf{P}}(k_v)$ is generated by $(U_\alpha)_{\alpha \in \Phi_P}$, there exists $\ell \in \mathbb{R}$ such that u belongs to the group generated by $(U_{\alpha, \ell})_{\alpha \in \Phi_P}$. Since $U_{\alpha, \ell}$ fixes the points in the half-space of A defined by $\alpha + \ell \geq 0$, choosing $x \in A$ such that $\alpha(x) \geq -\ell$ for all $\alpha \in \Phi_P$ suffices. \square

4. THE REDUCTIVE BOREL-SERRE AND SATAKE COMPACTIFICATIONS: THE S -ARITHMETIC CASE

We now consider a general S -arithmetic subgroup Γ and define a contractible space $X = X_S$ on which Γ acts properly. If the k -rank of \mathbf{G} is positive, as we shall assume, $\Gamma \backslash X$ is noncompact and it is important to compactify it. Borel and Serre [8] construct $\Gamma \backslash \overline{X}^{BS}$, the analogue of $\Gamma \backslash \overline{X}_\infty^{BS}$ from §2.3, and use it to study the cohomological finiteness of S -arithmetic subgroups. In this section we recall their construction and define several new compactifications of $\Gamma \backslash X$ analogous to those in §2.

4.1. The space $\Gamma \backslash X$ associated to an S -arithmetic group. Let S be a finite set of places of k containing the infinite places S_∞ and let $S_f = S \setminus S_\infty$. Define

$$G = G_\infty \times \prod_{v \in S_f} \mathbf{G}(k_v),$$

which is a locally compact group, and

$$X = X_\infty \times \prod_{v \in S_f} X_v,$$

where X_v is the Bruhat-Tits building associated to $\mathbf{G}(k_v)$ as described in §3. If we need to make clear the dependence on S , we write X_S . X is a locally compact metric space under the distance function induced from the factors. Since each factor is contractible (see §3.2), the same is true for X .

The group G acts isometrically on X . We view $\mathbf{G}(k) \subset G$ under the diagonal embedding. Any S -arithmetic subgroup $\Gamma \subset \mathbf{G}(k)$ is a discrete subgroup of G and acts properly on X [8, (6.8)]. It is known that the quotient $\Gamma \backslash X$ is compact if and only if the k -rank of \mathbf{G} is equal to 0. In the following, we assume that the k -rank of \mathbf{G} over k is positive. Then for every $v \in S_f$, the k_v -rank of \mathbf{G} is also positive.

4.2. **The Borel-Serre compactification** [8]. Define

$$\overline{X}^{BS} = \overline{X}_\infty^{BS} \times \prod_{v \in S_f} X_v ,$$

where \overline{X}_∞^{BS} is as in §2.3. This space is contractible and the action of $\mathbf{G}(k)$ on X extends to a continuous action on \overline{X}^{BS} . The action of any S -arithmetic subgroup Γ on \overline{X}^{BS} is proper [8, (6.10)]. When $S_f = \emptyset$ this is proved in [7] as mentioned in §2.3; in general, the argument is by induction on $|S_f|$. Using the barycentric subdivision of X_v , the key points are [8, (6.8)]:

(i) The covering of X_v by open stars $V(F)$ about the barycenters of faces F satisfies

$$\gamma V(F) \cap V(F) \neq \emptyset \iff \gamma \in \Gamma_F = \Gamma \cap \mathbf{G}(k_v)_F , \text{ and}$$

(ii) For any face $F \subset X_v$, Γ_F is an $(S \setminus \{v\})$ -arithmetic subgroup and hence by induction acts properly on $\overline{X}_{S \setminus \{v\}}^{BS}$.

Furthermore $\Gamma \backslash \overline{X}^{BS}$ is compact Hausdorff [8, (6.10)] which follows inductively from

(iii) There are only finitely many Γ -orbits of simplices in X_v for $v \in S_f$ and the quotient of \overline{X}_∞^{BS} by an arithmetic subgroup is compact.

4.3. **The reductive Borel-Serre compactification.** Define

$$\overline{X}^{RBS} = \overline{X}_\infty^{RBS} \times \prod_{v \in S_f} X_v .$$

There is a $\mathbf{G}(k)$ -equivariant surjection $\overline{X}^{BS} \rightarrow \overline{X}^{RBS}$ induced from the surjection in §2.4.

Proposition 4.1. *Any S -arithmetic subgroup Γ of $\mathbf{G}(k)$ acts discontinuously on \overline{X}^{RBS} with a compact Hausdorff quotient $\Gamma \backslash \overline{X}^{RBS}$.*

The proposition is proved similarly to the case of $\Gamma \backslash \overline{X}^{BS}$ outlined in §4.2; one replaces “proper” by “discontinuous” and begins the induction with Lemma 2.4. The space $\Gamma \backslash \overline{X}^{RBS}$ is the *reductive Borel-Serre compactification* of $\Gamma \backslash X$.

4.4. **Satake compactifications.** Let (τ, V) be a spherical representation of $\text{Res}_{k/\mathbb{Q}} \mathbf{G}$ as in §2.5 and define

$${}_{\mathbb{Q}}\overline{X}^\tau = {}_{\mathbb{Q}}\overline{X}_\infty^\tau \times \prod_{v \in S_f} X_v .$$

There is a $\mathbf{G}(k)$ -equivariant surjection $\overline{X}^{RBS} \rightarrow {}_{\mathbb{Q}}\overline{X}^\tau$ induced by $\overline{X}_\infty^{RBS} \rightarrow {}_{\mathbb{Q}}\overline{X}_\infty^\tau$ from §2.5.

Proposition 4.2. *Assume that the Satake compactification \overline{X}_∞^τ is geometrically rational. Then any S -arithmetic subgroup Γ acts discontinuously on ${}_{\mathbb{Q}}\overline{X}^\tau$ with a compact Hausdorff quotient $\Gamma \backslash {}_{\mathbb{Q}}\overline{X}^\tau$.*

The compact quotient $\Gamma \backslash {}_{\mathbb{Q}}\overline{X}^\tau$ is called the *Satake compactification* associated with (τ, V) .

5. THE FUNDAMENTAL GROUP OF THE COMPACTIFICATIONS

In this section we state our main result, Theorem 5.1, which calculates the fundamental group of the reductive Borel-Serre and the Satake compactifications of $\Gamma \backslash X$. The proof of Theorem 5.1 is postponed to §6.

Throughout we fix a spherical representation (τ, V) such that \overline{X}_∞^τ is geometrically rational.

In our situation of an S -arithmetic subgroup Γ acting on \overline{X}^{RBS} and ${}_{\mathbb{Q}}\overline{X}^\tau$, we denote Γ_{fix} by $\Gamma_{fix,RBS}$ and $\Gamma_{fix,\tau}$ respectively (see section 2.1 for the definition of Γ_{fix}). The main result of this paper is the following theorem.

Theorem 5.1. *For any S -arithmetic subgroup Γ , there exists a commutative diagram*

$$\begin{array}{ccc} \pi_1(\Gamma \backslash \overline{X}^{RBS}) & \xleftarrow{\cong} & \Gamma / \Gamma_{fix,RBS} \\ \downarrow & & \downarrow \\ \pi_1(\Gamma \backslash {}_{\mathbb{Q}}\overline{X}^\tau) & \xleftarrow{\cong} & \Gamma / \Gamma_{fix,\tau} \end{array}$$

where the horizontal maps are isomorphisms and the vertical maps are surjections induced by the Γ -equivariant projection $\overline{X}^{RBS} \rightarrow {}_{\mathbb{Q}}\overline{X}^\tau$ and the inclusion $\Gamma_{fix,RBS} \subseteq \Gamma_{fix,\tau}$.

The proof of the theorem will be given in §6. In the remainder of this section we will deduce from it more explicit computations of the fundamental groups. To do this we first need to calculate $\Gamma_{fix,RBS}$ and $\Gamma_{fix,\tau}$ which will require the information on stabilizers from §§2.4, 2.5, and 3.3.

Let \mathbf{P} be a parabolic k -subgroup of \mathbf{G} . The S -arithmetic subgroup Γ induces S -arithmetic subgroups $\Gamma_P = \Gamma \cap \mathbf{P}(k) \subseteq \mathbf{P}(k)$, $\Gamma_{N_P} = \Gamma \cap \mathbf{N}_{\mathbf{P}}(k) \subseteq \mathbf{N}_{\mathbf{P}}(k)$, and $\Gamma_{L_P} = \Gamma_P / \Gamma_{N_P} \subseteq \mathbf{L}_{\mathbf{P}}(k)$, as well as $\Gamma_{P_\tau} = \Gamma \cap \mathbf{P}_\tau(k) \subseteq \mathbf{P}_\tau(k)$ and $\Gamma_{H_{P,\tau}} = \Gamma_P / \Gamma_{P_\tau} \subseteq \mathbf{H}_{\mathbf{P},\tau}(k)$.

Let $E\Gamma \subseteq \Gamma$ be the subgroup generated by Γ_{N_P} as \mathbf{P} ranges over all parabolic k -subgroups of \mathbf{G} . Since $\gamma \mathbf{N}_{\mathbf{P}} \gamma^{-1} = \mathbf{N}_{\gamma \mathbf{P} \gamma^{-1}}$ for $\gamma \in \Gamma$, $E\Gamma$ is clearly normal. Let $E_\tau \Gamma \subseteq \Gamma$ be the subgroup generated by $\Gamma_{P_\tau} \cap \bigcap_{v \in S_f} K_v$ for every τ -saturated parabolic k -subgroup \mathbf{P} of \mathbf{G} and compact open subgroups $K_v \subset \mathbf{G}(k_v)$. As above, $E_\tau \Gamma$ is normal. Since Γ_{N_P} is generated by $\Gamma_{N_P} \cap \bigcap_{v \in S_f} K_v$ for various K_v by Lemma 3.3, it is easy to see that $E\Gamma \subseteq E_\tau \Gamma$.

A subgroup $\Gamma \subset \mathbf{G}(k)$ is *neat* if the subgroup of \mathbb{C} generated by the eigenvalues of $\rho(\gamma)$ is torsion-free for any $\gamma \in \Gamma$. Here ρ is a faithful representation $\mathbf{G} \rightarrow \mathbf{GL}_N$ defined over k and the condition is independent of the choice of ρ . Clearly any neat subgroup is torsion-free. Any S -arithmetic subgroup has a normal neat subgroup of finite index [4, §17.6]; the image of a neat subgroup by a morphism of algebraic groups is neat [4, §17.3].

Proposition 5.2. *Let Γ be an S -arithmetic subgroup. Then $E\Gamma \subseteq \Gamma_{fix,RBS}$ and $E_\tau \Gamma \subseteq \Gamma_{fix,\tau}$. If Γ is neat then equality holds for both.*

Proof. We proceed by induction on $|S_f|$. Suppose first that $|S_f| = 0$. By Lemma 2.3, Γ_{N_P} stabilizes every point of $X_P \subseteq \overline{X}_\infty^{RBS}$ for any parabolic k -subgroup \mathbf{P} , and hence $E\Gamma \subseteq \Gamma_{fix,RBS}$. Likewise by Lemma 2.5, Γ_{P_τ} stabilizes every point of $X_{P,\tau} \subset {}_{\mathbb{Q}}\overline{X}_\infty^\tau$ and so $E_\tau \Gamma \subseteq \Gamma_{fix,\tau}$.

If Γ is neat, then Γ_{L_P} and $\Gamma_{H_{P,\tau}}$ are likewise neat and hence torsion-free. The actions of Γ_{L_P} and $\Gamma_{H_{P,\tau}}$ are proper and hence $\Gamma_{L_P,z}$ and $\Gamma_{H_{P,\tau},z}$ are finite. Thus these stabilizer subgroups must be trivial. It follows then from Lemmas 2.3 and 2.5 that $E\Gamma = \Gamma_{fix,RBS}$ and $E_\tau \Gamma = \Gamma_{fix,\tau}$.

Now suppose that $|S_f| > 0$, pick $v \in S_f$ and let $S' = S \setminus \{v\}$. Write $\overline{X}^{RBS} = \overline{X}_{S'}^{RBS} \times X_v$. Suppose that $\gamma \in \Gamma_{N_{\mathbf{P}}}$ for some parabolic k -subgroup \mathbf{P} . By Lemma 3.3, $\gamma \in \mathbf{G}(k_v)_y$ for some $y \in X_v$. Thus $\gamma \in \Gamma' \cap \mathbf{N}_{\mathbf{P}}(k)$, where $\Gamma' = \Gamma \cap \mathbf{G}(k_v)_y$. Since $\mathbf{G}(k_v)_y$ is a compact open subgroup, Γ' is an S' -arithmetic subgroup. Since $|S'_f| < |S_f|$, we can apply induction to see that $\gamma = \gamma_1 \dots \gamma_m$, where $\gamma_i \in \Gamma'_{x_i}$ with $x_i \in \overline{X}_{S'}^{RBS}$. Since each $\gamma_i \in \Gamma_{(x_i, y)} \subset \Gamma_{fix, RBS}$, we see $E\Gamma \subseteq \Gamma_{fix, RBS}$. The proof that $E_\tau \Gamma \subseteq \Gamma_{fix, \tau}$ is similar since if $\gamma \in \Gamma_{P_\tau} \cap \bigcap_{v \in S_f} K_v$ then $\gamma \in \mathbf{G}(k_v)_y$ for some $y \in X_v$ [12, (3.2.4)].

Assume that Γ is neat. Let $(x, y) \in \overline{X}_{S'}^{RBS} \times X_v$, and let F be the face of X_v containing y . As above, $\Gamma_F = \Gamma \cap \mathbf{G}(k_v)_F$ is S' -arithmetic and, in this case, neat. So by induction, $\Gamma_{F, x} \subseteq E(\Gamma_F) \subseteq E\Gamma$. But since $\mathbf{G}(k_v)_y = \mathbf{G}(k_v)_F$, $\Gamma_{(x, y)} = \Gamma_{F, x}$. Therefore $\Gamma_{fix, RBS} \subseteq E\Gamma$. A similar argument shows that $\Gamma_{fix, \tau} \subseteq E_\tau \Gamma$. \square

We now can deduce several corollaries of Theorem 5.1 and Proposition 5.2.

Corollary 5.3. $\pi_1(\Gamma \backslash \overline{X}^{RBS})$ is a quotient of $\Gamma/E\Gamma$ and $\pi_1(\Gamma \backslash_{\mathbb{Q}} \overline{X}^\tau)$ is a quotient of $\Gamma/E_\tau \Gamma$. If Γ is neat, then $\pi_1(\Gamma \backslash \overline{X}^{RBS}) \cong \Gamma/E\Gamma$ and $\pi_1(\Gamma \backslash_{\mathbb{Q}} \overline{X}^\tau) \cong \Gamma/E_\tau \Gamma$.

Corollary 5.4. If k -rank $\mathbf{G} > 0$ and S -rank $\mathbf{G} \geq 2$, $\pi_1(\Gamma \backslash \overline{X}^{RBS})$ and $\pi_1(\Gamma \backslash_{\mathbb{Q}} \overline{X}^\tau)$ are finite.

Proof. Under the rank assumptions, $E\Gamma$ is S -arithmetic [27; 29, Theorem A, Corollary 1]. \square

6. PROOF OF THE MAIN THEOREM

In this section we give the proof of Theorem 5.1. If the two spaces \overline{X}^{RBS} and ${}_{\mathbb{Q}}\overline{X}^\tau$ were simply connected, and if the actions of Γ on them were free, then the path-lifting property of covering spaces would make computing the fundamental groups of the quotient spaces elementary. However, the actions of Γ are not free; in fact Lemmas 2.3 and 2.5 show that the Γ -stabilizers of points in these spaces may not be finite. Nonetheless, if Γ is neat the quotient maps do satisfy a weaker path-lifting property, *admissibility* (Proposition 6.8). This was introduced by Grosche in [17]. Provided that the spaces are simply connected, this property makes it possible to compute their fundamental groups ([17, Satz 5]).

To show that \overline{X}^{RBS} and ${}_{\mathbb{Q}}\overline{X}^\tau$ are simply connected, we prove that the natural surjections of \overline{X}^{BS} onto each of them also have this path-lifting property (Proposition 6.6). Since \overline{X}^{BS} is contractible it follows then that the two spaces are simply connected (Lemma 6.3).

To prove that the quotient maps are admissible, we use the fact that admissibility is a local property (Lemma 6.4). So we can cut a path into finitely many pieces, each of which lies in a “nice” neighborhood. We then push each such segment out to the boundary where the geometry is simpler. This technique is formalized in Lemma 6.5. The construction of suitable neighborhoods is via reduction theory.

It is easy to visualize this in the case where \mathbb{R} -rank $\mathbf{G} = 1$ and Γ is arithmetic. Then $\overline{X}^{RBS} = X \cup \{\text{cusps}\}$. Each cusp has horospherical neighborhoods which retract onto it. Suppose we have a path ω in \overline{X}^{RBS} . Let $y \in \overline{X}^{RBS}$ be a cusp and $U \subset \overline{X}^{RBS}$ be a horospherical neighborhood of y such that $\omega|_{[t_0, t_1]}$ lies in U with $x_0 = \omega(t_0), x_1 = \omega(t_1) \in X \cap U$. Let r_t be the deformation retraction of U onto y and set $\sigma_i(t) = r_t(x_i)$, $i = 0, 1$. Then $\omega|_{[t_0, t_1]} \cong \sigma_0^{-1} \cdot \sigma_1 \text{ rel}\{x_0, x_1\}$. Deforming paths in this way makes it easy to see that \overline{X}^{RBS} is simply connected and to compute the fundamental group of $\Gamma \backslash \overline{X}^{RBS}$.

Throughout homotopy of paths ω and η will always mean homotopy relative to the endpoints and will be denoted $\omega \cong \eta$. An action of a topological group Γ on a topological space Y will always be a continuous action.

Definition 6.1. (cf. [17, Definition 3]) A continuous surjection $p: Y \rightarrow X$ of topological spaces is *admissible* if for any path ω in X with initial point x_0 and final point x_1 and for any $y_0 \in p^{-1}(x_0)$, there exists a path $\tilde{\omega}$ in Y starting at y_0 and ending at some $y_1 \in p^{-1}(x_1)$ such that $p \circ \tilde{\omega}$ is homotopic to ω relative to the endpoints. An action of a group Γ on a topological space Y is *admissible* if the quotient map $Y \rightarrow \Gamma \backslash Y$ is admissible.

Proposition 6.2. *Let Y be a simply connected topological space and Γ a discrete group acting on Y . Assume that either*

- (i) *the Γ -action is discontinuous and admissible, or that*
- (ii) *the Γ -action is proper and Y is a locally compact metric space.*

Then the natural morphism $\Gamma \rightarrow \pi_1(\Gamma \backslash Y)$ induces an isomorphism $\Gamma / \Gamma_{fix} \cong \pi_1(\Gamma \backslash Y)$.

Proof. See [17, Satz 5] and [1] for hypotheses (i) and (ii) respectively. □

Proposition 6.3. *Let $p: Y \rightarrow X$ be an admissible continuous map of a simply connected topological space Y and assume that $p^{-1}(x_0)$ is path-connected for some $x_0 \in X$. Then X is simply connected.*

Proof. Let $\omega: [0, 1] \rightarrow X$ be a loop based at x_0 and let $\tilde{\omega}$ be a path in Y such that $p \circ \tilde{\omega} \cong \omega$ (relative to the basepoint). Let η be a path in $p^{-1}(x_0)$ from $\tilde{\omega}(1)$ to $\tilde{\omega}(0)$. Then the product $\tilde{\omega} \cdot \eta$ is a loop in the simply connected space Y and hence is null-homotopic. It follows that $\omega \cong p \circ \tilde{\omega} \cong p \circ (\tilde{\omega} \cdot \eta)$ is null-homotopic. □

Lemma 6.4. *A continuous surjection $p: Y \rightarrow X$ of topological spaces is admissible if and only if X can be covered by open subsets U such that $p|_{p^{-1}(U)}: p^{-1}(U) \rightarrow U$ is admissible.*

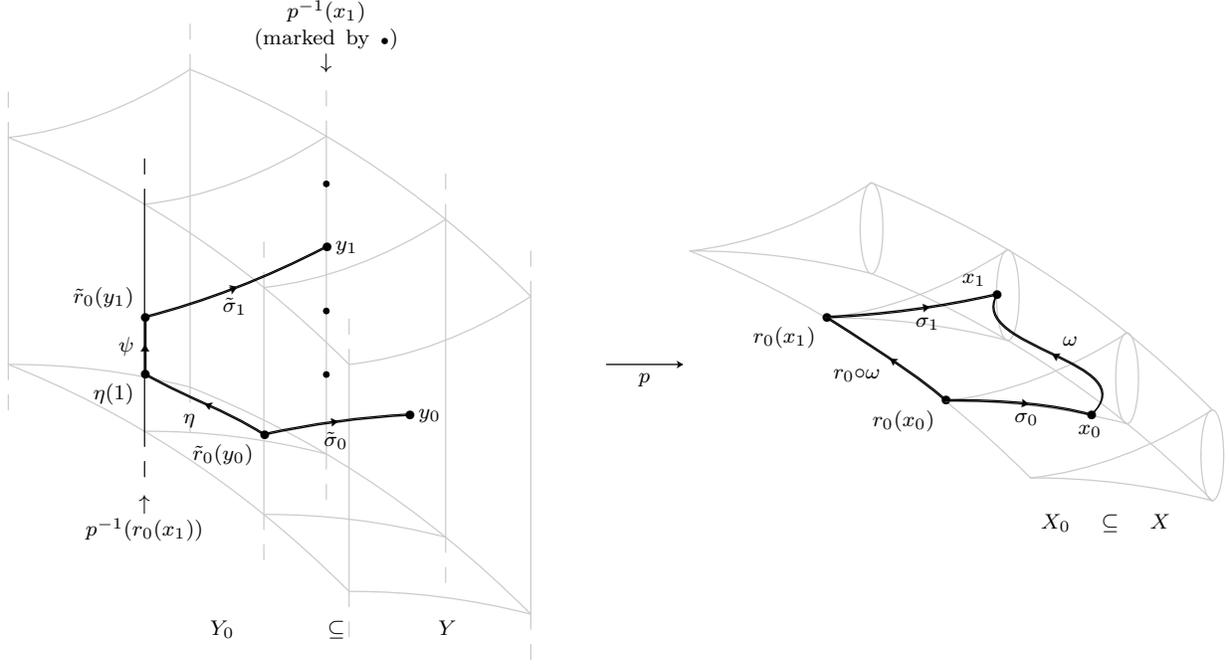
Proof. By the Lebesgue covering lemma, any path $\omega: [0, 1] \rightarrow X$ is equal to the product of finitely many paths, each of which maps into one of the subsets U . The lemma easily follows. □

Lemma 6.5. *Let $p: Y \rightarrow X$ be a continuous surjection of topological spaces. Assume there exist deformation retractions r_t of X onto a subspace X_0 and \tilde{r}_t of Y onto $Y_0 = p^{-1}(X_0)$ such that $p \circ \tilde{r}_t = r_t \circ p$. Also assume for all $x \in X$ that $\pi_0(p^{-1}(x)) \xrightarrow{\tilde{r}_{0*}} \pi_0(p^{-1}(r_0(x)))$ is surjective. Then p is admissible if and only if $p|_{Y_0}: Y_0 \rightarrow X_0$ is admissible.*

Proof. (See Figure 1.) Assume $p|_{Y_0}$ is admissible. If ω is a path in X from x_0 to x_1 , then $\omega \cong \sigma_0^{-1} \cdot (r_0 \circ \omega) \cdot \sigma_1$ where $\sigma_i(t) = r_t(x_i)$ for $i = 0, 1$. Pick $y_0 \in p^{-1}(x_0)$ and let $\eta(t)$ be a path in Y_0 starting at $\tilde{r}_0(y_0)$ such that $p \circ \eta \cong r_0 \circ \omega$. By assumption there exists $y_1 \in p^{-1}(x_1)$ such that $\tilde{r}_0(y_1)$ is in the same path-component of $p^{-1}(r_0(x_1))$ as $\eta(1)$; let ψ be any path in $p^{-1}(r_0(x_1))$ from $\eta(1)$ to $\tilde{r}_0(y_1)$. Set $\tilde{\omega} = \tilde{\sigma}_0^{-1} \cdot \eta \cdot \psi \cdot \tilde{\sigma}_1$, where $\tilde{\sigma}_i(t) = \tilde{r}_t(y_i)$. Then $p \circ \tilde{\omega} \cong \sigma_0^{-1} \cdot (r_0 \circ \omega) \cdot \sigma_1$ and thus p is admissible. □

Recall the $\mathbf{G}(k)$ -equivariant quotient maps $\overline{X}^{BS} \xrightarrow{p_1} \overline{X}^{RBS} \xrightarrow{p_2} \mathbb{Q}\overline{X}^\tau$ from §§4.3, 4.4.

Proposition 6.6. *The spaces \overline{X}^{RBS} and $\mathbb{Q}\overline{X}^\tau$ are simply connected.*

FIGURE 1. $p: Y \rightarrow X$ as in Lemma 6.5

Proof. For any finite place v , the building X_v is contractible. So we need only prove that $\overline{X}_\infty^{RBS}$ and ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ are simply connected (the case that $S_f = \emptyset$). By Proposition 6.3, Lemma 6.4, and the fact that \overline{X}_∞^{BS} is contractible, it suffices to find a cover of $\overline{X}_\infty^{RBS}$ by open subsets U over which p_1 (resp. $p_2 \circ p_1$) is admissible.

Consider first $\overline{X}_\infty^{RBS}$. The inverse image $p_1^{-1}(X_Q)$ of a stratum $X_Q \subseteq \overline{X}_\infty^{RBS}$ is $e(Q) = N_Q \times X_Q \subseteq \overline{X}_\infty^{BS}$. Set $\tilde{U} = \overline{A}_Q(1) \times N_Q \times X_Q \subseteq \overline{X}_\infty^{BS}$ (compare (7)) and $U = p_1(\tilde{U})$, a neighborhood of X_Q ; note $p_1^{-1}(U) = \tilde{U}$. Define a deformation retraction of \tilde{U} onto $e(Q)$ by

$$\tilde{r}_t(a, u, z) = \begin{cases} (\exp(\frac{1}{t} \log a), u, z) & \text{for } t \in (0, 1], \\ (o_Q, u, z) & \text{for } t = 0. \end{cases}$$

This descends to a deformation retraction r_t of U onto X_Q . Since $p_1|_{e(Q)}: N_Q \times X_Q \rightarrow X_Q$ is admissible and N_Q is path-connected, Lemma 6.5 shows that $p_1|_{\tilde{U}}$ is admissible.

Now consider ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ and a stratum $X_{Q,\tau}$, where \mathbf{Q} is τ -saturated. The inverse image $(p_2 \circ p_1)^{-1}(X_{Q,\tau})$ is $\coprod_{\mathbf{P}^\dagger = \mathbf{Q}} e(\mathbf{P}) \subseteq \overline{X}_\infty^{BS}$; it is an open subset of the closed stratum $\overline{e(Q)} = \coprod_{\mathbf{P} \subseteq \mathbf{Q}} e(\mathbf{P})$. For each \mathbf{P} such that $\mathbf{P}^\dagger = \mathbf{Q}$ (see (11)), we can write $e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}} = N_{\mathbf{P}} \times X_{Q,\tau} \times W_{\mathbf{P},\tau}$ by (10). Thus $(p_2 \circ p_1)^{-1}(X_{Q,\tau}) = Z_Q \times X_{Q,\tau}$, where $Z_Q = \coprod_{\mathbf{P}^\dagger = \mathbf{Q}} (N_{\mathbf{P}} \times W_{\mathbf{P},\tau})$. Note that $N_Q \times W_{Q,\tau}$ is dense in Z_Q , so Z_Q is path-connected.

For $X_{Q,\tau} \subset {}_{\mathbb{Q}}\overline{X}_\infty^\tau$, the construction of \tilde{U} is more subtle than in the case of $\overline{X}_\infty^{RBS}$. The theory of tilings [33, Theorem 8.1] describes a neighborhood in \overline{X}_∞^{BS} of the closed stratum $\overline{e(Q)}$ which is piecewise-analytically diffeomorphic to $\overline{A}_Q(1) \times \overline{e(Q)}$. (Note however that the induced decomposition on the part of this neighborhood in $X_\infty(Q)$ does *not* in general agree with that of (7) - see [33, §8, Remark (1)].) We thus obtain a neighborhood \tilde{U}

of $(p_2 \circ p_1)^{-1}(X_{Q,\tau}) = Z_Q \times X_{Q,\tau}$ in \overline{X}_∞^{BS} and a piecewise-analytic diffeomorphism $\tilde{U} \cong \overline{A}_Q(1) \times Z_Q \times X_{Q,\tau}$; let $U = p_2 \circ p_1(\tilde{U})$ and note $(p_2 \circ p_1)^{-1}(U) = \tilde{U}$. Since Z_Q is path-connected, we proceed as in the $\overline{X}_\infty^{RBS}$ case. \square

Remark 6.7. It is proved in [22] that every Satake compactification \overline{X}_∞^τ of a symmetric space X_∞ is a topological ball and hence contractible. Though the partial Satake compactification ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ is contained in \overline{X}_∞^τ as a subset, their topologies are different and this inclusion is not a topological embedding. Hence, it does not follow that ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ is contractible or that a path in ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ can be retracted into the interior. It can be shown, however, that ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$ is weakly contractible, i.e. that all of its homotopy groups are trivial.

Proposition 6.8. *For any neat S -arithmetic subgroup Γ , the action of Γ on \overline{X}^{RBS} and on ${}_{\mathbb{Q}}\overline{X}^\tau$ is admissible.*

Proof. Let $Y = \overline{X}^{RBS}$ or ${}_{\mathbb{Q}}\overline{X}^\tau$ and let $p: Y \rightarrow \Gamma \backslash Y$ be the quotient map, which in this case is open. It suffices to find for any point $x \in Y$ an open neighborhood U such that $p|_U$ is admissible. For then $p|_{\Gamma U}$ is admissible and hence, by Lemma 6.4, p is admissible.

We proceed by induction on $|S_f|$ and we suppose first that $S_f = \emptyset$.

Suppose x belongs to the stratum X_Q of $\overline{X}_\infty^{RBS}$. Since Γ is neat, Γ_{L_Q} is torsion-free. Thus we can choose a relatively compact neighborhood O_Q of x in X_Q so that $p|_{O_Q}: O_Q \rightarrow p(O_Q)$ is a homeomorphism. Let $U = p_1(\overline{A}_Q(s) \times N_Q \times O_Q) \subseteq \overline{X}_\infty^{RBS}$ where $s > 0$; this is a smaller version of the set U constructed in the proof of Proposition 6.6. By reduction theory, we can choose s sufficiently large so that the identifications induced by Γ on U agree with those induced by Γ_Q [44, (1.5)]. Since $\Gamma_Q \subseteq N_Q \widetilde{M}_Q$, it acts only on the last two factors of $\overline{A}_Q \times N_Q \times X_Q$. Thus the deformation retraction r_t of U onto O_Q (from the proof of Proposition 6.6) descends to a deformation retraction of $p(U)$ onto $p(O_Q) = O_Q$. Now apply Lemma 6.5 to see that $p|_U$ is admissible.

For x in the stratum $X_{Q,\tau}$ of ${}_{\mathbb{Q}}\overline{X}_\infty^\tau$, we again emulate the construction of U from the proof of Proposition 6.6. Specifically let $U = (p_2 \circ p_1)(\overline{A}_Q(s) \times Z_Q \times O_{Q,\tau})$ where $O_{Q,\tau}$ is a relatively compact neighborhood of x in $X_{Q,\tau}$ such that $p|_{O_{Q,\tau}}: O_{Q,\tau} \rightarrow p(O_{Q,\tau})$ is a homeomorphism; such a $O_{Q,\tau}$ exists since $\Gamma_{H_{Q,\tau}}$ is neat and hence torsion-free. By [33, Theorem 8.1], the identifications induced by Γ on U agree with those induced by Γ_Q and these are independent of the $\overline{A}_Q(s)$ coordinate. Thus the deformation retraction r_t descends to $p(U)$ and we proceed as above.

Now suppose that $|S_f| > 0$, pick $v \in S_f$ and let $S' = S \setminus \{v\}$. We consider $Y = \overline{X}^{RBS}$ which we write as $\overline{X}_{S'}^{RBS} \times X_v$; the case $Y = {}_{\mathbb{Q}}\overline{X}^\tau$ is identical. Following [8, (6.8)], for each face F of X_v let x_F be the barycenter of F and let $V(F)$ be the open star of x_F in the barycentric subdivision of X_v . The sets $V(F)$ form an open cover of X_v . For any $\gamma \in \Gamma$, $\gamma V(F) = V(\gamma F)$. If $F_1 \neq F_2$ are two faces with $\dim F_1 = \dim F_2$, then $V(F_1) \cap V(F_2) = \emptyset$. It follows that

$$\gamma V(F) \cap V(F) \neq \emptyset \iff \gamma \in \Gamma_F,$$

where $\Gamma_F = \Gamma \cap \mathbf{G}(k_v)_F$. It follows from §3.3 that Γ_F fixes F pointwise (since $\mathbf{G}(k_v)_F$ does) and is a neat S' -arithmetic subgroup (since $\mathbf{G}(k_v)_F$ is a compact open subgroup of $\mathbf{G}(k_v)$)

Let $U = \overline{X}_{S'}^{RBS} \times V(F)$ for some open face F of X_v . Define a deformation retraction r_t of U onto $\overline{X}_{S'}^{RBS} \times F$ by $r_t(w, z) = (w, tz + (1 - z)r_F(z))$, where $r_F(z)$ is the unique point in F

which is closest to $z \in V(F)$. The map r_t is Γ_F -equivariant since Γ_F fixes F pointwise and acts by isometries. So r_t descends to a deformation retraction of $p(U)$ onto $(\Gamma_F \backslash \overline{X}_{S'}^{RBS}) \times F$. The remaining hypothesis of Lemma 6.5 is satisfied since $r_0(\gamma w, \gamma z) = r_0(\gamma w, z)$ for $\gamma \in \Gamma_F$. Since $\overline{X}_{S'}^{RBS} \times F \rightarrow (\Gamma_F \backslash \overline{X}_{S'}^{RBS}) \times F$ is admissible by induction, the lemma implies that $p|_U$ is admissible. \square

We can now see that Theorem 5.1 holds if Γ is neat. According to Proposition 6.6, \overline{X}^{RBS} and ${}_{\mathbb{Q}}\overline{X}^\tau$ are simply connected. Proposition 6.8 shows that the actions of Γ on these spaces are admissible, and Propositions 4.1 and 4.2, that they are discontinuous. Therefore Proposition 6.2(i) applies to both spaces.

Corollary 6.9. *For any neat S -arithmetic subgroup Γ , the actions of $E\Gamma$ on \overline{X}^{RBS} and $E_\tau\Gamma$ on ${}_{\mathbb{Q}}\overline{X}^\tau$ are admissible.*

Proof. By Proposition 5.2 the action of $\Gamma/E\Gamma$ on $E\Gamma \backslash \overline{X}^{RBS}$ is free and by Proposition 4.1 it is discontinuous. It follows that $E\Gamma \backslash \overline{X}^{RBS} \rightarrow (\Gamma/E\Gamma) \backslash (E\Gamma \backslash \overline{X}^{RBS}) = \Gamma \backslash \overline{X}^{RBS}$ is a covering space (in fact a regular covering space) and thus $E\Gamma$ acts admissibly if and only if Γ acts admissibly. Now apply the proposition. The case of ${}_{\mathbb{Q}}\overline{X}^\tau$ is treated similarly. \square

Proof of Theorem 5.1. Let $\Gamma' \subseteq \Gamma$ be a normal neat subgroup of finite index. The idea in the general case is to factor $\overline{X}^{RBS} \rightarrow \Gamma \backslash \overline{X}^{RBS}$ as

$$\overline{X}^{RBS} \rightarrow E\Gamma' \backslash \overline{X}^{RBS} \rightarrow (\Gamma/E\Gamma') \backslash (E\Gamma' \backslash \overline{X}^{RBS}) = \Gamma \backslash \overline{X}^{RBS}$$

and apply Proposition 6.2(i) to the first map and Proposition 6.2(ii) to the second map.

By Proposition 5.2, $\Gamma'_{fix,RBS} = E\Gamma'$ and hence $(E\Gamma')_{fix,RBS} = E\Gamma'$. Arguing as above, Proposition 6.6 shows that \overline{X}^{RBS} is simply connected and Proposition 4.1 that Γ' acts discontinuously. It follows that $E\Gamma'$ acts discontinuously since $E\Gamma'$ contains all stabilizer subgroups Γ'_y . Corollary 6.9 shows that this action is admissible as well. Then Proposition 6.2(i) applies and proves that $E\Gamma' \backslash \overline{X}^{RBS}$ is simply connected.

We now claim that $E\Gamma' \backslash \overline{X}^{RBS}$ is locally compact. To see this, note that $E\Gamma' \backslash \overline{X}^{BS}$ is locally compact since it is triangulizable [8, (6.10)]. Furthermore the fibers of $p'_1: E\Gamma' \backslash \overline{X}^{BS} \rightarrow E\Gamma' \backslash \overline{X}^{RBS}$ have the form $\Gamma'_{N_P} \backslash N_P$ which are compact. The claim follows. Since $\Gamma'/E\Gamma'$ acts freely and Γ/Γ' is finite, the action of $\Gamma/E\Gamma'$ is proper. So we can apply Proposition 6.2(ii) to $\Gamma \backslash \overline{X}^{RBS} = (\Gamma/E\Gamma') \backslash (E\Gamma' \backslash \overline{X}^{RBS})$ and find that $\pi_1(\Gamma \backslash \overline{X}^{RBS}) \cong (\Gamma/E\Gamma') / (\Gamma/E\Gamma')_{fix,RBS} \cong \Gamma/\Gamma_{fix,RBS}$ as desired. Furthermore the proof shows that the isomorphism is induced by the natural morphism $\Gamma \rightarrow \pi_1(\Gamma \backslash \overline{X}^{RBS})$.

A similar proof using $E_\tau\Gamma'$ instead of $E\Gamma'$ treats the case of $\Gamma \backslash {}_{\mathbb{Q}}\overline{X}^\tau$; one only needs to observe that the fibers of $p'_2: E_\tau\Gamma' \backslash \overline{X}^{RBS} \rightarrow E_\tau\Gamma' \backslash {}_{\mathbb{Q}}\overline{X}^\tau$ have the form $\Gamma'_{L_{P,\tau}} \backslash \overline{W}_{P,\tau}^{RBS}$ which are compact. \square

7. APPENDIX

7.1. Computations of the congruence subgroup kernel. There is an intriguing similarity between our results on the fundamental group and computations of the congruence subgroup kernel $C(S, \mathbf{G})$.

For any nonzero ideal $\mathfrak{a} \subseteq \mathcal{O}$, set

$$\Gamma(\mathfrak{a}) = \{ \gamma \in \mathbf{G}(k) \mid \rho(\gamma) \in \mathbf{GL}_N(\mathcal{O}), \rho(\gamma) \equiv I \pmod{\mathfrak{a}} \} .$$

A subgroup $\Gamma \subset \mathbf{G}(k)$ is called an *S-congruence subgroup* if it contains $\Gamma(\mathfrak{a})$ as a subgroup of finite index for some ideal $\mathfrak{a} \subseteq \mathcal{O}$. In its simplest form, the *congruence subgroup problem* asks whether every *S*-arithmetic subgroup of $\mathbf{G}(k)$ is an *S*-congruence subgroup. The congruence subgroup kernel is a quantitative measure of how close this is to being true. We briefly outline its definition (see [38] and [28]).

Define a topology \mathcal{T}_c on $\mathbf{G}(k)$ by taking the set of *S*-congruence subgroups to be a fundamental system of neighborhoods of 1. Similarly define a topology \mathcal{T}_a by using the set of *S*-arithmetic subgroups. Let $\widehat{G}(c)$ and $\widehat{G}(a)$ denote the completions of $\mathbf{G}(k)$ in these topologies. Since every *S*-congruence subgroup is also *S*-arithmetic, \mathcal{T}_a is in general finer than \mathcal{T}_c and we have a surjective map

$$\widehat{G}(a) \longrightarrow \widehat{G}(c) .$$

The kernel of this map is called the *congruence subgroup kernel* $C(S, \mathbf{G})$.

From a more general perspective, the congruence subgroup problem is the determination of $C(S, \mathbf{G})$. The case when $C(S, \mathbf{G}) = 1$ is equivalent to every *S*-arithmetic subgroup being an *S*-congruence subgroup.

Assume that *k*-rank $\mathbf{G} > 0$ and *S*-rank $\mathbf{G} \geq 2$. Under these assumptions, it can be shown that (see [28, 29])

$$C(S, \mathbf{G}) \cong \varprojlim_{\mathfrak{a}} \Gamma(\mathfrak{a})/E\Gamma(\mathfrak{a})$$

and thus, in view of (1),

$$(12) \quad C(S, \mathbf{G}) = \varprojlim_{\mathfrak{a}} \pi_1(\Gamma(\mathfrak{a}) \backslash \overline{X}^{RBS}) .$$

Now set $\Gamma^*(\mathfrak{a}) = \bigcap_{\mathfrak{b} \neq 0} E\Gamma(\mathfrak{a}) \cdot \Gamma(\mathfrak{b})$ where \mathfrak{b} runs over nonzero ideals of \mathcal{O} . Clearly

$$E\Gamma(\mathfrak{a}) \subseteq \Gamma^*(\mathfrak{a}) \subseteq \Gamma(\mathfrak{a})$$

and $E\Gamma(\mathfrak{a}) = E\Gamma^*(\mathfrak{a})$. By Raghunathan's Main Lemma [28, (1.17)], for every nonzero ideal \mathfrak{a} there exists a nonzero ideal \mathfrak{a}' such that $\Gamma^*(\mathfrak{a}) \supseteq \Gamma(\mathfrak{a}')$. Thus $\Gamma^*(\mathfrak{a})$ is the smallest *S*-congruence subgroup containing $E\Gamma(\mathfrak{a})$. It follows that

$$C(S, \mathbf{G}) \cong \varprojlim_{\mathfrak{a}} \Gamma^*(\mathfrak{a})/E\Gamma(\mathfrak{a}) .$$

Raghunathan's main theorems in [28] and [29] show that $C(S, \mathbf{G})$ is finite under the rank assumptions. So the second limit will stabilize if we know that

$$\Gamma^*(\mathfrak{b})/E\Gamma(\mathfrak{b}) \longrightarrow \Gamma^*(\mathfrak{a})/E\Gamma(\mathfrak{a})$$

is surjective for $\mathfrak{b} \subset \mathfrak{a}$. This too follows from Raghunathan's Main Lemma [28, (1.17)] applied to \mathfrak{b} and from the definition of $\Gamma^*(\mathfrak{a})$. Thus

$$(13) \quad C(S, \mathbf{G}) \cong \Gamma^*(\mathfrak{a})/E\Gamma(\mathfrak{a}) \cong \pi_1(\Gamma^*(\mathfrak{a}) \backslash \overline{X}^{RBS}) ,$$

for any sufficiently small nonzero ideal \mathfrak{a} of \mathcal{O} .

It would be interesting to have an explanation for these topological interpretations of the congruence subgroup kernel.

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