CLASSIFICATION THEOREMS FOR FINITE GROUP ACTIONS USING THE EQUIVARIANT CUNTZ SEMIGROUP

EUSEBIO GARDELLA AND LUIS SANTIAGO

Abstract. We classify actions of finite groups on some class of C*-algebras with the Rokhlin property in terms of the Cuntz semigroup. An obstruction is obtained for the Cuntz semigroup of a C*-algebra allowing such an action. We also classify certain inductive limit actions of finite groups on a class of C*-algebras containing AI-algebras. This classification is done via the equivariant Cuntz semigroup.

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1. Introduction

Classification is a major subject in all areas of mathematics, and has attracted the attention of many talented mathematicians. In the category of C*-algebras, the program of classifying all amenable C*-algebras was initiated by George Elliott, first with the classification of AF-algebras, and later with the classification of certain simple C*-algebras of real rank zero. His work was followed by many other classification results for amenable C*-algebras, both in the stably finite and the purely infinite case.

The classification theory for von Neumann algebras precedes the classification program initiated by Elliott. In fact, the classification of amenable von Neumann algebras with separable pre-dual, which is due to Connes, Haagerup, Krieger and Takesaki, was completed more than 30 years ago. Connes moreover classified automorphisms of the type II$_1$ factor up to cocycle conjugacy in [7]. This can be regarded as the first classification result for actions on von Neumann algebras, and it was followed by his own work on the classification of pointwise outer actions of amenable groups on von Neumann algebras in [8]. (He in particular showed that they necessarily have the Rokhlin property.)

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Several people have since then tried to obtain similar classification results for actions on \( \mathrm{C}^* \)-algebras. Early results in this direction include work of Herman and Ocneanu in [18] on integer actions with the Rokhlin property on UHF-algebras, work of Fack and Maréchal in [12] and [13] for cyclic groups actions on UHF-algebras, and the work of Handelman and Rossmann [17] for locally representable compact group actions on AF-algebras. Other results have been obtained by Elliott and Su in [11] for direct limit actions of \( \mathbb{Z}_2 \) on AF-algebras, and by Izumi in [20] and [21], where he proved a number of classification results for actions with the Rokhlin property as well as for approximately representable actions. It should be emphasized that the classification of group actions on \( \mathrm{C}^* \)-algebras is a far less developed subject than the classification of \( \mathrm{C}^* \)-algebras and even farther less developed than the classification of group actions on von Neumann algebras.

It is worth noting that equivariant K-theory plays a crucial role in all of the results concerning actions on \( \mathrm{C}^* \)-algebras mentioned in the paragraph above: in some cases, like in [17], equivariant K-theory is in fact a complete invariant. (This terminology was not always used, though.) This is mainly due to the fact that the \( \mathrm{C}^* \)-algebras considered by these authors are completely classified by K-theory.

This paper contains classification results for two classes of finite group actions on certain stably finite \( \mathrm{C}^* \)-algebras: actions with the Rokhlin property, and locally representable actions. In both cases our invariants include some form of the equivariant Cuntz semigroup, as introduced in [16]. In fact, the purpose of this paper is to give an application to classification of actions of the equivariant Cuntz semigroup. (Though our results for finite group actions with the Rokhlin property are not stated in terms of the equivariant Cuntz semigroup, it can be seen that our invariant is equivalent to the equivariant Cuntz semigroup of the dual action.)

This paper is organized as follows. In Section 2, we collect a number of definitions and results that will be used throughout the paper. In particular, the construction of the equivariant Cuntz semigroup from [16] is recalled in Subsection 2.2. In Section 3, we define the equivalence relations up to which one may wish to classify actions of (locally) compact groups on \( \mathrm{C}^* \)-algebras, and prove a result that relates an action to its stabilization with the left regular representation from this viewpoint; see Theorem 3.1. When the group is abelian, this translates into a relationship between an action and its double dual. This result is our main tool to obtain exterior conjugacy of actions. In Section 4, we develop a general framework for classifying finite group actions with the Rokhlin property using what we call functors that classify homomorphisms. In Theorem 4.4, we classify actions of finite groups with the Rokhlin property on certain (not-necessarily unital) stably finite \( \mathrm{C}^* \)-algebras in terms of their induced action at the level of the semigroup \( \mathrm{Cu}^{-} \), as introduced by Leonel Robert in [33]. We actually prove a stronger result and show that equivariant homomorphisms at the level of \( \mathrm{Cu}^{-} \) lift to equivariant homomorphisms of the \( \mathrm{C}^* \)-algebras. In this sense, we obtain existence and uniqueness results for equivariant homomorphisms for actions with the Rokhlin property.

In Section 5, we study locally representable actions on the same class of stably finite \( \mathrm{C}^* \)-algebras, and show that the equivariant Cuntz semigroup is a complete invariant for exterior conjugacy of such actions; see Corollary 5.1. Moreover, in Theorem 5.1, we show that \( \mathrm{Cu}(G) \)-semimodule homomorphisms between the respective equivariant Cuntz semigroups lift to equivariant homomorphisms of the \( \mathrm{C}^* \)-algebras. In Section 6, we study absorption of UHF-algebras in relation to the Rokhlin property. Theorem 6.1 shows that if \( n \) is a natural number, then \( A \) and \( A \otimes M_{\infty} \) have the same Cuntz semigroup if and only if \( \mathrm{Cu}(A) \) is uniquely \( n \)-divisible (see Definition 6.1). We think this result is of independent interest and may have applications in classification in other contexts. Constraints for Rokhlin actions at the level of the Cuntz semigroup are shown in Theorem 6.2, and the Cuntz semigroup of the crossed product is computed in Corollary 6.2. As an application, we are able to characterize UHF-absorption in terms of existence of actions with the Rokhlin property in Theorem 6.3. This result moreover shows that for a given finite group \( G \), the model action \( \mu^G \)
constructed in Example 2.1 “generates” all actions of \( G \) with the Rokhlin property on a certain class of C*-algebras. (Similar results for Kirchberg algebras and simple separable C*-algebras with tracial rank zero had been shown by Izumi in [21].) In the case where the C*-algebra is stably finite, further K-theoretic obstructions are shown in Theorem 6.4 and the K-theory and Murray-von Neumann semigroup of the crossed product are computed in Corollary 6.4.

Acknowledgements:

2. Preliminary definitions and results

Let \( A \) be a C*-algebra. We denote by \( M(A) \) its multiplier algebra, by \( A^\dagger \) its unitization (that is, the C*-algebra obtained by adjoining a unit to \( A \), even if \( A \) is unital), and by \( A^\sim \) either \( A \) if \( A \) is unital or \( A^\dagger \) if \( A \) is not. If \( A \) is unital we denote by \( U(A) \) its unitary group. We denote by \( \text{Aut}(A) \) the automorphism group of \( A \). An automorphism \( \alpha \) of \( A \) is said to be approximately inner if for every finite subset \( F \subseteq A \) and every \( \varepsilon > 0 \), there exists a unitary \( u \in A^\sim \) such that \( \|\alpha(a) - uau^*\| < \varepsilon \) for all \( a \in F \). Two automorphisms \( \alpha \) and \( \beta \) of \( A \) are said to be approximately unitarily equivalent if \( \alpha^{-1} \circ \beta \) is approximately inner.

Topological groups are always assumed to be locally compact and Hausdorff. If \( G \) is a locally compact group and \( A \) is a C*-algebra, an action of \( G \) on \( A \) is a strongly continuous group homomorphism \( \alpha : G \to \text{Aut}(A) \). By strongly continuity we mean that for each \( a \) in \( A \), the map from \( G \) to \( A \) given by \( g \mapsto \alpha_g(a) \) is continuous with respect to the norm topology on \( A \).

A C*-algebra \( A \) is said to be a one-dimensional non-commutative CW-complex, abbreviated one-dimensional NCCW-complex, if \( A \) is given by a pullback diagram:

\[
\begin{array}{ccc}
A & \to & E \\
\downarrow & & \downarrow \\
C([0,1], F) & \to & F \oplus F,
\end{array}
\]

where \( E \) and \( F \) are finite dimensional C*-algebras and \( \text{ev}_x : C([0,1], F) \to F \) denotes the evaluation map at the point \( x \) in \([0,1]\). The Cuntz semigroup of a one-dimensional NCCW-complex was computed in the second example of [1, Section 4].

We denote by \( K \) the C*-algebra of compact operators on a separable Hilbert space. For a positive integer \( n \), we denote by \( Z_n \) the finite cyclic group of order \( n \). Finally, we take \( Z_+ = \{0, 1, 2, \ldots \} \) and \( \overline{Z}_+ = Z_+ \cup \{\infty\} \).

2.1. The Cuntz semigroup and the category \( \text{Cu} \). Let us briefly recall the definition of the Cuntz semigroup and of the category \( \text{Cu} \). Let \( A \) be a C*-algebra and let \( a, b \in A \) be positive elements. We say that \( a \) is Cuntz subequivalent to \( b \), denoted by \( a \precsim b \), if there is a sequence \( (d_n)_{n \in \mathbb{N}} \) in \( A \) such that \( d_n b d_n^* \to a \) as \( n \) goes to infinity. We say that \( a \) is Cuntz equivalent to \( b \), and denote this by \( a \sim b \), if \( a \precsim b \) and \( b \precsim a \). It is clear that \( \precsim \) is a preorder relation in the set of positive elements of \( A \) and thus \( \sim \) is an equivalence relation. We denote by \( [a] \) the Cuntz equivalence class of the element \( a \in A_+ \).

The first conclusion of the following lemma was proved in [34, Proposition 2.2] (see also [23, Lemma 2.2]). The second statement was shown in [32, Lemma 1].

**Lemma 2.1.** Let \( A \) be a C*-algebra and let \( a \) and \( b \) be positive elements in \( A \) such that \( \|a - b\| < \varepsilon \). Then \( (a - \varepsilon)_+ \precsim b \). More generally, if \( r \) is a non-negative real number, then \( (a - r - \varepsilon)_+ \precsim (b - r)_+ \).
The Cuntz semigroup of $A$, denoted by $\text{Cu}(A)$, is defined as the set of Cuntz equivalence classes of positive elements of $A \otimes \mathcal{K}$. Addition in $\text{Cu}(A)$ is given by

$$[a] + [b] = [a' + b'],$$

where $a', b' \in (A \otimes \mathcal{K})_+$ are such that $a' \sim a$, $b' \sim b$, and $a'b' = 0$. Furthermore, $\text{Cu}(A)$ becomes an ordered semigroup when equipped with the order $[a] \leq [b]$ if $a \preceq b$. If $\phi: A \to B$ is a *-homomorphism then $\phi$ induces an order-preserving map $\text{Cu}(\phi): \text{Cu}(A) \to \text{Cu}(B)$, with $\text{Cu}(\phi)([a]) = ([\phi \otimes \text{id}_\mathcal{K}](a)]$, where $\text{id}_\mathcal{K}: \mathcal{K} \to \mathcal{K}$ denotes the identity homomorphism.

It is shown in [9, Theorem 2] that the category $\text{Cu}$ is a functor from the category of C*-algebras to certain category of ordered abelian semigroups. We now proceed to define this category, which in this paper will be denoted by $\text{Cu}$.

Let $S$ be an ordered semigroup. Let $s, t \in S$. We say that $s$ is compactly contained in $t$, and denote this by $s \ll t$, if whenever $(t_n)_{n \in \mathbb{N}}$ is an increasing sequence in $S$ such that $t \leq \sup_{n \in \mathbb{N}} t_n$, then one has $s \leq t_k$ for some $k$. A sequence $(s_n)_{n \in \mathbb{N}}$ is said to be rapidly increasing if $s_n \ll s_{n+1}$ for all $n \in \mathbb{N}$.

**Definition 2.1.** An ordered abelian semigroup $S$ is an object in the category $\text{Cu}$ if it has a zero element and it satisfies the following properties:

O1 Every increasing sequence in $S$ has a supremum;

O2 For every $s \in S$ there exists a rapidly increasing sequence $(s_n)_{n \in \mathbb{N}}$ in $S$ such that $s = \sup_{n \in \mathbb{N}} s_n$.

O3 If $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ are increasing sequences in $S$, then

$$\sup_{n \in \mathbb{N}} s_n + \sup_{n \in \mathbb{N}} t_n = \sup_{n \in \mathbb{N}} (s_n + t_n);$$

O4 If $s_1, s_2, t_1, t_2$ are elements in $S$ such that $s_1 \ll t_1$ and $s_2 \ll t_2$, then $s_1 + s_2 \ll t_1 + t_2$.

Let $S$ and $T$ be semigroups in the category $\text{Cu}$. A order-preserving semigroup map $\varphi: S \to T$ is a morphism in the category $\text{Cu}$ if it preserves the zero element and it satisfies the following properties:

M1 If $(s_n)_{n \in \mathbb{N}}$ is an increasing sequence in $S$, then

$$\varphi \left( \sup_{n \in \mathbb{N}} s_n \right) = \sup_{n \in \mathbb{N}} \varphi(s_n);$$

M2 If $s$ and $t$ are elements in $S$ such that $s \ll t$, then $\varphi(s) \ll \varphi(t)$.

It is shown in [9, Theorem 2] that the category $\text{Cu}$ is closed under sequential inductive limits. The following description of inductive limits in the category $\text{Cu}$ follows from the proof of this theorem.

**Proposition 2.1.** Let $(S_n, \varphi_n)_{n \in \mathbb{N}}$, with $\varphi_n: S_n \to S_{n+1}$, be an inductive system in the category $\text{Cu}$. For $m \geq n$ in $\mathbb{N}$, let $\varphi_{n,m}: S_n \to S_m$ denote the composition $\varphi_{n,m} = \varphi_m \circ \cdots \circ \varphi_n$. A pair $(S, (\varphi_n, m)_{n \in \mathbb{N}})$, consisting of a semigroup $S$ and morphisms $\varphi_{n,\infty}: S_n \to S$ in the category $\text{Cu}$ satisfying $\varphi_{n+1,\infty} \circ \varphi_n = \varphi_{n,\infty}$ for all $n \in \mathbb{N}$, is the inductive limit of the system $(S_n, \varphi_n)_{n \in \mathbb{N}}$ if and only if

(i) For every $s \in S$ there exist elements $s_n \in S_n$, with $n \in \mathbb{N}$, such that $\varphi_n(s_n) \ll s_{n+1}$ for all $n \in \mathbb{N}$, and

$$s = \sup_{n \in \mathbb{N}} \varphi_{n,\infty}(s_n);$$

(ii) Whenever $s, s'$ and $t$ are elements in $S_n$ satisfying $\varphi_{n,\infty}(s) \leq \varphi_{n,\infty}(t)$ and $s' \ll s$, there exists $m \geq n$ such that $\varphi_{n,m}(s') \leq \varphi_{n,m}(t)$.
Lemma 2.2. Let $S$ be a semigroup in the category $\text{Cu}$, let $s$ be an element in $S$ and let $(s_n)_{n \in \mathbb{N}}$ be a rapidly increasing sequence in $S$ such that $s = \sup_{n \in \mathbb{N}} s_n$. Let $T$ be a subset of $S$ such that every element of $T$ is the supremum of a rapidly increasing sequence of elements of $T$. Suppose that for every $n \in \mathbb{N}$ there is $t \in T$ such that $s_n \ll t \leq s$. Then there exists an increasing sequence $(t_n)_{n \in \mathbb{N}} \subset T$ such that $s = \sup_{n \in \mathbb{N}} t_n$.

Proof. It is sufficient to construct a sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers and a sequence $(t_k)_{k \in \mathbb{N}}$ in $T$ such that $s_{n_k} \leq t_k \leq s_{n_{k+1}}$ for all $k \in \mathbb{N}$, since this implies that $s = \sup_{k \in \mathbb{N}} t_k$.

For $k = 1$, set $n_1 = 1$ and $s_1 = 0$. Assume inductively that we have constructed $n_j$ and $t_j$ for all $j < k$ and let us construct $n_{k+1}$ and $t_{k+1}$. By the assumptions of the lemma, there exists $t \in T$ such that $s_{n_k} \ll t \leq s$. Also by assumption, $t$ is the supremum of a rapidly increasing sequence of elements of $T$. Hence there exists $t' \in T$ such that $s_{n_k} \leq t' \ll s$. Use that $s = \sup_{n \in \mathbb{N}} s_n$ and $t' \ll s$, to choose $n_{k+1} \in \mathbb{N}$ such that $t' \leq s_{n_{k+1}} \ll s$. Set $t_{k+1} = t'$. Then $s_{n_{k+1}} \leq t_{k+1} \leq s_{n_{k+1}}$. This completes the proof of the lemma. □

Definition 2.2. Let $S$ be a semigroup in the category $\text{Cu}$ and let $T$ be a subset of $S$. We denote by $\mathcal{T}$ the union of $T$ with the set of all elements of $S$ that are the supremum of an increasing sequence of elements in $T$.

Lemma 2.3. Let $S$ be a semigroup in the category $\text{Cu}$. Let $\gamma_i: S \to S$, with $i \in I$, be a family of endomorphisms of $S$ in the category $\text{Cu}$. Then the set

$$S_\gamma = \left\{ s \in S: \exists (s_n)_{n \in \mathbb{N}} \text{ in } S: s_n \ll s_{n+1} \forall n \in \mathbb{N} \text{ and } s = \sup_{n \in \mathbb{N}} s_n, \right.$$ 

$$\gamma_i(s_n) = s_n \forall n \in \mathbb{N} \text{ and } \forall i \in I \right\},$$

is closed by taking suprema of increasing sequences. In other words, $S_\gamma = \overline{S_\gamma}$.

Proof. Let $(s_n)_{n \in \mathbb{N}}$ be an increasing sequence in $S_\gamma$. For each $n \in \mathbb{N}$, choose a rapidly increasing sequence $(s_{n,m})_{m \in \mathbb{N}} \subset S$ such that $s_n = \sup_{m \in \mathbb{N}} s_{n,m}$ and $\gamma_i(s_{n,m}) = s_{n,m}$ for all $i \in I$ and $m \in \mathbb{N}$. By the definition of the compact containment relation, there exist subsequences $(n_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ in $\mathbb{N}$ such that $s_{n_k} \ll s_{n_j, m_j}$ whenever $1 \leq k, l \leq j$. Let $s$ be the supremum of $(s_{n_j, m_j})$ in $S$. Then $s \in S_\gamma$, and it is straightforward to check using a diagonal argument that $s = \sup_{n \in \mathbb{N}} s_n$, thus showing that $\overline{S_\gamma} \subseteq S_\gamma$. The reverse inclusion is obvious. □

Let $A$ be a $\text{C}^*$-algebra, let $a \in A$ and let $\varepsilon > 0$. It can be checked that $[(a - \varepsilon)_+] \ll [a]$ and that $[a] = \sup_{\varepsilon > 0} [(a - \varepsilon)_+]$, thus showing that $\text{Cu}(A)$ satisfies Axiom O2.

Lemma 2.4. Let $A$ and $B$ be $\text{C}^*$-algebras and let $\varphi: \text{Cu}(A) \to \text{Cu}(B)$ be an order-preserving semigroup map. Suppose that for all $a$ in $(A \otimes K)_+$ one has

(i) $\varphi([a]) = \sup_{\varepsilon > 0} \varphi([(a - \varepsilon)_+])$ and

(ii) $\varphi([(a - \varepsilon)_+]) \ll \varphi([a])$ for all $\varepsilon > 0$.

Then $\varphi$ is a morphism in the category $\text{Cu}$; that is, it preserves suprema of increasing sequences and the compact containment relation.

Proof. Let $a$ be a positive element in $A \otimes K$ and let $(a_n)_{n \in \mathbb{N}} \subset (A \otimes K)_+$ be an increasing sequence such that $\sup_{n \in \mathbb{N}} [a_n] = [a]$ and $\varphi([a_n]) \leq \varphi([a])$ for all $n \in \mathbb{N}$. Suppose that $b \in (A \otimes K)_+$ is such that $[a_n] \ll [b]$ for all $n \in \mathbb{N}$ and let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $[(a - \varepsilon)_+] \ll [a_{n_0}]$ (here we are using the definition of the compact containment relation and that $[(a - \varepsilon)_+] \ll [a]$). By applying $\varphi$ to this inequality we get $\varphi([(a - \varepsilon)_+]) \leq \varphi([a_{n_0}]) \leq \varphi([b])$. Hence, by taking supremum in $\varepsilon > 0$ and applying (i) we get

$$\varphi([a]) = \sup_{\varepsilon > 0} \varphi([(a - \varepsilon)_+]) \leq \varphi([b]).$$

This shows that $\varphi([a])$ is the supremum of $(\varphi([a_n]))_{n \in \mathbb{N}}$. 5
Now let $a, b \in (A \otimes K)_+$ be such that $[a] \ll [c]$. Choose $\varepsilon > 0$ such that $[a] \leq [(b - \varepsilon)_+] \leq [b]$. It follows that $\varphi([a]) \leq \varphi([(b - \varepsilon)_+]) \leq \varphi([b])$. Now by (ii) applied to $[b]$ we get $\varphi([a]) \ll \varphi([b])$, which ends the proof.

The following lemma is a restatement of [32, Lemma 4].

**Lemma 2.5.** Let $A$ a C*-algebra, let $(x_j)_{j=0}^n$ be elements of $\text{Cu}(A)$ such that $x_{j+1} \ll x_j$ for all $j = 0, \ldots, n$, and let $\varepsilon > 0$. Then there exists $a \in (A \otimes K)_+$ such that

$$x_0 = [a] \gg x_1 \gg [(a - \varepsilon)_+] \gg x_2 \gg [(a - 2\varepsilon)_+] \gg x_3 \gg \cdots \gg x_{n-1} \gg [(a - (n-1)\varepsilon)_+] \gg x_n.$$ 

Let $A$ be a C*-algebra and let $p, q \in A$ be projections. We say that $p$ and $q$ are Murray-von Neumann equivalent, and denote this by $p \sim_{\text{MvN}} q$, if there exists a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$. The projection $p$ is said to be finite if whenever $q$ is a projection in $A$ such that $q \leq p$ and $q \sim_{\text{MvN}} p$, then $q = p$. If $A$ is unital then $A$ is said to be finite if its unit is a finite projection, $A$ is said to be stably finite if the unit of $M_n(A)$ is a finite projection for all $n \in \mathbb{N}$. If $A$ is not unital then we say that $A$ is (stably) finite if so is its unitization $A^\sim$.

**Lemma 2.6.** Let $A$ be a stably finite C*-algebra and let $p \in A \otimes K$ be a projection. Suppose that there are positive elements $a$ and $b$ in $A \otimes K$ such that $[p] = [a] + [b]$ in $\text{Cu}(A)$. Then $a$ and $b$ are Cuntz equivalent to projections in $A \otimes K$.

**Proof.** We have

$$[a] = \sup_{\varepsilon > 0} [(a - \varepsilon)_+] \text{ and } [b] = \sup_{\varepsilon > 0} [(b - \varepsilon)_+].$$

Since $[p] \ll [p]$, there exists $\varepsilon > 0$ such that $[p] = [(a - \varepsilon)_+] + [(b - \varepsilon)_+].$ Choose a function $f_\varepsilon \in C_0(0, \infty)$ that is zero on the interval $[\varepsilon, \infty)$, nonzero at every point of $(0, \varepsilon)$ and $\|f_\varepsilon\|_\infty \leq 1$. Then

$$[p] + [f_\varepsilon(a)] + [f_\varepsilon(b)] = [(a - \varepsilon)_+] + [f_\varepsilon(a)] + [(b - \varepsilon)_+] + [f_\varepsilon(b)] \leq [a] + [b] = [p].$$

Hence, $[p] + [f_\varepsilon(a)] + [f_\varepsilon(b)] = [p]$. Choose $c \in (A \otimes K)_+$ such that $[c] = [f_\varepsilon(a)] + [f_\varepsilon(b)]$ and $cp = 0$. Then $p + c \geq p$. By [23, Lemma 2.3 (iv)] it follows that for every $\delta > 0$ there exists $x \in A \otimes K$ such that

$$p + (c - \delta)_+ = x^*x, \quad xx^* \in p(A \otimes K)p.$$ 

Fix $\delta > 0$ and let $x$ be as above. Let $x = V|x|$ be the polar decomposition of $x$ in the bidual of $A \otimes K$. Set $p' = vpv^*$ and $c' = v(c - \delta)_+v^*$. Then $p'$ is a projection, $p'$ and $c'$ are orthogonal, $p$ and $p'$ are Murray-von Neumann equivalent, and $p' + c' \in pAp$. Using stable finiteness of $A$ we conclude that $p = p'$ and $c' = 0$. It follows that $(c - \delta)_+ = 0$ for all $\delta > 0$, and thus $c = 0$. Hence, $f_\varepsilon(b) = f_\varepsilon(a) = 0$ and in particular, $a$ and $b$ have a gap in their spectra. Therefore, they are Cuntz equivalent to projections. □

### 2.2. The Equivariant Cuntz semigroup.

In this subsection we present some important features of the equivariant Cuntz semigroup, which was introduced in [16]. This semigroup is a semimodule over a certain semiring whose definition we recall first.

Let $G$ be a compact group. Denote by $V(G)$ the semigroup of equivalence classes of finite dimensional representations of $G$, the operation given by direct sum. Recall that the representation ring of $G$, denoted $R(G)$, is the Grothendieck group of $V(G)$. The product structure on $R(G)$ is induced by the tensor product of representations. The construction of $R(G)$ resembles that of $K$-theory, while the object we define below is its Cuntz analog.

Recall that a semiring is a set with two operations $+$ and $\cdot$ which satisfy all axioms of a unitary ring except for the axiom demanding the existence of additive inverses.
Definition 2.3. The representation semiring of $G$, denoted by $\text{Cu}(G)$, is the set of all equivalence classes of unitary representations of $G$ on separable Hilbert spaces. The equivalence class of a representation $\mu: G \to U(H_\mu)$ is denoted by $[\mu]$. Addition on $\text{Cu}(G)$ is given by the direct sum of representations, while the product is given by the tensor product. That is, if $\mu: G \to U(H_\mu)$ and $\nu: G \to U(H_\nu)$ are unitary representations of $G$, then $[\mu] + [\nu]$ is represented by

$$\mu \oplus \nu: G \to U(H_\mu \oplus H_\nu) \quad (\mu \oplus \nu)(\xi, \eta) = (\mu_g\xi) \oplus (\nu_g\eta)$$

for $g$ in $G$, for $\xi$ in $H_\mu$ and for $\eta$ in $H_\nu$; while $[\mu] \cdot [\nu]$ is represented by

$$\mu \otimes \nu: G \to U(H_\mu \otimes H_\nu) \quad (\mu \times \nu)_g(\xi \otimes \eta) = (\mu_g\xi) \otimes (\nu_g\eta)$$

for $g$ in $G$, for $\xi$ in $H_\mu$ and for $\eta$ in $H_\nu$.

We endow $\text{Cu}(G)$ with the order $[\mu] \leq [\nu]$ if $\mu$ is unitary equivalent to a subrepresentation of $\nu$.

It is shown in Corollary 3.4 in [16] that $\text{Cu}(G)$ is an object in $\textbf{Cu}$. (It is in fact isomorphic to $\text{Cu}(C^*(G))$.)

Definition 2.4. Let $\mu: G \to U(H_\mu)$ and $\nu: G \to U(H_\nu)$ be unitary representations of $G$ on separable Hilbert spaces $H_\mu$ and $H_\nu$. Give $K(H_\mu, H_\nu)$ the $G$-action

$$(g \cdot T)(\xi) = (\nu_g \circ T \circ \mu_g^{-1})(\xi)$$

for $g$ in $G$, $T$ in $K(H_\mu, H_\nu)$ and $\xi$ in $H_\mu$. We give $K(H_\mu, H_\nu) \otimes A$ the obvious diagonal $G$-action.

Let $a$ in $(K(H_\mu) \otimes A)^G$ and $b$ in $(K(H_\nu) \otimes A)^G$ be positive elements. We say that $a$ is $G$-Cuntz subequivalent to $b$, and denote this by $a \lesssim_G b$, if there is a sequence $(d_n)$ in $(K(H_\mu, H_\nu) \otimes A)^G$ such that $d_nbd_n^* \rightarrow a$. We say that $a$ is $G$-Cuntz equivalent to $b$, and denote this by $a \sim_G b$, if $a \lesssim_G b$ and $b \lesssim_G a$. The $G$-Cuntz equivalence class of an element $a$ will be denoted by $[a]_G$.

The equivariant Cuntz semigroup of the dynamical system $(A, \alpha)$, denoted by $\text{Cu}_G(A, \alpha)$, is defined as the $G$-Cuntz equivalence classes of positive elements of $\bigsqcup_{\mu} (K(H_\mu) \otimes A)^G$ with addition given by $[a]_G + [b]_G = [a \oplus b]_G$, and the order $[a]_G \leq [b]_G$ if $a \lesssim_G b$.

The $\text{Cu}(G)$-semimodule structure of the equivariant Cuntz semigroup is defined as follows. Let $\mu: G \to U(H_\mu)$ and $\nu: G \to U(H_\nu)$ be unitary representations, and let $a \in (K(H_\nu) \otimes A)^G$ be a positive invariant element. Let $s_\mu \in K(H_\mu)$ be strictly positive $G$-invariant, and define $[\mu] \cdot [a]_G$ to be the class of the element

$$s_\mu \otimes a \in (K(H_\nu) \otimes K(H_\mu) \otimes A)^G \subseteq (K(H_\nu \otimes H_\mu) \otimes A)^G.$$

If $G$ is compact and abelian, then $\widehat{G}$ is discrete (and abelian) and $C^*(G) \cong \bigoplus_{\tau \in \widehat{G}} \mathbb{C}$. Thus,

$$\text{Cu}(G) \cong \text{Cu}(C^*(G)) \cong \{ f: \widehat{G} \to \mathbb{N} \}.$$

Equivalently, $\text{Cu}(G)$ consists of the suprema of all formal linear combinations of elements of $\widehat{G}$ with coefficients in $\mathbb{N}$, with addition and multiplication being the obvious ones. In particular, it follows that a $\text{Cu}(G)$-semimodule structure on a partially ordered abelian semigroup that is compatible with suprema, is necessarily completely determined by multiplication by the elements of $\widehat{G}$.

We summarize several results of [16] in the following theorem.

Theorem 2.1. Let $G$ be a compact second countable group, let $A$ be a $C^*$-algebra, and let $\alpha: G \to \text{Aut}(A)$ be an action. Then there is a natural $\text{Cu}$-isomorphism

$$\text{Cu}_G(A, \alpha) \cong \text{Cu}(A \rtimes_\alpha G).$$

In particular, $\text{Cu}(A \rtimes_\alpha G)$ has a natural $\text{Cu}(G)$-semimodule structure that makes the isomorphism above an isomorphism of $\text{Cu}(G)$-semimodules.
When \( G \) is abelian, the \( \text{Cu}(G) \)-semimodule structure on \( \text{Cu}(A \rtimes_{\alpha} G) \) is given by the dual action \( \hat{\alpha}: \hat{G} \to \text{Aut}(A \rtimes_{\alpha} G) \) as follows. If \( \tau \in \hat{G} \) and \( s \in \text{Cu}(A \rtimes_{\alpha} G) \), then
\[
\tau \cdot s = \text{Cu}(\hat{\alpha}_{\tau})(s).
\]

The following notion will be used throughout the paper.

**Definition 2.5.** Let \( G \) be a locally compact group, let \( A \) be a C*-algebra, and let \( \alpha \) be a continuous action of \( G \) on \( A \). An \( \alpha \)-cocycle is a function \( \omega: G \to U(M(A)) \) such that
\[
\begin{align*}
(i) & \quad \omega_{gh} = \omega_g \alpha_g(\omega_h) \text{ for all } g, h \in G \\
(ii) & \quad \text{for each } a \in A, \text{ the map } G \to A \text{ given by } g \mapsto \omega_g a \text{ is continuous.}
\end{align*}
\]
We define the \( \omega \)-perturbation of \( \alpha \), denoted \( \alpha^\omega \), by \( \alpha^\omega_g(a) = \omega_g \alpha_g(a) \omega_g^* \text{ for all } g \in G \text{ and all } a \in A. \)

It is not difficult to see that an action and any cocycle perturbation of it have naturally isomorphic crossed products, and that when the group is abelian, the isomorphism intertwines the dual actions. Even when the group is not abelian, we have the following.

**Proposition 2.2.** Let \( A \) be a C*-algebra, let \( G \) be a compact group, let \( \alpha: G \to \text{Aut}(A) \) be a continuous action, and let \( \omega: G \to U(M(A)) \) be an \( \alpha \)-cocycle. Then there is a natural isomorphism
\[
\text{Cu}_G(A, \alpha) \cong \text{Cu}_G(A, \alpha^\omega).
\]

2.3. Central sequence algebras and the Rokhlin property. Let \( A \) be a C*-algebra. Let \( \ell^\infty(N, A) \) denote the set of all bounded sequences \( (a_n)_{n \in \mathbb{N}} \) in \( A \) with the supremum norm
\[
\| (a_n)_{n \in \mathbb{N}} \| = \sup_{n \in \mathbb{N}} \| a_n \|
\]
and pointwise operations. Then \( \ell^\infty(N, A) \) is a C*-algebra, and it is unital when \( A \) is (the unit being the constant sequence \( 1_A \)). Let
\[
c_0(N, A) = \left\{ (a_n)_{n \in \mathbb{N}} \in \ell^\infty(N, A) : \lim_{n \to \infty} \| a_n \| = 0 \right\}.
\]
Then \( c_0(N, A) \) is an ideal in \( \ell^\infty(N, A) \), and we denote the quotient
\[
\ell^\infty(N, A)/c_0(N, A)
\]
by \( A^c \). Write \( \kappa_A: \ell^\infty(N, A) \to A^c \) for the quotient map. We identify \( A \) with the subalgebra of \( \ell^\infty(N, A) \) consisting of the constant sequences, and with a subalgebra of \( A^c \) by taking its image under \( \kappa_A \). We write \( A_\infty = A^c \cap A' \) for the relative commutant of \( A \) inside of \( A^c \).

It is also easy to check that the annihilator
\[
A^\perp = \{ b \in A_\infty : ba = 0 \text{ for all } a \in A \}
\]
is an ideal in \( A_\infty \). We write \( F(A) = A_\infty/A^\perp \) for the corresponding quotient. It is shown in [24] that \( F(A) \cong F(A \otimes K) \) for any C*-algebra \( A \). Moreover, \( F(A) = A_\infty \) if \( A \) is unital, and \( F(A) \) is unital if \( A \) is \( \sigma \)-unital.

If \( G \) is any locally compact group and \( \alpha: G \to \text{Aut}(A) \) is an action of \( G \) on \( A \), then there are actions of \( G \) on \( A^\infty \), on \( A_\infty \) and on \( F(A) \). For simplicity and ease of notation, and unless confusion is likely to arise, we denote these actions simply by \( \alpha \). Note that unless the group \( G \) is discrete, these actions will in general not be continuous.

**Lemma 2.7.** Let \( A \) be a unital C*-algebra and let \( u \) be a unitary in \( A_\infty \). Given \( \varepsilon > 0 \) and given a finite subset \( F \subseteq A \), there exists a unitary \( v \) in \( A \) such that \( \| va - \alpha v \| < \varepsilon \) for all \( a \in F \). If moreover \( A \) is separable, then there exists a sequence \( (u_n)_{n \in \mathbb{N}} \) of unitaries in \( A \) with
\[
\lim_{n \to \infty} \| u_n a - \alpha u_n \| = 0
\]
for all \( a \in A \), such that \( \kappa_A((u_n)_{n \in \mathbb{N}}) = u \) in \( A_\infty \).
Recall the definition of the Rokhlin property for actions of finite groups on \(\sigma\)-unital C*-algebras.

**Definition 2.6.** ([27, Definition 3.1]; also see [20, Definition 3.1]) Let \(A\) be a \(\sigma\)-unital C*-algebra, let \(G\) be a finite group and let \(\alpha: G \to \text{Aut}(A)\) be an action. We say that \(\alpha\) has the Rokhlin property if there exist orthogonal projections \(p_g\) in \(F(A)\) for \(g \in G\) satisfying

(i) \(\sum_{g \in G} p_g = 1\).

(ii) \(\alpha_g(p_h) = p_{gh}\) for all \(g, h \in G\).

**Remark 2.1.** An action \(\alpha: G \to \text{Aut}(A)\) as in Definition 2.6 has the Rokhlin property if and only if there exist orthogonal contractions \(r_g\) in \(A_{\infty}\) for \(g\) in \(G\) such that \(\sum_{g \in G} r_g\) is a unit for \(A\) and \(\alpha_g(r_h)a = r_{gh}a\) for all \(g, h \in G\) and for all \(a \in A\). This follows using that orthogonal positive contractions can be lifted ([26, Lemma 10.1.12]). Also, note that by the same lifting result the elements \(r_g\) for \(g \in G\) can be represented by orthogonal sequences of positive contractions in \(\ell^\infty(\mathbb{N}, A \otimes K)\).

We collect some results about finite group actions with the Rokhlin property. These results are well-known to the experts in the unital case. Part (i) of the following proposition is the noncommutative analog of the fact that a diagonal action of a group \(G\) on a product of spaces \(X \times Y\) is free if one of the factors is free.

**Proposition 2.3.** Let \(G\) be a finite group, let \(A\) be a C*-algebra, and let \(\alpha: G \to \text{Aut}(A)\) be an action with the Rokhlin property.

(i) If \(B\) is any C*-algebra and \(\beta: G \to \text{Aut}(B)\) is any action of \(G\) on \(B\), then the action \(\alpha \otimes \beta: G \to \text{Aut}(A \otimes B)\) defined by \(\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g\) for all \(g \in G\), for any tensor product on which it is defined, has the Rokhlin property.

(ii) If \(B\) is a C*-algebra and \(\phi: A \to B\) is an isomorphism, then the action \(g \mapsto \phi \circ \alpha_g \circ \phi^{-1}\) has the Rokhlin property.

(iii) If \(\beta: G \to \text{Aut}(A)\) is an action exterior equivalent to \(\alpha\), then \(\beta\) has the Rokhlin property.

We briefly comment what can go wrong when defining a tensor product action \(\alpha \otimes \beta\) on a tensor product \(A \otimes B\). Ask Chris.

The following example may be regarded as the “generating” Rokhlin action for a given finite group \(G\). For some classes of C*-algebras, it can be shown that every action of \(G\) with the Rokhlin property tensorially absorbs the action we construct below. See [21, Theorems 3.4 and 3.5] and Theorem 6.3 below.

**Example 2.1.** Let \(G\) be a finite group. Let \(\lambda: G \to U(\ell^2(G))\) be the left regular representation, and identify \(\ell^2(G)\) with \(\mathbb{C}^{|G|}\). Define an action \(\mu^G: G \to \text{Aut}(M_{|G|\infty})\) by

\[
\mu^G_g = \bigotimes_{n=1}^\infty \text{Ad}(\lambda_n)
\]

for all \(g \in G\). Then \(\alpha\) has the Rokhlin property. Note that \(\mu^G_g\) is approximately inner for all \(g \in G\).

It follows from Proposition 2.3 (i) that any action of the form \(\alpha \otimes \mu^G\) has the Rokhlin property. One of our main results, Theorem 6.3, states that in some circumstances, every action with the Rokhlin property has this form.

3. Classification of actions

One of the goals for introducing an equivariant version of the Cuntz semigroup is to apply the construction to obtain classification results for group actions on C*-algebras. When classifying actions, one usually has to restrict to a class of actions for which one has some control on the
possible crossed products, since cocycle equivalence implies that the crossed products are isomorphic (indeed, via an isomorphism that intertwines the dual actions). For the same reason, one should also restrict oneself to classifiable classes of C*-algebras.

We devote the rest of this paper to the classification certain finite abelian group actions on a class of stably finite C*-algebras. We assume that the actions either have the Rokhlin property or are locally representable. For finite group actions with the Rokhlin property, our results complement and extend those obtained by Izumi in [20]. On the other hand, our results on locally representable actions partially generalize those of Handelman and Rossmann in [17] to a much broader class of C*-algebras, and with essentially the same invariant. Our techniques seem to break down if the group is either non-finite or non-abelian, in contrast to the results in [17] where the authors allow the groups to be compact and not just finite.

We begin by defining the different notions up to which one may wish to classify actions of locally compact groups.

**Definition 3.1.** Let $G$ be a locally compact group, let $A$ be a C*-algebra and let $\alpha$ and $\beta$ be continuous actions of $G$ on $A$.

(i) We say that $\alpha$ and $\beta$ are **cocycle conjugate** if there exists a $\beta$ cocycle $\omega: G \to U(M(B))$ (see Definition 2.5), and an automorphism $\theta \in \text{Aut}(A)$ such that

$$\theta \circ \beta^\omega \circ \theta^{-1} = \alpha_g$$

for all $g \in G$.

(ii) We say that $\alpha$ and $\beta$ are **approximately inner cocycle conjugate** if they are cocycle conjugate and moreover the automorphism $\theta$ can be chosen to be approximately inner.

(iii) We say that $\alpha$ and $\beta$ are **exterior conjugate** if they are cocycle conjugate and moreover the automorphism $\theta$ as in (i) can be chosen to be the identity on $A$.

(iv) We say that $\alpha$ and $\beta$ are **conjugate** if they are cocycle conjugate, the automorphism $\theta$ as in (i) can be chosen to be approximately inner and (simultaneously) the cocycle $\omega$ as in (i) can be chosen to be $\omega_g = 1$ for all $g \in G$.

(v) We say that $\alpha$ and $\beta$ are **approximately inner conjugate** if they are cocycle conjugate, the automorphism $\theta$ as in (i) can be chosen to be approximately inner and (simultaneously) the cocycle $\omega$ as in (i) can be chosen to be $\omega_g = 1$ for all $g \in G$.

(vi) We say that $\alpha$ and $\beta$ are **inner conjugate** if they are cocycle conjugate, the automorphism $\theta$ as in (i) can be chosen to be inner and (simultaneously) the cocycle $\omega$ as in (i) can be chosen to be $\omega_g = 1$ for all $g \in G$.

**Remark 3.1.** We obviously have the following implications:

\[
\begin{array}{c}
\text{Cocycle Conjugacy} & \iff & \text{Approximately Inner Cocycle Conjugacy} & \iff & \text{Exterior Conjugacy} \\
\text{Conjugacy} & \iff & \text{Approximately Inner Conjugacy} & \iff & \text{Inner Conjugacy}.
\end{array}
\]

For actions of finite groups with the Rokhlin property, we will show that (approximately inner) cocycle conjugacy implies (approximately inner) conjugacy. See Proposition 4.2.

When the group in question is abelian, we would like to have a result relating equivalence of two actions with equivalence of their dual and double dual actions. It is well-known that if two actions are exterior conjugate, then their duals, and consequently their double duals, are conjugate. Theorem 3.1 below is a considerable generalization of this fact. We need an easy lemma first.
**Lemma 3.1.** Let $B$ be a unital C*-algebra, let $p$ be a projection in $B$ and set $A = pBp$. Let $\varphi : B \to B$ satisfy $\varphi(p) = p$, and denote by $\phi : A \to A$ the restriction of $\varphi$ to $A$. If $\varphi$ is (approximately) inner, then so is $\phi$.

**Proof.** Assume first that $\varphi$ is approximately inner. Let $\varepsilon > 0$ and let $F$ be a finite subset of $A$. Normalize $F$ so that $\|a\| \leq 1$ for all $a \in F$. Since $p$ is the unit of $A$, we will denote it by $1_A$. Choose $\delta > 0$ such that whenever $v \in A$ satisfies $\|v^*v - 1_A\| < \delta$ and $\|vv^* - 1_A\| < \delta$, there exists a unitary $w$ in $A$ such that $\|v - w\| < \varepsilon / 3$. Set $\varepsilon_0 = \min \{\varepsilon / 3, \delta\}$. Since $\varphi$ is approximately inner, there exists a unitary $u$ in $B$ such that $\|uau^* - \varphi(a)\| < \varepsilon_0$ for all $a \in F \cup \{1_A\}$. Set $v = pwp \in A$. Then

$$\|v^*v - p\| = \|pu^*ppu - p\| \leq \|pu - up\| + \|pu^*up - p\| < \varepsilon_0 < \delta,$$

and analogously, $\|vv^* - p\| < \delta$. It follows that there is a unitary $w$ in $A$ with $\|v - w\| < \varepsilon / 3$. Finally,

$$\|waw^* - \phi(a)\| = \|waw^* - \varphi(a)\| \leq 2\|v - w\| + \|uau^* - \varphi(a)\| < \varepsilon,$$

showing that $\phi$ is approximately inner.

If $\varphi$ is inner, say $\varphi = \text{Ad}(w)$ for some unitary $w$ in $B$, then $wp = pw = pwp$ is a unitary in $A$, and it follows that $\phi = \text{Ad}(pwp)$. \hfill \Box

We point out that the hypotheses of the lemma above can be relaxed: it is enough to assume that $p$ is a positive element in $B$. We do not need this strengthening, so we do not prove it either.

**Theorem 3.1.** Let $G$ be a finite group, let $A$ and $B$ be unital C*-algebras and let $\alpha$ and $\beta$ be actions of $G$ on $A$ and $B$ respectively. Denote by $\lambda : G \to U(\ell^2(G))$ the left regular representation, and denote by $e \in \mathcal{K}(\ell^2(G))$ the projection onto the constant functions on $G$.

(i) Assume that $A = B$. Then the actions $\alpha$ and $\beta$ are exterior conjugate if and only if $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$ are inner conjugate.

(ii) There are a unital homomorphism $\phi : A \to B$ and a $\beta$-cocycle $\omega : G \to U(B)$ such that

$$\phi \circ \alpha_g = \beta^\omega_g \circ \phi$$

for all $g \in G$ if and only if there is a homomorphism $\theta : A \otimes \mathcal{K}(\ell^2(G)) \to B \otimes \mathcal{K}(\ell^2(G))$ such that $\theta(1_A \otimes e)$ is unitarily equivalent to $1_B \otimes e$ and

$$\theta \circ (\alpha_g \otimes \text{Ad}(\lambda_g)) = (\beta_g \otimes \text{Ad}(\lambda_g)) \circ \theta$$

for all $g \in G$.

(iii) With the notation used in (ii) above, the homomorphism $\phi$ is invertible if and only if so is $\theta$.

(iv) Assume that $A = B$. With the notation used in (ii) above, the homomorphism $\phi$ is approximately inner if and only if so is $\theta$, in which case the condition that $\theta(1_A \otimes e)$ be unitarily equivalent to $1_A \otimes e$ is automatic.

**Proof.** In the proof of this theorem, whenever $A$ is assumed to be equal to $B$, we will denote its unit by 1. We will distinguish their units otherwise.

(i) Assume that $\alpha$ and $\beta$ are exterior conjugate and let $\omega : G \to U(\ell^2(G))$ be a cocycle in $A$ such that $\alpha^\omega = \beta$. Let $u = \sum_{g \in G} \omega_g \otimes e_{g,g} \in A \otimes \mathcal{K}(\ell^2(G))$. We check that $u$ is a unitary:

$$uu^* = \left( \sum_{g \in G} \omega_g \otimes e_{g,g} \right) \left( \sum_{h \in G} \omega_h^* \otimes e_{h,h} \right) = 1,$$
using that $e_{g,h}e_{h,h} = 0$ whenever $g \neq h$; similarly, $u^* u = 1$. We claim that the inner automorphism $\text{Ad}(u)$ intertwines the actions $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$. For $a \in A$ and $g, h, k \in G$, we have

\[
\begin{align*}
(u(\alpha_g \otimes \text{Ad}(\lambda_g))(u^*(a \otimes e_{h,k})u)u^* &= u(\alpha_g \otimes \text{Ad}(\lambda_g))(\omega_g^* a \omega_k \otimes e_{h,k})u^* \\
&= u(\alpha_g(\omega_h)^* \alpha_g(a) \alpha_g(\omega_k) \otimes e_{gh,gtk})u^* \\
&= \omega_{gh} \alpha_g(\omega_h)^* \alpha_g(a) \alpha_g(\omega_k) \omega_{gk}^* \otimes e_{gh,gtk} \\
&= \beta_g(a) \otimes e_{gh,gtk} \\
&= (\beta_g \otimes \text{Ad}(\lambda_g))(a \otimes e_{h,k}),
\end{align*}
\]

as desired.

Conversely, assume that $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$ are unitarily equivalent, and let $v \in A \otimes K(\ell^2(G))$ be a unitary such that $\text{Ad}(v)$ intertwines $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$. Set $f = v(1 \otimes e)v^*$ and denote by

\[
\varphi: (1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e) \to f(A \otimes K(\ell^2(G)))(1 \otimes e)
\]

the restriction of $\text{Ad}(v)$ to the corner $(1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e)$. Since $\text{Ad}(v)$ intertwines $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$, it is clear that $\varphi$ intertwines the restriction of $\alpha \otimes \text{Ad}(\lambda)$ to the corner $(1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e)$ and the restriction of $\beta \otimes \text{Ad}(\lambda)$ to the corner $f(A \otimes K(\ell^2(G)))(1 \otimes e)$. It follows that $\alpha$ and $(\beta \otimes \text{Ad}(\lambda))|_{f(A \otimes K(\ell^2(G)))}$ are conjugate via $\varphi$.

Define

\[
\varphi \circ \iota_e: (A, \alpha) \to (f(A \otimes K(\ell^2(G)))(1 \otimes e), (\beta \otimes \text{Ad}(\lambda))|_{f(A \otimes K(\ell^2(G)))})
\]

is an equivariant isomorphism.

Consider the $\beta$-cocycle $\omega: G \to U(A)$ given by

\[
\omega_g \otimes e = (1 \otimes e)u^*(\beta \otimes \text{Ad}(\lambda))(v(1 \otimes e))
\]

for $g \in G$. We claim that $(\beta \otimes \text{Ad}(\lambda))|_{f(A \otimes K(\ell^2(G)))}$ and $(\beta^\omega \otimes \text{Ad}(\lambda))|_{(1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e)}$ are intertwined by the restriction of $\text{Ad}(v^*)$ to $f(A \otimes K(\ell^2(G)))(1 \otimes e)$, which is just $\varphi^{-1}$.

Given $a \in A$ and $g \in G$, we have:

\[
(\beta^\omega \otimes \text{Ad}(\lambda))_g(a \otimes e) = (\omega_g \beta_g(a) \omega_g^*) \otimes e
\]

\[
= (\omega_g \otimes e)(\beta \otimes \text{Ad}(\lambda))_g(a \otimes 1)(\omega_g \otimes e)^*
\]

\[
= (1 \otimes e)v^*(\beta \otimes \text{Ad}(\lambda))(v(1 \otimes e)(a \otimes e)(1 \otimes e)v^*) v(1 \otimes e),
\]

and thus

\[
(\text{Ad}(v^*) \circ (\beta^\omega \otimes \text{Ad}(\lambda))_g \circ \text{Ad}(v))(a \otimes e) = f(\beta \otimes \text{Ad}(\lambda))_g f(f(1 \otimes e)f),
\]

proving the claim.

Finally, notice that the composition $\text{Ad}(v^*) \circ \text{Ad}(v) = \text{id}_{A \otimes K(\ell^2(G))}$ restricted to the invariant corner $(1 \otimes e)(A \otimes K(\ell^2(G)))(1 \otimes e)$ intertwines $\alpha$ and $\beta^\omega$, showing that $\alpha = \beta^\omega$.

(ii). Let $\phi: A \to B$ be a unital homomorphism and let $\omega: G \to U(B)$ be a $\beta$-cocycle in $B$ such that $\phi \circ \alpha_g = \beta^\omega \circ \phi$ for all $g \in G$. The homomorphism $\phi \otimes \text{id}_{K(\ell^2(G))}: A \otimes K(\ell^2(G)) \to B \otimes K(\ell^2(G))$ clearly intertwines the actions $\alpha \otimes \text{Ad}(\lambda)$ and $\beta^\omega \otimes \text{Ad}(\lambda)$. Moreover, part (1) implies that there is a unitary $u$ in $B \otimes K(\ell^2(G))$ such that $\text{Ad}(u) \circ (\beta_g^\omega \otimes \text{Ad}(\lambda_g)) = (\beta \otimes \text{Ad}(\lambda)) \circ \text{Ad}(u)$ for all $g \in G$. It follows that $\theta = \text{Ad}(u) \circ (\phi \otimes \text{id}_{K(\ell^2(G))})$ intertwines $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$. Finally

\[
\theta(1_A \otimes e) = (\text{Ad}(u) \circ (\phi \otimes \text{id}_{K(\ell^2(G))}))(1_A \otimes e) = \text{Ad}(u)(1_B \otimes e)
\]
is clearly unitarily equivalent to $1_B \otimes e$. Thus $\theta$ is the desired homomorphism.

Conversely, assume that $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$ are conjugate via a homomorphism

$$\theta: A \otimes \mathcal{K}(\ell^2(G)) \to A \otimes \mathcal{K}(\ell^2(G))$$

for which $\theta(1_A \otimes e)$ is unitarily equivalent to $1_B \otimes e$. Let $v \in U(B \otimes \mathcal{K}(\ell^2(G)))$ be a unitary implementing the equivalence. Set $f = \theta(1_A \otimes e)$ and denote by

$$\varphi: e(A \otimes \mathcal{K}(\ell^2(G)))e \to f(B \otimes \mathcal{K}(\ell^2(G)))f$$

the restriction of $\theta$ to the corner $e(A \otimes \mathcal{K}(\ell^2(G)))e \cong A$. As in the proof of (i), consider the $\beta$-cocycle $\omega: G \to U(B)$ given by $\omega_g \otimes e = (1_B \otimes e)v^*(\beta \otimes \text{Ad}(\lambda))g(v(1_B \otimes e))$ for $g \in G$. A computation similar to the one carried out in the proof of (i) now shows that if we let $\phi$ be the restriction of $\text{Ad}(v^*) \circ \theta$ to $(1_A \otimes e)(A \otimes \mathcal{K}(\ell^2(G)))(1_A \otimes e)$, then $\phi$ intertwines the restrictions of $\alpha \otimes \text{Ad}(\lambda)$ to the corner $(1_A \otimes e)(A \otimes \mathcal{K}(\ell^2(G)))(1_A \otimes e)$ and the restriction of $\beta^\omega \otimes \text{Ad}(\lambda)$ to the corner $(1_B \otimes e)(B \otimes \mathcal{K}(\ell^2(G)))(1_B \otimes e)$. Using that $(1_A \otimes e)(A \otimes \mathcal{K}(\ell^2(G)))(1_A \otimes e)$ is naturally identified with $A$ via $\iota_e(a) = a \otimes e$, and that this identification sends $\alpha$ to the corresponding restriction of $\alpha \otimes \text{Ad}(\lambda)$, and similarly with $\beta^\omega$ on $B$, we conclude that the homomorphism $\phi$ of $A$ intertwines the actions $\alpha$ and $\beta^\omega$.

(iii). Assume that $\phi$ is invertible. Then the proof of (ii) shows that there is a unitary $u$ in $A \otimes \mathcal{K}(\ell^2(G))$ such that the homomorphism $\theta$ has the form $\theta = \text{Ad}(u) \circ (\phi \otimes \text{id}_{\mathcal{K}(\ell^2(G))})$. It follows that $\theta$ is also invertible. Conversely, assume that $\theta$ is invertible and adopt the notation of the proof of part (ii) above. Then the composition $\text{Ad}(v^*) \circ \theta$, whose restriction to the invariant corner $(1 \otimes e)(A \otimes \mathcal{K}(\ell^2(G)))(1 \otimes e)$ is $\phi$, is invertible.

(iv). Assume that $\phi$ is approximately inner, then so is $\theta = \text{Ad}(u) \circ (\phi \otimes \text{id}_{\mathcal{K}(\ell^2(G))})$. Conversely, assume that $\theta$ is approximately inner, and adopt the notation of the proof of part (ii) above. Then the composition $\text{Ad}(v^*) \circ \theta$, whose restriction to the invariant corner $(1 \otimes e)(A \otimes \mathcal{K}(\ell^2(G)))(1 \otimes e)$ is $\phi$, is approximately inner. It follows from Lemma 3.1 that $\phi$ is approximately inner.

The last claim follows from the fact that two projections that are sufficiently close are unitarily equivalent.

Corollary 3.1. Adopt the notation of Theorem 3.1. Then:

(i) The actions $\alpha$ and $\beta$ are cocycle conjugate if and only if $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$ are conjugate via an isomorphism $\theta: A \otimes \mathcal{K}(\ell^2(G)) \to B \otimes \mathcal{K}(\ell^2(G))$ for which $\theta(1_A \otimes e)$ is unitarily equivalent to $1_B \otimes e$.

(ii) If $A = B$, then the actions $\alpha$ and $\beta$ are approximately inner cocycle conjugate if and only if $\alpha \otimes \text{Ad}(\lambda)$ and $\beta \otimes \text{Ad}(\lambda)$ are approximately inner conjugate.

Proof. Follows from parts (ii), (iii) and (iv) in Theorem 3.1.

Corollary 3.2. Adopt the notation of Theorem 3.1 and assume that $G$ is moreover abelian. Denote by $T_\alpha: (A \rtimes_\alpha G) \rtimes \hat{G} \to A \otimes \mathcal{K}(\ell^2(G))$ and $T_\beta: (B \rtimes_\beta G) \rtimes \hat{G} \to B \otimes \mathcal{K}(\ell^2(G))$ the isomorphisms given by Takai duality.

(i) The actions $\alpha$ and $\beta$ are cocycle conjugate if and only if $\hat{\alpha}$ and $\hat{\beta}$ are conjugate via an isomorphism

$$\theta: (A \rtimes_\alpha G) \rtimes \hat{G} \to (B \rtimes_\beta G) \rtimes \hat{G}$$

for which the image of $1_A \otimes e$ under $T_\alpha^{-1} \circ \theta \circ T_\beta$ is unitarily equivalent to $1_B \otimes e$.\[13]
(A ⋊_\alpha G) ⋊_{\tilde{\alpha}} \hat{G} \xrightarrow{\theta} (B ⋊_\beta G) ⋊_{\beta} \hat{G}

T_{\alpha} \downarrow \hspace{1cm} \downarrow T_{\beta}
A \otimes \mathcal{K}(\ell^2(G)) \xrightarrow{T_{\alpha}^{-1} \circ \theta \circ T_{\beta}} B \otimes \mathcal{K}(\ell^2(G)).

(ii) Assume that $A = B$. Then the actions $\alpha$ and $\beta$ are exterior conjugate if and only if $\tilde{\alpha}$ and $\tilde{\beta}$ are inner conjugate.

(iii) Assume that $A = B$. Then the actions $\alpha$ and $\beta$ are approximately inner cocycle conjugate if and only if $\tilde{\alpha}$ and $\tilde{\beta}$ are approximately inner conjugate.

Proof. This follows from the fact that the double crossed product $(A ⋊_\alpha G) ⋊_{\tilde{\alpha}} \hat{G}$ can be naturally identified via $T_{\alpha}$ with $A \otimes \mathcal{K}(\ell^2(G))$ in such a way that the double dual action $\tilde{\alpha}$ is taken to $\alpha \otimes \text{Ad}(\lambda)$, and similarly for the action $\beta$, and using Theorem 3.1 and Corollary 3.1. \hfill \square

4. Actions with the Rokhlin Property and Functors that Classify Homomorphisms.

Let $G$ be a locally compact group and let $A$ denote the category of C*-algebras. Let us denote by $A^G$ the category whose objects are $G$-C*-dynamical systems $(A, \alpha)$, this is, $A$ is a C*-algebra and $\alpha: G \to \text{Aut}(A)$ is a strongly continuous action; and whose morphisms are equivariant *-homomorphisms. This is, if $(A, \alpha)$ and $(B, \beta)$ are objects in $A^G$, then

$$\text{Hom}_{A^G}((A, \alpha), (B, \beta)) = \{ \psi \in \text{Hom}_A(A, B) : \psi \circ \alpha_g = \beta_g \circ \psi \text{ for all } g \in G \}.$$ 

If $B$ is a subcategory of $A$, we denote by $B^G$ the subcategory of $A^G$ whose objects are dynamical systems $(A, \alpha)$ such that $A$ is an object in $B$, and with morphisms are given by

$$\text{Hom}_{B^G}((A, \alpha), (B, \beta)) = \text{Hom}_{A^G}((A, \alpha), (B, \beta)).$$

Whenever the terminology from Category Theory is not needed, we will say ‘$(A, \alpha)$ is a C*-dynamical system’, instead of ‘$(A, \alpha)$ is an object in $A^G$’. Likewise, in these situations we will say that a homomorphism $\phi: (A, \alpha) \to (B, \beta)$ is equivariant instead of saying that it is a morphism in $A^G$.

Definition 4.1. Let $G$ be a locally compact group, let $(A, \alpha)$ and $(B, \beta)$ be objects in $A^G$ and let $\phi$ and $\psi$ be morphisms in $\text{Hom}_{A^G}((A, \alpha), (B, \beta))$. We say that $\phi$ and $\psi$ are (equivariantly) approximately unitarily equivalent, and denote this by $\phi \sim_{G-\text{au}} \psi$, if for any finite subset $F \subseteq A$ and for any $\varepsilon > 0$ there exists a unitary $u \in (B^\beta)^\sim$ such that

$$\|\phi(a) - u^* \psi(a) u\| < \varepsilon,$$

for all $a \in F$.

Note that when $G$ is the trivial group, this definition agrees with the standard definition of approximate unitary equivalence of *-homomorphisms. In this case we will omit the group $G$ in the notation for the equivalence relation $\sim_{G-\text{au}}$, and write simply $\sim_{\text{au}}$.

Let $C_1$ and $C_2$ be two categories. Recall that a functor $F: C_1 \to C_2$ is said to be sequentially continuous if for any countable direct system $(C_n, \tau_n)_{n \in \mathbb{N}}$ in $C_1$ whose direct limit $C = \varinjlim(C_n, \tau_n)$ exists in $C_1$, the direct limit of the system $(F(C_n), F(\tau_n))_{n \in \mathbb{N}}$ exists in $C_2$ and moreover one has

$$F(\varinjlim(C_n, \tau_n)) = \varinjlim(F(C_n), F(\tau_n)).$$

More generally, a functor is called continuous if it preserves arbitrary direct limits.
**Definition 4.2.** Let $G$ be a locally compact group, let $B$ be a subcategory of $A$, let $C$ be a category in which inductive limits of sequences exist and let $B^G$ be as before. Let $F: B^G \to C$ be a sequentially continuous functor.

(i) We say that the functor $F$ **classifies isomorphisms** if it has the following properties:

(a) for every pair of objects $(A, \alpha)$ and $(B, \beta)$ in $B^G$ and for every isomorphism

$$\lambda: F(A, \alpha) \to F(B, \beta)$$

in $C$, there exists an isomorphism $\phi: (A, \alpha) \to (B, \beta)$ in $B^G$ such that $F(\phi) = \lambda$.

(b) for every pair of objects $(A, \alpha)$ and $(B, \beta)$ in $B^G$ and for every pair of isomorphisms $\phi, \psi: F(A, \alpha) \to F(B, \beta)$ in $C$, one has that $F(\phi) = F(\psi)$ if and only if $\phi \sim_{G-\text{au}} \psi$.

(ii) We say that the functor $F$ **classifies homomorphisms** if it has the following properties:

(a) for every pair of objects $(A, \alpha)$ and $(B, \beta)$ in $B^G$ and for every morphism

$$\lambda: F(A, \alpha) \to F(B, \beta)$$

in $C$, there exists a morphism $\phi: (A, \alpha) \to (B, \beta)$ in $B^G$ such that $F(\phi) = \lambda$.

(b) for every pair of objects $(A, \alpha)$ and $(B, \beta)$ in $B^G$ and for every pair of morphisms $\phi, \psi: F(A, \alpha) \to F(B, \beta)$ in $C$, one has that $F(\phi) = F(\psi)$ if and only if $\phi \sim_{G-\text{au}} \psi$.

As before, when $G$ is the trivial group we will omit the superscript $G$.

The following theorem is a consequence of the definition of functor that classifies isomorphisms and Elliott’s intertwining argument. See, for example, the Theorem in [10].

**Is there a reference for the following theorem? if not, maybe we should prove it?**

**Theorem 4.1.** Let $G$ be a locally compact group, let $B$ be a subcategory of $A$, let $C$ be a category in which inductive limits of sequences exist and let $B^G$ be as before. Let $F: B^G \to C$ be a functor that classifies homomorphisms. Then $F$ classifies isomorphisms.

**Proposition 4.1.** Let $G$ be a finite group, let $(A, \alpha)$ and $(B, \beta)$ be $C^*$-dynamical systems such that $\beta$ has the Rokhlin property, let $\phi, \psi: (A, \alpha) \to (B, \beta)$ be equivariant homomorphisms, and assume moreover that $\phi$ and $\psi$ are approximately unitarily equivalent (in the category $A$ of $C^*$-algebras). Then $\phi$ and $\psi$ are equivariantly approximately unitarily equivalent (in the category $A^G$ of $C^*$-dynamical systems).

**Proof.** Let $F$ be a finite subset of $A$ and let $\varepsilon > 0$. We have to show that there exists a unitary $w \in (B^\beta)^{-}$ such that

$$\|\phi(a) - w^*\psi(a)w\| < \varepsilon,$$

for all $a \in F$. Set $F' = \bigcup_{g \in G} \alpha_g(F)$, which is again a finite subset of $A$. Since $\phi$ and $\psi$ are approximately unitarily equivalent, there exists a unitary $u \in B^{-}$ such that

$$\|\phi(b) - u^*\psi(b)u\| < \varepsilon$$

for all $b$ in $F'$. If $u = x + \lambda 1_{B^{-}}$ for some $x$ in $B$ and some $\lambda$ in $C$ of modulus 1, then equation (4.1) above is satisfied if one replaces $u$ with $\lambda u$. Thus, we may assume that the unitary $u$ has the form $u = x + 1_{B^{-}}$ for some $x$ in $B$. Fix $g$ in $G$ and fix $a$ in $F$. Then $b = \alpha_{g^{-1}}(a)$ belongs to $F'$. Using equation (4.1) and the fact that $\phi$ and $\psi$ are equivariant, we get

$$\|\beta_{g^{-1}}(\phi(a)) - u^*\beta_{g^{-1}}(\psi(a))u\| < \varepsilon.$$ 

By applying $\beta_g$ to the equation above, we conclude that

$$\|\phi(a) - \beta_g(u)^*\psi(a)\beta_g(u)\| < \varepsilon$$

for all $a$ in $F$ and all $g$ in $G$.

Choose positive orthogonal contractions $r_g$ in $B^\infty$ for $g$ in $G$ as in the definition of the Rokhlin
property for $\beta$, and set $v = \sum_{g \in G} \beta_g(x)r_g + 1_{B^\sim}$. Using that $x_g + 1_{B^\sim}$ is a unitary in $B^\sim$, one checks that

$$v^*v = \sum_{g \in G} (\beta_g(x^*)x_g^2 + \beta_g(x)r_g + \beta_g(x)r_g^*) + 1_{B^\sim} = 1_{B^\sim}.$$ 

Analogously, we have $vv^* = 1_{B^\sim}$ and hence $v$ is a unitary in $B^\sim$. Also, note that for every $b \in B$ we have

$$v^*bv = \sum_{g \in G} r_g \beta_g(u)^*b \beta_g(u).$$

Therefore,

$$\|\phi(a) - v^*\psi(a)v\| = \left\| \sum_{g \in G} r_g \phi(a) - \sum_{g \in G} r_g \beta_g(u)^*\psi(a)\beta_g(u) \right\| < \varepsilon,$$

for all $a$ in $F$. Since $v = \sum_{g \in G} \beta_g(x)e_x + 1_{B^\sim}$, we have $v \in ((B^\beta)^\sim)^\infty \subseteq (B^\sim)^\infty$. By Lemma 2.7, we can choose a unitary $w$ in $(B^\beta)^\sim$ such that

$$\|\phi(a) - w^*\psi(a)w\| < \varepsilon,$$

for all $a$ in $F$. \hfill \Box

The following result is probably standard. Since we have not been able to find a proof in the literature, we present it here.

**Lemma 4.1.** Let $A$ and $B$ be C*-algebras and let $\psi: A \to B$ be a *-homomorphism. Suppose there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of unitaries in $B^\sim$ such that the sequence $(v_n\phi(x)v_n^*)_{n \in \mathbb{N}}$ converges in $B$ for all $x$ in a dense subset of $A$. Then there exists a *-homomorphism $\psi: A \to B$ which is unitarily equivalent to $\phi$ and such that

$$\lim_{n \to \infty} v_n\phi(x)v_n^* = \psi(x)$$

for all $x$ in $A$.

**Proof.** Let

$$S = \{ x \in A : (v_n\phi(x)v_n^*)_{n \in \mathbb{N}} \text{ converges in } B \} \subseteq A.$$ 

Then $S$ is a dense *-subalgebra of $A$. For each $x$ in $S$, denote by $\psi_0(x)$ the limit of the sequence $(v_n\phi(x)v_n^*)_{n \in \mathbb{N}}$. The map $\psi_0: S \to B$ is linear, multiplicative, preserves the adjoint operation, and is bounded by $\|\phi\|$, so it therefore extends by continuity to a *-homomorphism $\psi: A \to B$. Given $a \in A$ and given $\varepsilon > 0$, use density of $S$ in $A$ to choose $x \in S$ such that $\|a - x\| < \varepsilon/3$. Choose $N \in \mathbb{N}$ such that $\|v_N\phi(x)v_N^* - \psi(x)\| < \varepsilon/3$. Then

$$\|\psi(a) - v_N\phi(a)v_N^*\| \leq \|\psi(a - x)\| + \|\psi(x) - v_N\phi(x)v_N^*\| + \|v_N\phi(x)v_N^* - v_N\phi(a)v_N^*\|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

It follows that $\psi(a) = \lim_n v_n\phi(a)v_n^*$ for all $a$ in $A$, as desired. \hfill \Box

The unital case of following proposition is [20, Lemma 5.1]. The proof for arbitrary C*-dynamical systems follows similar ideas.

**Proposition 4.2.** Let $G$ be a finite group and let $(A, \alpha)$ and $(B, \beta)$ be C*-dynamical systems. Suppose that $A$ is separable and suppose that $\beta$ has the Rokhlin property. Let $\phi: A \to B$ be a *-homomorphism such that $\beta_g \circ \phi$ and $\phi \circ \alpha_g$ are approximately unitarily equivalent for all $g \in G$. Then:
(i) For any $\varepsilon > 0$ and for any finite set $F \subseteq A$ there exists a unitary $u \in B^\sim$ such that
\begin{equation}
\| (\beta_g \circ \operatorname{Ad}(w) \circ \phi)(x) - (\operatorname{Ad}(w) \circ \phi \circ \alpha_g)(x) \| < \varepsilon, \quad \text{for all } g \in G \text{ and for all } x \in F,
\end{equation}
for all $x \in F$.

(ii) There exists an equivariant $*$-homomorphism $\psi : A \to B$ that is approximately unitarily equivalent to $\phi$.

**Proof.** (i) Let $F$ be a finite subset of $A$ and let $\varepsilon > 0$. Set $F' = \bigcup_{g \in G} \alpha_g(F)$, which is a finite subset of $A$. Since $\beta_g \circ \phi$ is unitarily equivalent to $\phi \circ \alpha_g$ for all $g$ in $G$, there exist unitaries $u_g$ in $B^\sim$ for $g$ in $G$ such that
\[ \| (\beta_g \circ \phi)(a) - (\operatorname{Ad}(u_g) \circ \phi \circ \alpha_g)(a) \| < \frac{\varepsilon}{2}, \]
for all $a$ in $F'$ and all $g$ in $G$. Upon replacing $u_g$ with a scalar multiple of it, one can assume that there are $x_g$ in $B$ for $g$ in $G$ such that $u_g = x_g + 1_{B^\sim}$ for all $g$ in $G$. For $a$ in $F$ and for $g$ and $h$ in $G$, we have
\begin{align*}
\| (\operatorname{Ad}(u_g) \circ \phi \circ \alpha_h)(a) & - (\beta_h \circ \operatorname{Ad}(u_{h^{-1}g}) \circ \phi)(a) \| \\
& = \| (\operatorname{Ad}(u_g) \circ \phi \circ \alpha_h)(\alpha_{g^{-1}h}(a)) - (\beta_h \circ \operatorname{Ad}(u_{h^{-1}g}) \circ \phi \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(a)) \| \\
& \leq \| (\operatorname{Ad}(u_g) \circ \phi \circ \alpha_h)(\alpha_{g^{-1}h}(a)) - (\beta_g \circ \phi)(\alpha_{g^{-1}h}(a)) \| \\
& + \| (\beta_g \circ \phi)(\alpha_{g^{-1}h}(x)) - (\beta_h \circ \operatorname{Ad}(u_{h^{-1}g}) \circ \phi \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(x)) \|
\end{align*}
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Choose positive orthogonal contractions $r_g$ in $B_{\infty}$ for $g$ in $G$ as in the definition of the Rokhlin property for $\beta$, and set
\[ u = \sum_{g \in G} r_g x_g + 1_{B^\sim} \in (B^\sim)_{\infty}. \]

Using that $x_g + 1_{B^\sim}$ is a unitary in $B^\sim$, one checks that
\[ u^* u = 1_{B^\sim} + \sum_{g \in G} (r_g^* x_g^* x_g + r_g x_g + r_g x_g^*) = 1_{B^\sim}. \]

Analogously, one also checks that $uu^* = 1_{B^\sim}$, thus showing that $u$ is a unitary in $(B^\sim)_{\infty}$. The map $\operatorname{Ad}(u)$ can be written in terms of the maps $\operatorname{Ad}(u_g)$ as follows:
\[ (\operatorname{Ad}(u))(x) = u x u^* = \sum_{g \in G} (u_g x u_g^*) r_g = \sum_{g \in G} (\operatorname{Ad}(u_g))(x) r_g, \]
for all $x$ in $A$. Now, for $a$ in $F$ we have the following identities
\begin{align*}
(\beta_h \circ \operatorname{Ad}(u) \circ \phi)(a) &= \sum_{g \in G} r_{hg}(\beta_h \circ \operatorname{Ad}(u_g) \circ \phi)(a) = \sum_{g \in G} r_g (\beta_h \circ \operatorname{Ad}(u_{h^{-1}g}) \circ \phi)(a), \\
(\operatorname{Ad}(u) \circ \phi \circ \alpha_h)(a) &= \sum_{g \in G} r_g (\operatorname{Ad}(u_g) \circ \phi \circ \alpha_h)(a).
\end{align*}

Therefore,
\[ \| (\beta_h \circ \operatorname{Ad}(u) \circ \phi)(a) - (\operatorname{Ad}(u) \circ \phi \circ \alpha_h)(a) \| \leq \sup_{g \in G} \| (\operatorname{Ad}(u_g) \circ \phi \circ \alpha_h)(a) - (\beta_h \circ \operatorname{Ad}(u_{h^{-1}g}) \circ \phi)(a) \| < \varepsilon. \]
This in turn implies that

\[
\|(\text{Ad}(u) \circ \phi)(a) - \phi(a)\| = \left\| \sum_{g \in G} r_g((\text{Ad}(u_g) \circ \phi)(a) - \phi(a)) \right\| \\
\leq \sup_{g \in G} \|(\text{Ad}(u_g) \circ \phi)(a) - \phi(a)\| \\
\leq \sup_{g \in G} \left[ \|(\text{Ad}(u_g) \circ \phi \circ \alpha_g)(\alpha_{g^{-1}}(a)) - (\beta_g \circ \phi)(\alpha_{g^{-1}}(a))\| + \|(\beta_g \circ \phi \circ \alpha_{g^{-1}})(a) - \phi(a)\| \right] \\
\leq \varepsilon + \sup_{g \in G} \|(\beta_g \circ \phi \circ \alpha_{g^{-1}})(a) - \phi(a)\|.
\]

We have shown that the inequalities in (4.2) hold for a unitary \( u \in (B^\sim)^\infty \). By Lemma 2.7, we can replace \( u \) with a unitary in \( w \) in \( B^\sim \) in such a way that both inequalities still hold for \( w \) in place of \( u \).

(ii) Let \((F_n)_{n \in \mathbb{N}}\) be an increasing sequence of finite subsets of \( A \) whose union is dense in \( A \). Upon replacing each \( F_n \) with \( \bigcup_{g \in G} \alpha_g(F_n) \), we may assume that \( \alpha_g(F_n) = F_n \) for all \( g \) in \( G \) and for all \( n \) in \( \mathbb{N} \). Set \( \phi_1 = \phi \) and find a unitary \( u_1 \) in \( B^\sim \) such that the conclusion of the first part of the proposition is satisfied with \( \phi_1 \) and \( \varepsilon = 1 \). Set \( \phi_2 = \text{Ad}(u_1) \circ \phi_1 \), and find a unitary \( u_2 \) in \( B^\sim \) such that the conclusion of the first part of the proposition is satisfied with \( \phi_2 \) and \( \varepsilon = 1/2 \). Iterating this process, there exist *-homomorphisms \( \phi_n: A \to B \) with \( \phi_1 = \phi \) and unitaries \((u_n)_{n \in \mathbb{N}}\) in \( B^\sim \) such that \( \phi_{n+1} = \text{Ad}(u_n) \circ \phi_n \), for all \( n \) in \( \mathbb{N} \), which moreover for all \( n \) in \( \mathbb{N} \) satisfy

\[
\|(\beta_g \circ \phi_n)(x) - (\phi_n \circ \alpha_g)(x)\| < \frac{1}{2^n}
\]

for all \( g \) in \( G \) and for all \( x \) in \( F_n \), and

\[
\|\phi_{n+1}(x) - \phi_n(x)\| < \frac{3}{2^n}
\]

for all \( x \) in \( F_n \). Set \( v_n = u_n \cdots u_1 \) for \( n \) in \( \mathbb{N} \). Then the sequence \((v_n)_{n \in \mathbb{N}}\) of unitaries in \( B^\sim \) and the *-homomorphism \( \phi: A \to B \) satisfy the hypotheses of Lemma 4.1, so it follows that the sequence \((\phi_n)_{n \in \mathbb{N}}\) of *-homomorphisms converges to a *-homomorphism \( \psi: A \to B \) that satisfies \( \beta_g \circ \psi = \psi \circ \alpha_g \) for all \( g \) in \( G \); this is, \( \psi \) is equivariant. Since each \( \phi_n \) is unitarily equivalent to \( \phi \), we conclude that \( \phi \) and \( \psi \) are approximately unitarily equivalent.

\[\square\]

Particular cases of the following proposition have been proven by Izumi in [20, Theorem 3.5], where only unital C*-algebras were considered, and more recently by Nawata in [27, Theorem 3.5], where he extended Izumi’s result to C*-algebras of almost stable rank one. The proof of the general case is a modification of Nawata’s proof.

**Proposition 4.3.** Let \( G \) be a finite group, let \( A \) be separable C*-algebra and let \( \alpha \) and \( \beta \) be actions of \( G \) on \( A \) with the Rokhlin property. Assume that \( \alpha_g \) and \( \beta_g \) are unitarily approximately equivalent for all \( g \in G \). Then there exists an approximately inner automorphism \( \theta \) of \( A \) such that \( \theta \circ \alpha_g = \beta_g \circ \theta \) for all \( g \in G \).

**Proof.** The proof of [27, Theorem 3.5] uses [27, Lemma 3.4], which requires \( A \) to have almost stable rank one. It follows immediately by taking \( A = B \) and \( \phi = \text{id}_A \) in Proposition 4.2 above that this assumption is unnecessary. The result now follows with the same argument as in [27, Theorem 3.5]. \[\square\]

**Remark 4.1.** In view of [27, Remark 3.6], it may be worth pointing out that one can directly modify the proof of [27, Lemma 3.4] to get rid of the assumption that \( A \) have almost stable rank one. Indeed,
one just needs to replace the element \( w \) in the proof by the unitary \( w' = \sum_{g \in G} (v_g - \lambda_g 1_A) f_g + 1_A \), where \( \lambda_g \in \mathbb{C} \) is such that \( v_g - \lambda_g 1_A \in A \).

**Definition 4.3.** Let \( G \) be a locally compact group. Let \( C \) be a category and let \( C^G \) denote the category whose objects are pairs \((C, \gamma)\), where \( C \) is an object in \( C \) and \( \gamma: G \to \text{Aut}(C) \) is a group homomorphism, also called an action of \( G \) on \( C \). We do not require any kind of continuity for this action since \( C \) does not a priori have a topology. The morphisms of \( C^G \) consist of the morphisms of \( C \) that are equivariant.

Let \( B \) be a subcategory of the category \( A \) of \( C^* \)-algebras and let \( F: B \to C \) be a functor. Recall the definition of \( B^G \) from the beginning of this section: its objects are \( G \)-\( C^* \)-dynamical systems \((A, \alpha)\) and its morphisms are equivariant homomorphisms of \( C^* \)-algebras. Then \( F \) induces a functor \( F^G: B^G \to C^G \) as follows.

(i) For an object \((A, \alpha)\) in \( B^G \) define an action \( F(\alpha): G \to \text{Aut}(F(A)) \) by \((F(\alpha))_g = F(\alpha_g)\) for all \( g \in G \). We then set \( F^G(A, \alpha) = (F(A), F(\alpha)) \);

(ii) For a morphism \( \phi \in \text{Hom}_B((A, \alpha), (B, \beta)) \), we set \( F^G(\phi) = F(\phi) \).

Let \( R^G \) denote the subcategory of \( B^G \) consisting of those \( C^* \)-dynamical systems \((A, \alpha)\) such that \( A \) is separable and \( \alpha \) has the Rokhlin property.

**Example 4.1.** Let \( G \) be a compact abelian group. It follows from Theorem 2.1 that the equivariant Cuntz semigroup \( Cu_G \), regarded as a functor from the category \( A \) of all \( C^* \)-dynamical systems to the category \( C^G \), is equivalent to the composition

\[
(A, \alpha) \mapsto (A \rtimes_\alpha G, \hat{\alpha}) \mapsto (Cu(A \rtimes_\alpha G), Cu(\hat{\alpha})).
\]

Hence \( Cu_G \) is the composition of the crossed product functor with the functor \( C^G \) as described in Definition 4.3 above.

**Lemma 4.2.** Let \( G \) be a locally compact group, let \( \Lambda \) be a directed set and let \( C \) be a category where inductive limits over \( \Lambda \) exist. Let \( C^G \) be the associated category as above. The following statements hold:

(i) Inductive limits over \( \Lambda \) exist in \( C^G \).

(ii) If \( D \) is a category where inductive limits over \( \Lambda \) exist and \( F: C \to D \) is a functor that preserves direct limits over \( \Lambda \), then the associated functor \( F^G: C^G \to D^G \) also preserves direct limits over \( \Lambda \).

**Proof.** (i) Let \((C_\lambda, \alpha_\lambda)_{\lambda \in \Lambda}, (\gamma_{\lambda, \mu})_{\lambda, \mu \in \Lambda, \lambda < \mu}\) be a direct system in \( C^G \) over \( \Lambda \), where \( \gamma_{\lambda, \mu}: (C_\lambda, \alpha_\lambda) \to (C_\mu, \alpha_\mu) \) for \( \lambda < \mu \) is a morphism in \( C^G \). Let \((C, (\gamma_{\lambda, \infty})_{\lambda \in \Lambda})\), with \( \gamma_{\lambda, \infty}: C_\lambda \to C \), be its direct limit in the category \( C \). For each \( \lambda \) in \( \Lambda \) and for each \( g \) in \( G \), consider the maps \( \gamma_{\lambda, \infty} \circ \alpha_\lambda(g) \). Then

\[
(\gamma_{\mu, \infty} \circ \alpha_\mu(g)) \circ \gamma_{\lambda, \mu} = \gamma_{\lambda, \infty} \circ \alpha_\lambda(g)
\]

for all \( \mu \) in \( \Lambda \) such that \( \lambda < \mu \). Hence, by the universal property of the inductive limit \((C, (\gamma_{\lambda, \infty})_{\lambda \in \Lambda})\), there exists a unique \( C \)-morphism \( \alpha(g): C \to C \) that satisfies \( \alpha(g) \circ \gamma_{\lambda, \infty} = \gamma_{\lambda, \infty} \circ \alpha_\lambda(g) \) for all \( \lambda \) in \( \Lambda \). Note that for \( g \) and \( h \) in \( G \), one has

\[
(\alpha(g) \circ \alpha(h)) \circ \gamma_{\lambda, \infty} = \gamma_{\lambda, \infty} \circ \alpha_\lambda(g) \circ \alpha(h) = \gamma_{\lambda, \infty} \circ \alpha_\lambda(gh).
\]

Therefore, by uniqueness of the morphism \( \alpha_\lambda(gh) \), it follows that \( \alpha(g) \circ \alpha(h) = \alpha(gh) \) for all \( g \) and \( h \) in \( G \). This implies that \( \alpha(g) \) is an automorphism of \( C \) and that \( \alpha: G \to \text{Aut}(C) \) is an action. Thus \((C, \alpha)\) is an object in \( C^G \).

We claim that \((C, \alpha)\) is the inductive limit of \((C_\lambda, \alpha_\lambda)_{\lambda \in \Lambda}, (\gamma_{\lambda, \mu})_{\lambda, \mu \in \Lambda, \lambda < \mu}\) in the category \( C^G \). For \( \lambda \) in \( \Lambda \), The map \( \gamma_{\lambda, \infty} \) is equivariant since \( \gamma_{\lambda, \infty} \circ \alpha_\lambda(g) = \alpha(g) \circ \gamma_{\lambda, \infty} \) for all \( g \) in \( G \) and all for all \( \lambda \in \Lambda \). Let \((D, \beta)\) be an object in \( C^G \) and for \( \lambda \) in \( \Lambda \), let \( \rho_\lambda: (C_\lambda, \alpha_\lambda) \to (D, \beta) \) be an
equivariant morphism. By the universal property of the inductive limit $C$, there exists a unique morphism $\rho: C \to D$ satisfying $\rho_\lambda = \rho \circ \gamma_{\lambda,\infty}$ for all $\lambda$ in $\Lambda$. We therefore have
\[
(\beta(g)^{-1} \circ \rho \circ \alpha(g)) \circ \gamma_{\lambda,\infty} = \beta^{-1}(g) \circ \rho \circ \gamma_{\lambda,\infty} \circ \alpha_\lambda(g) = \beta^{-1}(g) \circ \rho_{\lambda,\infty} \circ \alpha_\lambda(g) = \rho_{\lambda,\infty},
\]
for all $g$ in $G$ and for all $\lambda$ in $\Lambda$. Hence by uniqueness of $\rho$, we conclude that $\beta^{-1}(g) \circ \rho \circ \alpha(g) = \rho$ for all $g$ in $G$. In other words, $\rho$ is equivariant. We have shown that $(C, \alpha)$ has the universal property of the inductive limit in $C^G$, thus proving the claim and part (i).

(ii) Let $((C_\lambda, \alpha_\lambda)_{\lambda \in \Lambda}, (\gamma_{\lambda,\mu})_{\lambda,\mu \in \Lambda, \lambda < \mu})$ be a direct system in $C^G$ and let $(C, \alpha)$ be its inductive limit in $C^G$, which exists by the first part of this lemma. We claim that $(F(C), F(\alpha))$ is the inductive limit of $((F(C_\lambda), F(\alpha_\lambda))_{\lambda \in \Lambda}, (F(\gamma_{\lambda,\mu}))_{\lambda,\mu \in \Lambda, \lambda < \mu})$ in the category $D^G$. Let $(D, \delta)$ be an object in $D^G$ and for $\lambda$ in $\Lambda$, let $\rho_\lambda: (F(C_\lambda), F(\alpha_\lambda)) \to (D, \delta)$ be an equivariant morphism satisfying $\rho_\mu = F(\gamma_{\lambda,\mu}) \circ \rho_\lambda$ for all $\mu$ in $\Lambda$ with $\lambda < \mu$. Since $F$ is continuous by assumption, we have
\[
F(C) = \lim_{\lambda \in \Lambda, \lambda < \mu} ((F(C_\lambda)), (F(\gamma_{\lambda,\mu}))_{\lambda,\mu \in \Lambda, \lambda < \mu})
\]
in $D$. By the universal property of the inductive limit $F(C)$ in $D$, there exits a unique morphism $\rho: F(C) \to D$ in the category $D$ satisfying $\rho \circ F(\gamma_{\lambda,\infty}) = \rho_\lambda$. It follows that
\[
(\delta(g)^{-1} \circ \rho \circ F(\alpha(g))) \circ F(\gamma_{\lambda,\infty}) = \delta(g)^{-1} \circ \rho_\lambda \circ F(\alpha_\lambda(g)) = \rho_\lambda,
\]
for all $g$ in $G$ and for all $\lambda$ in $\Lambda$. By uniqueness of the morphism $\rho$, we conclude that $\delta(g)^{-1} \circ \rho \circ F(\alpha(g)) = \rho$ for all $g$ in $G$. That is, $\rho: (F(C), F(\alpha)) \to (D, \delta)$ is equivariant. This shows that $(F(C), F(\alpha))$ has the universal property of inductive limits in $D^G$. \qed

Adopt the notation of the lemma above. It follows that if the category $C$ is closed under arbitrary inductive limits, then so is $B^G$ and the induced functor $F: B^G \to C^G$ preserves arbitrary inductive limits. Nevertheless, we will be concerned mainly with the category $Cu$ and the functor $Cu$ which is sequentially continuous by [9, Theorem 2], and will therefore mostly use Lemma 4.2 with $\Lambda = \mathbb{N}$.

The next theorem asserts that the functor on $RB^G$ induced by functor that classifies homomorphisms on a subcategory $B$ of $C^*$-algebras again classifies homomorphisms.

**Theorem 4.2.** Let $G$ be a finite group, let $B$ be a subcategory of $A$ that is closed under inductive limits of sequences, and let $C$ be a category where inductive limits of sequences exist. Let $B^G$, let $C^G$ and let $RB^G$ be as in Definition 4.3. Let $F: B \to C$ be a functor that classifies homomorphisms.

(i) Let $(A, \alpha)$ be an object in $B^G$ and let $(B, \beta)$ be an object in $RB^G$. Assume that $A$ and $B$ are separable as $C^*$-algebras. Then

(a) For every morphism $\gamma: (F(A), F(\alpha)) \to (F(B), F(\beta))$ in $C^G$ there exists a morphism $\phi: (A, \alpha) \to (B, \beta)$ in $B^G$ such that $F^G(\phi) = \gamma$.

(b) If $\phi, \psi: (A, \alpha) \to (B, \beta)$ are morphisms in $B^G$ such that $F^G(\phi) = F^G(\psi)$, then $\phi$ and $\psi$ are equivariantly unitarily approximately equivalent.

(ii) The restriction of the functor $F^G$ to $RB^G$ classifies homomorphisms.

**Proof.** (i) Let $(A, \alpha)$ be an object in $B^G$ and let $(B, \beta)$ be an object in $RB^G$. Assume that $A$ is separable as a $C^*$-algebra.

(a) Let $\gamma: (F(A), F(\alpha)) \to (F(B), F(\beta))$ be a morphism in $C^G$. Using that $F: B \to C$ classifies homomorphisms, choose a $*$-homomorphism $\psi: A \to B$ such that $F(\psi) = \gamma$. Note that
\[
F(\psi \circ \alpha_g) = F(\psi) \circ F(\alpha_g) = F(\beta_g) \circ F(\psi) = F(\beta_g \circ \psi),
\]
for all $g \in G$. Using again that $F$ classifies homomorphisms, we conclude that $\psi \circ \alpha_g$ and $\beta_g \circ \psi$ are approximately unitarily equivalent for all $g$ in $G$. Therefore, by part (ii) of Proposition 4.2 there
exists an equivariant \( \ast \)-homomorphism \( \phi: (A, \alpha) \rightarrow (B, \beta) \) such that \( \phi \) and \( \psi \) are approximately unitarily equivalent. Thus \( \phi \) is a morphism in \( B^G \) and
\[
F^G(\phi) = F(\phi) = F(\psi) = \gamma,
\]
as desired.

(b) Let \( \phi, \psi: (A, \alpha) \rightarrow (B, \beta) \) be morphisms in \( B^G \) such that \( F^G(\phi) = F^G(\psi) \). Then \( \phi \) and \( \psi \) are approximately unitarily equivalent because \( F \) classifies homomorphisms and \( F \) agrees with \( F^G \) on morphisms. It then follows from Proposition 4.1 that \( \phi \) and \( \psi \) are equivariantly approximately unitarily equivalent, as desired.

(ii) Since \( B \) is assumed to be closed under inductive limits of sequences, it follows from Lemma 4.2 above that inductive limits of sequences exist in the category \( C^G \) as well, and that the functor \( F^G \) is sequentially continuous. Now the first part of the theorem implies that \( F^G \) classifies homomorphisms. \( \square \)

The next theorem asserts that the functor on \( RB^G \) induced by functor that classifies isomorphisms on a subcategory \( B \) of \( C^* \)-algebras again classifies isomorphisms. Compare with Theorem 4.2 above.

**Theorem 4.3.** Let \( G \) be a finite group, let \( B \) be a subcategory of \( A \) that is closed under inductive limits of sequences, and let \( C \) be a category where inductive limits of sequences exist. Let \( B^G \), let \( C^G \) and let \( RB^G \) be as in Definition 4.3. Let \( F: B \rightarrow C \) be a functor that classifies isomorphisms.

(i) Let \( (A, \alpha) \) and \( (B, \beta) \) be objects in \( RB^G \). Then

(a) For every isomorphism \( \gamma: (F(A), F(\alpha)) \rightarrow (F(B), F(\beta)) \) in \( C^G \) there exists an isomorphism \( \phi: (A, \alpha) \rightarrow (B, \beta) \) in \( B^G \) such that \( F^G(\phi) = \gamma \).

(b) If \( \phi, \psi: (A, \alpha) \rightarrow (B, \beta) \) are isomorphisms in \( B^G \) such that \( F^G(\phi) = F^G(\psi) \), then \( \phi \) and \( \psi \) are equivariantly approximately unitarily equivalent.

(ii) The restriction of the functor \( F^G \) to \( RB^G \) classifies isomorphisms.

(iii) The actions \( \alpha \) and \( \beta \) are conjugate if and only if there is an isomorphism \( \gamma: F^G(A, \alpha) \rightarrow F^G(B, \beta) \) in \( C^G \).

(iv) The actions \( \alpha \) and \( \beta \) are (equivariantly) approximately inner conjugate if and only if \( F(\alpha_g) = F(\beta_g) \) for all \( g \in G \).

**Proof.** (i) Let \( (A, \alpha) \) and \( (B, \beta) \) be objects in \( RB^G \).

(a) Let \( \gamma: F^G(A, \alpha) \rightarrow (B, \beta) \) be an isomorphism in \( RB^G \). Since \( F: B \rightarrow C \) classifies isomorphisms, it follows that there exists an isomorphism \( \psi: A \rightarrow B \) in \( B \) such that \( F(\psi) = \gamma \). For \( g \) in \( G \), it follows that

\[
F(\psi^{-1} \circ \beta_g \circ \psi) = F(\psi^{-1}) \circ F(\beta_g) \circ F(\psi) = \gamma^{-1} \circ F(\beta_g) \circ \gamma = \gamma^{-1} \circ \gamma \circ F(\alpha_g) = F(\alpha_g).
\]

Using again that \( F \) classifies isomorphisms, we conclude that \( \alpha_g \) and \( \psi^{-1} \circ \beta_g \circ \psi \) are approximately unitarily equivalent for all \( g \) in \( G \). Using Proposition 4.3, choose an approximately inner automorphism \( \theta: A \rightarrow A \) such that

\[
\theta \circ \alpha_g = \psi^{-1} \circ \beta_g \circ \psi \circ \theta
\]

for all \( g \) in \( G \). Set \( \phi = \psi \circ \theta \). Then \( \phi: A \rightarrow B \) is equivariant, so \( \phi: (A, \alpha) \rightarrow (B, \beta) \) is a morphism in \( RB^G \). Finally,

\[
F^G(\phi) = F(\phi) = F(\psi) \circ F(\theta) = F(\psi) = \gamma,
\]
as desired.

(b) Follows immediately from (a) and the fact that the functor \( F \) classifies isomorphisms.

(ii) Since \( B \) is assumed to be closed under inductive limits of sequences, it follows from Lemma 4.2 that inductive limits of sequences exist in the category \( C^G \) as well, and that the functor \( F^G \) is
Let \( \text{Theorem 4.5}. \) functor \( (\text{Cu} \sim \text{rank one}) \) the functors \( \text{Cu} \) such that \( A \) be a semigroup \( \text{Cu} \) trivial \( K \)

Therefore (i) and (ii) follow from part (ii) of Theorem 4.2 applied to the category \( B \) prescribed strictly positive element \( \sim \) as \( \text{C}^*\)-algebras. By [33, Theorem 1], the functor consisting of \( \text{Cu} \) \( K \) algebras that can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial \( s \)

When \( A \) and \( B \) are unital, the functor \( \text{Cu} \sim \) can be replaced by the Cuntz semigroup \( \text{Cu} \), and \( s_A \) and \( s_B \) can be taken to be the units of \( A \) and \( B \), respectively.

**Proof.** Let \( B \) denote the subcategory of the category \( A \) of \( \text{C}^*\)-algebras consisting of those \( \text{C}^*\)-algebras that can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial \( K_1\)-groups. Then \( B \) is clearly closed under inductive limits, and the objects of \( B \) are separable as \( \text{C}^*\)-algebras. By [33, Theorem 1], the functor consisting of \( \text{Cu} \sim \) together with the class of a prescribed strictly positive element \( s \). in the algebra, is a functor that classifies homomorphisms. Therefore (i) and (ii) follow from part (ii) of Theorem 4.2 applied to the category \( B \) and to the functor \( (\text{Cu} \sim, [s \cdot]) \).

The last statement of the theorem follows from the fact that for unital \( \text{C}^*\)-algebras of stable rank one, the functors \( \text{Cu} \sim \) and \( \text{Cu} \) are equivalent. See [33, Subsection 3.1]. In other words, the semigroup \( \text{Cu} \sim(A) \) can be recovered functorially from \( \text{Cu}(A) \) and vice-versa.

**Theorem 4.4.** Let \( G \) be a finite group. Let \((A, \alpha)\) and \((B, \beta)\) be separable \( \text{C}^*\)-dynamical systems such that \( A \) can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial \( K_1\)-groups and such that \( B \) has stable rank one. Assume that \( \beta \) has the Rokhlin property.

(i) Fix strictly positive elements \( s_A \) and \( s_B \) of \( A \) and \( B \) respectively. Let \( \gamma : \text{Cu} \sim(A) \to \text{Cu} \sim(B) \) be a morphism in the category \( \text{Cu} \) such that

\[
\gamma([s_A]) \leq [s_B] \text{ and } \gamma \circ \text{Cu} \sim(\alpha_g) = \text{Cu} \sim(\beta_g) \circ \gamma
\]

for all \( g \in G \). Then there exists an equivariant *-homomorphism

\[
\phi : (A, \alpha) \to (B, \beta) \text{ such that } \text{Cu} \sim(\phi) = \gamma.
\]

(ii) If \( \phi, \psi : (A, \alpha) \to (B, \beta) \) are equivariant *-homomorphisms such that \( \text{Cu} \sim(\phi) = \text{Cu} \sim(\psi) \), then \( \phi \) and \( \psi \) are equivariantly unitarily approximately equivalent.

When \( A \) and \( B \) are unital, the functor \( \text{Cu} \sim \) can be replaced by the Cuntz semigroup \( \text{Cu} \), and \( s_A \) and \( s_B \) can be taken to be the units of \( A \) and \( B \), respectively.

**Proof.** Let \( B \) denote the subcategory of the category \( A \) of \( \text{C}^*\)-algebras consisting of those \( \text{C}^*\)-algebras that can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial \( K_1\)-groups. Then \( B \) is clearly closed under inductive limits, and the objects of \( B \) are separable as \( \text{C}^*\)-algebras. By [33, Theorem 1], the functor consisting of \( \text{Cu} \sim \) together with the class of a prescribed strictly positive element \( s \). in the algebra, is a functor that classifies homomorphisms. Therefore (i) and (ii) follow from part (ii) of Theorem 4.2 applied to the category \( B \) and to the functor \( (\text{Cu} \sim, [s \cdot]) \).

The last statement of the theorem follows from the fact that for unital \( \text{C}^*\)-algebras of stable rank one, the functors \( \text{Cu} \sim \) and \( \text{Cu} \) are equivalent. See [33, Subsection 3.1]. In other words, the semigroup \( \text{Cu} \sim(A) \) can be recovered functorially from \( \text{Cu}(A) \) and vice-versa.

**Theorem 4.5.** Let \( G \) be a finite group, and let \((A, \alpha)\) and \((B, \beta)\) be separable dynamical systems such that \( A \) and \( B \) can be written as inductive limits of 1-dimensional NCCW-complexes with trivial \( K_1\)-groups. Suppose that \( \alpha \) and \( \beta \) have the Rokhlin property.

(i) Fix strictly positive elements \( s_A \) and \( s_B \) of \( A \) and \( B \) respectively. Then the actions \( \alpha \) and \( \beta \) are conjugate if and only if there exists an isomorphism \( \gamma : \text{Cu} \sim(A) \to \text{Cu} \sim(B) \) with \( \gamma([s_A]) = [s_B] \), such that \( \gamma \circ \text{Cu} \sim(\alpha_g) = \text{Cu} \sim(\beta_g) \circ \gamma \) for all \( g \in G \).

(ii) Assume that \( A = B \). Then the actions \( \alpha \) and \( \beta \) are strongly conjugate if and only if \( \text{Cu} \sim(\alpha_g) = \text{Cu} \sim(\beta_g) \) for all \( g \in G \).

When \( A \) and \( B \) are unital, the functor \( \text{Cu} \sim \) can be replaced by \( \text{Cu} \), and \( s_A \) and \( s_B \) can be taken to be the units of \( A \) and \( B \), respectively.

**Proof.** Let \( B \) denote the subcategory of the category \( A \) of \( \text{C}^*\)-algebras consisting of those \( \text{C}^*\)-algebras that can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial \( K_1\)-groups. By part (ii) of Theorem 4.3, the functor on \( \text{RB} G \) induced by \( \text{Cu} \sim \) together with the class of a prescribed strictly positive element \( s \). in the algebra, is a functor that classifies homomorphisms. The result follows.
Let $G$ be a finite group. Recall that the action $\mu^G: G \to \text{Aut}(M_{|G|^\infty})$ constructed in Example 2.1 has the Rokhlin property and $\mu^G_g$ is approximately inner for all $g$ in $G$.

**Corollary 4.1.** Let $G$ be a finite group and let $(A, \alpha)$ and $(A, \beta)$ be separable C*-dynamical systems such that $A$ can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial $K_1$-groups. Suppose that $\text{Cu}^\sim(\alpha_g) = \text{Cu}^\sim(\beta_g)$ for all $g \in G$. Then $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ are conjugate.

**Proof.** The actions $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ have the Rokhlin property by part (i) of Lemma 2.3. Note that $\mu^G_g$ is approximately inner for all $g$ in $G$. Thus,

$$\text{Cu}^\sim(\alpha \otimes \mu^G_g) = \text{Cu}^\sim(\alpha \otimes \text{id}_{M_{|G|^\infty}}) = \text{Cu}^\sim(\beta \otimes \text{id}_{M_{|G|^\infty}}) = \text{Cu}^\sim(\beta \otimes \mu^G_g)$$

for all $g$ in $G$. It follows from part (i) in Theorem 4.5 that $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ are conjugate. □

5. **Locally representable actions and the equivariant Cuntz semigroup**

We now proceed to classify a different class of actions of finite abelian groups on certain stably finite algebras using the equivariant Cuntz semigroup. A related and more restrictive class of actions has been studied by Handelman and Rossmann in [17], while Izumi in [20] investigated a somewhat more general class, on a different class of C*-algebras (for example, his methods only allow for simple C*-algebras). We proceed to define the class of actions we will be interested in.

**Definition 5.1.** Let $A$ be a unital C*-algebra that can be written as a direct limit $A = \lim\limits_{n \to \infty} (A_n, \tau_n)$ where for each $n$ in $\mathbb{N}$, the C*-algebra $A_n$ is a unital one-dimensional NCCW-complex with $K_1(A_n) = 0$ and $\tau_n: A_n \to A_{n+1}$ is a unital map. Let $G$ be a finite group and let $\alpha: G \to \text{Aut}(A)$ be an action. We say that $\alpha$ is **locally representable** if there exist group homomorphisms $u^{(n)}: G \to U(A_n)$ for $n$ in $\mathbb{N}$ such that $\alpha_g = \lim\limits_{n \to \infty} \text{Ad} \left( u^{(n)}_g \right)$ for all $g$ in $G$. In other words, $\alpha$ is a direct limit of inner actions on the building blocks of $A$.

We briefly compare our notion of local representability with other similar ones previously studied. In [17], the authors considered actions of compact groups on AF-algebras that can be written as direct limits of inner actions on finite-dimensional C*-algebras. The actions they studied are clearly locally representable in our sense, although our definition is likely to be more general even for actions on AF-algebras. See Remark 5.1 below for comments on a related example.

On the other hand, the class of actions Izumi studied in [20, Subsection 3.2] is the class of pointwise outer approximately representable actions, which we define below. (Notice that this notion has so far only been defined for discrete abelian groups, while the others make sense for arbitrary locally compact groups.) Izumi only considered simple C*-algebras, and hence needed to restrict himself to pointwise outer actions.

**Definition 5.2.** Let $G$ be a discrete abelian group, let $A$ be a unital C*-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action. We say that $\alpha$ is **approximately representable** if there exists a group homomorphism $u: G \to U((A_\infty)^{\alpha_\infty})$ such that

$$(\alpha_\infty)_g(a) = u_g a u_g^*$$

for all $g \in G$ and all $a \in A$.

It follows in particular that approximately representable actions are pointwise approximately inner.

**Remark 5.1.** It is straightforward to show that locally representable actions as in Definition 5.1 are approximately representable as in Definition 5.2. The converse is known to be false. In [3], Blackadar constructed an example of a $\mathbb{Z}_2$ action on the CAR algebra $M_{2\infty}$ whose crossed product is
not an AF-algebra. Blackadar’s example is approximately representable essentially by construction, but is not locally representable. In fact, it is easy to check that the crossed product of an AF-algebra by a locally representable action must have trivial $K_1$.

It was shown in [20, Lemma 5.1] that for finite abelian group actions, approximate representability is the notion dual to the Rokhlin property. This is, if $\alpha$ is an action of an abelian group $G$ on a unital $C^*$-algebra $A$, then $\alpha$ is approximately representable (has the Rokhlin property) if and only if its dual action $\hat{\alpha} : \hat{G} \to \text{Aut}(A \rtimes_\alpha G)$ has the Rokhlin property (is approximately representable).

Handelman and Rossmann classified locally representable actions on AF-algebras using equivariant K-theory (although they did not use this terminology). See [17, Theorem III.1]. More recently, Izumi classified approximately representable actions of finite abelian groups in terms of the crossed products. Notice however that the invariant he considered is not functorial and is in general too difficult to compute.

Our classification results for locally representable actions extend those of Izumi, at least on a certain class of $C^*$-algebras, with the advantage that we use the equivariant Cuntz semigroup as the invariant, which is both a functor and generally easier to compute than the crossed product itself.

With the aid of pre-existing classification results for $C^*$-algebras (as opposed to actions), we are able to obtain strong classification of the actions, meaning that homomorphisms of the invariant lift to equivariant homomorphisms of the objects. We are also able to characterize when two such actions are approximately inner (cocycle) conjugate.

Let $\alpha : G \to \text{Aut}(A)$ be an action of a compact group $G$ on a $C^*$-algebra $A$. Denote by $u : G \to U(M(A \rtimes_\alpha G))$ the canonical unitary representation of $G$ associated to the crossed product. We denote by $e_\alpha \in A \rtimes_\alpha G$ the projection $e_\alpha = \int_G u_g \, dg$.

Theorem 5.1. Let $G$ be a finite abelian group, let $A$ and $B$ be unital $C^*$-algebras that can be written as inductive limits of one-dimensional NCCW-complexes with trivial $K_1$, and let $\alpha$ and $\beta$ be locally representable actions of $G$ on $A$ and $B$ respectively. Denote by $e \in \mathcal{K}(\ell^2(G))$ the projection onto the constant functions on $G$.

For every $\text{Cu}(G)$-semimodule morphism $\rho : \text{Cu}_G(A, \alpha) \to \text{Cu}_G(B, \beta)$ sending the class of the unit of $A$ in $\text{Cu}_G(A, \alpha)$ to the class of the unit of $B$ in $\text{Cu}_G(B, \beta)$, there are a $\beta$-cocycle $\omega : G \to U(B)$ and a unital equivariant homomorphism $\phi : (A, \alpha) \to (B, \beta^\omega)$ such that the composition $\text{Cu}_G(A, \alpha) \xrightarrow{\phi} \text{Cu}_G(B, \beta^\omega) \xrightarrow{\cong} \text{Cu}_G(B, \beta)$ is $\rho$, where the isomorphism $\text{Cu}_G(B, \beta^\omega) \cong \text{Cu}_G(B, \beta)$ is the one associated with the exterior equivalence of $\beta$ and $\beta^\omega$ as in Proposition 2.2.

Moreover:

(i) The homomorphism $\phi$ is unique up to equivariant approximate unitary equivalence.
(ii) The cocycle $\omega$ is trivial if and only if $\rho$ maps the class of the projection $[e_\alpha]$ in $\text{Cu}_G(A, \alpha)$ to the class of the projection $[e_\beta]$ in $\text{Cu}_G(B, \beta)$.

Proof. By considering the direct limit decompositions of the crossed products $A \rtimes_\alpha G$ and $B \rtimes_\beta G$, we conclude that these are again inductive limit of one-dimensional NCCW complexes with trivial $K_1$. Moreover, the dual actions $\hat{\alpha}$ and $\hat{\beta}$ have the Rokhlin property by [20, Lemma 3.8]. Via Theorem 2.1, the $\text{Cu}(G)$-semimodule morphism $\rho : \text{Cu}_G(A, \alpha) \to \text{Cu}_G(B, \beta)$ can be regarded as an equivariant morphism between the $\hat{G}$-dynamical systems $\hat{\rho} : (\text{Cu}(A \rtimes_\alpha G), \text{Cu}(\hat{\alpha})) \to (\text{Cu}(B \rtimes_\beta G), \text{Cu}(\hat{\beta}))$. By
Theorem 4.5, the morphism $\rho$ lifts to an equivariant unital $\ast$-homomorphism $\varphi: (A \rtimes_\alpha G, \hat{G}) \to (B \rtimes_\beta G, \hat{G})$ that satisfies $\text{Cu}(\varphi) = \hat{\rho}$. Applying the crossed product functor, we obtain an equivariant unital $\ast$-homomorphism $\psi: (A \rtimes_\alpha G) \rtimes_\hat{\alpha} \hat{G} \to (B \rtimes_\beta G) \rtimes_\hat{\beta} \hat{G}$. By Takai duality, this translates into an equivariant unital $\ast$-homomorphism $\hat{\psi}: A \otimes K(\ell^2(G)) \to B \otimes K(\ell^2(G))$. We claim that the homomorphism $\psi$ satisfies $\text{Cu}(\psi)([1_A \otimes e]) = [1_B \otimes e]$ in $\text{Cu}(B \otimes K(\ell^2(G)))$.

To see this, note first that the inclusions
\[
A \hookrightarrow A \rtimes_\alpha G \quad \text{and} \quad A \rtimes_\alpha G \hookrightarrow (A \rtimes_\alpha G) \rtimes_\hat{\alpha} \hat{G}
\]
are unital (thus so is their composition). Moreover, $\text{Cu}(\varphi)([1_A \rtimes_\alpha G]) = \hat{\rho}([1_A \rtimes_\alpha G]) = [1_B \rtimes_\beta G]$ since $\rho$ itself maps the class of the unit of $A$ to the class of the unit of $B$. Analogously,
\[
\text{Cu}(\psi)([(A \rtimes_\alpha G) \rtimes_\hat{\alpha} \hat{G}]) = 1_{(B \rtimes_\beta G) \rtimes_\hat{\beta} \hat{G}}.
\]
Identify $A \rtimes_\alpha G) \rtimes_\hat{\alpha} \hat{G}$ and $A \otimes K(\ell^2(G))$ via the isomorphism given by Takai duality. Then the embedding of $A$ into the double crossed product is simply the unital inclusion of $A$ as the first tensor factor. Set $n = |G|$. Then $[1_A \otimes 1_{K(\ell^2(G))}] = n[1_A \otimes e]$ in $\text{Cu}(A \otimes K(\ell^2(G)))$, and similarly $[1_B \otimes 1_{K(\ell^2(G))}] = n[1_B \otimes e]$ in $\text{Cu}(B \otimes K(\ell^2(G)))$. Hence,
\[
n\text{Cu}(\psi)([1_A \otimes e]) = n[1_B \otimes e]
\]
in $\text{Cu}(A \otimes K(\ell^2(G)))$. Since multiplication by $n$ on $\text{Cu}(A \otimes K(\ell^2(G)))$ is an order-embedding, we conclude that $\text{Cu}(\psi)([1_A \otimes e]) = [1_B \otimes e]$ as desired.

Since $B$ has stable rank one, so does $B \otimes K(\ell^2(G))$, and this tensor product is therefore stably finite. In particular, $1_B \otimes e$ and $\psi(1_A \otimes e)$ are Murray-von Neumann equivalent in $B \otimes K(\ell^2(G))$. Moreover, since $B \otimes K(\ell^2(G))$ has cancellation of projections, it follows that $1_B \otimes e$ and $\psi(1_A \otimes e)$ actually are unitarily equivalent in $B \otimes K(\ell^2(G))$. It follows from Corollary 3.1 that there are a $\beta$-cocycle $\omega: G \to U(B)$ and an equivariant homomorphism $\phi: (A, \alpha) \to (B, \beta^\omega)$.

We claim that $\text{Cu}_G(\phi)$ becomes $\rho$ under the identification $\text{Cu}_G(B, \beta^\omega) \cong \text{Cu}_G(B, \beta)$ associated with the exterior equivalence of $\beta$ and $\beta^\omega$.

We introduce some notation first. Let $D$ be a unital C*-algebra and let $\delta: G \to \text{Aut}(D)$ be an action of $G$ on $D$.

- Denote by $\varphi_\delta: \text{Cu}_G(D, \delta) \to \text{Cu}(D \rtimes_\delta G)$ the natural isomorphism given by Julg’s Theorem.
- Denote by $T_\delta: (D \rtimes_\delta G)_\delta \hat{G} \to D \otimes K(\ell^2(G))$ the natural isomorphism given by Takai duality.
- Denote by $E_\delta: \text{Cu}_G(D \otimes K(\ell^2(G)), \delta \otimes \text{Ad}(\lambda)) \to \text{Cu}_G(D \otimes K(\ell^2(G)), \delta \otimes \text{id}_K)$ be the isomorphism given by exterior equivalence of $\delta \otimes \text{Ad}(\lambda)$ and $\delta \otimes \text{id}_K$.
- Denote by $S_\delta: \text{Cu}_G(D \otimes K(\ell^2(G)), \delta \otimes \text{Ad}(\lambda)) \to \text{Cu}_G((D, \delta)$ the natural isomorphism associated with stability of the equivariant Cuntz semigroup functor.
Consider the following diagram

\[
\begin{array}{cccc}
\text{Cu}_G(A, \alpha) & \xrightarrow{\text{Cu}_G(\phi)} & \text{Cu}_G(B, \beta) & \xrightarrow{\cong} & \text{Cu}_G(B, \beta) \\
S^{-1}_\alpha & & S^{-1}_\beta & & \\
\text{Cu}_G((A \otimes \mathcal{K}(\ell^2(G))), \alpha \otimes \text{Ad}(\lambda)) & \xrightarrow{\text{Cu}_G(\psi)} & \text{Cu}_G((B \otimes \mathcal{K}(\ell^2(G))), \beta \otimes \text{Ad}(\lambda)) & & \\
E_\alpha & & E_\beta & & \\
\text{Cu}_G((A \otimes \mathcal{K}(\ell^2(G))), \alpha \otimes \text{id}_\mathcal{K}) & & \text{Cu}_G((B \otimes \mathcal{K}(\ell^2(G))), \beta \otimes \text{id}_\mathcal{K}) & & \\
\varphi_\alpha \otimes \text{id}_\mathcal{K} & & \varphi_\beta \otimes \text{id}_\mathcal{K} & & \\
\text{Cu}((A \rtimes_\alpha G) \otimes \mathcal{K}(\ell^2(G))) & \xrightarrow{\text{Cu}(\varphi)\sim \tilde{\rho}} & \text{Cu}(B \rtimes_\beta G) & & \\
\text{Cu}(e) & & \text{Cu}(\lambda) & & \\
\varphi^{-1} & & \varphi^{-1} & & \\
\text{Cu}_G(A, \alpha) & \xrightarrow{\rho} & \text{Cu}_G(B, \beta). & & \\
\end{array}
\]

It is readily checked that the diagram is commutative, using naturality of all the isomorphisms involved. One also shows that the composition of the vertical arrows on the left-hand side equals the identity map on \(\text{Cu}_G(A, \alpha)\). Similarly, the composition of the vertical arrows on the right-hand side equals the identity on \(\text{Cu}_G(B, \beta)\). This shows that \(\text{Cu}_G(\phi)\) becomes \(\rho\) under the appropriate (canonical) identifications.

(i) If \(\phi\) and \(\phi'\) are lifts of \(\rho\), it follows that \(\text{Cu}(\phi) = \text{Cu}(\phi') = \rho\) and hence \(\phi\) and \(\phi'\) are unitarily equivalent by classification.

(ii) Assume that \(\omega_g = 1\) for all \(g \in G\). Denote by \(u_g\) for \(g \in G\) the canonical unitaries in \(A \rtimes_\alpha G\) and by \(v_g\) for \(g \in G\) the canonical unitaries in \(B \rtimes_\beta G\). Then \(\phi\) induces a homomorphism \(\tilde{\phi}: A \rtimes_\alpha G \to B \rtimes_\beta G\) given by \(\tilde{\phi}(a) = \phi(a)\) for \(a \in A\) and \(\tilde{\phi}(u_g) = v_g\) for \(g \in G\). It follows that \(\tilde{\phi}(e_\alpha) = e_\beta\), and thus \(\rho([e_\alpha]) = [e_\beta]\). Conversely, assume that \(\rho([e_\alpha]) = [e_\beta]\). Under the usual identifications

\[B \rtimes_\beta G \hookrightarrow (B \rtimes_\beta G) \rtimes_{\beta} \tilde{G} \cong B \otimes \mathcal{K}(\ell^2(G)),\]

the subalgebra \(B \rtimes_\beta G\) is identified with the fixed point algebra \((B \otimes \mathcal{K}(\ell^2(G)))^{\beta \otimes \text{Ad}(\lambda)}\), and the projection \(e_\beta\) is sent to \(1_B \otimes e\). It follows that if \(\rho([e_\alpha]) = [e_\beta]\), then \(\theta(1_A \otimes e)\) and \(1_B \otimes e\) are unitarily equivalent via a unitary \(v\) in \((B \otimes \mathcal{K}(\ell^2(G)))^{\beta \otimes \text{Ad}(\lambda)}\). (Note that \(v\) is fixed by \(\beta \otimes \text{Ad}(\lambda)\).) Hence the cocycle \(\omega: G \to U(B)\) in Theorem 3.1, which was defined as

\[\omega_g \otimes e = (1 \otimes e)v^* ((\beta \otimes \text{Ad}(\lambda))\gamma(v(1 \otimes e)))\]

for \(g\) in \(G\), is just the trivial cocycle, and the claim follows.

(iii) If \(\phi\) is approximately inner, it is clear that \(\rho\) becomes the identity under the identifications mentioned in the statement. Conversely, the induced map \(\tilde{\rho}: \text{Cu}(A \otimes \mathcal{K}(\ell^2(G))) \to \text{Cu}(A \otimes \mathcal{K}(\ell^2(G)))\) is the identity, then the homomorphism \(\theta: A \otimes \mathcal{K}(\ell^2(G)) \to A \otimes \mathcal{K}(\ell^2(G))\) is approximately inner by classification. It follows from part (iii) of Theorem 3.1 that \(\phi\) is approximately inner as well. \(\square\)

In particular, we are able to determine when two actions as in Theorem 5.1 are (approximately inner) conjugate or (approximately inner) cocycle conjugate.
Corollary 5.1. Let $G$ be a finite abelian group, let $A$ and $B$ be $C^*$-algebras that can be written as inductive limits of one-dimensional NCCW-complexes with trivial $K_1$, and let $\alpha$ and $\beta$ be locally representable actions of $G$ on $A$ and $B$ respectively, obtained as direct limit actions on such direct limit decompositions. Denote by $e \in K(\ell^2(G))$ the projection onto the constant functions on $G$.

(i) The actions $\alpha$ and $\beta$ are cocycle conjugate if and only if there is an $\text{Cu}(G)$-semimodule isomorphism $\rho: \text{Cu}_G(A, \alpha) \rightarrow \text{Cu}_G(B, \beta)$ sending the unit of $A$ in $\text{Cu}_G(A, \alpha)$ to the class of the unit of $B$ in $\text{Cu}_G(B, \beta)$.

(ii) Assume that $A = B$. Then $\alpha$ and $\beta$ are approximately inner cocycle conjugate if and only if the induced $\text{Cu}(\hat{G})$-semimodule morphism
\[
\hat{\rho}: \text{Cu}_{\hat{G}, \hat{\alpha}}(A \rtimes_{\alpha} G) \rightarrow \text{Cu}_{\hat{G}, \hat{\beta}}(A \rtimes_{\beta} G)
\]
becomes the identity under the following natural identifications:

(a) The natural isomorphisms $\varphi_{\hat{\alpha}}: \text{Cu}_{\hat{G}}(A \rtimes_{\alpha} G, \hat{\alpha}) \rightarrow \text{Cu}((A \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \hat{G})$ and $\varphi_{\hat{\beta}}: \text{Cu}_{\hat{G}}(A \rtimes_{\beta} G, \hat{\beta}) \rightarrow \text{Cu}((A \rtimes_{\beta} G) \rtimes_{\hat{\beta}} \hat{G})$ given by Julg’s Theorem.

(b) The Takai duality isomorphism $T_{\alpha}: (A \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \hat{G} \rightarrow A \otimes K(\ell^2(G))$, under which $\hat{\alpha}$ is sent to $\alpha \otimes \text{Ad}(\lambda)$.

(c) The Takai duality isomorphism $T_{\beta}: (A \rtimes_{\beta} G) \rtimes_{\hat{\beta}} \hat{G} \rightarrow A \otimes K(\ell^2(G))$, under which $\hat{\beta}$ is sent to $\beta \otimes \text{Ad}(\lambda)$.

(d) The natural $\text{Cu}$-isomorphism $\text{Cu}(A) \rightarrow \text{Cu}(A \otimes K(\ell^2(G)))$ induced by the inclusion $\iota_e: A \rightarrow A \otimes K(\ell^2(G))$ given by $\iota_e(a) = a \otimes e$ for $a \in A$.

\[
\begin{array}{ccc}
\text{Cu}_{\hat{G}}(A \rtimes_{\alpha} G, \hat{\alpha}) & \overset{\rho}{\longrightarrow} & \text{Cu}_{\hat{G}}(A \rtimes_{\beta} G, \hat{\beta}) \\
\downarrow \varphi_{\hat{\alpha}} & & \downarrow \varphi_{\hat{\beta}} \\
\text{Cu}((A \rtimes_{\alpha} G) \rtimes_{\hat{\alpha}} \hat{G}) & & \text{Cu}((A \rtimes_{\beta} G) \rtimes_{\hat{\beta}} \hat{G}) \\
\downarrow \text{Cu}(T_{\alpha}) & & \downarrow \text{Cu}(T_{\beta}) \\
\text{Cu}(A \otimes K) & & \text{Cu}(A \otimes K) \\
\downarrow \text{Cu}(\iota_e)^{-1} & & \downarrow \text{Cu}(\iota_e)^{-1} \\
\text{Cu}(A) & \overset{\text{id}_{\text{Cu}(A)}}{\longrightarrow} & \text{Cu}(A).
\end{array}
\]

(iii) The actions $\alpha$ and $\beta$ are conjugate if and only if there is an $\text{Cu}(G)$-semimodule isomorphism $\rho: \text{Cu}_G(A, \alpha) \rightarrow \text{Cu}_G(B, \beta)$ sending the unit of $A$ in $\text{Cu}_G(A, \alpha)$ to the class of the unit of $B$ in $\text{Cu}_G(B, \beta)$, and moreover sending the class of $[e_{\alpha}]$ in $\text{Cu}_G(A, \alpha)$ to the class of $[\rho(e_{\alpha})]$ in $\text{Cu}_G(B, \beta)$.

(iv) The actions $\alpha$ and $\beta$ are approximately inner conjugate if and only if there is an $\text{Cu}(G)$-semimodule isomorphism $\rho: \text{Cu}_G(A, \alpha) \rightarrow \text{Cu}_G(B, \beta)$ sending the unit of $A$ in $\text{Cu}_G(A, \alpha)$ to the class of the unit of $B$ in $\text{Cu}_G(B, \beta)$ and sending the class of $[e_{\alpha}]$ in $\text{Cu}_G(A, \alpha)$ to the class of $[\rho(e_{\alpha})]$ in $\text{Cu}_G(B, \beta)$, such that the induced $\text{Cu}(\hat{G})$-semimodule morphism
\[
\hat{\rho}: \text{Cu}_{\hat{G}, \hat{\alpha}}(A \rtimes_{\alpha} G, \hat{\alpha}) \rightarrow \text{Cu}_{\hat{G}, \hat{\beta}}(A \rtimes_{\beta} G, \hat{\beta})
\]
becomes the identity under the natural identifications listed in (ii).

Proof. Adopt the notation of Theorem 5.1 and its proof. We claim that $\phi$ is approximately inner if and only if the $\text{Cu}(\hat{G})$-morphism $\hat{\rho}$ becomes the identity on $\text{Cu}(A)$ under the identifications listed in condition (ii) of the statement of this corollary.
If $\phi$ is approximately inner, it is clear that $\hat{\rho}$ becomes the identity on $Cu(A)$. Conversely, if the induced map $\hat{\rho} : Cu(A \otimes K(\ell^2(G))) \to Cu(A \otimes K(\ell^2(G)))$ becomes the identity on $Cu(A)$, then the homomorphism $\theta: A \otimes K(\ell^2(G)) \to A \otimes K(\ell^2(G))$ is approximately inner by classification. It follows from part (3) of Theorem 3.1 that $\phi$ is approximately inner as well.

(i) Follows from the main (un-numbered) statement in Theorem 5.1 by considering $\rho$ and its inverse $\rho^{-1}$.

(ii) Follows from part (i) above and the claim at the beginning of the proof.

(iii) Follows combining part (i) of this theorem and part (i) in Theorem 5.1.

(iv) Follows combining part (ii) of this theorem and part (i) in Theorem 5.1.  

\[\square\]


In this section we study absorption of UHF-algebras in relation to the Rokhlin property. We show that for a certain class of C*-algebras, absorption of a UHF-algebra of infinite type is equivalent to existence of an action with the Rokhlin property that is pointwise approximately inner. (The cardinality of the group is related to the type of the UHF-algebra.) Moreover, in this case, not only the C*-algebra absorbs the corresponding UHF-algebra, but also the action in question absorbs the model action constructed in Example 2.1. Thus, Rokhlin actions allow us to prove that certain algebras are (equivariantly) UHF-absorbing.

As a crucial step in proving UHF-absorption, we characterize those C*-algebras whose Cuntz semigroup is isomorphic to the Cuntz semigroup of its UHF-stabilization; see Theorem 6.1.

6.1. Unique $n$-divisibility. The goal of this section is to show that for C*-algebras that can be expressed as direct limits of one-dimensional NCCW-complexes with trivial $K_1$ group, absorption of the UHF-algebra of type $n^\infty$ is equivalent to its Cuntz semigroup being $n$-divisible. Along the way, we show that for a C*-algebra $A$, the Cuntz semigroups of $A$ and of $A \otimes M_n$ are isomorphic if and only if $Cu(A)$ is uniquely $n$-divisible. (The “uniquely” part is automatic if $A$ can be expressed as direct limits of one-dimensional NCCW-complexes with trivial $K_1$ group.) We believe the results in this section are of independent interest, and may be useful even if one is not concerned with classification of actions.

We begin defining the main notion of this section.

**Definition 6.1.** Let $n \in \mathbb{N}$. An ordered semigroup $S$ is said to be uniquely $n$-divisible if multiplication by $n$ on $S$ is an order-preserving semigroup isomorphism.

If $S$ is an ordered semigroup and $\varphi : S \to S$ is an endomorphism, we say that $\varphi$ is an order embedding if $\varphi(s) \leq \varphi(t)$ implies $s \leq t$ for all $s$ and $t$ in $S$.

**Remark 6.1.** Given $n$ in $\mathbb{N}$, it is easy to check that an ordered semigroup is uniquely $n$-divisible if and only if

(i) (Order embedding) If $s$ and $t$ in $S$ are such that $ns \leq nt$, then $s \leq t$;

(ii) (n-divisibility) For every $s \in S$ there is a unique $t \in S$ such that $nt = s$.

**Theorem 6.1.** Let $A$ be a C*-algebra and let $n$ in $\mathbb{N}$ be non-zero. Then $Cu(A) \cong Cu(A \otimes M_n)$ if and only if $Cu(A)$ is uniquely $n$-divisible.

**Proof.** Using the inductive limit decomposition $M_n = \varinjlim M_{n^k}$ with connecting maps given by $a \mapsto a \otimes 1_n$ for $a$ in $M_{n^k}$, we can write $A \otimes M_n$ as the inductive limit

\[
A \xrightarrow{\phi_1} M_n(A) \xrightarrow{\phi_2} M_{n^2}(A) \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_k} A \otimes M_{n^k},
\]

where for all $k$ in $\mathbb{N}$, the map $\phi_k : M_{n^k}(A) \to M_{n^k}(A)$ is given by $\phi_k(a) = a \otimes 1_n$ for all $a \in M_{n^k}(A)$. By continuity of the functor $Cu$ (see [9, Theorem 2]), the semigroup $Cu(A \otimes M_n)$ is
isomorphic to the inductive limit in the category $\text{Cu}$ of the following sequence:

\begin{equation}
\begin{align*}
\text{Cu}(A) & \xrightarrow{\phi_1} \text{Cu}(M_n(A)) \xrightarrow{\phi_2} \text{Cu}(M_{n^2}(A)) \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_{n-1}} \text{Cu}(A \otimes M_{n^\infty}).
\end{align*}
\end{equation}

By [9, Appendix 6], the inclusion $i_k : A \rightarrow M_{n^k}(A)$ from $A$ onto the upper left corner of $M_{n^k}(A)$ induces an isomorphism between the Cuntz semigroup of $A$ and the Cuntz semigroup of $M_{n^k}(A)$. For $k$ in $\mathbb{N}$, let $\varphi_k : \text{Cu}(A) \rightarrow \text{Cu}(A)$ be given by

$$\varphi_k = \text{Cu}(i_{k+1})^{-1} \circ \text{Cu}(\phi_{k+1}) \circ \text{Cu}(i_k).$$

The sequence (6.1) implies that $\text{Cu}(A \otimes M_{n^\infty})$ is the inductive limit of

\begin{equation}
\begin{align*}
\text{Cu}(A) & \xrightarrow{\varphi_1} \text{Cu}(A) \xrightarrow{\varphi_2} \text{Cu}(A) \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-1}} \text{Cu}(A \otimes M_{n^\infty}),
\end{align*}
\end{equation}

Fix $k$ in $\mathbb{N}$. We claim that $\varphi_k(x) = nx$, for all $x \in \text{Cu}(A)$. For $j = 1, \ldots, n$, denote by $e_{j,j} \in M_n(\mathbb{C})$ the matrix with a 1 in the $(j,j)$-entry and zeros everywhere else. Let $a$ be a positive element in $M_{n^k}(A \otimes \mathcal{K})$ and set $b = a \otimes e_{1,1} \in M_{n^{k+1}}(A \otimes \mathcal{K})$. Then

$$\varphi_k([a]) = (\text{Cu}(i_{k+1})^{-1} \circ \text{Cu}(\phi_{k+1}))([b]) = \text{Cu}(i_{k+1})^{-1}([b \otimes 1_n]) = \sum_{j=1}^n \text{Cu}(i_{k+1})^{-1}([b \otimes e_{j,j}]).$$

Since $e_{j,j}$ is Cuntz equivalent to $e_{1,1}$ for all $j = 1, \ldots, n$, it follows that $b \otimes e_{j,j}$ is Cuntz equivalent to $b \otimes e_{1,1}$ for all $j = 1, \ldots, n$. Hence,

$$\varphi_k([a]) = \sum_{j=1}^n \text{Cu}(i_{k+1})^{-1}([b \otimes e_{j,j}]) = n \text{Cu}(i_{k+1})^{-1}([b \otimes e_{1,1}]) = n[a].$$

In other words, each of the maps $\varphi_k$ is multiplication by $n$.

We proceed to show that if $A$ is such that $\text{Cu}(A) \cong \text{Cu}(A \otimes M_{n^\infty})$, then $\text{Cu}(A)$ is uniquely $n$-divisible.

We first show that multiplication by $n$ on $\text{Cu}(A)$ is an order-embedding. So let $x, y \in \text{Cu}(A \otimes M_{n^\infty})$ be such that $nx \leq ny$. We claim that $x \leq y$. By the inductive limit decomposition given in the sequence (6.2) and by part (i) of Proposition 2.1, it follows that there are sequences $(x_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ in $\text{Cu}(A)$ such that

$$\varphi_k(x_k) \ll x_{k+1} \text{ for all } k \in \mathbb{N} \quad \text{and} \quad x = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(x_k)$$

$$\varphi_k(y_k) \ll y_{k+1} \text{ for all } k \in \mathbb{N} \quad \text{and} \quad y = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(y_k).$$

Since for all $k \in \mathbb{N}$ one has

$$\varphi_{k+1,\infty}(nx_{k+1}) = n \varphi_{k+1,\infty}(x_{k+1}) \ll nx \leq ny = \sup_{k \in \mathbb{N}} \varphi_{l,\infty}(ny_l),$$

it follows from the definition of compact containment that for each $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $\varphi_{k+1,\infty}(nx_{k+1}) \leq \varphi_{l,\infty}(ny_l)$. Using this and the fact that $\varphi_k(nx_k) \ll nx_{k+1}$, it follows that there exists $m \geq k, l$ such that $\varphi_{k,m}(nx_k) \leq \varphi_{l,m}(ny_l)$. In particular,

$$n^{m-k+1} x_k \leq n^{m-l+1} y_l.$$

We therefore obtain

$$\varphi_{k,m+1}(x_k) = n^{m-k+1} x_k \leq n^{m-l+1} y_l \leq \varphi_{l,m+1}(y_l),$$

and thus $\varphi_{k,\infty}(x_k) \leq \varphi_{l,\infty}(y_l) \leq y$. Since $k$ is arbitrary and $x = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(x_k)$ we conclude that $x \leq y$, showing that multiplication by $n$ on $\text{Cu}(A)$ is an order-embedding, as desired.
We now show that $\text{Cu}(A)$ is $n$-divisible. Let $x \in \text{Cu}(A \otimes M_n炎)$ and let $x_k \in \text{Cu}(A)$ for $k \in \mathbb{N}$ be as above. For each $k \in \mathbb{N}$ we have

$$\varphi_{k,k+2}(x_k) = n^2 x_k = n \varphi_{k+1,k+2}(x_k).$$

It follows that $\varphi_{k,\infty}(x_k) = ny_k$ where $y_k = \varphi_{k+1,\infty}(x_k)$. Since $(\varphi_{k,\infty}(x_k))_{k \in \mathbb{N}}$ is an increasing sequence, $(ny_k)_{k \in \mathbb{N}}$ is an increasing sequence as well, and hence so is $(y_k)_{k \in \mathbb{N}}$. Let $y$ denote the supremum of $(y_k)_{k \in \mathbb{N}}$. Then

$$x = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(x_k) = \sup_{k \in \mathbb{N}} ny_k = n \sup_{k \in \mathbb{N}} y_k = ny,$$

which shows that $\text{Cu}(A)$ is $n$-divisible. This finishes the proof of the “only if” direction.

Conversely, suppose that $\text{Cu}(A)$ is $n$-divisible then the endomorphism of $\text{Cu}(A)$ given by multiplication by $n$ is an order-preserving isomorphism. It then follows from the sequence (6.2) it follows that $\text{Cu}(A) \cong \text{Cu}(A \otimes M_n炎)$.

\textbf{Remark 6.2.} Let $\mathcal{Q}$ denote the universal UHF-algebra. Then using the same ideas as in the proof of the previous proposition one can show that $\text{Cu}(A) \cong \text{Cu}(A \otimes \mathcal{Q})$ if and only if $\text{Cu}(A)$ is uniquely $n$-divisible for every $n \in \mathbb{N}$. (Equivalently, $\text{Cu}(A)$ is $p$-divisible for every prime number $p$.)

We now turn to direct limits of one-dimensional NCCW-complexes. The following lemma will allow us to reduce to the case where the algebra itself is a one-dimensional NCCW-complex when proving that multiplication by $n$ is an order embedding.

\textbf{Lemma 6.1.} Let $(S_k, \varphi_k)_{k \in \mathbb{N}}$ be an inductive system in the category $\text{Cu}$, with $\varphi_k: S_k \to S_{k+1}$, and let $S = \lim\limits_{\leftarrow} (S_k, \varphi_k)$ be its inductive limit in $\text{Cu}$. Let $n \in \mathbb{N}$. If multiplication by $n$ on $S_k$ is an order-embedding for all $k$ in $\mathbb{N}$, then the same holds for $S$.

\textbf{Proof.} For $l \geq k$, denote by $\varphi_{k,l}: S_k \to S_{l+1}$ the composition $\varphi_{k,l} = \varphi_l \circ \cdots \circ \varphi_k$, and denote by $\varphi_{k,\infty}: S_k \to S$ the canonical maps as in the definition of inductive limit. Let $s, t \in S$ be such that $ns \leq nt$. By part (i) of Proposition 2.1, for each $k \in \mathbb{N}$ there exist $s_k, t_k \in S_k$ such that

$$\varphi_k(s_k) \ll s_{k+1}, \quad s = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(s_k),$$

$$\varphi_k(t_k) \ll t_{k+1}, \quad t = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(t_k).$$

Note in particular that $\varphi_{k,\infty}(s_k) \ll \varphi_{k+1,\infty}(s_{k+1})$ and $\varphi_{k,\infty}(t_k) \ll \varphi_{k+1,\infty}(t_{k+1})$ for all $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. Then

$$\varphi_{k,\infty}(ns_k) \ll \varphi_{k+1,\infty}(ns_{k+1}) \ll \sup_{j \in \mathbb{N}} \varphi_{j,\infty}(nt_j).$$

It follows by the definition of the compact containment relation that there exists $j$ such that

$$\varphi_{k,\infty}(ns_k) \ll \varphi_{k+1,\infty}(ns_{k+1}) \leq \varphi_{j,\infty}(nt_j).$$

By part (ii) of Proposition 2.1, there exists $l$ such that

$$n \varphi_{k,l}(s_k) = \varphi_{k,l}(ns_k) \leq \varphi_{j,l}(nt_j) = n \varphi_{j,l}(t_j).$$

Using now that the endomorphism of $S_k$ given by multiplication by $n$ is an order-embedding we conclude that $\varphi_{k,l}(s_k) \leq \varphi_{j,l}(t_j)$. In particular, $\varphi_{k,\infty}(s_k) \leq \varphi_{j,\infty}(t_j) \leq t$. Since $k \in \mathbb{N}$ is arbitrary and $s = \sup_{k \in \mathbb{N}} \varphi_{k,\infty}(s_k)$, we conclude that $s \leq t$. \hfill $\square$

\textbf{Proposition 6.1.} Let $A$ be a C*-algebra that can be written as the inductive limit of 1-dimensional NCCW-complexes. Let $x, y \in \text{Cu}(A)$ and $n \in \mathbb{N}$ be such that $nx \leq ny$, then it follows that $x \leq y$. 

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Proof. By Lemma 6.1, it is sufficient to show the proposition in the case when $A$ is a 1-dimensional NCCW-complex. Let $E = \bigoplus_{j=1}^{r} M_{k_j}(\mathbb{C})$ and $F = \bigoplus_{j=1}^{s} M_{l_j}(\mathbb{C})$ be finite dimensional $C^*$-algebras, and for $x \in [0, 1]$ denote by $ev_x: C([0, 1], F) \to F$ the evaluation map at the point $x$. Assume that $A$ is given by the pullback decomposition

$$
\begin{array}{ccc}
A & \longrightarrow & E \\
\downarrow & & \downarrow \\
C([0, 1], F) & \xrightarrow{ev_0 \oplus ev_1} & F \oplus F,
\end{array}
$$

By the second example of [1, Section 4], the Cuntz semigroup of $A$ is order-isomorphic to a subsemigroup of $Lsc([0, 1], \mathbb{N}) \oplus \mathbb{N}$. Since $nx \leq ny$ in $Lsc([0, 1], \mathbb{N}) \oplus \mathbb{N}$ implies $x \leq y$, the same holds in $Cu(A)$.

In the following corollary, we need to assume that the building blocks have trivial $K_1$ group to use Robert’s classification.

**Corollary 6.1.** Let $A$ be a unital inductive limit of 1-dimensional NCCW-complexes with trivial $K_1$-groups and let $n \in \mathbb{N}$. Suppose that $Cu(A)$ is $n$-divisible (this is, for every $x \in Cu(A)$ there exists $y \in Cu(A)$ such that $x = ny$). Then $A \cong A \otimes M_{n\infty}$.

**Proof.** Notice that since $Cu(A)$ is $n$-divisible and $A \neq \{0\}$, we must have $n \neq 0$. It then follows from Proposition 6.1 together with the assumptions in this corollary that the semigroup $Cu(A)$ is uniquely $n$-divisible. Hence, by Theorem 6.1, we have $Cu(A) \cong Cu(A \otimes M_{n\infty})$. This implies by the classification theorem [33, Theorem 1] that there is an isomorphism $A \otimes K \cong A \otimes M_{n\infty} \otimes K$. Using now that $M_{n\infty}$-absorption is inherited by hereditary $C^*$-subalgebras (see [37, Corollary 3.1]), we conclude that $A \cong A \otimes M_{n\infty}$. \qed

6.2. **Cuntz semigroup constrains and absorption of the model action.** Let $G$ be a group and let $(S, \gamma)$ be an object in the category $\text{Cu}^G$, this is, $S$ is a semigroup in the category $\text{Cu}$ and $\gamma: G \to \text{Aut}(S)$ is an action of $G$ on $S$. We denote by $S_\gamma$ the set

$$S_\gamma = \left\{ x \in S : \exists (x_n)_n \in \mathbb{N} \text{ in } Cu(A) : \begin{array}{l}
x_n \ll x_{n+1} \forall n \in \mathbb{N} \text{ and } x = \sup_{n \in \mathbb{N}} x_n, \\
Cu(\alpha_g)(x_n) = x_n \forall g \in G, \forall n \in \mathbb{N}
\end{array} \right\}.
$$

**Theorem 6.2.** Let $A$ be a $C^*$-algebra and let $\alpha$ be an action of a finite group $G$ on $A$ with the Rokhlin property. Let $i: A^\alpha \to A$ be the inclusion map. Then $Cu(i): Cu(A^\alpha) \to Cu(A)$ is an order-embedding and its range is given by

$$\text{Im}(Cu(i)) = \text{Im} \left( \sum_{g \in G} Cu(\alpha_g) \right) = Cu(A)_{Cu(\alpha)}.$$

**Proof.** Let us start by showing that $Cu(i)$ is an order-embedding. For convenience, let us denote the action $\alpha \otimes \text{id}_K$ by $\tilde{\alpha}$. Note that $\tilde{\alpha}$ has the Rokhlin property by part (i) of Proposition 2.3. Let $a$ and $b$ in $(A \otimes K)^{\tilde{\alpha}} = A^\alpha \otimes K$ be positive elements such that $a \preceq b$ in $(A \otimes K)_+$. Let $\varepsilon > 0$. By Rørdam’s Lemma there exists $c$ in $A \otimes K$ such that $(a - \varepsilon)_+ = c b c^*$. Fix $g$ in $G$. Applying $\tilde{\alpha}_g$ on both sides of this identity and using that $a$ and $b$ are invariant under $\tilde{\alpha}$ yields

$$(a - \varepsilon)_+ = \tilde{\alpha}_g(c) b \tilde{\alpha}_g(c)^*. $$

Using the Rokhlin property for $\tilde{\alpha}$, choose orthogonal positive contractions $r_g$ in $(A \otimes K)_\infty$ such that $\sum_{g \in G} r_g$ is a unit for $A$ and $(\tilde{\alpha}_g(r_h) - r_{gh})(x) = 0$ for all $g$ and $h$ in $G$ and all $x$ in $A \otimes K$. Set
\[d = \sum_{g \in G} r_g \tilde{\alpha}_g(c) \in (A \otimes K)^\infty.\] Then
\[\text{Im}(\sum_{g \in G} r_g \tilde{\alpha}_g(c)) \subseteq \text{Cu}(\alpha) \subseteq \text{Cu}(A) \subseteq \text{Im}(\text{Cu}(i)).\]

The second inclusion is straightforward since \(\text{Cu}(\alpha_h) \circ \sum_{g \in G} \text{Cu}(\alpha_g) = \sum_{g \in G} \text{Cu}(\alpha_g)\) for all \(h \in G\), and \(\text{Cu}(A) \subseteq \text{Cu}(\alpha)\) by Lemma 2.3. We proceed to show the first inclusion. Fix a positive element \(x\) in \(A^\alpha \otimes K\) and let \(\varepsilon > 0\). As before and using the Rokhlin property for \(\tilde{\alpha}\), choose orthogonal positive contractions \(c_g\) in \(A \otimes K\) for \(g \in G\) such that
\[
\left\| x - \sum_{g \in G} c_g x c_g \right\| < \varepsilon \quad \text{and} \quad \left\| \alpha_g(c_e x c_e) - c_g x c_g \right\| < \varepsilon,
\]
for all \(g \in G\). Using the first inequality above and using Lemma 2.1 applied to the elements \(x\) and \(\sum_{g \in G} c_g x c_g\), we obtain
\[
[(x - 4\varepsilon)_+] \leq \left[ \sum_{g \in G} c_g x c_g - 3\varepsilon \right]_+ \leq \left[ \sum_{g \in G} c_g x c_g - \varepsilon \right]_+ \leq [x].
\]
Furthermore, using the second inequality in (6.5) and using Lemma 2.1 applied this time to the elements \(\alpha_g(c_e x c_e)\) and \(c_g x c_g\), we deduce that
\[
\left[ (c_g x c_g - 3\varepsilon)_+ \right] \leq \left[ (\alpha_g(c_e x c_e) - 2\varepsilon)_+ \right] \leq \left[ (c_g x c_g - \varepsilon)_+ \right].
\]
By adding the previous inequalities for \(g \in G\) and using that \(\text{Cu}(\alpha_g)(c_e a c_e - 2\varepsilon)_+ = [(\alpha_g(c_e a c_e) - 2\varepsilon)_+]\), we conclude that
\[
[(a - 4\varepsilon)_+] \leq \sum_{g \in G} \text{Cu}(\alpha_g)(c_e a c_e - 2\varepsilon)_+ \leq [a].
\]
We have shown that for every \(\varepsilon > 0\) there is an element \(x\) in \(\text{Im}\left( \sum_{g \in G} \text{Cu}(\alpha_g) \right)\) such that
\[
[(a - \varepsilon)_+] \leq x \leq [a].
\]
It follows now by Lemma 2.2 applied to \([a] = \sup_{\varepsilon > 0} [(a - \varepsilon)_+]\) and to the set \(S = \text{Im}\left( \sum_{g \in G} \text{Cu}(\alpha_g) \right)\), that \([a]\) is the supremum of an increasing sequence in \(\text{Im}\left( \sum_{g \in G} \text{Cu}(\alpha_g) \right)\), showing that the first inclusion in (6.4) holds.

In order to complete the proof, let us show that the third inclusion in (6.4) is also true. Fix \(x\) in \(F_{Cu(\alpha)}\). Choose a rapidly increasing sequence \((x_n)_{n \in \mathbb{N}}\) in \(Cu(A)\) such that \(\text{Cu}(\alpha_g)(x_n) = x_n\) for
all $n \in \mathbb{N}$ and all for all $g$ in $G$. Fix $m \in \mathbb{N}$ and consider the elements $x_n$ with $n \geq m$. Note that $x_m \ll x_{m+1} \ll \cdots \ll x$. By Lemma 2.5, there is a positive element $a \in A \otimes K$ such that

$$x_m \ll [(a - 3\varepsilon)\,] \ll x_{m+1} \ll (a - 2\varepsilon)\, \ll x_{m+2} \ll \cdots \ll [a] = x.$$ 

Note that this implies that

$$[\alpha(a)] = \text{Cu}(\alpha_g)[a] = \text{Cu}(\alpha_g)(x) = x = [a] \leq [a]$$ 

and

$$[(a - 2\varepsilon)\,] \leq x_2 = \text{Cu}(\alpha_g)(x_2) \leq \text{Cu}(\alpha_g)[(a - \varepsilon)\,] = [\alpha((a - \varepsilon)\,)]$$

for every $g \in G$. By the definition of Cuntz subequivalence, there are elements $f_g$ and $h_g \in A \otimes K$ for $g$ in $G$ such that

$$\|\alpha(a) - f_g a f_g^*\| < \frac{\varepsilon}{|G|}$$

and

$$||(a - 2\varepsilon)\, - h_g(\alpha(a) - \varepsilon)\,h_g^*\| < \frac{\varepsilon}{|G|}.$$

Choose positive orthogonal contractions $r_g$ in $(A \otimes K)_\infty$ for $g$ in $G$ as in the definition of the Rokhlin property for $\alpha$. Set $f = \sum_{g \in G} f_g r_g$ and $h = \sum_{g \in G} h_g r_g$. Then

$$\left\| \sum_{g \in G} r_g \alpha_g(a) r_g - f a f^* \right\| = \left\| \sum_{g \in G} r_g (\alpha_g(a) - f_g a f_g^*) \right\| < |G| \cdot \frac{\varepsilon}{|G|} = \varepsilon.$$

Similarly,

$$\left\| (a - 2\varepsilon) - h \left( \sum_{g \in G} \alpha_g((a - \varepsilon)\,)^h_g \right) \right\| < \varepsilon.$$

Using that $r_g$ commutes with $\alpha(a)$ and that $r_g \alpha(a) = r_g \alpha(a)$ for all $g$ in $G$ and all $a \in A \otimes K$, one easily shows that

$$\sum_{g \in G} r_g \alpha_g(a) r_g = \sum_{g \in G} \alpha(r_g a r_g)$$

and

$$r_g (\alpha_g((a - \varepsilon)\,)) r_g = (r_g \alpha_g(a) r_g - \varepsilon)\, +$$

for all $g \in G$ and all $a$ in $A \otimes K$. We thus have

$$\sum_{g \in G} r_g (\alpha_g((a - \varepsilon)\,)) r_g = \sum_{g \in G} r_g (\alpha_g(a) - \varepsilon)\, + r_g$$

$$= \sum_{g \in G} (r_g \alpha_g(a) r_g - \varepsilon)\,$$

$$= \left( \sum_{g \in G} r_g \alpha_g(a) r_g - \varepsilon \right)\, +$$

$$= \left( \sum_{g \in G} \alpha(r_g a r_g) - \varepsilon \right)\, +.$$

Therefore, we conclude that

$$\left\| \sum_{g \in G} \alpha(r_g a r_g) - f a f^* \right\| < \varepsilon$$
and
\begin{equation}
\left\| (a - 2\varepsilon)_+ - h \left( \sum_{g \in G} \bar{\alpha}(rar) - \varepsilon \right) + h^* \right\| < \varepsilon.
\end{equation}

It follows that there is a positive element \( r \in A \otimes K \) such that the inequalities
\begin{equation}
\left\| \sum_{g \in G} \bar{\alpha}(rar) - faf^* \right\| < \varepsilon
\end{equation}

and
\begin{equation}
\left\| (a - 2\varepsilon)_+ - h \left( \sum_{g \in G} \bar{\alpha}(rar) - \varepsilon \right) + h^* \right\| < \varepsilon
\end{equation}

hold in \( A \otimes K \). By Lemma 2.1 applied to the elements \( \sum_{g \in G} \bar{\alpha}(rar) \) and \( faf^* \), and to the elements \( (a - 2\varepsilon)_+ \) and \( h \left( \sum_{g \in G} \bar{\alpha}(raf) - \varepsilon \right) + h^* \), we deduce that
\begin{equation}
[(a - 3\varepsilon)_+] \leq \left[ \left( \sum_{g \in G} \bar{\alpha}_g(rar) - \varepsilon \right) \right] \leq [a].
\end{equation}

Therefore,
\begin{equation}
x_m \ll \left[ \left( \sum_{g \in G} \bar{\alpha}_g(rar) - \varepsilon \right) \right] \ll x.
\end{equation}

Note that the element \( \left( \sum_{g \in G} \bar{\alpha}_g(rar) - \varepsilon \right)_+ \) belongs to \((A \otimes K)^\alpha\) and so it is in the image of the inclusion map \( i \otimes \text{id}_K : (A \otimes K)^\alpha \to A \otimes K \). Since \( m \) is arbitrary, we deduce that \( x \) is the supremum of an increasing sequence in \( \text{Im}(\text{Cu}(i)) \) by Lemma 2.2. Choose a sequence \( (y_n)_{n \in \mathbb{N}} \) in \( \text{Cu}(A^\alpha) \) such that \( (\text{Cu}(i)(y_n))_{n \in \mathbb{N}} \) is increasing in \( \text{Cu}(A) \) and \( x = \sup(\text{Cu}(i)(y_n)) \). Since \( \text{Cu}(i) \) is an order-embedding, it follows that \( (y_n)_{n \in \mathbb{N}} \) is itself increasing in \( \text{Cu}(A^\alpha) \). Set \( y = \sup_{n \in \mathbb{N}} y_n \). Then \( \text{Cu}(y) = x \) because \( \text{Cu}(i) \) preserves suprema of increasing sequences. This concludes the proof of the theorem. \( \square \)

**Corollary 6.2.** Let \( A \) be a C*-algebra and let \( \alpha \) be an action of a finite group \( G \) on \( A \) with the Rokhlin property. Then there is a natural \( \text{Cu} \)-isomorphism
\begin{equation}
\text{Cu}(A \rtimes_\alpha G) \cong \left\{ x \in \text{Cu}(A) : \exists (x_n)_{n \in \mathbb{N}} \text{ in } \text{Cu}(A) : x_n \ll x_{n+1} \forall n \in \mathbb{N} \text{ and } x = \sup_{n \in \mathbb{N}} x_n, \right. \left. \text{Cu}(\alpha_g)(x_n) = x_n \forall g \in G, \forall n \in \mathbb{N} \right\}.
\end{equation}

**Proof.** Since \( \alpha \) has the Rokhlin property, the fixed point algebra \( A^\alpha \) is Morita equivalent to the crossed product \( A \rtimes_\alpha G \) by [30, Theorem 2.8]. Therefore, there is a natural isomorphism \( \text{Cu}(A \rtimes_\alpha G) \cong \text{Cu}(A^\alpha) \). Denote by \( i : A^\alpha \to A \) the natural embedding. By Theorem 6.2, the semigroup \( \text{Cu}(A^\alpha) \) can be naturally identified with its image under the order-embedding \( \text{Cu} \)-morphism \( \text{Cu}(i) \), which is \( \text{Cu}(A)_{\text{Cu}(\alpha)} \) again by Theorem 6.2. The result follows. \( \square \)

**Corollary 6.3.** Let \( A \) be a C*-algebra, let \( \alpha \) be an action of a finite group \( G \) on \( A \) with the Rokhlin property, and set \( n = |G| \). Suppose that \( \text{Cu}(\alpha_g) = \text{id}_{\text{Cu}(A)} \) for every \( g \in G \) and that multiplication by \( n \) on \( \text{Cu}(A) \) is an order-embedding. (We remind the reader that this means that whenever \( x \) and \( y \) are elements in \( \text{Cu}(A) \) that satisfy \( nx \leq ny \), one has \( x \leq y \).) Then \( \text{Cu}(A) \) is uniquely \( n \)-divisible.
Proof. It suffices to show that for all \( x \) in \( \text{Cu}(A) \), there exists \( y \) in \( \text{Cu}(A) \) such that \( x = ny \).

By Theorem 6.2 we have

\[
\text{Im} \left( \sum_{g \in G} \text{Cu}(\alpha_g) \right) = \text{Cu}(A)_{\text{Cu}(\alpha)}.
\]

Since \( \text{Cu}(\alpha_g) = \text{id}_{\text{Cu}(A)} \) for all \( g \) in \( G \), this identity can be rewritten as

\[
\overline{n \text{Cu}(A)} = \text{Cu}(A).
\]

In particular, if \( x \) is an element in \( \text{Cu}(A) \), then there exists a sequence \( (y_k)_{k \in \mathbb{N}} \) in \( \text{Cu}(A) \) such that \( (ny_k)_{k \in \mathbb{N}} \) is increasing and \( x = \sup_{k \in \mathbb{N}} (ny_k) \). Since \( (ny_k)_{k \in \mathbb{N}} \) is increasing, it follows from our assumptions that \( (y_k)_{k \in \mathbb{N}} \) is increasing as well. Set \( y = \sup_{k \in \mathbb{N}} y_k \). Then

\[
x = \sup_{k \in \mathbb{N}} (ny_k) = n \sup_{k \in \mathbb{N}} y_k = ny,
\]

and the claim follows. \( \square \)

Compare the following result with [21, Theorems 3.4 and 3.5].

**Theorem 6.3.** Let \( G \) be a finite group and let \( A \) be a unital C*-algebra that can be written as direct limit of a sequence of 1-dimensional NCCW-complexes with trivial \( K_1 \)-group. Then the following statements are equivalent:

(i) The C*-algebra \( A \) absorbs the UHF-algebra \( M_{|G|\infty} \).

(ii) There is an action \( \alpha: G \to \text{Aut}(A) \) with the Rokhlin property such that \( \text{Cu}(\alpha_g) = \text{id}_{\text{Cu}(A)} \) for all \( g \in G \).

(iii) There are actions of \( A \) with the Rokhlin property, and for any action \( \beta: G \to \text{Aut}(A) \) with the Rokhlin property and for any action \( \delta: G \to \text{Aut}(A) \) such that \( \text{Cu}(\beta_g) = \text{Cu}(\delta_g) \) for all \( g \) in \( G \), one has \( (A, \beta) \cong (A \otimes M_{|G|\infty}, \delta \otimes \mu^G) \), that is, there is an isomorphism \( \varphi: A \to A \otimes M_{|G|\infty} \) such that

\[
\varphi \circ \beta_g = (\delta \otimes \mu^G)_g \circ \varphi
\]

for all \( g \) in \( G \).

In particular, if the above statements hold for \( A \), and if \( \alpha: G \to \text{Aut}(A) \) is an action with the Rokhlin property such that \( \text{Cu}(\alpha_g) = \text{id}_{\text{Cu}(A)} \) for all \( g \in G \), then \( (A, \alpha) \cong (A \otimes M_{|G|\infty}, \text{id}_A \otimes \mu^G) \).

**Proof.** (i) implies (ii). Fix an isomorphism \( \varphi: A \to A \otimes M_{|G|\infty} \) and define an action \( \alpha: G \to \text{Aut}(A) \) by \( \alpha_g = \varphi^{-1} \circ (\text{id}_A \otimes \mu^G)_g \circ \varphi \) for all \( g \) in \( G \). For a fixed group element \( g \) in \( G \), the automorphism \( \text{id}_A \otimes \mu^G_g \) of \( A \otimes M_{|G|\infty} \) is approximately inner, and hence so is \( \alpha_g \). It follows that \( \text{Cu}(\alpha_g) = \text{id}_{\text{Cu}(A)} \) for all \( g \) in \( G \), as desired.

(ii) implies (i). Assume that there is an action \( \alpha: G \to \text{Aut}(A) \) with the Rokhlin property such that \( \text{Cu}(\alpha_g) = \text{id}_{\text{Cu}(A)} \) for all \( g \in G \). Then \( A \cong A \otimes M_{|G|\infty} \) by Proposition 6.1, Corollary 6.1 and Corollary 6.3.

(i) and (ii) imply (iii). Let \( \beta \) and \( \delta \) be actions of \( G \) on \( A \) as in the statement. Since \( M_{|G|\infty} \) is a strongly self-absorbing algebra, there exists an isomorphism \( \phi: A \to A \otimes M_{|G|\infty} \) that is approximately unitarily equivalent to the map \( \iota: A \to A \otimes M_{|G|\infty} \) given by \( \iota(a) = a \otimes 1_{M_{|G|\infty}} \) for \( a \) in \( A \).

In particular, one has \( \text{Cu}(\phi) = \text{Cu}(\iota) \). Hence, for every \( a \in (A \otimes \mathcal{K})_+ \) we have

\[
(Cu(\phi) \circ Cu(\beta_g))(\{a\}) = Cu(\iota)((\beta_g \otimes \text{id}_\mathcal{K})(a)) = \left[ ((\beta_g \otimes \text{id}_\mathcal{K})(a)) \otimes 1_{M_{|G|\infty}} \right]
\]

\[
(Cu(\delta_g \otimes \mu^G) \circ Cu(\phi))(\{a\}) = Cu(\delta_g \otimes \mu^G)(\left[ a \otimes 1_{M_{|G|\infty}} \right]) = \left[ ((\delta_g \otimes \text{id}_\mathcal{K})(a)) \otimes 1_{M_{|G|\infty}} \right].
\]
Since $\text{Cu}(\beta_g) = \text{Cu}(\delta_g)$ for all $g$ in $G$, it follows that $\text{Cu}(\phi) \circ \text{Cu}(\beta_g) = \text{Cu}(\delta_g \otimes \mu_g) \circ \text{Cu}(\phi)$ for all $g$ in $G$. In other words, the $\text{Cu}$-isomorphism $\text{Cu}(\phi): \text{Cu}(A) \to \text{Cu}(A \otimes M_{G(\infty)})$ is equivariant. Therefore, by the unital case of Theorem 4.5, there exists an isomorphism $\varphi: A \to A \otimes M_{G(\infty)}$ such that $\varphi \circ \beta_g = (\delta \otimes \mu^G)_g \circ \varphi$ for all $g \in G$, showing that $\beta$ and $\delta \otimes \mu^G$ are conjugate.

(iii) implies (i). The existence of an action $\beta: G \to \text{Aut}(A)$ with the Rokhlin property implies the existence of an isomorphism $A \to A \otimes M_{G(\infty)}$, simply by taking $\delta = \beta$.

The last claim follows immediately from (iii). \qed

Let $A$ be a C*-algebra and let $p$ and $q$ be projections in $A$. We say that $p$ is Murray-von Neumann subequivalent to $q$, and denote this by $p \precsim_{\text{MVN}} q$, if there is a projection $p'$ in $A$ such that $p \sim_{\text{MVN}} p'$ and $p' \leq q$. Note that in general $p \precsim_{\text{MVN}} q$ and $q \precsim_{\text{MVN}} p$ do not imply that $p \sim_{\text{MVN}} q$. Nevertheless, it is well known that this is the case whenever $A$ is finite.

Recall further that the Murray-von Neumann semigroup of $A$, denoted by $V(A)$, is defined as the quotient of the set of projections of $A \otimes K$ by the Murray-von Neumann equivalence relation.

Also note that in arbitrary C*-algebras, it is true that $p \precsim_{\text{Cu}} q$ if and only if $p \precsim_{\text{MVN}} q$. In particular, if $A$ is finite then $p \sim_{\text{Cu}} q$ if and only if $p \sim_{\text{MVN}} q$. As is the case for the Cuntz semigroup, the Murray-von Neumann semigroup construction gives rise to a functor $V$ from the category of C*-algebras to the category of semigroups. By the discussion above, if $A$ is stably finite then the semigroup $V(A)$ can be identified with the ordered subsemigroup of $\text{Cu}(A)$ consisting of the Cuntz equivalence classes of projections of $A \otimes K$.

Recall that if $S$ is a semigroup in $\text{Cu}$ and let $x$ and $y$ are elements of $S$, we say that $x$ is compactly contained in $y$, and denote this by $x \ll y$, if for every increasing sequence $(y_n)_{n \in \mathbb{N}}$ in $S$ such that $y = \sup_{n \in \mathbb{N}} y_n$, there exists $n_0 \in \mathbb{N}$ such that $x \leq y_n$ for all $n \geq n_0$.

**Definition 6.2.** Let $S$ be a semigroup in $\text{Cu}$ and let $x$ be an element of $S$. We say that $x$ is compact if $x \ll x$. Equivalently, $x$ is compact if whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in $S$ such that $x = \sup_{n \in \mathbb{N}} x_n$, then there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for all $n \geq n_0$.

It is easy to check that the Cuntz class $[p] \in \text{Cu}(A)$ of any projection $p$ in a C*-algebra $A$ (or in $A \otimes K$) is a compact element in $\text{Cu}(A)$. Moreover, when $A$ is stably finite then every compact element of $\text{Cu}(A)$ is the Cuntz class of a projection in $A \otimes K$ by [4, Theorem 3.5]. In particular, $V(A)$ can be identified with the semigroup of compact elements of $\text{Cu}(A)$ if $A$ is a stably finite C*-algebra.

When studying stably finite C*-algebras in connection with finite group actions with the Rokhlin property, the following lemma is often times useful. The result may be interesting in its own right, and could have been proved in [28] since it is a direct application of their methods.

**Lemma 6.2.** Let $G$ be a finite group, let $A$ be a unital stably finite C*-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then the crossed product $A \rtimes_\alpha G$ and the fixed point algebra $A^\alpha$ are stably finite.

**Proof.** The fixed point algebra $A^\alpha$, being a unital subalgebra of $A$, is stably finite. On the other hand, the crossed $A \rtimes_\alpha G$, being stably isomorphic to $A^\alpha$ by [30, Theorem 2.8], must itself also be stably finite. \qed

One would hope that crossed products by actions of finite groups with the Rokhlin property would also preserve the (smaller) class of finite C*-algebras. (Note that the above argument breaks down in this case.) However, this is not true. Let $A$ be any unital C*-algebra such that $A$ is finite but $M_2(A)$ is not. (For example, the C*-algebra constructed by Clarke in [6].) Let $\alpha: \mathbb{Z}_2 \to \text{Aut}(A \oplus A)$ be the action determined by $\alpha_{-1}(a, b) = (b, a)$ for all $a$ and $b$ in $A$. (We use multiplicative notation, so $\mathbb{Z}_2 = \{-1, 1\}$.) It is straightforward to check that $\{(1_A, 0), (0, 1_A)\}$ is a family of Rokhlin
projections for $\alpha$ for any choice of a finite subset $F \subseteq A$ and any choice of $\varepsilon > 0$, and thus $\alpha$ has the Rokhlin property. Moreover, $A \oplus A$ is finite while the crossed product $(A \oplus A) \rtimes_\alpha \mathbb{Z}_2 \cong M_2(A)$ is not.

**Theorem 6.4.** Let $A$ be a stably finite $C^*$-algebra and let $\alpha$ be an action of a finite group $G$ with the Rokhlin property. Let $i: A^\alpha \to A$ be the inclusion map. The following statements hold:

(i) The map $V(i): V(A^\alpha) \to V(A)$ is an order-embedding and

$$ \text{Im}(V(i)) = \text{Im}\left( \sum_{g \in G} V(\alpha_g) \right) = \{ x \in V(A) : V(\alpha_g)(x) = x \text{ for all } g \in G \}.$$

(ii) If $A$ is unital, then $K_0(i): K_0(A^\alpha) \to K_0(A)$ is an order-embedding and

$$ \text{Im}(K_0(i)) = \text{Im}\left( \sum_{g \in G} K_0(\alpha_g) \right) = \{ x \in K_0(A) : K_0(\alpha_g)(x) = x \text{ for all } g \in G \}.$$

**Proof.** (i) The fact that $V(i)$ is an order-embedding is a consequence of Theorem 6.2 and the remarks before and after 6.2. Let us now show the inclusions

$$ \text{(6.6)} \quad \text{Im}(V(i)) \subseteq \text{Im}\left( \sum_{g \in G} V(\alpha_g) \right) \subseteq \{ x \in V(A) : V(\alpha_g)(x) = x \text{ for all } g \in G \} \subseteq \text{Im}(V(i)).$$

Let $p \in A^\alpha \otimes \mathcal{K}$ be a projection. By Theorem 6.2 there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $(A \otimes \mathcal{K})_+$ such that $\left( \sum_{g \in G} \text{Cu}(\alpha_g)([a_n]) \right)_{n \in \mathbb{N}}$ is increasing and

$$\lceil i(p) \rceil = \sup_{n \in \mathbb{N}} \left( \sum_{g \in G} \text{Cu}(\alpha_g)([a_n]) \right).$$

Since $\lceil i(p) \rceil$ is a compact element in $\text{Cu}(A)$, it follows that there exists $n_0 \in \mathbb{N}$ such that $\lceil i(p) \rceil = \sum_{g \in G} \text{Cu}(\alpha_g)([a_n])$ for all $n \geq n_0$. Fix $m \geq n_0$. It is easy to check that if $S$ is a semigroup in the category $\text{Cu}$, then a sum of elements in $S$ is compact if and only if each summand is compact. It follows that $\text{Cu}(\alpha_g)([a_m])$ is compact for all $g \in G$. In particular, $[a_m] = \text{Cu}(\alpha_g)([a_m])$ is compact. Since $A$ is stably finite by assumption, there exists a projection $q$ in $A \otimes \mathcal{K}$ such that $[q] = [a_m]$. Thus

$$V(i)([p]) = \sum_{g \in G} V(\alpha_g)([q]) \in \text{Im}\left( \sum_{g \in G} V(\alpha_g) \right),$$

showing that the first inclusion in (6.6) holds.

Using the fact that $\alpha_h \circ \left( \sum_{g \in G} \alpha_g \right) = \sum_{g \in G} \alpha_g$ for all $h$ in $G$, it is easy to check that

$$\text{Im}\left( \sum_{g \in G} V(\alpha_g) \right) \subseteq \{ x \in V(A) : V(\alpha_g)(x) = x \text{ for all } g \in G \},$$

thus showing that the second inclusion also holds.

We proceed to prove the third inclusion. Let $x$ be an element in $V(A)$ that satisfies $V(\alpha_g)(x) = x$ for all $g \in G$. Note that $x$ is compact as an element in $\text{Cu}(A)$. It follows that $\text{Cu}(\alpha_g)(x) = x$ for all $g$ in $G$ and hence by Theorem 6.2 there exists $a \in (A^\alpha \otimes \mathcal{K})_+$ such that $\text{Cu}(i)([a]) = x$. Since the map $\text{Cu}(i)$ is an order-embedding again by Theorem 6.2, one concludes that $[a]$ is compact.

Finally, the fixed point algebra $A^\alpha$ is stably finite by Lemma 6.2 and thus there is a projection
p in \(A^a \otimes K\) such that \([p] = [a]\) in \(\text{Cu}(A^a)\). It follows that \(\text{Cu}(i([p]) = x\), showing that the third inclusion in (6.6) is also true.

(ii) Follows using the first part and the fact that the \(K_0\)-group of a unital \(C^*\)-algebra is the Grothendieck group of the Murray-von Neumann semigroup of the algebra.

In the following corollary, the picture of \(V(A \rtimes \alpha G)\) is valid even if \(A\) does not have a unit.

**Corollary 6.4.** Let \(A\) be a stably finite unital \(C^*\)-algebra and let \(\alpha\) be an action of a finite group \(G\) on \(A\) with the Rokhlin property. Then there are isomorphisms

\[
\begin{align*}
V(A \rtimes \alpha G) & \cong \{x \in V(A) : V(\alpha_g)(x) = x \text{ for all } g \in G\}, \\
K_*(A \rtimes \alpha G) & \cong \{x \in K_*(A) : K_*(\alpha_g)(x) = x \text{ for all } g \in G\}.
\end{align*}
\]

**Proof.** Recall that if \(\alpha\) has the Rokhlin property, then the fixed point algebra \(A^\alpha\) and the crossed product \(A \rtimes \alpha G\) are Morita equivalent, and hence have isomorphic \(K\)-theory and Murray-von Neumann semigroup. The isomorphisms for \(V(A \rtimes \alpha G)\) and \(K_0(A \rtimes \alpha G)\) then follow from Theorem 6.4 above.

Denote \(B = A \otimes C(S^1)\) and give \(B\) the diagonal action \(\beta = \alpha \otimes \text{id}_{C(S^1)}\) of \(G\). Note that \(B\) is stably finite and unital, and that \(\beta\) has the Rokhlin property by part (i) of Proposition 2.3. Moreover, there are natural isomorphisms \(B \rtimes_{\beta} G \cong (A \rtimes_{\alpha} G) \otimes C(S^1)\). Applying the Künneth formula in the first step, together with the conclusion of this proposition for \(K_0\) (which was shown to hold in the paragraph above) in the second step, and again the Künneth formula in the fourth step, we obtain

\[
\begin{align*}
\{x \in K_*(A) : K_*(\alpha_g)(x) = x \text{ for all } g \in G\} & \cong \{x \in K_0(B) : K_0(\beta_g)(x) = x \text{ for all } g \in G\} \\
& \cong K_0(B \rtimes_{\beta} G) \\
& \cong K_0((A \rtimes_{\alpha} G) \otimes C(S^1)) \\
& \cong K_*(A \rtimes_{\alpha} G),
\end{align*}
\]

as desired. \(\square\)

**References**


Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA.