

SHALIKA GERMS FOR TAMELY RAMIFIED ELEMENTS IN GL_n

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ABSTRACT. Degenerating the action of the elliptic Hall algebra on the Fock space, we give a combinatorial formula for the Shalika germs of tamely ramified regular semisimple elements γ of GL_n over a nonarchimedean local field. As a byproduct, we compute the weight polynomials of affine Springer fibers in type A and orbital integrals of tamely ramified regular semisimple elements.

We conjecture that the Shalika germs of γ correspond to residues of torus localization weights of a certain quasi-coherent sheaf \mathcal{F}_γ on the Hilbert scheme of points on \mathbb{A}^2 , thereby finding a geometric interpretation for them.

As corollaries, we obtain the polynomiality in q of point-counts of compactified Jacobians of planar curves, as well as a virtual version of the Cherednik-Danilenko conjecture on their Betti numbers. Our results also provide further evidence for the ORS conjecture relating compactified Jacobians and HOMFLY-PT invariants of algebraic knots.

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¹The genesis of this manuscript is as follows: In 2018, CCT showed OK his unpublished computations of Shalika germs of GL_n . These were in many examples checked to match predictions from the knot theory side by OK, which led to the strategy pursued in this paper.

CCT subsequently told OK that Waldspurger had given an algorithm for the germs in [69, 70], which OK later found to match a recursion on the knot theory/symmetric function side. The final outcome is this paper, mostly written by the first author. CCT is nevertheless included as the second author, as neither understanding the proof method nor deciphering the difficult papers [69, 70] would have been possible without his generous aid. The application to compactified Jacobians is also originally CCT's idea.

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1. INTRODUCTION

Shalika germs are a family of functions on a neighborhood of the identity in an algebraic group G defined over a nonarchimedean field F . They were introduced in [64] and further studied in for example [18, 42, 60, 69]. We refer to [41] for a survey. More precisely, given $f \in C_c^\infty(G(F))$ an Iwahori bi-invariant function on G , the *Shalika germ expansion* [18, 64] states that

Proposition 1.1.

$$I_\gamma(f) = \sum_{\mathcal{O} \in G(0)} \Gamma_{\mathcal{O}}(\gamma) I_{\mathcal{O}}(f)$$

Where $G(0)$ is the set of unipotent orbits in G , $I_\gamma, I_{\mathcal{O}}$ are orbital integrals and $\Gamma_{\mathcal{O}} : U \rightarrow \mathbb{C}$ are functions called the *Shalika germs*, defined on an open subset $\gamma \in U \subseteq G$.

See Corollary 2.7 for a more precise version we will use. Computing the Shalika germs for a given group G is an important, in general open problem, which refines formulas for regular semisimple orbital integrals as well as character values of supercuspidal representations [51]. In the present paper, we find a new mathematical incarnation of the Shalika germs of tamely ramified elements in terms of the Hilbert scheme of points on $\text{Hilb}^n(\mathbb{A}^2)$. From now on, we let γ be tamely ramified, meaning $F(\gamma)$ is a product of tamely ramified extensions of F .

In [70], Waldspurger gives a rather complicated inductive formula for (almost, see Theorem 2.19 and Section 6) the Shalika germs of γ in this case, built inductively from those of elements of smaller *depth*. Here depth is meant in the sense of Moy-Prasad theory, see Definition 2.48. The algorithm in [70] rests on a clever choice of test functions and a version of "Kazhdan's lemma", using which one can bootstrap computation of the germs to what is essentially just linear algebra. A similar idea is used again by Waldspurger in [71] and also by the second author in [67] based on a lemma of Kim-Murnaghan [39] to obtain less precise results for general groups. Waldspurger mentions similar strategies due to Kazhdan, Henniart, and others. The issue is that the resulting linear algebra is usually cumbersome to carry out and many steps of the algorithm have no obvious conceptual meaning. As noted by Waldspurger in [70]:

L'auteur est convaincu qu'il existe une bonne combinatoire, moins naïve que celle utilisée ici, qui devrait permettre de calculer les germes.
- J.-L. Waldspurger [70]

Our method of computation of the germs will use Waldspurger's techniques from [69,70], the most notable difference being in our use of the Lie algebra \mathfrak{gl}_n in place of GL_n , and replacing, or rather extending, his PSH-algebra calculations by the *elliptic Hall algebra*. This greatly clarifies the resulting combinatorics, giving a formula which is essentially computable "by hand" ².

The elliptic Hall algebra (EHA) has appeared in many guises related to automorphic forms, starting with the original article of Burban and Schiffmann [10]. For us, it appears through a "shuffle algebra" action on the algebra of symmetric functions in infinitely many variables [54,62]. More precisely, much of this paper is concerned with a symmetric function we denote $\mathbf{f}_\gamma = \mathbf{f}_{\bar{p},\bar{q}}$, which can be constructed using this action, as we will explain in Section 5. We have called $\mathbf{f}_{\bar{p},\bar{q}}$ "the master symmetric function" for lack of a better name. In modern language, it is the $t \rightarrow 1$ limit of the $n \rightarrow \infty$ limit of a symmetric polynomial in n variables, which appears in the definition of the *superpolynomial* for an iterated torus knot [12,29].

In [69,70], Waldspurger already essentially introduces this symmetric function, but mostly as a bookkeeping tool which turns out to be helpful because of the relation to calculations in the Hall algebra of $GL_n(\mathbb{F}_q)$. Apparently, the relationship of Shalika germs to this Hall algebra was first suggested to him by B. Srinivasan.

We make the case that the master symmetric function is a completely natural object and arises as a vector in the Fock space representation of the elliptic Hall algebra. By the results of [6,63] the elliptic Hall algebra is the decategorification of a form of induction-restriction functors for a coherent realization affine character sheaves, and one expects this to be mirrored on the constructible side of Langlands duality. Indeed, in an appropriate sense each γ gives rise to $G(F)$ -equivariant constructible sheaf on $\mathfrak{g}(F)$ by taking the extension by zero of the constant sheaf on the conjugacy class and one may view our induction as some shadow of yet-to-be-defined induction-restriction functors for affine character sheaves of this sort (meaning that we impose no singular support condition). While we only work with the elements γ themselves, it would be compelling to make the induction more precise on the level of affine character sheaves.

On the decategorified level, the appearance of the Fock space representation on the harmonic analysis side seems to stem from the natural "cyclotomic" quotient map from the affine Hecke algebra down to the finite one in type A. Note that these maps respect parabolic induction. This is exactly the same phenomenon as in the skein-theoretic version [50].

1.1. The formulas. Let us now state our main results in some detail. Let $G = GL_n$, $\mathfrak{g} = Lie(G) = \mathfrak{gl}_n$, $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$ and $\gamma \in \mathfrak{g}(\mathcal{K})$ be a regular semisimple (elliptic) element. We endow \mathcal{K} and its extensions with the standard valuation. Later on we will also be working with an arbitrary non-archimedean local field denoted F . We will only be interested in conjugation-invariant notions, but it will be helpful for us to choose a nice representative in the conjugacy class of γ . Further, we will only be concerned with the orbital integrals (and some related geometric objects like affine Springer fibers) of γ , which further allows us to mod out by a large power of the uniformizer t .

²Computer code available at <https://www.math.toronto.edu/salomon/Shalika.zip> and in the arXiv submission.

Let

$$(1.1) \quad u = \begin{pmatrix} 0 & 0 & \cdots & 0 & t \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and note that $u^n = t$.

Lemma 1.2. *For any topologically nilpotent elliptic regular semisimple $\gamma' \in \mathfrak{gl}_n(\mathcal{K})$ there exists an element $\gamma \in \mathfrak{gl}_n(\mathcal{K})$ of the form*

$$(1.2) \quad \gamma = \sum_{i=1}^d a_i u^{nr_i}$$

for some $d \geq 0$, $a_i \in \mathbb{C}$ and positive rational numbers r_1, \dots, r_d with $nr_i \in \mathbb{Z}$, such that the corresponding affine Springer fibers satisfy $\mathrm{Sp}_\gamma \cong \mathrm{Sp}_{\gamma'}$ (meaning that there is an isomorphism of underlying reduced ind-projective varieties). Without loss of generality, we may take $r_1 > r_2 > \cdots > r_d$. Additionally, the set $\{r_i\}_{i=1}^d$ coincides with the set of root valuations of γ , in the sense of [26].

Proof. The regular conjugacy classes in $\mathfrak{gl}_n(\mathcal{K})$ are parametrized by the characteristic polynomials. Write $\mathrm{char}(\gamma) = x^n - b_1 x^{n-1} + \cdots + b_n$ for $b_i \in \mathcal{K}$.

The Newton-Puiseux algorithm (see e.g. [20] for a reference relevant to this paper) allows one to solve $\mathrm{char}(\gamma) = 0$ for x as a Puiseux series $x = \sum_i a_i t^{r_i}$ in t . The exponents in this Puiseux series are all rational numbers whose denominator divides n . To get the assertion $\mathrm{Sp}_\gamma \cong \mathrm{Sp}_{\gamma'}$ we may use Artin approximation, see e.g. [52, Section 3.6.] for this particular case. In effect, we may take $b_i \in \mathbb{C}[t^\pm]$ which also makes this Puiseux series finite. Replacing $t^{1/n}$ by u and using Cayley-Hamilton gives the first claim.

As to the last claim, note that u^{nr_i} has all root valuations equal to r_i and over the field of Puiseux series, all the terms in the expansion of γ may be simultaneously diagonalized. Now the result is easy to check using standard properties of the valuation and the Puiseux expansion. \square

Remark 1.3. Lemma 1.2 works over other residue fields as well, and in fact any non-archimedean F , provided $F(\gamma)$ is a tamely (and totally) ramified extension of F [14, Chapter IV.6.].

Definition 1.4. Writing the exponents r_i appearing in Eq. (1.2) as $r_1 = m_1/n_1, r_2 = m_2/n_1 n_2$ etc. gives the *Puiseux pairs* $(m_1, n_1), \dots, (m_d, n_d)$ of γ . Rewriting the Puiseux expansion of $\mathrm{char}(\gamma)$ as

$$x = t^{q_d/p_d} (a_d + t^{q_{d-1}/p_d p_{d-1}} (a_{d-1} + \cdots (a_2 + a_1 t^{q_1/p_1 \cdots p_d})) \cdots)$$

we get the *Newton pairs* $(p_1, q_1), \dots, (p_d, q_d)$ of γ . We will write (\bar{p}, \bar{q}) for this sequence. We will also impose the (harmless) requirement $q_d \geq p_d$. (Note that ours is the opposite of the conventions used in e.g. [20]).

Remark 1.5. Knowledge of the Puiseux (equivalently, Newton) pairs is equivalent to knowing the *topological type* of the singularity $\{\mathrm{char}(\gamma) = 0\}$. Recall that this is by definition the knot in S^3 determined by intersecting $\{\mathrm{char}(\gamma) = 0\} \subset \mathbb{C}^2$ with a small three-sphere centered at the origin.

Example 1.6. Let $n = 4$ and

$$\gamma = u^6 + u^7 = \begin{pmatrix} 0 & t^2 & t^2 & 0 \\ 0 & 0 & t^2 & t^2 \\ t & 0 & 0 & t^2 \\ t & t & 0 & 0 \end{pmatrix}$$

Then $r_1 = 7/4, r_2 = 3/2$ and

$$\text{char}(\gamma) = x^4 - 2t^3x^2 - 4t^5x - t^7 + t^6$$

The Puiseux pairs are $(m_1, n_1) = (7, 2), (m_2, n_2) = (3, 2)$ and the Newton pairs are $(p_1, q_1) = (1, 2), (p_2, q_2) = (2, 3)$. The link is the "(2, 13)-cable of the trefoil". This example features also for example in [20, p. 58].

Recall that *rational Dyck paths of slope m/n* are lattice paths in an m by n rectangle fitting under the diagonal. See Definition 3.6.

As explained in Section 4, one may define a homomorphism from the ring of symmetric functions to itself $\varphi_{m/n} : \text{Sym}_q \rightarrow \text{Sym}_q$ by sending each e_k to

$$E_{m,n,k} = \sum_{\pi \in \mathbb{D}_{km, kn}} q^{\text{area}(\pi)} e_\pi$$

see Section 3 for the notation. Our first main theorem is the following.

Theorem 1.7. *Let $\gamma \in \mathfrak{gl}_n(\mathcal{K})$ be an elliptic regular semisimple element. Let $(p_1, q_1), \dots, (p_d, q_d)$ be the Newton pairs of γ . Then the master symmetric function $\mathbf{f}_\gamma = \mathbf{f}_{\vec{p}, \vec{q}}$ only depends on the Newton (or Puiseux) pairs of γ and is given by*

$$\mathbf{f}_\gamma = \varphi_{q_d/p_d}(\cdots \varphi_{q_1/p_1}(e_1) \cdots).$$

There is a version of this Theorem for not necessarily elliptic elements. Essentially, if γ belongs to a Levi $L(\lambda)$, we take \mathbf{f}_γ to be the product of the master symmetric functions of the factors, see Definition 6.28. This is uniquely determined by the Puiseux series of the branches of the spectral curve (which correspond to the different blocks of γ). We may also work with arbitrary F whenever $F(\gamma)$ is tamely and totally ramified.

As explained in Section 6, the Shalika germs are obtained from the master symmetric function \mathbf{f}_γ by expanding it in the plethystically transformed homogeneous symmetric functions \tilde{h}_λ (Lemma 3.8). Similarly, one gets what we call "Steinberg germs" from Theorem 2.19 by expanding in the untransformed homogeneous symmetric functions.

On the harmonic analysis side, our recursion boils down to writing $\gamma^\zeta = \gamma - a_d u^{nr_d}$. It lies in the center of the centralizer of γ , which is F'^* for some tamely ramified extension F'/F of degree e (see Section 6). Let d be the F' -valuation of $\gamma - \gamma^\zeta$. We can use the Shalika expansion for γ^ζ and compare the two.

This translates the Dyck path recursion to the following formula for the Shalika germs using compositions, see Theorem 6.20.

Theorem 1.8. *The transition matrix between the Shalika germs for γ and γ^ζ is given by*

$$M_{\lambda, \lambda'} = \left(c_{\lambda'} \sum_{\mu \vdash n'} \frac{|S_\lambda \cap C_\mu|}{b_\mu \lambda!} \prod_{i=1}^{\ell(\mu)} \left(\sum_{\alpha \in e\mu_i} \text{wt}(\alpha)_{d/e} \tilde{h}_\alpha \right) \right) \Big|_{\tilde{h}_\lambda}$$

where $|\tilde{h}_\lambda$ means we pick the coefficient of \tilde{h}_λ . Here $\text{wt}(\alpha)_{d|e}$ is as in Eq. (5.11), $|S_\lambda \cap C_\mu|$ is the number of elements in the symmetric group S_n which belong to both the Young subgroup S_λ and have cycle type μ , and

$$c_{\lambda'} = (1-q)^{n'} [\lambda']_q!, \quad b_\mu = \prod_{i=1}^{\ell(\mu)} (1-q^{\mu_i})$$

Remark 1.9. So far, we have been restricting to the case where F'/F is totally ramified. In the case where the residue field is $k = \mathbb{F}_q$ and our extension is unramified, the above theorem also works with appropriate modifications, as explained in Section 6. Of course, in this case one needs to modify the statement of Theorem 1.7 to account for intermediate unramified extensions.

Finally, the point counts of affine Springer fibers/their weight polynomials/integrals of characteristic functions of standard parahorics over the orbit of γ are obtained easily from the master symmetric function by pairing it with elementary symmetric functions:

Theorem 1.10. *Let $\lambda \vdash n$ be a partition and $\mathbf{1}_\lambda$ the characteristic function of the Lie algebra of the associated standard parahoric. Then*

$$I_\gamma(\mathbf{1}_\lambda) = q^{\dim \text{Sp}_\gamma} \langle \mathbf{f}_\gamma, e_\lambda \rangle|_{q \rightarrow q^{-1}}$$

where we pair using the usual Hall inner product and Sp_γ is the affine Springer fiber of γ . See Remark 5.11 for the normalizations.

The following is more or less obvious from above and has been a folklore conjecture for quite long. We give details in Section 7.2.

Corollary 1.11. *The point-counts of (local) compactified Jacobians of plane curves are polynomials in q and only depend on the Newton-Puiseux pairs of γ . In addition, they are polynomials with nonnegative integer coefficients.*

Proof. Theorem 1.10 combined with Proposition 7.10 implies that

$$q^{\dim \text{Sp}_\gamma} \langle \mathbf{f}_\gamma, e_\lambda \rangle|_{q \rightarrow q^{-1}}$$

is the number of points of the projective variety $X_\gamma := \text{Sp}_\gamma^\lambda / \Lambda$ after spreading out and modding out by q outside a finite set of primes. On the other hand, it is well known that $\langle e_\mu, e_\lambda \rangle$ counts the number of certain nonnegative integer matrices with row sums μ and column sums λ and in particular is a nonnegative integer. Since $\mathbf{f}_\gamma = \sum a_\gamma^\lambda(q) e_\lambda$ with $a_\gamma^\lambda(q) \in \mathbb{N}[q]$ by Theorem 1.7, we get

$$|X_\gamma(\mathbb{F}_q)| \in \mathbb{N}[q]$$

as desired. □

Note that the element γ does not have to be elliptic for this to hold. Indeed, as explained in Section 2 the point-counts on regular semisimple affine Springer fibers can be reduced to those of elliptic elements. The above is also in line with the expectation that all local compactified Jacobians (for elliptic γ , say) are paved by affines.

Our final result follows from our method using the elliptic Hall algebra. In Section 8 we connect this Theorem to the geometry of Hilbert schemes of points on \mathbb{A}^2 .

Theorem 1.12. *The master symmetric function \mathbf{f}_γ admits a canonical t -deformation $\widehat{\mathbf{f}}_\gamma$ which admits a t -deformed version of the Shalika germ expansion:*

$$\widehat{\mathbf{f}}_\gamma = \sum_{\lambda \vdash n} \widetilde{\Gamma}_\lambda(\gamma) \widetilde{H}_\lambda$$

where \widetilde{H}_λ are the modified Macdonald polynomials.

It is unlikely the methods in this paper will yield results for other groups, in that the elliptic Hall algebra seems to be confined to work with $G = GL_n$ only. There are also a number of geometric simplifications in this case for the affine Springer fibers. It is however interesting to ponder what part of the theory carries through to other G .

Another direction of generalization is to the wildly ramified elements. For some of our methods, especially on the harmonic analysis side, there is no issue in using other nonarchimedean fields, in particular extensions of \mathbb{Q}_p . The geometric interpretation of orbital integrals from [25, 38, 44] still goes through but the geometry is now replaced by the Witt vector affine flag varieties of [73]. It would be interesting to consider the geometry of affine Springer fibers in this case.

1.2. Outline of the paper. In Section 2, we review some general theory of orbital integrals and various versions of the Shalika germ expansion. In Section 3 we introduce background on symmetric functions, and in Section 4 we define and study a degenerate version of the Elliptic Hall Algebra. Section 5 is devoted to making the connection of our results to HOMFLY type knot invariants precise. It appears before the technical heart of the paper, Section 6, because results of the latter are strongly guided by the computation of the knot superpolynomials. Finally, we discuss some applications in Section 7 and the relationship of our results to the Hilbert scheme of points on \mathbb{A}^2 in Section 8.

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2. ORBITAL INTEGRALS

In this section, we fix k a finite field and let F be a non-archimedean local field with residue field k and \mathcal{O} its ring of integers. The group G will be GL_n or a product $\prod GL_{n_i}$ for which one can reduce the discussion to the former case. For any $\gamma \in G(F)$ and $f \in C_c^\infty(G(F))$ (henceforth complex-valued), the orbital integral is

$$I_\gamma(f) := \int_{g \in C_{G(F)}(\gamma) \backslash G(F)} f(g^{-1}\gamma g) dg$$

Definition 2.1. For γ semisimple, the measure dg is defined as follows: On $G(F)$ we have the up to a scalar unique Haar measure, which we will choose to be normalized so that $G(\mathcal{O})$ has measure 1. $C_{G(F)}(\gamma)$ is a product of general linear groups over extensions F' of F , and we use the same normalization. The same is done for $\gamma \in \mathfrak{g}(F)$ semisimple.

For general $\gamma \in G(F)$ or $\gamma \in \mathfrak{g}(F)$, the orbit of γ is locally closed and the centralizer unimodular, therefore the orbit of γ under the adjoint action admits an invariant measure. A way to see this is to identify the orbit with the orbit inside $\mathfrak{gl}_n(F)^*$ via the natural equivariant embedding of varieties $GL_n \hookrightarrow \mathfrak{gl}_n$ and the Killing form. The coadjoint orbit admits a natural symplectic structure (even in this non-archimedean setting), whose top wedge is an invariant volume form.

We will not attempt to fix a normalization for arbitrary γ . However when $\gamma \in G(F)$ is unipotent, we will normalize the measure as follows, following [36, 69]. Let $\lambda \vdash n$ and consider the standard (Richardson) parabolic $P = P(\lambda^\dagger)$ for the unipotent orbit associated to λ . Let N be its unipotent radical, M its Levi factor and for $f \in C_c^\infty(G(F))$ let $f^P \in C_c^\infty(M(F))$ be defined by

$$(2.1) \quad f^P(m) = \delta_P(m)^{1/2} \int_{G(\mathcal{O}) \times N(F)} f(k^{-1}mnk) dndk$$

where δ_P is the modular function for P . By [36, Proposition 5], the linear forms given by the unipotent orbital integrals $I_\lambda^G(-)$ for $\lambda \vdash n$ are proportional to $f \mapsto f^P(1)$. We normalize the measure on the unipotent orbit so that $I_\lambda^G(f) = f^P(1)$. This discussion applies verbatim with G replaced by \mathfrak{g} , etc., with the resulting formula being

$$(2.2) \quad f^{\mathfrak{p}}(X) = \int_{G(\mathcal{O}) \times \mathfrak{n}(F)} f(\text{ad}(k^{-1})(X + Y)) dY dk$$

and the required proportionality being to $I_\lambda^{\mathfrak{g}}(f) = f^{\mathfrak{p}}(0)$. We will drop the superscripts G, \mathfrak{g} when understood from the context. This is even more merited in view of

Lemma 2.2. *Let $f \in C_c^\infty(\mathfrak{g}(F)/\text{Lie}(\mathbf{I}))$ be the characteristic function of a standard parahoric, or more generally $f \in C_c^\infty(\mathfrak{g}(\mathcal{O})/\text{Lie}(\mathbf{I}))$. Then the restriction of f to $G(\mathcal{O}) \subset \mathfrak{g}(\mathcal{O})$ equals the characteristic function of the corresponding standard parabolic subgroup, and*

$$f^P(1) = f^{\mathfrak{p}}(0)$$

because $\text{Lie}(\mathbf{P}_\lambda) \cap G(\mathcal{O}) = \mathbf{P}_\lambda$ in this case.

2.1. Shalika germs. For any subset $\Omega \subset G(F)$ we denote by $J(\Omega)$ the space of invariant distributions on $G(F)$ supported on elements of the form $g^{-1}\gamma g$ with $g \in G(F)$, $\gamma \in \Omega$. The famous Howe conjecture states that

Theorem 2.3 ([36], [32], [16], [5]). *For any compact subset $\Omega \subset G(F)$ and an open subgroup $K \subset G(F)$, the restriction of $J(\Omega)$ to $C_c(G(F)/K)$ is finite-dimensional.*

The same is true when we replace Ω by a compact subset in $\mathfrak{g}(F)$, K by an open sub- \mathcal{O} -module in $\mathfrak{g}(F)$ and $C_c(G(F)/\Omega)$ by $C_c(\mathfrak{g}(F)/K)$. In this article we will make use of precise versions of the above finiteness. A particularly important one is the following theorem, proved by Hales [30, Thm. 1] for the span of regular semisimple orbital integrals, and which follows in general from Proposition 2.32 (see also Courtés [17, Thm 1.10]):

Theorem 2.4. *Recall that $G = GL_n$. Let $\mathcal{U} \subset G(F)$ be the (F -points of the) unipotent variety and $\mathbf{I} \subset G(F)$ be an Iwahori subgroup. Then the restriction of $J(G(\mathcal{O}))$ to $C_c(G(F)/\mathbf{I})$ is equal to that of $J(\mathcal{U})$ to $C_c(G(F)/\mathbf{I})$. Both restrictions have a basis given by unipotent orbital integrals.*

Remark 2.5. Hales [30] works over characteristic zero F , as does Waldspurger [69, 70]. However, the most essential ingredient for Hales is the original Shalika expansion [64], which also works in positive characteristic assuming finiteness of unipotent orbits and convergence of the unipotent orbital integrals. This was proved in [49] when the characteristic is *good* for G . In particular, for GL_n it holds in arbitrary characteristic. Moreover, the more general Proposition 2.32 works in arbitrary characteristic, so we do not have to worry about this issue.

Remark 2.6. As far as invariant distributions are concerned, any test function in $C_c(G(F)/K)$ can be averaged by K -conjugation into $C_c(K \backslash G(F)/K)$. Likewise in Theorem 2.4 we can replace $C_c(G(F)/\mathbf{I})$ by $C_c(\mathbf{I} \backslash G(F)/\mathbf{I})$.

The above theorem is a variant of the so-called Shalika germ expansion, reinterpreted as:

Corollary 2.7. *For any $\gamma \in G(\mathcal{O})$, there exists constants $\Gamma_\lambda(\gamma)$ where λ runs over unipotent orbits of $G(F)$ (i.e. partitions of n) such that for any $f \in C_c(G(F)/\mathbf{I})$, we have*

$$(2.3) \quad I_\gamma(f) = \sum_\lambda \Gamma_\lambda(\gamma) I_\lambda(f).$$

For the Lie algebra case one has the following which works also for arbitrary connected reductive group G provided that $\text{char } k \gg \text{rank } G$.

Theorem 2.8. *[Thm. 2.1.5., [18]] Let $\mathcal{N} \subset \mathfrak{g}(F)$ be the (F -points of the) nilpotent cone and $\text{Lie } \mathbf{I} \subset \mathfrak{g}(F)$ be an Iwahori subalgebra. Then the restriction of $J(\mathfrak{g}(\mathcal{O}))$ to $C_c(\mathfrak{g}(F)/\text{Lie } \mathbf{I})$ is equal to that of $J(\mathcal{N})$ to $C_c(\mathfrak{g}(F)/\text{Lie } \mathbf{I})$. Both restrictions have a basis given by nilpotent orbital integrals.*

In particular, the given bound when $G = GL_n$ is $\text{char } k > 2n$. Whenever the theorem works, it asserts the existence of unique functions $\Gamma_\lambda(\gamma)$ such that (2.3) holds for $f \in C_c(\mathfrak{g}(F)/\text{Lie } \mathbf{I})$ where $I_\lambda(f)$ is the integral on nilpotent orbits this time; by abuse of language we will denote the germs again by $\Gamma_\lambda(\gamma)$, thanks to the following proposition:

Proposition 2.9. *Let $\gamma \in \mathfrak{g}(\mathcal{O})$. Fix a nilpotent orbit λ in $\mathfrak{g}(F)$ – it corresponds to a unipotent orbit in $G(F)$ under $x \mapsto 1 + x$. Under this matching, the following are equal:*

- (1) *The Lie algebra Shalika germ $\Gamma_\lambda(\gamma)$ from Theorem 2.8.*
- (2) *The Lie algebra Shalika germ $\Gamma_\lambda(c + \gamma)$ from Theorem 2.8, for any $c \in \mathcal{O}$.*
- (3) *The Lie group Shalika germ $\Gamma_\lambda(c + \gamma)$ from Theorem 2.4 and Corollary 2.7, for any $c \in \mathcal{O}$ such that $c + \gamma \in G(\mathcal{O})$.*

Proof. (1) equals (2) since the central translation doesn't affect orbital integrals.

Just like the unipotent orbital integrals on $G(F)$ may be computed using Eq. (2.1), so may the nilpotent ones on $\mathfrak{g}(F)$ using Eq. (2.2). By Lemma 2.2 the restriction to $G(\mathcal{O})$ of the characteristic function of any standard parahoric subalgebra of $\mathfrak{g}(F)$ is the the characteristic function of the corresponding standard parahoric subgroup and the nilpotent orbital integral equals the unipotent orbital integral of the restriction. By [69, Corollaire 4.4.], the nilpotent and unipotent orbital integrals are determined as distributions on Iwahori-(bi-)invariant functions by their values on $\mathbf{1}_{\text{Lie}(\mathbf{P}_\lambda)} \in C_c(\mathfrak{g}(\mathcal{O})/\text{Lie } \mathbf{I})$ where $\lambda \vdash n$. Therefore (2) and (3) have the same Shalika germs, either for the group or the algebra. Therefore (2) equals (3). \square

Remark 2.10. Theorem 2.8 is expected to hold even if $\text{char } k \leq 2n$. For example, in [46, §5] it is shown that (2.3) holds for any fixed function and for fixed γ which is *quasi-regular* (see the abstract to [46], and note in particular that when $\text{char } k > n$ quasi-regularity is equivalent to regular semisimplicity) and in a small enough neighborhood $V_f \ni \gamma$. Consequently, (2.3) holds for quasi-regular elements in a small enough neighborhood and for all functions in $C_c(\mathfrak{g}(\mathcal{O})/\text{Lie } \mathbf{I})$, since the latter is finite-dimensional.

This is what we used in the proof of Proposition 2.9, and thus our Shalika germ results for tamely ramified regular semisimple γ will also work for Lemaire’s germs. Even more generally, in [46, §5.2, pp. 505] Lemaire defines normalized Shalika germs \tilde{b}_i defined on the set of all quasi-regular elements using homogeneity. Remark 6.18 together with propositions 2.11 and 2.9 show that such \tilde{b}_i also agree with the Lie algebra/group Shalika germs and in particular the above results, as well as the computations in Section 6 extend to this case as well given $f \in C_c(\mathfrak{g}(\mathcal{O})/\text{Lie } \mathbf{I})$.

In the Lie algebra setup, for a nilpotent orbit $\mathbb{O} \subset \mathfrak{g}(F)$ the Shalika germs enjoy the following homogeneity property.

Proposition 2.11. *For $G = GL_n$, we have*

$$\Gamma_{\mathbb{O}}(t\gamma) = |t|^{-\frac{1}{2} \dim \mathbb{O}} \Gamma_{\mathbb{O}}(\gamma).$$

Remark 2.12. For general reductive group when $\text{char } k$ is very good we have $\Gamma_{\mathbb{O}}(t^2\gamma) = |t|^{-\dim \mathbb{O}} \Gamma_{t^{-2}\mathbb{O}}(\gamma) = |t|^{-\dim \mathbb{O}} \Gamma_{\mathbb{O}}(\gamma)$ since \mathbb{O} and $t^{-2}\mathbb{O}$ are the same orbit in $\mathfrak{g}(F)$. However \mathbb{O} and $t^{-1}\mathbb{O}$ are typically different orbits and the identity in Proposition 2.11 does not hold in general.

Remark 2.13. In terms of partitions, if $\mathbb{O} \leftrightarrow \lambda \vdash n$,

$$\frac{1}{2} \dim \mathbb{O} = \sum_{i=1}^{\ell(\lambda^t)} (i-1)\lambda_i^t =: n(\lambda^t)$$

2.2. Steinberg germs. Another way to make use of Theorem 2.4, analogous to that of Corollary 2.7, is proposed by Waldspurger [70, Prop. 2.4.]. In fact, Waldspurger goes further to study the space of distributions $J(G(F)_c)$, where $G(F)_c$ is defined as follows.

Definition 2.14. Let us say an element $g \in G(F) = GL_n(F)$ is *compact mod center* if all its eigenvalues have the same valuation; we will suppress the “mod center” and just call them *compact* when no confusion should arise. Let $G(F)_c \subset G(F)$ be the subset of all compact elements. We will denote its characteristic function by $\mathbf{1}_c$.

Let us also call $\gamma \in \mathfrak{g}(F)$ compact if it belongs to some parahoric subalgebra, or equivalently that it is conjugate to an element in $\mathfrak{g}(\mathcal{O})$.

Remark 2.15. Note that in the group case, not all compact-modulo-center elements are literally central translations of compact elements. Instead, they become compact in the usual sense under the map to PGL_n .

We will also need the following alternative characterization of the compact modulo center elements in GL_n :

Lemma 2.16. *The set $G(F)_c$ coincides with the union of conjugates of normalizers of standard parahorics. In particular, for GL_n we have*

$$G(F)_c = \text{Ad}(G(F)) \left(\bigcup_{\substack{e|n, \alpha \vdash n/e, \\ d \in \mathbb{Z}}} u^{nd/e} \mathbf{P}_{\alpha^e} \right)$$

where u is the matrix from the introduction and \mathbf{P}_{α^e} is a standard parahoric subgroup.

Remark 2.17. Note that $u^{nd/e}, d \in \mathbb{Z}$, normalizes \mathbf{P}_{α^e} . Compare also to the beginning of [70, Section 4].

Definition 2.18. Let St_n be the Steinberg representation of $G(F) = GL_n(F)$. More generally, for λ any partition of n (Definition 3.1) let $P(\lambda) \subset GL_n(F)$ be the corresponding parabolic and $L(\lambda)$ its Levi subgroup. Let St_λ be the parabolic induction³ of the Steinberg representation of $L(\lambda)$ to $G(F)$

We will henceforth identify St_λ with its character, an invariant distribution on $G(F)$. We denote by $\text{St}_{\lambda,c}$ the restriction of St_λ to $G(F)_c$, i.e. the distribution which is truncated to be 0 outside $G(F)_c$. Denote by $J_{\text{St},c} \subset J(G(F)_c)$ the subspace spanned by $\text{St}_{\lambda,c}$. Write $C_c(\mathbf{I} \backslash G(F) / \mathbf{I}) = \bigoplus_{k \in \mathbb{Z}} C_c(\mathbf{I} \backslash G(F)^{\text{val}=k} / \mathbf{I})$ where $G(F)^{\text{val}=k}$ is the subset of elements whose determinant has valuation k . Its characteristic function is denoted $\mathbf{1}_k$. In [69, Proposition 2.4.] and [70, Proposition III 4.], when $\text{char } F = 0$, Waldspurger proved:

Theorem 2.19. *For $k \in \mathbb{Z}$ the restriction of $\text{St}_{\lambda,c}$ to $C_c(\mathbf{I} \backslash G(F)^{\text{val}=k} / \mathbf{I})$ is non-zero iff λ is divisible by $n / \gcd(k, n)$. The restriction of $J_{\text{St},c}$ to $C_c(\mathbf{I} \backslash G(F)^{\text{val}=k} / \mathbf{I})$ has a basis given by the restrictions of these $\text{St}_{\lambda,c}$.*

Theorem 2.20. *For any regular semisimple $\gamma \in G(F)^{\text{val}=d}$ that is compact mod center, there exists unique constants $\Gamma_\lambda^{\text{St}}(\gamma)$ where $\lambda \vdash \gcd(d, n)$, so that for any $f \in C_c(\mathbf{I} \backslash G(F)^{\text{val}=d} / \mathbf{I})$ we have*

$$(2.4) \quad I_\gamma(f) = \sum_\lambda \Gamma_\lambda^{\text{St}}(\gamma) \text{St}_{n',\lambda,c}(f)$$

where $n' := n / \gcd(d, n)$. We shall call the constants $\Gamma_\lambda^{\text{St}}(\gamma)$ the Steinberg germs of γ .

Remark 2.21. A few remarks are in order. The most important one is that in [69, 70] there is an assumption on $\text{char } F = 0$. However, it is easy to see that the proof of [69, Proposition 2.4.] only uses characteristic-independent facts about the representation theory of $G(F)$. In the next section, Section 2.3, we will in particular construct unique distributions $\text{St}_{\lambda,c}$ satisfying Theorems 2.19, 2.20 and [70, V 11, V12]. If one were able to carry out Clozel's work in [16] in positive characteristic, then these $\text{St}_{\lambda,c}$ could safely be identified with the truncated characters of parabolic inductions of Steinberg representations. While it is somewhat awkward we cannot do this right now, it will not affect the computation of the Shalika germs themselves.

Further, we note that the "St"-superscript stands for Steinberg, and should not be confused with the notion of "stability" in the automorphic forms literature.

³Parabolic inductions are normalized by the usual modulus character. But in fact, the modulus character is trivial on compact (mod center) elements because it is a homomorphism to (\mathbb{R}^+, \times) that is trivial on center. Since we will immediately restrict to compact elements, one can ignore the normalization issue.

2.3. Study of germs via rigid cocenters of affine Hecke algebras. This subsection is almost independent from the rest; here we digress to mention an approach to the Howe conjecture using cocenter of the (extended) affine Hecke algebra

$$\mathcal{H} := C_c(\mathbf{I}\backslash G(F)/\mathbf{I})$$

following [15, 33].

Note that as an abstract algebra, \mathcal{H} only depends on the *residue field* of F (more precisely the size), through for example the well-known description by generators and relations. In particular, the results in this subsection apply in all characteristics and can be viewed as alternative proofs for results in the previous two subsections, as well as strengthening those in [33, Section 5]. If the reader is more geometrically inclined, it does no harm to skip this subsection and black-box the transfer of characteristic zero results from the previous chapter to (possibly very large) positive characteristic using e.g. the theory of "nearby fields" (corps proches) or the model-theoretic apparatus.

Consider the space $J(G(F)_c)$ of invariant distributions supported on the compact mod center elements of $G(F)$. There is a natural map

$$J(G(F)_c) \rightarrow \mathcal{H}^*$$

to the linear dual of the AHA given by evaluating the distributions on the functions. The $G(F)$ -invariance of the distributions amounts to this map factoring through the *cocenter*

$$(2.5) \quad J(G(F)_c) \rightarrow (\mathcal{H}/[\mathcal{H}, \mathcal{H}])^* \rightarrow \mathcal{H}^*$$

We will soon see that the map in Eq. (2.5) can be further shown to factor through the dual of the *rigid* cocenter introduced in [15]. First, note that \mathcal{H} and $\mathrm{tr}(\mathcal{H}) := \mathcal{H}/[\mathcal{H}, \mathcal{H}]$ are graded by the valuation of the determinant, i.e. as before, we have the decomposition

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} C_c(\mathbf{I}\backslash G(F)^{\mathrm{val}=k}/\mathbf{I})$$

and similarly for the cocenter. Let us denote $\mathcal{H}^{\mathrm{val}=k}$, $\mathrm{tr}(\mathcal{H})^{\mathrm{val}=k} := \mathrm{tr}(\mathcal{H}^{\mathrm{val}=k})$ the corresponding subspaces.

This is further refined by the *Newton decomposition* of the group $G(F)$ as well as \mathcal{H} from [33]. From the Cartan decomposition, we have $\mathbf{I}\backslash G(F)/\mathbf{I} \cong \widetilde{W}$, where $\widetilde{W} \cong \mathbb{Z}^n \rtimes S_n$ is the extended affine Weyl group associated to $G = GL_n$. We also write $W^{\mathrm{fin}} := S_n$ in the above. We will identify $X_*(T) \cong \mathbb{Z}^n$ in a standard way. We have a decomposition $\widetilde{W} \cong \Omega \rtimes W^{\mathrm{aff}}$ where W^{aff} is the affine Weyl group and $\Omega \cong \mathbb{Z}$. Consider the conjugation action of \widetilde{W} on itself.

Definition 2.22. The *Kottwitz map* is the projection $\kappa : \widetilde{W} \rightarrow \Omega$. The *Newton map* $\nu' : \widetilde{W} \rightarrow \frac{1}{n}\mathbb{Z}^n$ is defined as follows. If $w^k \cdot x = \lambda + x$ we let $\nu'(w) = \lambda/k$. By sorting, we may take this to be the unique dominant element in the W^{fin} -orbit of $\nu'(w)$, which gives another map $\nu^+ : \widetilde{W} \rightarrow (\frac{1}{n}\mathbb{Z}^n)_+$.

Together, we get a map

$$\pi := (\kappa, \nu^+) : \widetilde{W} \rightarrow \Omega \times (\frac{1}{n}\mathbb{Z}^n)_+ =: \mathfrak{K}$$

Definition 2.23. For $\nu \in \mathfrak{K}$, the *Newton stratum* of $G(F)$ is

$$G(\nu) := \bigcup_{w \in \widetilde{W} \text{ minimal length, } \pi(w) = \nu} \mathrm{Ad}(G(F))\mathbf{I}w\mathbf{I}$$

Theorem 2.24 (Theorem A, [33]). *We have the Newton decomposition of $G(F)$ is*

$$G(F) = \bigsqcup_{\nu \in \mathfrak{N}} G(\nu)$$

i.e. the group decomposes as a disjoint union of Newton strata.

For the algebraic/combinatorial point of view, we have the following theorem of He and Nie [34, Thm. 6.7], which asserts

Theorem 2.25. *The cocenter $\mathrm{tr}(\mathcal{H})$ has a basis consisting of (the image of) characteristic functions $T_w := [\mathbf{I}w\mathbf{I}]$ for any set of choices of one minimal-length representative w from each conjugacy class in \widetilde{W} ; the image of T_w in $\mathrm{tr}(\mathcal{H})$ is independent of the choice of such representative.*

Based on the results in [34], He proved the following theorem

Theorem 2.26 (Theorem C and Theorem 11, [33]). *The cocenter of the Iwahori-Hecke algebra \mathcal{H} is spanned by the images of T_w for minimal length representatives for conjugacy classes in \widetilde{W} and we have a Newton decomposition*

$$(2.6) \quad \mathrm{tr}(\mathcal{H}) = \bigoplus_{\nu \in \mathfrak{N}} \mathrm{tr}(\mathcal{H})_\nu$$

where $\mathrm{tr}(\mathcal{H})_\nu$ is spanned by the images of the Iwahori-Matsumoto generators T_w where $\pi(w) = \nu \in \mathfrak{N}$ and w is of minimal length.

We finally define the *rigid cocenter* following [15] and its relation to $J(G(F)_c)$. Let \mathcal{A} be the apartment for $T \subset GL_n$. \mathcal{A} can be identified with the metric space \mathbb{R}^n decorated with the affine hyperplanes $\{x_i - x_j = m\}$ for any $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $m \in \mathbb{Z}$, so that \widetilde{W} acts on \mathcal{A} by affine isometries. This action is transitive on the alcoves, which are by definition the connected components of the complement of all the affine hyperplanes.

Fix an alcove $C \subset \mathcal{A}$. For example, one can take the standard alcove

$$C = \{(x_1, \dots, x_n) \in \mathcal{A} \mid x_1 < \dots < x_n < x_1 + 1\}$$

In the previous semi-direct product $\widetilde{W} \cong W^{\mathrm{aff}} \rtimes \Omega$ we have W^{aff} is the affine Weyl group - the subgroup generated by reflections about the hyperplanes, and $\Omega = \mathrm{Stab}_{\widetilde{W}}(C) \cong \mathbb{Z}$. Note that the Kottwitz map $\kappa : \widetilde{W} \rightarrow \Omega$ together with the Cartan decomposition induces a homomorphism $G(F) \rightarrow \Omega$, which is, abusing notation slightly, the so-called Kottwitz map κ on $G(F)$. This map can be identified with

$$\kappa := \mathrm{val} \circ \det : G(F) \rightarrow \mathbb{Z}$$

as before. Note that the the image of the matrix u from Eq. (1.1) under κ can be identified with a generator of Ω .

Definition 2.27. We say $w \in \widetilde{W}$ is *compact* if $w.x = x + k$ for some $x \in \mathcal{A}$, $k \in \frac{1}{n}\mathbb{Z}$. In other words, w has a "fixed point mod center".

For example, any $w = (w', x) \in S_n \rtimes \mathbb{Z}^n$ with w' elliptic in the usual sense, is compact. In the other extreme, if $w' = 1$ then w is compact iff x has all coordinates equal. Equivalently, it is easy to see that compactness of $w \in \widetilde{W}$ is equivalent to centrality of the Newton point $\nu'(w) \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Note also that when w is compact, $nk \in \mathbb{Z}$ agrees with the image of w in Ω .

Remark 2.28. This terminology is not used in [15, 33] but we will soon see it coheres well with the rigid cocenter.

Definition 2.29. The *rigid cocenter* of \mathcal{H} is

$$\mathrm{tr}(\mathcal{H})^{rig} := \bigoplus_{\substack{\nu=(\kappa, \nu') \in \mathfrak{N}, \\ \nu' \text{ central}}} \mathrm{tr}(\mathcal{H})_\nu$$

By [15], $\mathrm{tr}(\mathcal{H})^{rig}$ is exactly the image of the subspace of \mathbf{I} -bi-invariant C^∞ -functions on $G(F)$ represented by functions supported on the compact-mod-center elements. More precisely, we have

Proposition 2.30. *The set $G(F)_c \subset G(F)$ of compact-mod-center elements in $G(F)$ is exactly*

$$G(F)_c = \bigsqcup_{\substack{\nu=(\kappa, \nu') \in \mathfrak{N}, \\ \nu' \text{ central}}} G(F)_\nu$$

Note further that $\mathrm{tr}(\mathcal{H})^{rig}$ is still graded by $k \in \Omega$ and this is the restriction of the grading coming from $\mathrm{val} \circ \det$. The previous Proposition gives us

Corollary 2.31. *Identify $\mathrm{tr}(\mathcal{H})^{rig}$ as a direct summand of $\mathrm{tr}(\mathcal{H})$ (as vector spaces) using (2.6). Then the map (2.5) factors as*

$$J(G(F)_c) \rightarrow (\mathrm{tr}(\mathcal{H})^{rig})^* \hookrightarrow (\mathrm{tr}(\mathcal{H}))^*$$

In particular, the image of a distribution in $J(G(F)_c)$ under the map (2.5) is determined by its image in $(\mathrm{tr}(\mathcal{H})^{rig})^$.*

We will be interested in understanding the restrictions of the unipotent orbital integrals $I_\lambda(-)$ and the truncated Steinberg characters $\mathrm{St}_{\lambda, c}$ to $\mathrm{tr}(\mathcal{H}^{rig})$. Further, we want to understand the truncations of the latter for a fixed $k \in \Omega$. Combining Theorem 2.25 and Corollary 2.31, we get

Proposition 2.32. *The image of any $D \in J(G(F)_c)$ under (2.5) is determined by $D([\mathbf{I}w\mathbf{I}])$ where w runs over a set of minimal length representatives for conjugacy classes in \widetilde{W} s.t. the Newton factor of $\pi(w)$ is central.*

Imitating [70, IV 1.] we define

Definition 2.33. Suppose $\lambda \in P(n/e)$. Let $f_\alpha^{d, e} \in \mathcal{H}^{\mathrm{val}=\frac{nd}{e}}$ be the characteristic function of $u^{nd/e}\mathbf{P}_{\alpha^e}$ where \mathbf{P}_{α^e} is the standard parahoric associated to α^e . Here u is the matrix in Eq. (1.1) where t is a uniformizer for F .

Note that by Lemma 2.16 $f_\alpha^{d, e}$ is supported on the compact-mod-center elements and therefore its image in the cocenter lies in the rigid part.

Let us now fix $k \in \mathbb{Z} \cong \Omega$ and only look at the span $[\mathbf{I}w\mathbf{I}]$ for $w \in W^{\mathrm{aff}} \times \{k\}$ that are compact, i.e. restrict to $\mathrm{tr}(\mathcal{H})^{rig, \mathrm{val}=k}$. Let $e = n/\mathrm{gcd}(n, k)$ and $d = ke/n$, so that $k = nd/e$. Note that as $\Omega \subset \widetilde{W}$ we can also view k as a length zero element in \widetilde{W} . From Proposition 2.30 it is then not hard to see the following:

Proposition 2.34. *There is a basis of $\mathrm{tr}(\mathcal{H})^{rig, \mathrm{val}=k}$ given by (images in the cocenter of) characteristic functions $[\mathbf{I}w_i\mathbf{I}]$ where $w_i = (\sigma_i, k) \in W^{\mathrm{aff}} \times \Omega$ with $\sigma_i \in S_{n/e}$ a set of minimal length representatives for conjugacy classes in $S_{n/e}$. Here we embed (as sets)*

$$S_{n/e} \hookrightarrow S_n \hookrightarrow W^{\mathrm{aff}} \times \{k\} \subset \widetilde{W}$$

where the first inclusion is given by permutation of the first n/e elements. In particular, this space has dimension the number of partitions of n/e .

Proof. By construction, each w_i is compact, of minimal length, and the elements are in distinct conjugacy classes. For each $k \in \Omega$ we would like to know the number of compact conjugacy classes in $W^{\text{aff}} \times \{k\}$. If it is the number of partitions of n/e we are done.

Note that the map $G(F) \twoheadrightarrow \Omega$ is given by $g \mapsto \text{val}(\det(g)) \in \mathbb{Z} \cong \Omega$, and there is a section $\widetilde{W} \hookrightarrow GL_n(F)$ with image generated by permutation matrices and diagonal matrices with diagonal entries in $t^{\mathbb{Z}}$, where $t \in F$ is a fixed uniformizer. It's easy to see that an element in \widetilde{W} is compact iff its image in $GL_n(F)$ under this section is compact, which is the case iff all eigenvalues for a $g \in GL_n(F)$ have the same valuation.

When $\text{val}(\det(g)) = k$, that g is compact is equivalent to that all eigenvalues have valuations $k/n = d/e$ (the latter is the irreducible fraction). For compact g in the image of the section $\widetilde{W} \subset GL_n(F)$, we need each cycle of the permutation to have length divisible by $e = n/\text{gcd}(k, n)$. Conversely, for every partition of n for which all parts are divisible by e , we have a unique compact conjugacy class in $W^{\text{aff}} \times \{k\}$ mapping to k . Hence the space has dimension equal to the number of partitions of n/e . \square

Example 2.35. If $n = 4$, $e = 2$, $d = 1$, $k = 2$, then $S_{n/e} = S_2$ which is abelian. The elements $(1, 2), (s, 2) \in W^{\text{aff}} \times \Omega$ send $(a, b, c, d) \in \mathbb{R}^4$ to $(c + 1, d + 1, a, b)$ and $(d + 1, c + 1, a, b)$ respectively. As elements of $\widetilde{W} = S_n \times \mathbb{Z}^n$ we have $w_1 = ((13)(24), (0, 0, 1, 1))$ and $w_2 = ((1324), (0, 0, 1, 1))$.

Corollary 2.36. *The images of the functions $f_\alpha^{d,e}$ for $\alpha \vdash n/e$ also give a basis of $\text{tr}(\mathcal{H})^{\text{rig, val}=k}$ indexed by partitions of n/e .*

Proof. The linear independence is clear by imitating e.g. [69, Corollaire 4.4.] again. By Proposition 2.34 the dimension is the number of partitions of n/e . \square

By the Corollary, for any $w \in W^{\text{aff}} \times \{k\} \subset \widetilde{W}$ we can write

$$[\mathbf{I}w\mathbf{I}] = \sum_{\alpha \vdash \frac{n}{e}} c(w, \alpha) f_\alpha^{d,e}$$

and for chosen w_i as above, the matrix with entries $c(w_i, \alpha)$ is a change-of-basis matrix. By the well-known generators-and-relations description of the Iwahori-Hecke algebra \mathcal{H} , where $[\mathbf{I}w\mathbf{I}]$ corresponds to the "standard basis" T_w , together with the Cartan decomposition of $G(F)$, we see that $c(w, \alpha)$ are rational functions in q that depend only on n , k and α , but *not* on the local field F . We forego the explicit computation of these rational functions, although it should be an interesting exercise.

By Lemma 2.16 combined with Definition 2.33 and Proposition 2.30 we also have that the image of a distribution $D \in J(G(F)_c)$ in $(\text{tr}(\mathcal{H})^{\text{rig, val}=k})^*$ is also determined by $D(f_\alpha^{d,e})$ where $d, e, k \in \mathbb{Z}$ are as before, and $\alpha \vdash n/e$ is a partition.

Consider now the unipotent orbital integrals I_λ and let $k = 0$. Let $f_\alpha = f_\alpha^{0,1}$ be the characteristic function of the standard parahoric \mathbf{P}_α . Pairing with f_α for varying α, λ , Proposition 3.13 (which is basically computed in [69, Proposition 4.2] whose proof is characteristic-independent) shows that we get an invertible matrix,

in particular the I_λ give a basis of $J(G(\mathcal{O})) = J(G(F)_c)^{\text{val}=0}$. Combined with Eq. (2.5) and Proposition 2.34, we get a new proof of Theorem 2.4. We also have

Theorem 2.37. *Fix n as before and let $d, e \in \mathbb{Z}$, with $e|n$ be varying. There exist unique element $\tilde{\text{St}}_{\lambda,c,k} \in (\text{tr}(\mathcal{H})^{\text{rig, val}=k})^*$ whose pairing with the family of test functions $f_\alpha^{d,e}$ is given by the right-hand-side of Proposition 3.15, that is*

$$\tilde{\text{St}}_{\lambda,c,k}(f_\alpha^{d,e}) = (-1)^{nd-k} \langle e_\alpha, h_\lambda \rangle$$

When $\text{char } F = 0$ the elements $\tilde{\text{St}}_{\lambda,c} = \sum_k \tilde{\text{St}}_{\lambda,c,k}$ coincide with the image of the truncated Steinberg characters $\text{St}_{\lambda,c}$ under (2.5). Moreover, these $\tilde{\text{St}}_{\lambda,c,k}$ for $\lambda \vdash n/e$ form a basis of $(\text{tr}(\mathcal{H})^{\text{rig, val}=k})^*$.

Proof. Theorem 2.19 combined with Proposition 3.15 shows the first assertion. The second follows from the invertibility of the square matrix given by the pairings $\tilde{\text{St}}_{\lambda,c,k}(f_\alpha^{d,e}) = (-1)^{nd-k} \langle e_\alpha, h_\lambda \rangle$ for varying α, λ . \square

Combining the results for all $k \in \mathbb{Z}$, we get an element $\tilde{\text{St}}_{\lambda,c} \in (\text{tr}(\mathcal{H})^{\text{rig}})^*$ that would serve as the image of version $\text{St}_{\lambda,c}$ under (2.5) satisfying Theorems 2.19 and 2.20.

Finally, let us note that given such $\tilde{\text{St}}_{\lambda,c}$, the essentially combinatorial, characteristic-independent proof of [70, Lemme V 12] goes through with the truncated Steinberg characters replaced by these $\tilde{\text{St}}_{\lambda,c}$. In particular, Proposition 6.9 goes through in arbitrary characteristic for tamely ramified γ .

2.4. Parabolic induction.

Definition 2.38. Suppose $M \subset P = MN \subset G$ are compatible Levi subgroup and parabolic subgroup defined over \mathcal{O} . For $f \in C_c^\infty(G(F))$, we define its parabolic restriction $\text{Res}_M^G(f) \in C_c^\infty(M(F))$ as

$$\text{Res}_M^G(f)(m) := \int_{G(\mathcal{O})} \int_{N(F)} f(gmng^{-1}) dn dg$$

where the measure is normalized so that $G(\mathcal{O})$ and $N(\mathcal{O})$ have measure 1.

Definition 2.39. Let M, N, P be as above and $\mathfrak{m} := \text{Lie } M$. For $f \in C_c^\infty(\mathfrak{g}(F))$, we define its parabolic restriction $\text{Res}_\mathfrak{m}^g(f) \in C_c^\infty(\mathfrak{m}(F))$ as

$$\text{Res}_\mathfrak{m}^g(f)(X) := \int_{G(\mathcal{O})} \int_{\text{Lie } N(F)} f(\text{Ad}(g)(X + Y)) dY dg$$

where the measure is normalized so that $G(\mathcal{O})$ and $\text{Lie } N(\mathcal{O})$ have measure 1.

Recall that $G(F)_c^{\text{val}=0} \subset G(F)$ is the subset of elements whose eigenvalues all have valuation 0. As $G = GL_n$, we realize $G(F)_c^{\text{val}=0}$ also as a subset of $\mathfrak{g}(F)$.

Lemma 2.40. *If $f \in C_c^\infty(G(F)_c^{\text{val}=0})$, then Definition 2.38 and 2.39 agree; we have $\text{Res}_M^G(f) = \text{Res}_\mathfrak{m}^g(f)$.*

Proof. In both definitions, the resulting $\text{Res}_M^G(f)$ and $\text{Res}_\mathfrak{m}^g(f)$ is evidently supported on $M(F)_c^{\text{val}=0}$. Here one may view M as a product of general linear groups and $M(F)_c^{\text{val}=0}$ is again the subset of elements whose all eigenvalues have valuation 0. For $m \in M(F)_c^{\text{val}=0}$, $\text{Ad}(m) : N(F) \rightarrow N(F)$ preserves the Haar measure on $N(F)$. This shows

$$\int_{N(F)} f(gmng^{-1}) dn = \int_{n(F)} f(g(m+n)g^{-1}) dn$$

and thus the two definitions agree. \square

Example 2.41. One has obviously that $\text{Res}_M^G(\mathbf{1}_{G(\mathcal{O})}) = \mathbf{1}_{M(\mathcal{O})}$ from Definition 2.38, and hence also $\text{Res}_{\mathfrak{m}}^{\mathfrak{g}}(\mathbf{1}_{G(\mathcal{O})}) = \mathbf{1}_{M(\mathcal{O})}$.

Proposition 2.42. *Suppose $\gamma \in M(F)$ is G -regular, meaning γ is regular when viewed as an element of G . Then*

$$I_{\gamma}^G(f) = |\det(\text{Ad}(\gamma)|_{\text{Lie } G/\text{Lie } M} - \text{id}_{\text{Lie } G/\text{Lie } M})|^{-1/2} \cdot I_{\gamma}^M(\text{Res}_M^G(f))$$

Proposition 2.43. *Suppose $\gamma \in \mathfrak{m}(F)$ is G -regular, meaning γ is regular when viewed as an element of \mathfrak{g} . Then*

$$I_{\gamma}^G(f) = |\det(\text{ad}(\gamma)|_{\text{Lie } G/\text{Lie } M})|^{-1/2} \cdot I_{\gamma}^M(\text{Res}_{\mathfrak{m}}^{\mathfrak{g}}(f))$$

Definition 2.44. We define $\text{Ind}_{\mathfrak{m}}^{\mathfrak{g}} : C_c^{\infty}(\mathfrak{m}(F))^* \rightarrow C_c^{\infty}(\mathfrak{g}(F))^*$ to be the adjoint of Res_M^G . It is called *parabolic induction*.

In particular, Proposition 2.43 effectively says that we have the equality of invariant distributions

$$|\det(\text{ad}(\gamma)|_{\text{Lie } G/\text{Lie } M})|^{-1/2} \cdot \text{Ind}_M^G I_{\gamma}^M(-) = I_{\gamma}^G(-)$$

Proposition 2.45. *Suppose \mathbb{O} is a nilpotent orbit of $\mathfrak{m}(F)$ and $\tilde{\mathbb{O}}$ the induced orbit in the sense of Lusztig-Spaltenstein, i.e. $\tilde{\mathbb{O}}$ contains an open dense subset of $\mathbb{O} + \text{Lie } N(F)$. Then $\text{Ind}_{\mathfrak{m}}^{\mathfrak{g}} I_{\mathbb{O}}^M = I_{\tilde{\mathbb{O}}}^G$.*

Remark 2.46. In terms of partitions (in the sense of Definition 3.1), if $M = GL_{n_1} \times \cdots \times GL_{n_r}$, and \mathbb{O} is a unipotent orbit corresponding to a sequence of partitions

$$\lambda^{(1)} \vdash n_1, \dots, \lambda^{(r)} \vdash n_r$$

the induced orbit is

$$\tilde{\mathbb{O}} \leftrightarrow (\lambda_1^{(1)} + \cdots + \lambda_1^{(r)}, \dots, \lambda_k^{(1)} + \cdots + \lambda_k^{(r)})$$

where k is the length of the longest $\lambda^{(i)}$. For example, when $M = T$, the zero orbit in T induces to the principal one in GL_n .

Corollary 2.47. *For $\gamma \in \mathfrak{m}(F)$ which is G -regular, we have*

$$\Gamma_{\tilde{\mathbb{O}}}^G(\gamma) = \begin{cases} 0 & \text{if } \tilde{\mathbb{O}} \text{ is not induced from } M, \\ |\det(\text{ad}(\gamma)|_{\text{Lie } G/\text{Lie } M})|^{-1/2} \cdot \Gamma_{\mathbb{O}}^M(\gamma) & \text{if } \tilde{\mathbb{O}} \text{ is induced from } \mathbb{O} \subset \mathfrak{m}(F). \end{cases}$$

where $\Gamma_{\mathbb{O}}^G(\gamma)$ is as in Theorem 2.8 and $\Gamma_{\mathbb{O}}^M(\gamma)$ is likewise but for $G = M$.

2.5. Moy-Prasad theory. Introducing some Moy-Prasad theory will be convenient for the statement of our results – we will do it in this section. However, this section may be skipped on a first reading, and is not essential for any of our proofs. We will only sketch this for $G = GL_n$. For any point the Bruhat-Tits building of G , denoted $x \in \mathfrak{B}$, as well as a real number $r \in \mathbb{R}$, Moy and Prasad define subspaces $\mathfrak{g}_{x, \geq r} \subset \mathfrak{g}(F)$ (and also $\mathfrak{g}_{x, r}, \mathfrak{g}_{x, > r}$, etc.). Suppose x lies in the apartment of our chosen maximal torus, then $\mathfrak{g}_{x, \geq r}$ can be described as

$$\mathfrak{gl}_n(F)_{x, \geq r} = \bigoplus_{\langle \alpha, x \rangle + i \geq r} t^i \mathfrak{g}_{\alpha}$$

e.g. for generic x this is an Iwahori subalgebra, and for $x = 0$ the hyperspecial $\mathfrak{g}(\mathcal{O})$.

There are analogous notions for the group G itself, which we denote $G_{x,\geq r}$. Basically, they can be defined using the exponential map between root subalgebras and subgroups. Again, if $x = 0$ (in our chosen coordinates) we get the hyperspecial and if x is generic, say in the fundamental alcove, we get the "standard" Iwahori.

Definition 2.48. The *depth* of $\gamma \in G(F)$ (respectively, $\gamma \in \mathfrak{g}(F)$) is the smallest r for which there exists $x \in \mathfrak{B}$ so that $\gamma \in G_{x,\geq r}$ (resp. $\gamma \in \mathfrak{g}_{x,\geq r}$).

Remark 2.49. For GL_n , the depth of γ coincides with its smallest root valuation.

3. SYMMETRIC FUNCTIONS AND COMBINATORICS

In this section, we review some theory of symmetric functions relevant to the computation of Shalika germs. The theory is very well covered in many sources, see for example [31, Section 3].

3.1. Combinatorics. We begin with two combinatorial definitions.

Definition 3.1. A *partition* of an integer $n > 0$, written $\lambda \vdash n$ or $\lambda \in P(n)$ is a nonincreasing sequence of positive integers

$$\lambda_1 \geq \dots \geq \lambda_k > 0, \quad \sum_i \lambda_i = n$$

and a *composition* of n , written $\alpha \vDash n$ is an ordered collection $(\alpha_1, \dots, \alpha_k)$ of positive integers such that $\sum_i \alpha_i = n$. In both cases, we write $\ell(\lambda) = \ell(\alpha) = k$ for the length of the composition or partition and denote by λ^t, α^t the conjugate partition (resp. composition).

We will draw the Young/Ferrers diagrams of partitions in French notation. We think of them as lying in $\mathbb{Z}_{\geq 0}^2$ with the first box always at $(0,0)$. For a box $\square \in \lambda$ with coordinates (i, j) we denote

$$(3.1) \quad a(\square) = \lambda_i - i - 1, \quad l(\square) = \lambda_j^t - j - 1, \quad a'(\square) = i, \quad l'(\square) = j$$

the arm, leg, coarm, and coleg lengths of the box. The q, t -content of a box is defined to be $q^{a'(\square)} t^{l'(\square)}$. Finally, we have

Definition 3.2. For two partitions (or compositions) λ, μ define

$$\mathbf{M}(\lambda, \mu)$$

to be the set of nonnegative integer matrices (of size $\ell(\lambda) \times \ell(\mu)$) whose rows sum to λ and columns sum to μ .

Definition 3.3. A *standard Young tableau* is a filling of the Ferrers diagram of $\lambda \vdash n$ with the letters $1, \dots, n$ such that the letters increase in columns and rows.

Given a Young tableau and a box \square_i labeled i , we define the arm length as $a(\square_i)$ and so on. We let z_i be the q, t -content of the box \square_i .

We will also need the following Lemma in Sections 5, 6.

Lemma 3.4. *To each composition $\alpha \vDash n$ is associated a unique Young tableau $T(\alpha)$ defined as follows. To each α_i we assign the sequence of numbers $\sum_{j=1}^{i-1} \alpha_j + 1, \sum_{j=1}^{i-1} \alpha_j + 2, \dots, \sum_{j=1}^i \alpha_j$ and form a tableau by taking one-row diagrams with these fillings, and then dropping them on top of each other, with the rule that gravity brings boxes as low as possible. In particular, the tableau decomposes as a sequence of horizontal α_i -strips.*

Example 3.5. To the compositions $4 = 2 + 2$, $4 = 1 + 2 + 1$ and $4 = 1 + 3$ we assign the tableaux

$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}$$

The final combinatorial gadget we will need are Dyck paths.

Definition 3.6. Let $m, n, k \geq 1$, $(m, n) = 1$. Then the set

$$\mathbb{D}_{km, kn}$$

will be the collection of lattice paths in a $kn \times km$ -rectangle in $\mathbb{Z}_{\geq 0}^2$, fitting under the diagonal (which has slope m/n).

The *area* of a Dyck path $D \in \mathbb{D}_{km, kn}$ is defined to be the number of full squares between the path and the diagonal.

3.2. The ring of symmetric functions. Let $\text{Sym}_{q,t}$ be the ring of symmetric functions over $\mathbb{Q}(q, t)$ in the alphabet $\{X_1, \dots, X_n, \dots\}$ and denote the five usual bases of monomial, homogeneous, elementary, Schur, and power sum symmetric functions by

$$\{m_\lambda\}, \{h_\lambda\}, \{e_\lambda\}, \{s_\lambda\}, \{p_\lambda\}$$

Here λ is a partition in the sense of Definition 3.1. Note that the first four are also bases of $\text{Sym}_{\mathbb{Z}}$ while the last one needs a ring containing \mathbb{Q} .

Recall that the *modified Macdonald polynomials* $\tilde{H}_\lambda[X; q, t]$, $\lambda \vdash n$ are the unique symmetric functions with the properties

$$(3.2) \quad \tilde{H}_\mu[X(1-q); q, t] \in \mathbb{Q}(q, t)\{s_\lambda \mid \lambda \geq \mu\}$$

$$(3.3) \quad \tilde{H}_\mu[X(1-t); q, t] \in \mathbb{Q}(q, t)\{s_\lambda \mid \lambda \geq \mu^t\}$$

$$(3.4) \quad \langle \tilde{H}_\mu[X; q, t], s_{(n)} \rangle = 1$$

Here the last pairing is the Hall inner product, defined in Definition 3.10.

We do not require much of the advanced theory of Macdonald polynomials, but let us note down the following definition as well as some specializations.

Definition 3.7. The operator ∇ of Bergeron and Garsia scales by definition each \tilde{H}_λ by $q^{n(\lambda)}t^{n(\lambda^t)}$ where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$.

From [31, Proposition 3.5.8.] we have

Lemma 3.8 (The limit of \tilde{H}_λ as $q \rightarrow 1$). *The modified Macdonald symmetric function \tilde{H}_λ at $q = 1$ is given by*

$$(3.5) \quad \tilde{H}_\lambda(X; 1, t) = (1-t)^{|\lambda|} [\lambda^t]_t! h_{\lambda^t}[X/(1-t)] =: \tilde{h}_{\lambda^t}[X; t]$$

in other words a plethystically transformed homogeneous symmetric function, up to normalization.

From the q, t -symmetry of \tilde{H}_λ we immediately have

Corollary 3.9 (The limit as $t \rightarrow 1$).

$$\tilde{H}_\lambda(X; q, 1) = \tilde{h}_\lambda[X; q]$$

We will denote $\tilde{h}_\lambda := \tilde{H}_\lambda[X; q, 1]$ and call these the *specialized Macdonald symmetric functions* or the *plethystically transformed homogeneous symmetric functions*. Later on, we will also need the prefactor

$$(3.6) \quad c_\lambda(q) := (1-q)^{|\lambda|} [\lambda]_q! = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (q^j - 1)$$

in Eq. (3.5).

Note that compared to the Macdonald polynomials, \tilde{h}_λ are much simpler in behaviour. For instance, they are multiplicative:

$$\tilde{h}_\lambda \tilde{h}_\mu = \tilde{h}_{\lambda+\mu}$$

and one can deduce combinatorial expansions for them in terms of the other standard bases via known relations between h_λ and these bases. This will turn out to be important in the proof of Theorem 6.20. Further, the ∇ -operator from Definition 3.7 becomes a ring homomorphism on symmetric functions in this limit.

We will also need a few different inner products on the ring of symmetric functions, the interplay of whom turns out to play a key role. We remark that by "inner product" we simply mean a symmetric bilinear form valued in $\mathbb{Q}(q, t)$.

Definition 3.10. (1) The *Hall inner product* is the inner product on $\text{Sym}_{q,t}$ defined by

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

(2) The *q-inner product* is

$$\langle f, g \rangle_q := \langle f, g[X/(1-q)] \rangle$$

(3) The *q, t-inner product* is

$$\langle f, g \rangle_{q,t} := \langle f, g\left[\frac{(1-q)}{(1-t)}X\right] \rangle$$

(4) The *geometric inner product* is

$$(f, g) = -q^{\deg f} \langle (\nabla^{-1}(f))[X(1-t^{-1})], g[X(1-t^{-1})] \rangle_{q,t^{-1}}$$

Remark 3.11. There is a natural Frobenius characteristic map from the direct sum of the representation rings of symmetric groups

$$\bigoplus_{n \geq 0} \text{Rep}(S_n) \rightarrow \text{Sym}_{\mathbb{Z}}$$

Endowing the source with the natural inner product on characters, and the target with the Hall inner product ((1) in the above Definition), this map is an isomorphism of Hopf algebras and an isometry.

Similarly, if we let

$$\text{Hall}(GL(\mathbb{F}_q))$$

be the Hall algebra of the general linear groups as n ranges from 0 to ∞ there is a natural inner product on this space, again coming from the convolution product on characters, and a map

$$\text{Hall}(GL(\mathbb{F}_q)) \rightarrow \text{Sym}_q$$

which is an isomorphism of Hopf algebras and an isometry with respect to the inner product (2) $\langle \cdot, \cdot \rangle_q$.

Remark 3.12. The last inner product will be used in Section 8 and is the one naturally arising from the geometry of Hilbert schemes of points. It can be more easily characterized as the unique inner product satisfying

$$(\tilde{H}_\lambda, \tilde{H}_\mu) = \delta_{\lambda\mu} g_\lambda$$

where

$$(3.7) \quad g_\lambda = \prod_{\square \in \lambda} (1 - q^{a(\square)} t^{-l(\square)-1}) (1 - q^{-a(\square)-1} t^{l(\square)})$$

3.3. Orbital integrals and symmetric functions. Let us review how symmetric functions arise in the theory of orbital integrals on GL_n , following [69, 70]. The following two propositions will be important to us.

Proposition 3.13 (Proposition 4.2. [69]). *Let ψ_λ be the characteristic function of the standard parahoric subgroup $\mathbf{P}_\lambda \subset GL_n(F)$ corresponding to the partition λ . Let $I_\mu(-)$ be the orbital integral over the unipotent class of type μ . Then we have*

$$(3.8) \quad I_\lambda(\psi_\mu) = c_\lambda(q) \langle e_\mu, h_\lambda \rangle_q$$

where $\langle -, - \rangle_q$ is the q -inner product from Definition 3.10. Note that in terms of the usual Hall scalar product,

$$I_\lambda(\psi_\mu) = \langle e_\mu, \tilde{h}_\lambda \rangle$$

Remark 3.14. Warning: Waldspurger uses the notation φ_λ for our ψ_λ . For him, ψ_λ denotes a different function. Note also that Waldspurger has an additional $c_\mu(q)/c_{(n)}(q)$ in the formula, this stems from a slightly different normalization for Shalika germs and the inner product.

Proposition 3.15. *Suppose $e \geq 1$, $(d, e) = 1$ and $\lambda \in P(n/e)$. Let $f_\alpha^{d,e}$ be the characteristic function of $u^{nd/e} \mathbf{P}_\lambda$ where \mathbf{P}_λ is as above. Here u is the matrix in Eq. (1.1).*

Then

$$\text{St}_{e\lambda}(\mathbf{1}_{nd/e} \mathbf{1}_c f_\mu^{d,e}) = (-1)^{nd-nd/e} \langle e_\mu, h_\lambda \rangle$$

where the pairing is the Hall inner product.

In particular, when $d = 0$, we have $\text{St}_{e\lambda}(\mathbf{1}_0 \mathbf{1}_c f_\mu^{0,e}) = \langle e_\mu, h_\lambda \rangle$ and $\mathbf{1}_0 \mathbf{1}_c f_\mu^{0,e} = \psi_\mu$.

4. THE ELLIPTIC HALL ALGEBRA

In this section, we define the elliptic Hall algebra (EHA) and recall some necessary facts about it. Apart from Theorem 4.9 results in this section are contained in [22, 29, 53–56, 61–63, 66]. For the basic theory, our main references are [22, 53, 66] and for the results on symmetric functions, one may refer to [53–56, 62].

For most of the paper, in particular for the application in the proofs of our main results in Section 2, we want to understand the $t \rightarrow 1$ degeneration of the Fock space representation of the EHA.

Definition 4.1. The *elliptic Hall algebra* (quantum toroidal \mathfrak{gl}_1) is the \mathbb{C} -algebra $\mathcal{E} = \mathcal{E}_{q_1, q_2, q_3}$ depending on $q_1, q_2, q_3 \in \mathbb{C}^\times$, $q_1 q_2 q_3 = 1$, generated by elements

$$P_{m,n}, (m, n) \in \mathbb{Z}^2 \setminus (0, 0)$$

and satisfying the relations

$$[P_{m_1, n_1}, P_{m_2, n_2}] = 0$$

if $(m_1, n_1), (m_2, n_2)$ lie on the same line through the origin, and

$$[P_{m_1, n_1}, P_{m_2, n_2}] = \frac{\theta_{m_1+m_2, n_1+n_2}}{\alpha_1}$$

if $(m_1, n_1), (m_2, n_2), (m_1 + m_2, n_1 + n_2)$ form a quasi-empty triangle. Here

$$\exp\left(\sum_{k=1}^{\infty} P_{km, kn} \alpha_k x^k\right) = \sum_{\ell=1}^{\infty} \theta_{\ell m, \ell n} x^\ell$$

for $(m, n) = 1$ and

$$\alpha_k = \frac{(q_1^k - 1)(q_2^k - 1)(q_3^k - 1)}{k}$$

Proposition 4.2 (Triangular decomposition). *Let $\mathcal{E}^>$ be the subalgebra generated by the $P_{1,n}, n \in \mathbb{Z}$, $\mathcal{E}^<$ be the subalgebra generated by the $P_{-1,n}, n \in \mathbb{Z}$, and \mathcal{E}^0 be the subalgebra generated by $P_{0,\pm k}, k \in \mathbb{Z}_{>0}$. The multiplication map gives a \mathbb{C} -linear isomorphism*

$$\mathcal{E}^< \otimes \mathcal{E}^0 \otimes \mathcal{E}^> \rightarrow \mathcal{E}$$

For the rest of this paper, we may as well restrict our attention to the positive part $\mathcal{E}^>$ of the EHA, or rather the nonnegative part \mathcal{E}^{\geq} which is by definition generated by $P_{1,n}, n \in \mathbb{Z}, P_{0,k}, k > 0$. Further, we wish to study the $q_3 \rightarrow 1$ limit of this algebra and the Fock space representation. The relationship of the parameters q_1, q_2, q_3 to the Macdonald theory parameters is $q_1 = q, q_3 = t^{-1}$ so that this limit amounts to setting $t = 1$. We will use these identifications freely.

Note that since the definition of \mathcal{E} is symmetric in the q_i , this choice is immaterial for many things. Importantly, it does matter for the Fock space representation (to be introduced soon), whose definition is not symmetric in the q_i .

Let us now remark on the structure of \mathcal{E} in the limit $t \rightarrow 1$ as an abstract algebra, although this will not be important for us. Consider the quantum torus in one variable, or in other words the algebra of q -difference operators on \mathbb{C}^\times . It is the $\mathbb{C}[q^\pm]$ -algebra

$$\mathfrak{D} := \mathbb{C}[q^\pm]\langle X^\pm, D^\pm \rangle / DX - qXD.$$

Considering this associative algebra as a Lie algebra we get a 2-dimensional central extension \mathfrak{D}_{c_1, c_2} [22] with central charges $c_1, c_2 \in \mathbb{C}$ defined as

$$[X^{i_1} D^{j_1}, X^{i_2} D^{j_2}] = (q^{j_1 i_2} - q^{j_2 i_1}) X^{i_1 + i_2} D^{j_1 + j_2} - \delta_{(i_1, j_1), (-i_2, -j_2)} q^{i_1 j_1} (i_1 c_1 + j_1 c_2).$$

By [22] we may view \mathcal{E} as a quantization of the universal enveloping algebra of \mathfrak{D}_{c_1, c_2} , and taking the $q_3 \rightarrow 1$ limit recovers just this universal enveloping algebra, at least up to a completion.

For example for the limit $q_3 \rightarrow 1$, we have (see [66, Proposition 5.6.]) that

$$(1 - q)P_{1,m} = D^m X, (q^{-1} - 1)P_{-1,m} = X^{-1} D^m$$

and

$$(1 - q^{-m})P_{0,m} = D^m.$$

Remark 4.3. It is not possible to directly set $q_3 \rightarrow 1$ in the defining relations of the EHA as given above. A way to circumvent this is to redefine:

$$\exp\left((1 - q_3)^{-1} \sum_{k=1}^{\infty} P_{km, kn} \alpha_k x^k\right) = \sum_{\ell=1}^{\infty} \theta_{\ell m, \ell n} x^\ell$$

or alternatively to rescale the generators of $\mathcal{E}^<$ by $1 - q_2$ and those of $\mathcal{E}^>$ by $1 - q_1$. Effectively, this gives an integral form of \mathcal{E} in the sense of Lusztig. See e.g. [66, Section 5.4.] and [57] for details.

As the name suggests, \mathcal{E} specializes to the Hall algebra of coherent sheaves on an elliptic curve over a finite field (when q_1 is the Frobenius eigenvalue on H^1 and q_2 its conjugate). In that setting, the slope of vector bundles gives rise to natural Hall subalgebras. These lift to \mathcal{E} , and are by definition the commutative subalgebras "living on lines through the origin".

Definition 4.4. Let $m, n \in \mathbb{Z}_{\geq 0}^2$, $(m, n) = 1$. The slope $\frac{m}{n}$ -subalgebra of $\mathcal{E}^{\geq 0}$ is the subalgebra $\mathcal{E}^{m/n}$ generated by $P_{km, kn}$, $k \geq 0$

Theorem 4.5 ([53]). Let $\text{Sym}_{q,t}$ be the algebra of symmetric functions over $\mathbb{C}(q, t)$ as introduced in Section 3. There is an algebra isomorphism

$$\varphi_{m/n} : \text{Sym}_{q,t} \rightarrow \mathcal{E}^{m/n}$$

sending $p_k \mapsto P_{km, kn}$.

We will call this homomorphism the *slope m/n plethysm*.

4.0.1. The Fock space.

Definition 4.6. The *Fock space* is the $\mathbb{C}(q, t)$ -vector space \mathcal{F} spanned by the basis

$$\{|\lambda\rangle\}_{\lambda \vdash n, n \geq 0}$$

Recall that \mathcal{F} appears naturally from the Hilbert scheme of points on \mathbb{A}^2 or symmetric functions over $\mathbb{C}(q, t)$. We will freely identify \mathcal{F} with the space of symmetric functions $\text{Sym}_{q,t}$ (see Section 3) so that the basis $|\lambda\rangle$ corresponds to the Macdonald basis \tilde{H}_λ . The reason for our usage of the Fock space as opposed to just $\text{Sym}_{q,t}$ will become clear in Section 8.

Theorem 4.7 ([23, 62]). There is an action of $\mathcal{E}_{q_1, q_2, q_3}$ on \mathcal{F} by so called *shuffle algebra operations*.

We will be interested in the action of the operators $P_{km, kn} \in \mathcal{E}^{\geq 0}$ and more generally the slope m/n subalgebras $\mathcal{E}^{m/n}$ in the Fock space, especially in the $t \rightarrow 1$ limit. For example, the operators $P_{0,m}$ act as multiplication by the symmetric functions p_m , and the operator $P_{1,0}$ is a so called Macdonald eigenoperator.

In [54] the matrix coefficients of the operators $P_{km, kn}$ in the basis $|\lambda\rangle$ are computed (see also [23]). Below the orthogonalizing inner product $\langle \lambda | \mu \rangle = \delta_{\lambda\mu} g_\lambda$ corresponds to the geometric inner product $(-, -)$ on symmetric functions, see Definition 3.10.

Theorem 4.8 ([54], see Eq. (37) in [29]). We have

$$\begin{aligned} \langle \lambda | P_{km, kn} | \mu \rangle &= \frac{\gamma^{kn}}{[k]_q} \cdot \frac{g_\lambda}{g_\mu} \sum_{\mu = \lambda + \square_1 + \dots + \square_{kn}}^{\text{SYT}} \left[\sum_{j=0}^{k-1} (qt)^j \frac{z_{n(k-1)+1} z_{n(k-2)+1} \dots z_{n(k-j)+1}}{z_{n(k-1)} z_{n(k-2)} \dots z_{n(k-j)}} \right] \\ &\cdot \frac{\prod_{i=1}^{kn} z_i^{S_{m/n}^{(i)}} (qt\chi_i - 1)}{\left(1 - qt \frac{z_2}{z_1}\right) \dots \left(1 - qt \frac{z_{kn}}{z_{kn-1}}\right)} \prod_{1 \leq i < j \leq kn} \omega'^{-1} \left(\frac{z_j}{z_i} \right) \prod_{1 \leq i \leq kn}^{\square \in \lambda} \omega'^{-1} \left(\frac{z(\square)}{z_i} \right) \end{aligned}$$

where

$$\omega'(x) = \frac{(x-1)(x-qt)}{(x-q)(x-t)}, \quad \gamma = \frac{(q-1)(t-1)}{qt(qt-1)}$$

and

$$S_{m/n}(i)' := \lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$$

Although we do not need the full strength of the formula in Theorem 4.8, it is recorded here for our computations in Section 5 and possible generalizations. The $t \rightarrow 1$ limit of this formula for $\mu = \emptyset$ is studied in Proposition 5.17.

We will now begin to study the degeneration of the representation on \mathcal{F} as $t \rightarrow 1$. The most important fact about the $t = 1$ limit is the following.

Proposition 4.9. *In the Fock representation at $t = 1$, the positive half $\mathcal{E}_{q,1/q,1}^{\geq}$ acts by multiplication operators.*

Proof. As shown in [57], the operators $P_{m,n}$ for $(m,n) \in \mathbb{Z} \times \mathbb{N}$ generate \mathcal{E}^{\geq} over $\mathbb{Z}[q_1^{\pm}, q_2^{\pm}]$, as do the operators $H_{m,n}$ which are defined by the identity

$$1 + \sum_{s=1}^{\infty} \frac{H_{sm,sn}}{x^s} = \exp\left(\sum_{s=1}^{\infty} \frac{P_{sm,sn}}{sx^s}\right)$$

By [56, Theorem 2.15.] one can write the action of either $H_{m,n}$ or $P_{m,n}$ as a contour integral, for example:

$$H_{m,n} \cdot f[X] = \int_{0 < X < |z_n| < \dots < |z_1| < \infty} \frac{z_i^{S'_{m/n}(i)}}{\prod_{i=1}^{n-1} (1 - qt \frac{z_{i+1}}{z_i}) \prod_{i < j} \omega'(z_j/z_i)} \wedge^{\bullet} \left(-\frac{X}{z_1}\right) \cdots \wedge^{\bullet} \left(-\frac{X}{z_n}\right) \cdot f\left[X - (1-q)(1-t) \sum_{i=1}^n z_i\right] \prod_{a=1}^n \frac{dz_a}{2\pi i z_a}$$

Where $\wedge^{\bullet}(-\frac{X}{z}) = \sum_{k=0}^{\infty} \frac{h_k}{z^k}$. Here the contours are concentric circles in the prescribed order and are contained between the poles $0, x_1, \dots, \infty$, see e.g. [54, 56] for details.

Now the plethystic operator

$$f[X] \mapsto f[X \pm (1-q)(1-t)z] = \exp\left[\pm \sum_{k=1}^{\infty} \frac{p_k^{\dagger} z^k}{k}\right] \cdot f[X]$$

at $t = 1$ becomes just the identity, so that this is a multiplication operator. \square

Remark 4.10. We note that this Proposition is conjectured in [8, 9].

Remark 4.11. The operators $P_{1,n}, n \in \mathbb{Z}$ can be described as follows, see e.g. [9]. In plethystic notation, their action on the Fock space is given by

$$P_{1,n} \cdot f[X] = f\left[X + \frac{(1-t)(1-q)}{z}\right] \sum_{i \geq 0} (-z)^i e_i[X] \Big|_{z^n}$$

where by $|_{z^n}$ we mean extracting the coefficient of z^n in this series. At $t = 1$ this becomes just multiplication by e_n . It is however not true that the algebra generated by these operators over $\mathbb{Z}[q_1^{\pm}, q_2^{\pm}]$ is all of \mathcal{E}^{\geq} anymore.

In addition to the $P_{m,n}$ we want to understand the elements $E_{km, kn} := \varphi_{m/n}(e_k)$ from [55, 61, 62] in the limit $t \rightarrow 1$.

Proposition 4.12. *Suppose that $m, n > 0$ and $\gcd(m, n) = 1$. At $t = 1$ the operator $\varphi_{m/n}(e_k)|_{t=1}$ becomes a multiplication operator by the symmetric function:*

$$E_{m,n,k} := \sum_{D \in \mathbb{D}_{km, kn}} q^{\text{area}(D)} e_D.$$

Here D is a Dyck path in $(km \times kn)$ rectangle below the diagonal, $\text{area}(D)$ is the area between D and the diagonal, and $e_D := \prod_{\text{horizontal steps } h_i(D) \text{ of } D} e_{h_i(D)}$.

Proof. Given Proposition 4.9, this is [8, Eq. (4.5.4)] (see also [7]). \square

Remark 4.13. In fact, according to [22] while the construction of the limit $t = q_3 \rightarrow 1$ of the algebra \mathcal{E} is independent of our choice in q_1, q_2, q_3 , the construction of the Fock representation \mathcal{F} naturally breaks the symmetry (in physics, this is related to the threefold symmetry of the refined topological vertex). The action of the skein algebra of the torus on that of the solid torus made explicit in [50] corresponds to the $q_2 = (qt)^{-1} \rightarrow 1$ limit, and can be thought of as a "rotation" of our representation by 120 degrees.

4.1. Double affine Hecke algebras. In order to define the superpolynomials in the next section, it will be relevant for us to treat \mathcal{E} as the limit of the spherical double affine Hecke algebras as $n \rightarrow \infty$, and the Fock space representation as a limit of the polynomial representations of the spherical DAHA. This point of view is adopted in e.g. [61].

Definition 4.14. The *double affine Hecke algebra* (DAHA) \mathbb{H}_n is the $\mathbb{Q}(q, t)$ -algebra generated by

$$X_1^\pm, \dots, X_n^\pm, Y_1^\pm, \dots, Y_n^\pm, T_1, \dots, T_n$$

with the relations

$$(4.1) \quad [X_i, X_j] = 0$$

$$[Y_i, Y_j] = 0$$

$$(4.2) \quad (T_i - t)(T_i + t^{-1}) = 0 \quad [T_i, T_j] = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$(4.3) \quad T_i X_j = X_j T_i \quad T_i Y_j = Y_j T_i$$

$$(4.4) \quad T_i X_i T_i = X_{i+1} \quad T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$$

$$(4.5) \quad Y_1 X_1 \cdots X_n = q X_1 \cdots X_n Y_1 \quad Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2$$

where $|i - j| > 1$.

The *spherical DAHA* is the subalgebra

$$\mathbb{SH}_n := \mathbf{e} \mathbb{H}_n \mathbf{e}$$

where $\mathbf{e} := \frac{1}{[n]_t!} \sum_{w \in S_n} t^{\ell(w)} T_w$ is the symmetrizing idempotent for the finite Hecke algebra. Note also that \mathbb{H}_n contains two affine Hecke algebras of S_n as subalgebras, namely one generated by the T_i, X_i and another one generated by the T_i, Y_i . We will denote these by $\mathcal{H}_n^{\text{aff}, X}, \mathcal{H}_n^{\text{aff}, Y}$.

The following is proved in [12, 29, 61] and will be essential for our computations:

Lemma 4.15. *There is an action of the braid group $B_3 = \widehat{SL_2(\mathbb{Z})}$ on $\mathbb{H}_{q,t}$ by algebra automorphisms.*

Proof. See [12, Section 1.3]. \square

The generators of this action are

$$\tau_+ := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and they act by

$$\begin{aligned} \tau_+ : T_i &\mapsto T_i, X_i \mapsto X_i, Y_i \mapsto Y_i X_i (T_1 \cdots T_{i-1})^{-1} (T_{i-1} \cdots T_1)^{-1} \\ \tau_- : T_i &\mapsto T_i, X_i \mapsto X_i Y_i (T_{i-1} \cdots T_i) (T_1 \cdots T_{i-1}), Y_i \mapsto Y_i \end{aligned}$$

Next, let

$$(4.6) \quad P_{0,k}^{(n)} = \mathbf{e} \left(\sum_{i=1}^n Y_i^k \right) \mathbf{e} \in \mathbb{S}\mathbb{H}_n$$

For arbitrary integers $(a, b) \in \mathbb{Z}^2 \setminus \{0\}$ we have the following.

Proposition 4.16 (Section 2.2., [61]). *Let $k = \gcd(a, b)$ and $\gamma_{a/k, b/k}$ be any matrix of the form*

$$\gamma_{a/k, b/k} = \begin{pmatrix} * & a/k \\ * & b/k \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

Then the elements

$$(4.7) \quad P_{(a,b)}^{(n)} := \gamma_{a/k, b/k} (P_{0,k}^{(n)})$$

are well-defined, i.e. do not depend on the chosen matrix.

Proposition 4.17. *The elements $P_{a,b}^{(n)}$ generate $\mathbb{S}\mathbb{H}_n$ as an algebra.*

Further, one can show the $P_{a,b}^{(n)} \in \mathbb{S}\mathbb{H}_n$ satisfy relations similar to those of $P_{a,b} \in \mathcal{E}$. In fact, by [61, Theorem 4.6.], we have

Proposition 4.18. *There is a surjective algebra homomorphism*

$$(4.8) \quad \mathcal{E} \twoheadrightarrow \mathbb{S}\mathbb{H}_n$$

for all n , sending

$$P_{a,b} \mapsto P_{a,b}^{(n)}$$

This map restricts to a surjection

$$\mathcal{E}^> \twoheadrightarrow \mathbb{S}\mathbb{H}_n^+$$

where $\mathbb{S}\mathbb{H}_n^+$ is generated by $P_{a,b}^{(n)}$ with $a > 0$ or $a = 0, b > 0$.

It remains to connect these facts to the Fock space. Recall the following

Definition 4.19. The polynomial representation of \mathbb{H}_n is

$$\mathrm{Ind}_{\mathcal{H}^{aff, Y, n}}^{\mathbb{H}_n} 1 \cong \mathbb{C}(q, t)[X_1^\pm, \dots, X_n^\pm]$$

The polynomial representation of $\mathbb{S}\mathbb{H}_n$ is $\mathbf{e} \mathbb{C}(q, t)[X_1^\pm, \dots, X_n^\pm]$.

It is clear from the above that by restricting the action of $\mathbb{S}\mathbb{H}_n$ on the polynomial representation to the positive part $\mathbb{S}\mathbb{H}_n^+$ we get an action on symmetric polynomials in n variables $\mathbb{C}(q, t)[X_1, \dots, X_n]^{S_n}$.

Theorem 4.20 (Section 5.1, [61]). *This action together with the Fock representation of \mathcal{E} intertwine the surjections $\mathrm{Sym}_{q,t} \twoheadrightarrow \mathbb{C}(q, t)[X_1, \dots, X_n]^{S_n}$ and $\mathcal{E}^> \twoheadrightarrow \mathbb{S}\mathbb{H}_n^+$.*

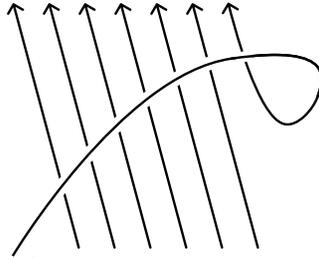


FIGURE 1. The Coxeter braid \mathbf{cox}_7 .

5. KNOT INVARIANTS

5.1. **Algebraic knots.** Recall from Definition 1.4 and Remark 1.5 that to any (reduced) germ of a plane curve $\{f = 0\} \subset \mathbb{C}^2$ we may associate both a Puiseux expansion and the link $\text{Link}_0(f) \subset S^3$. To simplify the discussion, we let f be irreducible, although appropriately interpreted all our results hold for any f . These correspond to each other as follows. For a single Newton pair, we have the torus knot $T(p, q)$. It is the braid closure of the q :th power of the Coxeter braid \mathbf{cox}_p (see Fig. 1).

Next, for knots L_1, L_2 in the solid torus, or more precisely elements in the skein algebra of the annulus, we define the *satellite* of L_1 by L_2 , denoted $L_1 * L_2$ by thickening L_1 to an annulus and placing the diagram of L_2 inside this annulus. Note that this operation is "acting on the right". Denote T_p^q the annular closure of the diagram of \mathbf{cox}_p^q shown in Fig. 1 (in the blackboard framing). Finally, for a given sequence (\vec{p}, \vec{q}) define the iterated torus knot

$$(5.1) \quad T(\vec{p}, \vec{q}) := T_{p_d}^{q_d} * (T_{p_{d-1}}^{q_{d-1}} * (\dots * (T_{p_1}^{q_1}) \dots))$$

where we think of these as links in S^3 by filling the core of the thickened annulus.

Remark 5.1. The sequence, or pair of sequences (\vec{p}, \vec{q}) is denoted (\vec{r}, \vec{s}) in [12]. Note that it can be *any* sequence of (coprime) integers, in which generality we obtain *iterated torus knots*. However, the Newton pairs are always positive and eventually have $p_k = 1$.

An alternative way to produce the iterated torus link is by *cabling* (see [20, Appendix A]), for which we need yet another sequence (\vec{p}, \vec{a}) where $a_d = q_d, a_i := a_{i+1}p_{i+1}p_i + q_i, 1 \leq i < d$. For a pair of coprime integers (p, a) we let the (p, a) -cable of a link $L \subset S^3$ be the link $\text{Cab}(p, a)(L)$ formed by thickening L to a small solid torus and placing the torus knot $T(p, a)$ inside it. Note that this operation is "acting on the left". Then it is an instructive exercise to check that

$$T(\vec{p}, \vec{q}) = \text{Cab}(p_1, a_1) \cdots (\text{Cab}(p_d, a_d)(\bigcirc) \cdots)$$

Remark 5.2. We are again opposite to the conventions in [12, 20, 47]. Note that in [12] the notation (\vec{a}, \vec{p}) is used instead. For the satellite construction, we refer to [47, Section 4].

5.2. Superpolynomials. The *superpolynomial* for links in S^3 has been proposed as a three-variable polynomial specializing to the HOMFLY-PT polynomial [19]. There are two main definitions for it:

1) as the Poincaré polynomial

$$P_L(a, q, t) := \sum_{i,j,k} q^i t^j a^k \dim \text{HHH}^{i,j,k}(L)$$

of the *triplely graded Khovanov-Rozansky* homology (or HOMFLY homology) $\text{HHH}(L)$. This is a homology theory for knots and links in S^3 defined using a braid presentation of L and Soergel bimodules. For more details, we refer to [40].

2) For iterated torus links, a definition of $P_L(a, q, t)$ was given by Cherednik-Danilenko [12] using double affine Hecke algebras; see also [4, 29] and will be repeated in Definition 5.5.

The first and second definitions are known to agree for torus knots (see [28] for a survey with more references) and conjectured to agree in general, but this is still unproven at the time of writing. We will use the second definition in this paper, but this also comes with a caveat. Namely, the polynomial is defined using a cabling presentation as in Eq. 5.1 and the topological invariance is not clear.

More precisely, it is not known whether there exist two distinct presentations of some iterated torus knot (/link) L as iterated cables, so that the resulting polynomials are different (see e.g. [50, p. 6]). In other words, this second version of the superpolynomial is not immediately a topological invariant of L . On the other hand, in this paper we only care about algebraic knots, where any ambiguity in the resulting isotopy type of the link is fixed by setting $q_1 > p_1$ (this is reflected in the choice of a coordinate in the Puiseux expansion). In other words, we may speak of $P_L(a, q, t)$ as an *invariant of the algebraic knot*. See also [13, Theorem 4.3.(ii)].

Next, we will recall the approaches of Cherednik-Danilenko and Gorsky-Negut [12, 29] to superpolynomials of iterated torus knots and how they degenerate at $t = 1$. In fact [29] only work out the torus knot case, while [12] do not use the elliptic Hall algebra, so one should regard what is below as a mixture of the two.

In [29], the approach is as follows. For a sequence of pairs of coprime integers $(p_1, q_1), \dots, (p_d, q_d)$ we have an iterated torus knot $T(\vec{p}, \vec{q}) = T_{q_d}^{p_d} * \dots * T_{q_1}^{p_1}$ as above. By Theorem 4.5, we have also have algebra homomorphisms $\text{Sym}_{q,t} \rightarrow \mathcal{E}^{q_i/p_i}, i = 1, \dots, d$ sending $p_k \mapsto P_{kq_i, kp_i}$. By Theorem 4.7 the algebra \mathcal{E} acts on the Fock space $\mathcal{F} \cong \text{Sym}_{q,t}$ by shuffle algebra operations. we denote the action of $E \in \mathcal{E}$ on $f \in \text{Sym}_{q,t}$ by $E \cdot f$.

Definition 5.3. The *full, or deformed master symmetric function* associated to (\vec{p}, \vec{q}) is

$$(5.2) \quad \widehat{\mathbf{f}}_{\vec{p}, \vec{q}} = \varphi_{q_d/p_d}(\dots(\varphi_{q_2/p_2}(P_{q_1, p_1} \cdot 1) \cdot 1) \dots) \cdot 1$$

A recursive description is thus as follows. Set $f_{(p_1, q_1)} = P_{q_1, p_1} \cdot 1$, and for $j = 2, \dots, d$ define $f_{(p_1, q_1), \dots, (p_j, q_j)}$ as follows. First, expand $f_{(p_1, q_1), \dots, (p_{j-1}, q_{j-1})}$ in terms of the power sum symmetric functions p_k and replace all p_k in the resulting expression by the operators $P_{q_i k, p_i k}$, then act on $1 \in$ the Fock representation. The result is a symmetric function.

Define the *evaluation vector* from [29, Eq. (39)]:

$$(5.3) \quad \mathbf{v}(a) = \sum_{\mu \vdash n} \frac{\widetilde{H}_\mu}{g_\mu} \prod_{\square \in \mu} (1 - aq^{a'(\square)} t^{l'(\square)})$$

where

$$g_\mu := \prod_{\square \in \mu} (1 - q^{a(\square)} t^{-l(\square)-1}) \prod_{\square \in \mu} (1 - q^{-a(\square)-1} t^{l(\square)})$$

and $a'(\square), l'(\square)$ denote the coarm and coleg of a box in the Ferrers diagram.

Remark 5.4. The factor g_μ is the product of the weights of the \mathbb{G}_m^2 -representation $\Lambda^\bullet T_\mu^\vee \text{Hilb}^n(\mathbb{A}^2)$, where the variable a encodes the exterior degree.

Definition 5.5. Let $L = T(\vec{p}, \vec{q})$ be an iterated torus link. The *superpolynomial* of L is defined to be

$$P_L(a, q, t) := (\mathbf{f}_{\vec{p}, \vec{q}}, \mathbf{v}(a))$$

where $\mathbf{v}(a)$ is defined in Eq. (5.3). Note that we are using the geometric inner product.

Remark 5.6. We make three remarks on the above definitions.

- (1) Note that the full master symmetric function and the superpolynomial depend on three variables q, t, u . At $u = 0$ the evaluation vector simplifies to

$$\mathbf{v}(0) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu}{g_\mu}$$

and it is indeed the quantity

$$(\mathbf{f}_{\vec{p}, \vec{q}}, \mathbf{v}(0)) = \langle \mathbf{f}_{\vec{p}, \vec{q}}, e_n \rangle$$

at $t = 1$ that gives the spherical orbital integrals in Section 2.

- (2) The homogeneity property of Shalika germs from Proposition 2.11 is reflected in the \tilde{H}_λ -expansion of $\mathbf{f}_{\vec{p}, \vec{q}}$, as we will prove in the next section (see Remark 6.18). Namely, the operator ∇ of Bergeron and Garsia scales each \tilde{H}_λ by $q^{n(\lambda)} t^{n(\lambda^t)}$ where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$, and in the $t = 1$ limit this will turn into scaling the \tilde{h}_λ in the expansion of \mathbf{f}_γ by $q^{n(\lambda)}$. Note that on the level of the link of the singularity, this scaling $\gamma \mapsto t\gamma$ corresponds to adding a full twist.
- (3) Note also that the full master symmetric function could be decorated by a partition or even a sequence of partitions, by replacing the first vacuum state "1" in the Fock space by a modified Macdonald polynomial \tilde{H}_λ (resp. replacing all of the vacuum states by $\tilde{H}_{\lambda_d}, \dots, \tilde{H}_{\lambda_1}$). In principle, our methods give formulas for these cases as well at $t = 1, a = 0$.

Let us now discuss how the above connects to the approach in [12]. In *loc. cit.* the following "evaluation homomorphism" or "coinvariant" $\{-\}_{ev} : \mathbb{H}_n \rightarrow \mathbb{C}(q, t)$ on the DAHA is defined:

$$\{-\}_{ev} : X_a \mapsto q^{-(\rho, a)}, Y_b \mapsto q^{-(\rho, b)}, T_i \mapsto t$$

Recall also from the discussion around Definition 4.19 that the DAHA acts on its polynomial representation

$$\mathbb{C}(q, t)[X_1^\pm, \dots, X_n^\pm]$$

and this restricts to an action of $\mathbb{S}\mathbb{H}_n^+$ on symmetric polynomials in n variables. Recall also the $\widehat{\text{SL}}_2(\mathbb{Z})$ -action and the elements $P_{a,b}^{(n)} \in \mathbb{S}\mathbb{H}_n$. The *DAHA-Jones* polynomial of Cherednik-Danilenko is defined in [12, Eq. (4.26)] (with slightly different notation) as

$$(5.4) \quad JD_{\vec{p}, \vec{q}}^{(n)}(q, t) = \{\gamma_{q_1/p_1}(\dots(\gamma_{q_{d-1}/p_{d-1}}(P_{q_d, p_d}^{(n)} \cdot 1) \cdot 1) \dots) \cdot 1\}_{ev}$$

This is related to the superpolynomial P_L from above by

Theorem 5.7 (Cherednik's stabilization conjecture, Section 3.4., [29]). *We have*

$$JD_{\vec{p}, \vec{q}}^{(n)}(q, t) = P_L(t^n, q, t)$$

Remark 5.8. When $m = 0$, the superpolynomials coincide with the Poincaré polynomials of triply graded Khovanov-Rozansky homologies of L by the explicit computations in [21, 35, 48]. It is an important open problem to verify this for algebraic links and more general iterated torus links.

We now come to explicit combinatorial formulas for the superpolynomials and the master symmetric functions. We will first recall the full torus knot case as in [29], and then work out the general formula at $a = 0, t = 1$.

The following formula for the full master symmetric function of a torus knot is given by [29, Theorem 1.1]. Let T be a standard Young tableau on n letters. For the box labeled i in its diagram, denote by z_i the q, t -content of the box. Recall also that

$$\omega'(x) = \frac{(x-1)(x-qt)}{(x-q)(x-t)}$$

and

$$S'_i := S_{q/p}(i)' := \lfloor \frac{iq}{p} \rfloor - \lfloor \frac{(i-1)q}{p} \rfloor$$

and define $\nu := \frac{(1-q)(1-t)}{(1-qt)}$. Then we have the following.

Theorem 5.9 (Theorem 1.1. and Eq. (37), [29]).

$$(5.5) \quad \widehat{\mathbf{f}}_{m,n} = \sum_{\lambda \vdash n} \nu^n \frac{\widetilde{H}_\lambda}{g_\lambda} \sum_{\text{SYT}(\lambda)} \frac{\prod_{i=1}^n z_i^{S'_{m/n}(i)} (qtz_i - 1)}{\left(1 - qt \frac{z_2}{z_1}\right) \cdots \left(1 - qt \frac{z_n}{z_{n-1}}\right)} \prod_{i < j} \omega' \left(\frac{z_j}{z_i} \right)^{-1}$$

Note that to pass from this description to the case of iterated cables is rather cumbersome. Namely, one should expand each \widetilde{H}_λ in the power sum symmetric functions (or perhaps the elementary ones [55]), replace each p_k by operators of the form $P_{kq, kp}$, use the formula [29, Eq. (37)], and proceed, but it is not obvious if this gives rise to any simple combinatorial formula.

However, at $t = 1$ the above formula massively simplifies and we may write down the general result. To do this, let us introduce some notation.

Definition 5.10. The *degenerate master symmetric function* (or just *master symmetric function* in the rest of the text) of an iterated torus knot $T(\vec{p}, \vec{q})$, is the $t = 1$ specialization of $\widehat{\mathbf{f}}_{\vec{p}, \vec{q}}$:

$$\mathbf{f}_{\vec{p}, \vec{q}} := \widehat{\mathbf{f}}_{\vec{p}, \vec{q}}|_{t=1}$$

Remark 5.11. This definition stems from a somewhat unfortunate notation clash between knot homology, symmetric functions and point-counting on affine Springer fibers – it would possibly be more appropriate to define $\mathbf{f}_{\vec{p}, \vec{q}} = \widehat{\mathbf{f}}_{\vec{p}, \vec{q}}|_{q=1}$ and then replace t by q everywhere. Since the q, t -formulas are transposition-symmetric under switching q, t , this will affect our formulas by a transposition and a $q \rightarrow q^{-1}$ in the Shalika germ expansion as well as the orbital integrals.

Definition 5.12. We call the coefficient of \widetilde{H}_λ in Eq. (5.5) for a fixed $T \in \text{SYT}(\lambda)$ the (q, t) -weight of the SYT T . We will denote it by $\widehat{\text{wt}}_{m/n}(T)$. Note that the weight depends on m/n .

Lemma 5.13. *By comparison to Eq. (5.5), a convenient formula for the weight is given by*

$$(5.6) \quad \widehat{\text{wt}}_{m/n}(T) = \frac{\prod_{i=1}^n z_i^{S_{m/n}(i)-1}}{\prod_{i=2}^n (1 - \frac{1}{z_i})(1 - qt \frac{z_{i-1}}{z_i})} \prod_{i < j} \omega\left(\frac{z_i}{z_j}\right)$$

where

$$\omega(x) = \frac{(1-x)(1-qt x)}{(1-qx)(1-tx)}$$

and

$$S_{m/n}(i) = \lceil \frac{im}{n} \rceil - \lceil \frac{(i-1)m}{n} \rceil$$

Proof. First, note that $S_{m/n}(n-i) = S'_{m/n}(i)$ and $\omega'(x^{-1}) = \omega(x)$. In particular, reversing the labeling on the z_i we see

$$\text{wt}_{m/n}(T) = \frac{\nu^n \prod_{i=1}^n z_i^{S_{m/n}(i)} \prod_i (qt z_i - 1)}{g_\lambda \prod_{i=2}^n (1 - qt \frac{z_{i-1}}{z_i})} \prod_{i > j} \omega\left(\frac{z_i}{z_j}\right)^{-1}$$

On the other hand, one can check that

$$g_\lambda = z_1 \cdots z_n \frac{\nu^n \prod_{i=1}^n (1 - z_i^{-1})(1 - qt z_i)}{\prod_{i < j} \omega\left(\frac{z_i}{z_j}\right) \omega\left(\frac{z_j}{z_i}\right)}$$

Plugging the latter equation into the former one, we are done. \square

Let us now study the limit as $t \rightarrow 1$.

Proposition 5.14.

- (1) *Let $T \in \text{SYT}(\lambda)$. then the order of vanishing of the weight $\text{wt}(T)_{m/n}$ at $t = 1$ equals*

$$(5.7) \quad v(T) = |\lambda| - \ell(\lambda) - \pi(T)$$

where $\pi(T)$ is the number of pairs of consecutive boxes \square_i, \square_{i+1} in T s.t. they lie in consecutive columns. Note that this number is always ≥ 0 and independent of m/n .

- (2) *Suppose that $v(T) = 0$, so that the weight does not vanish at $t = 1$. Then it is equal to*

$$\frac{\prod_{i=1}^n z_i^{S_{m/n}(i)-1}}{\prod_{i=2}^n (1 - z_i^{-1})(1 - q \frac{z_{i-1}}{z_i})}$$

Here z_i are (q, t) -contents of boxes in T now specialized at $t = 1$ and as in [29], we simply ignore $\ell(\lambda) - 1 + n(T)$ zero factors in the denominator.

Proof. From the formula for the weight in Eq. (5.6) independence of the subscript m/n is clear. Looking at the denominator, we have the claimed factor and the factors

$$\omega\left(\frac{z_i}{z_j}\right) = \frac{(1 - \frac{z_i}{z_j})(1 - qt \frac{z_i}{z_j})}{(1 - q \frac{z_i}{z_j})(1 - t \frac{z_i}{z_j})}, \quad i < j$$

At a first glance it looks like this factor is always just 1 at $t = 1$. However, this only holds if $\frac{z_i}{z_j} \notin \{\frac{1}{q}, \frac{1}{t}, \frac{1}{qt}\}$ (wheel conditions). For example, if $\frac{z_i}{z_j} = 1/q$, then we get

$$\omega\left(\frac{z_i}{z_j}\right) = \frac{(1-1/q)(1-t)}{(1-t/q)}$$

which gives a zero at $t = 1$ or order 1. Similarly, for $\frac{z_i}{z_j} = 1/t$ we get a zero of order 1 from $\frac{(1-1/t)(1-q)}{(1-q/t)}$, and at $\frac{z_i}{z_j} = 1/qt$ we get a pole of order 1 from $\frac{(1-1/qt)}{(1-1/t)(1-1/q)}$. Since each box which has a box to the right of it contributes a zero, each box with a box above it contributes a zero, and a box with a box diagonally above it contributes a pole, we see that this gives a total number of $|\lambda| - 1$ zeroes.

The factors $(1 - z_i^{-1})$ in the denominator vanish at $t = 1$ iff $z_i = t^k$ for some $k \geq 1$, i.e. \square_i is at the beginning of a row and we ignore the first factor. Therefore, they contribute a pole of order $\ell(\lambda) - 1$.

Finally, the factors $(1 - qtz_{i-1}/z_i)$ contribute a pole of order $\pi(T)$ at $t = 1$, since this factor vanishes at $t = 1$ iff $z_i = qz_{i-1}$. The result follows. \square

Remark 5.15. The first author thanks Eugene Gorsky for explanations related to the combinatorics of this Proposition.

Recall from Lemma 3.4 that to each composition $\alpha \vDash n$ there is associated an unique Young tableau.

Lemma 5.16. *The weight $\text{wt}(T)_{m/n}$ does not vanish iff T comes from a composition.*

Proof. We need to show that only the tableaux coming from compositions have $v(T) := |\lambda| - \ell(\lambda) - \pi(T) = 0$, cf. Eq. (5.7). Note that $v(T) = 0$ iff $\pi(T) = |\lambda| - \ell(\lambda)$. Additionally, the latter is an upper bound (i.e. the condition is satisfied for every box except the ends of the rows, which is obviously the maximum number of boxes), so we are looking to maximize the number of consecutive pairs of boxes in consecutive columns.

For a tableau of shape λ , coming from a composition $\alpha = \alpha_1 + \dots + \alpha_r$ we have exactly

$$\pi(T) = (\alpha_1 - 1) + \dots + (\alpha_r - 1) = |\alpha| - \ell(\alpha)$$

by construction. Conversely, if $\pi(T) = |\lambda| - \ell(\lambda)$, the horizontal strip coming from the top boxes in the diagram must have consecutive labels. Stripping it away gives α_r , and we continue inductively to build $\alpha_1 + \dots + \alpha_r$. \square

At $t = 1$ we then finally have

Proposition 5.17.

$$(5.8) \quad P_{m,n} \cdot 1 = \sum_{\alpha \vDash n} \text{wt}(\alpha)_{m/n} \tilde{h}_\alpha = \sum_{\alpha \vDash n} \frac{(-1)^{n-\ell(\alpha)} z_1^{S_1} \dots z_n^{S_n}}{c_{\alpha-1}(q)} \tilde{h}_\alpha$$

where $S_i = \lceil \frac{im}{n} \rceil - \lfloor \frac{(i-1)m}{n} \rfloor$. Here $(m, n) = 1$.

Proof. From the second part of Proposition 5.14 we have that the denominator of $\text{wt}(\alpha)_{m/n}$ is

$$\prod_{i=2}^n \frac{1}{(1 - z_i^{-1})(1 - qz_{i-1}/z_i)}$$

with the convention that zero factors are ignored.

Arranging the q, t -contents at $t = 1$ into a vector by reading the tableau box by box, we write

$$z(\alpha) = (1, q, \dots, q^{\alpha_1-1}, 1, q, \dots, q^{\alpha_2-1}, \dots, 1, \dots, q^{\alpha_r-1})$$

and by definition $z(\alpha)_i = z_i$.

It is easy to see that the $1 - qz_i/z_{i+1}$ factors are only nonvanishing at the ends of the parts of α so this becomes

$$\prod_{j=1}^r \frac{1}{(1-q^{-1}) \cdots (1-q^{-\alpha_j-1})} \prod_{j=1}^{r-1} \frac{1}{1-q^{\alpha_j}} = \frac{(-1)^{n-\ell(\alpha)} z_1 \cdots z_n}{(1-q)^{n-1} [\alpha_1]! \cdots [\alpha_{r-1}]! [\alpha_r-1]!} = \frac{(-1)^{n-\ell(\alpha)} z_1 \cdots z_n}{c_{\alpha-1}(q)}$$

where $[m]! = \prod_{i=1}^m (1-q^i)/(1-q)$ and $\alpha-1$ is the composition where we remove 1 from the last part.

Since the numerator was $z_1^{S_{m/n}(1)-1} \cdots z_n^{S_{m/n}(n)-1}$ we get the result. For further reference, we will denote the coefficient for a fixed α the *weight of α* :

$$(5.9) \quad \text{wt}(\alpha)_{m/n} = \frac{(-1)^{n-\ell(\alpha)} z_1^{S_{m/n}(1)} \cdots z_n^{S_{m/n}(n)}}{c_{\alpha-1}(q)}$$

□

In order to write down the transition matrix of Shalika germs, we will also need the case when m, n are not coprime. This is the $t \rightarrow 1$ limit of the formula in Theorem 4.8 at $\mu = \emptyset$.

Proposition 5.18.

$$(5.10) \quad P_{km, kn} \cdot 1 = \sum_{\alpha \vDash kn} \text{wt}(\alpha)_{m/n} \tilde{h}_\alpha$$

where for $\alpha \vDash kn$

$$(5.11) \quad \text{wt}(\alpha)_{m/n} = \left(1 + \sum_{j=1}^{k-1} q^j \frac{z_{(k-j)n} \cdots z_{(k-1)n}}{z_{(k-j)n+1} \cdots z_{(k-1)n+1}} \right) \frac{(-1)^{n-\ell(\alpha)} z_1 \cdots z_n}{c_{\alpha-1}(q)}$$

Proof. Comparing to Theorem 4.8 and Proposition 5.17 this is proved exactly in the same way, but we also have the coefficient

$$\sum_{j=0}^{k-1} q^j \frac{z_{n(k-1)+1} \cdots z_{n(k-j)+1}}{z_{n(k-1)} \cdots z_{n(k-j)}}$$

where the first summand is to be just read as 1. □

6. THE COMBINATORIAL FORMULAS

In this section, we state and prove the inductive combinatorial formula for the Shalika germs and the Steinberg germs, as well as the orbital integrals themselves. Our method is on the harmonic analysis side heavily based on results of [70], in particular the combinatorial result Lemme V. 12. therein. Currently, it can be regarded as the most technical part of our computations, but we also hope the results in this section give insight to the rather brute-force approach in [70].

Definition 6.1. Let $m \geq n \geq 0$, let $\lambda \vdash m$ have n parts and $\mu \vdash n$ and consider the set $\Upsilon_\lambda^\mu \subset \mathbb{Z}^n$ defined by

$$\Upsilon_\lambda^\mu := \left\{ (d_1, \dots, d_n) \mid \left(\sum_{i=1}^k d_i \right) \left(\sum_{i=k+1}^n \lambda_i \right) - \left(\sum_{i=1}^k \lambda_i \right) \left(\sum_{i=k+1}^n d_i \right) \right. \\ \left. \begin{cases} = 0, & \text{if there is a } j \text{ so that } k = \sum_{i=1}^j \mu_i, \quad k = 1, \dots, n-1 \\ > 0, & \text{otherwise} \end{cases} \right\}$$

Example 6.2. Let $m = n$, $\mu = (n)$ and $\lambda = (1^n)$. Then

$$\Upsilon_{(1^n)}^{(n)} = \left\{ \vec{d} \mid (n-k) \cdot \sum_{i=1}^k d_i - k \cdot \sum_{i=k+1}^n d_i > 0, k = 1, \dots, n-1 \right\}$$

Proposition 6.3. The subset $D_{d,n} \subseteq \Upsilon_{(1^n)}^{(n)}$ given by \vec{d} with $\sum_i d_i = d$ is in bijection with the slope d/n rational Dyck paths $\mathbb{D}_{d,n}^>$ strictly under the diagonal.

Proof. Giving a Dyck path is the same as giving the sequence of its horizontal steps. If we are looking at d/n -Dyck paths, these steps have to sum to d and there are at most n steps, which gives us a sequence (d_1, \dots, d_n) . Since such a Dyck path lies above the line with slope d/n , we must have

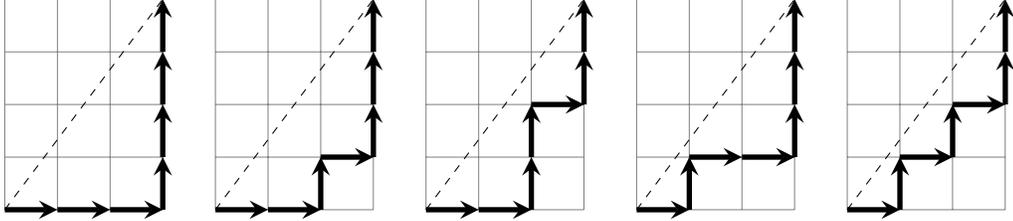
$$(n-k) \left(\sum_{i=1}^k d_i \right) - k \left(d - \sum_{i=1}^k d_i \right) = n \sum_{i=1}^k d_i - kd > 0$$

for all k (if we allowed all Dyck paths, for noncoprime d, n equality could also hold). The converse is clear. □

Example 6.4. Let $n = 4$, $d = 3$. The allowed sequences are

$$(3, 0, 0, 0), (2, 1, 0, 0), (2, 0, 1, 0), (1, 2, 0, 0), (1, 1, 1, 0)$$

and these correspond to the Dyck paths



Now let $P = \sum_{\vec{d}} a_{\vec{d}} X^{\vec{d}} \in \mathbb{Z}[[X_1, \dots, X_n]]_{\prod X_i}$ and define

$$(6.1) \quad \Upsilon_\lambda^\mu(P) := \sum_{\vec{d} \in \Upsilon_\lambda^\mu} a_{\vec{d}} X^{\vec{d}}$$

Remark 6.5. The notation used in [70, I 10] for Υ_λ^μ is Γ_λ^μ , but we have avoided this notation in order not to get it confused with the shalika germs $\Gamma_\lambda(-)$. Note also that on pages 856 and 878 of *loc. cit.* the confusing notation $\Gamma_\lambda^{\mu P}$ is used, but this seems to be due to a printing/typographical error.

In general, one should think of the elements of Υ_λ^μ Lie-theoretically as follows. \mathbb{Z}^n is the weight lattice of GL_n , and each collection of n integers $\lambda_1, \dots, \lambda_n$ gives

a linear form on \mathbb{Z}^n as defined above. On the other hand, μ gives a parabolic subgroup of GL_n , and the (in)equalities above decide that this linear form should (not) vanish on the relative root subspaces of the corresponding Levi subgroup. This cuts out a cone in the apartment of T . Fixing the coordinatewise sum is intersecting this cone with an affine hyperplane. We remark that the definition of Υ_λ^μ is related to the definition of "Hecke-regular functions" in [3, Section 4].

6.1. Comparison to Waldspurger's recursion. Let us recall the setup of [70, Section VI–VII], in slightly simplified form (the simplification being that for us $E = F$ and $r = 1$ in the notation of *loc. cit.*). Now F is a nonarchimedean local field of characteristic zero (but see Remark 2.5), F'/F is a tamely ramified extension of F , and $n' = n/ef$ where f is the residual degree and e the ramification index. Let $G' = GL_{n'}(F')$. Consider $\gamma \in GL_n(F)$ that is elliptic.

Definition 6.6. We call $\delta \in F'^*$ cuspidal for F'/F if $(\text{val}_{F'}(\delta), e) = 1$ and if the reduction of $t^{-\text{val}_{F'}(\delta)}\delta^e$ in the residue field of $\mathcal{O}_{F'}$ generates the residue field over that of \mathcal{O}_F .

Let $I_{G'}^1$ be the unipotent radical of the Iwahori of G' . We can always write $\gamma = \delta\gamma'$ where δ is $F' := F(\delta)/F$ -cuspidal and $\gamma' \in I_{G'}^1$ (see Lemme VI 3. in *loc. cit.*). Note that γ', δ are not uniquely determined but $n', e, f, \text{val}_{F'}(\delta)$ are, and so is the extension F' , up to isomorphism.

Remark 6.7. If we are working with a *totally ramified* extension, in terms of the Puiseux expansion, this is like writing

$$a_d u^{nr_d} + \dots + a_1 u^{nr_1} = a_d u^{nr_d} \left(1 + \frac{a_{d-1}}{a_d} u^{n(r_{d-1}-r_d)} + \dots + \frac{a_1}{a_d} u^{n(r_1-r_d)} \right)$$

which is analogous to our $\gamma - a_d u^{nr_d} = \gamma^\lessdot$. Namely, we inductively reduce to elements of smaller depth in the smaller group $GL_{n'}$, cf. Definition 2.48.

Following *loc. cit.* if we further suppose $\gamma \equiv 1 \pmod{\varpi_{F(\gamma)}\mathcal{O}_{F(\gamma)}}$ where ϖ is some chosen uniformizer, for example if $\gamma = 1 + a_d u^{nr_d} + \dots + a_1 u^{nr_1}$ then we may write

$$\gamma = \eta(1 + \delta\gamma')$$

where $\eta \equiv 1 \pmod{\varpi_F\mathcal{O}_F}$ where ϖ_F is a chosen uniformizer, e.g. t in the function field case. Write then $\delta\gamma' =: \gamma''$. This is also an analog of our γ^\lessdot , but note that since Γ^{St} are only defined on the group, which is where Waldspurger is working, we need to pass through the map $x \mapsto 1 + x$ as in Proposition 2.9. This, with homogeneity, will ensure $\Gamma_\lambda(\gamma^\lessdot) = \Gamma_\lambda(\gamma')$, but we tacitly avoid keeping track of group vs. Lie algebra Shalika germs in order to not overburden the notation.

Definition 6.8. Let $X(T) = \sum_{i \geq 0} h_i T^i \in \text{Sym}_{q,t}[[T]]$. Fix $\lambda' \vdash n'$ and number the squares in its diagram $1, \dots, n'$. Let

$$P = \prod_{k=1}^{n'} X(X_k)$$

and let $x_n(d\lambda', q)$ be the coefficient of T^n in the series

$$\Upsilon^{\lambda'}(P)$$

evaluated at $X_\square = q^{i-(\lambda'_j-1)/2}T$, where $\Upsilon^{\lambda'}$ is as defined in Eq. (6.1) and where i, j run over the coordinates of the boxes in λ' . See [70, Section I 10., V 12.].

In [70, Lemme VII 5.], Waldspurger proves the following.

Proposition 6.9. *We have*

$$(6.2) \quad \sum_{\lambda \vdash n} \Gamma_{\lambda}^{St}(\gamma, q) h_{\lambda} = q^{n'nd/2-n'} \sum_{\lambda' \vdash n'} (-1)^{\frac{n}{e} - \ell(\lambda')} \Gamma_{\lambda'}^{St}(\gamma'', q) x_n(d\lambda', q)$$

where $x_n(d\lambda', q)$ is as defined in Definition 6.8 and the Steinberg germs are as in Theorem 2.19.

We refer the reader to [70, Lemme VII 5., Lemme V 12.] for details. We warn the interested reader that there are some printing errors in the latter lemma, e.g. the second displayed equation on p. 880 should have a subscripted X_2 , a sentence after it there seems to be an extra "A" in front of λ , and on line 7 of p. 881 another subscript seems to have gone astray. Also in the statement the "q" should not be a subscript of $t(\lambda)$ but on the same line as (-1) .

Definition 6.10. Let $\gamma \in GL_n(\mathcal{O})$ be elliptic. By Corollary 2.7 we have the group Shalika germs $\Gamma_{\lambda}(\gamma)$ for γ and the Steinberg germs $\Gamma_{\lambda}^{St}(\gamma)$. Let the *master symmetric function of γ* be

$$\mathbf{f}_{\gamma} := \sum_{\lambda} \Gamma_{\lambda}(\gamma) \tilde{h}_{\lambda}$$

or equivalently by Propositions 3.13, 3.15 $\mathbf{f}_{\gamma} = \sum_{\lambda} \Gamma_{\lambda}^{St}(\gamma) h_{\lambda}$. If $\text{val}_F(\det(\gamma)) = d$, we define

$$\mathbf{f}_{\gamma} := \sum_{\lambda} \Gamma_{\lambda}^{St}(\gamma) h_{\lambda}$$

but note that there is no obvious analog of the Shalika germs in this case. Of course, one may define these via a change of basis a posteriori, but the harmonic analysis meaning is unclear.

Similarly, if $\gamma \in \mathfrak{gl}_n(F)$ define

$$\mathbf{f}_{\gamma} := \sum_{\lambda} \Gamma_{\lambda}(\gamma) \tilde{h}_{\lambda}$$

It is clear from Proposition 2.9 that this coincides with \mathbf{f}_{γ} above if for example γ is of the form $1 + \gamma'$ where γ' is topologically nilpotent. In this case we also have $\mathbf{f}_{\gamma} = \mathbf{f}_{\gamma'}$.

Definition 6.11. Let γ be as above. The coefficients of the expansion of \mathbf{f}_{γ} in the elementary symmetric polynomials are called the *Dyck germs* of γ . This is also the definition for γ non-elliptic, where \mathbf{f}_{γ} is as in Definition 6.28.

By construction, our recursion for $\mathbf{f}_{\bar{p}, \bar{q}}$ from Section 5 which is essentially the totally ramified case yields

$$(6.3) \quad \sum_{\lambda \vdash n} \sigma_{\lambda}(\gamma) e_{\lambda} = \sum_{\lambda' \vdash n/e} \sigma_{\lambda'}(\gamma'') E_{d, e, \lambda'}$$

Where σ_{λ} are the "Dyck germs" of Definition 6.11. A good example to keep in mind here is $\gamma = 1 + u^6 + u^7 = 1 \cdot (1 + u^6(1 + u)) \in GL_4(\mathcal{O})$, with $\gamma'' = \delta\gamma'$, $\delta = u^6$, $\gamma' = (1 + u)$ and $\eta = 1$. Also in this case, $f = 1, e = 2, n' = 4/2 = 2, d = 3$.

The main goal of this section is to compare the construction of $\mathbf{f}_{\bar{p}, \bar{q}}$ from Section 5 to \mathbf{f}_{γ} introduced above. We start with a Lemma.

Lemma 6.12. *We have $x_{e\lambda'}(d\lambda', q) = q^{n'nd/2-n'/2} \prod_{i=1}^{\ell(\lambda')} \left(\sum_{\pi \in \mathbb{D}_{e\lambda_i, d\lambda_i}^{\leq}} q^{\text{area}(\pi)} e_{\pi} \right)$ where we only sum over Dyck paths strictly under the diagonal.*

Proof. Since

$$x_{e\lambda'}(d\lambda', q) = \prod_{i=1}^{\ell(\lambda')} x_{e\lambda'_i}(\lambda'_i d, q)$$

(see e.g. the second displayed equation of p. 883 in [70]), we may restrict to a single factor in the product. Up to replacing $q^{i-(\lambda'_j-1)/2}T$ by $q^i T$ in Definition 6.8, this follows from Proposition 6.3. \square

Our main theorem is

Theorem 6.13. *The right-hand sides of Eqs. (6.2) and (6.3) are equal. That is,*

$$q^{n'nd/2-n/2} \sum_{\lambda' \vdash n'} (-1)^{\frac{n}{e}-\ell(\lambda')} \Gamma_{\lambda'}^{St}(\gamma'') x_n(d\lambda', q) = \sum_{\lambda' \vdash n'} \sigma_{\lambda'}(\gamma'') E_{e,d,\lambda'}$$

In particular, the left-hand sides are also equal.

Proof. We will prove this by induction. It is clearly true for $n/e = 1$. Start by writing

$$\sum_{\lambda' \vdash n/e} \sigma_{\lambda'}(\gamma'') E_{d,e,\lambda'}$$

where $E_{e,d,\lambda'} := \prod_{i=1}^{\ell(\lambda')} E_{d,e,\lambda'_i}$ and

$$E_{d,e,\lambda'_i} = \sum_{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i}} q^{\text{area}(\pi)} e_\pi$$

as before.

The analogous equation on the left is [70, p. 883]

$$x_n(d\lambda', q) = \prod_{i=1}^{\ell(\lambda')} x_{e\lambda'_i}(d\lambda'_i, q)$$

where

$$x_{e\lambda'_i}(d\lambda'_i, q) = \sum_{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i}^<} q^{\text{area}(\pi)} e_\pi$$

by Lemma 6.12.

Let $E_{d,e,\lambda'_i}^< = \sum_{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i}^<} q^{\text{area}(\pi)} e_\pi$, which is the same as E_{d,e,λ'_i} where we only sum over Dyck paths strictly under the diagonal. This is again by Lemma 6.12 the same as $x_{e\lambda'_i}(d\lambda'_i, q)$ up to a factor. In order to get rid of the factor, we prefer to again renormalize P from Definition 6.8 by replacing $q^{i-(\lambda'_j-1)/2}T$ by $q^i T$, replacing the exponent by the arm length of the Dyck path thought of a partition inside a staircase shape (note that the arm sequence of a Dyck path determines the area). For example when λ' is a single row, it is easy to check the two series differ exactly by $q^{(ed(n')^2-en')/2} = q^{(n'nd-n)/2}$.

Now, note that from the equation $\sum_{k=1}^n (-1)^k h_{n-k} e_k = 0$ it follows that

$$e_n = \sum_{\alpha \models n} (-1)^{\ell(\alpha)} h_\alpha$$

where we sum over all compositions of n . On the other hand, we have

$$E_{e,d,\lambda'_i} = \sum_{\alpha \models \lambda'_i} \sum_{\substack{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i} \\ \text{touch } \pi = \alpha}} q^{\text{area}(\pi)} e_\pi$$

where $\text{touch}(\pi) = \alpha$ specifies that π touches the diagonal at α .

Suppose by induction then that

$$\sum_{\lambda' \vdash n/e} \Gamma_{\lambda'}^{St}(\gamma'', q) h_{\lambda'} = \sum_{\lambda' \vdash n/e} \sigma_{\lambda'}(\gamma'') e_{\lambda'}$$

Writing $\lambda' = (\lambda'_1, \dots, \lambda'_{\ell(\lambda')})$ we then have

$$e_{\lambda'} = \prod_{i=1}^{\ell(\lambda')} \left(\sum_{\alpha \in \lambda'_i} (-1)^{\ell(\alpha)} h_{\alpha} \right)$$

Now, for two arbitrary compositions denote by $\alpha + \beta$ their concatenation. By sorting, this gives a partition of $|\alpha| + |\beta|$ of length $\ell(\alpha) + \ell(\beta)$. Given a collection $\vec{\alpha}$ of compositions $\alpha^{(1)} \vdash \lambda'_1, \dots, \alpha^{(\ell(\lambda'))} \vdash \lambda'_{\ell(\lambda')}$ write $\vec{\alpha} \leftrightarrow \lambda'$. In particular, for a fixed $\mu' \vdash n/e$ collecting all collections of compositions whose sum has associated partition μ' we see that

$$\Gamma_{\mu'}^{St}(\gamma'') = \sum_{\substack{\lambda' \leftrightarrow \vec{\alpha} \\ \text{sort}(\alpha^{(1)} + \dots + \alpha^{(\ell)}) = \mu'}} (-1)^{\frac{n}{e} - \ell(\lambda')} \sigma_{\lambda'}(\gamma'').$$

It now remains to replace $E_{d,e,\lambda'}$ by a similar expansion. Indeed, by definition we have

$$E_{d,e,\lambda'} = \prod_{i=1}^{\ell(\lambda')} \left(\sum_{\alpha \in \lambda'_i} \sum_{\substack{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i} \\ \text{touch } \pi = \alpha}} q^{\text{area}(\pi)} e_{\pi} \right)$$

but also that

$$\sum_{\substack{\pi \in \mathbb{D}_{e\lambda'_i, d\lambda'_i} \\ \text{touch } \pi = \alpha}} q^{\text{area}(\pi)} e_{\pi} = \prod_{i=1}^{\ell(\alpha)} E_{d,e,\alpha_i}^<$$

Again collecting all $\vec{\alpha} \leftrightarrow \mu'$ and by our inductive assumption, we see that

$$\sum_{\lambda' \vdash n/e} (-1)^{\frac{n}{e} - \ell(\lambda')} \Gamma_{\lambda'}^{St}(\gamma'') x_n(d\lambda', q) = \sum_{\lambda' \vdash n/e} \sigma_{\lambda'}(\gamma'') E_{d,e,\lambda'}$$

□

Example 6.14. We illustrate this theorem in the simplest nontrivial example, $n = 4, e = 2, d = \text{odd}$. Then we can write the RHS of Eq. (6.3) as

$$\begin{aligned} \sigma_{11} E_{d,e,11} + \sigma_2 E_{d,e,2} &= \sigma_{11} E_{d,e,1}^2 + \sigma_2 E_{d,e,2} \\ &= \sigma_{11} (E_{d,e,1}^<)^2 + \sigma_2 (E_{d,e,2}^< + (E_{d,e,1}^<)^2) = (\sigma_{11} + \sigma_2) (E_{d,e,1}^<)^2 + \sigma_2 E_{d,e,1}^<. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sigma_2 e_2 + \sigma_{11} e_{11} &= \sigma_2 (h_{11} - h_2) + \sigma_{11} h_{11} \\ &= (\sigma_2 + \sigma_{11}) h_{11} + (-1)^{2-1} \sigma_2 h_2. \end{aligned}$$

This does not quite yet cover the induction for general tamely ramified γ over nonarchimedean local F as outlined in [70, Section VII 7.], in which each intermediate step may carry some unramified extension.

For example, in the notation introduced above and in [70, VII 1.], namely $\gamma = \delta\gamma'$, suppose $F(\delta)/F$ has residue degree $f \geq 1$, $\text{val}_F(\det_G(\delta)) = nd/e =: a$. By Theorem 2.19 there is a Steinberg germ Γ_{λ}^{St} for each $\lambda \vdash \frac{n}{e}$. Note that even though we are working on GL_n , these are partitions of n/e . Similarly, we have germs $\Gamma_{\lambda'}^{St}(\gamma')$ for

$\lambda \vdash n/ef$. The associated master symmetric functions are $\mathbf{f}_\gamma = \sum_\lambda \Gamma_\lambda^{St}(\gamma)h_\lambda$ and similarly for $\mathbf{f}_{\gamma'}$. From [70, Proposition VII 2.] we then have

Proposition 6.15.

$$\mathbf{f}_\gamma = (-1)^{nd-nd/e} \mathbf{f}_{\gamma'}[X^f]$$

We will also denote this latter plethysm/Adams operation by $\tau_r : f[X] \mapsto f[X^r]$. Note that when $f = 1$ this allows us to compare Steinberg germs of γ and γ' directly.

Let us finally note that in [69, Théorème 1.3.] the following situation, which is implicit in the above, is addressed. Suppose $F' := F(\gamma)/F$ is unramified of degree f , and write $\gamma = 1 + t^a X + t^b Y$ where $a < b$, $X \in \mathcal{O}_{F(\gamma)}$ generates the residue field of F' and Y is such that $F'(Y)$ is a degree $n' = n/f$ extension. Suppose also $\gamma' := 1 + t^{b-a} Y$. Translating the notation to ours, we have

Theorem 6.16.

$$\mathbf{f}_\gamma = (-1)^{n-n'} \nabla|_{t=1}^a \tau_f(\mathbf{f}_{\gamma'})$$

where ∇ is the Macdonald eigenoperator from Definition 3.7

Proof. One notices that in order to compare the Shalika germs $s_\lambda(\gamma)$ in *loc. cit.* to ours, there is a factor of $c_\lambda(q)$ and another of $c_{\lambda'}(q^f)$ inside the plethysm. This is explained by the fact that there is a mismatch between [69, 70], namely we use the master symmetric function in the latter whereas in the former paper the Shalika germs are collected into a generating function

$$\sum_\lambda s_\lambda(\gamma, q) c_\lambda(q) h_\lambda$$

instead of $\mathbf{f}_\gamma = \sum_\lambda \Gamma_\lambda(\gamma) \tilde{h}_\lambda$ which has an additional plethysm $X \mapsto X/(1-q)$. Composing this with τ_f explains the power $q \mapsto q^f$ as well as the sign.

Finally, on the LHS of the Theorem in *loc. cit.* we have factors of the form $q^{an(\lambda^t)}$ which are exactly the ones coming from homogeneity of Shalika germs as observed in Remark 5.6. \square

Remark 6.17. In [70] an unramified character and some "twisted" Steinberg germs appear. While these are not studied in the present paper, we expect them to have nice expressions and combinatorics in terms of symmetric functions. For example, we do not know what the fundamental lemma proved in [70] looks like in our language.

Remark 6.18. As can be seen in 6.16 or by changing the first Newton pair from (p, q) to $(p, q+p)$, the ∇ -operator at $t = 1$ corresponds to multiplying the element γ in the Lie algebra or taking $1 + \gamma \rightarrow 1 + t\gamma$ in the group. Comparing to our formulas, this actually yields homogeneity of Shalika germs for the tamely ramified elements (compare to the proof of [69, Lemme 1.2.]). See Remark 2.5 for more discussion.

Let us end this section with introducing a canonical t -deformation of \mathbf{f}_γ as defined above for either $\gamma \in G(F)$ or $\gamma \in \mathfrak{g}(F)$. Note that by induction, as explained in [70, VII 7.], \mathbf{f}_γ is constructed using the steps in Theorem 6.13 as well as Proposition 6.15 (or Theorem 6.16), which are operations on symmetric functions, namely compositions of slope m/n plethysms $\varphi_{m/n} : \text{Sym}_q \rightarrow \text{Sym}_q$, the specialized nabla operator $\nabla|_{t=1}$, scalar multiplication, and the Adams operations τ_f . Promoting $\nabla|_{t=1}$ to $\nabla : \text{Sym}_{q,t} \rightarrow \text{Sym}_{q,t}$, the slope m/n plethysms to a family of operators

coming from $\text{Sym}_{q,t} \rightarrow \mathcal{E}^{m/n}$, and keeping the τ_f as they are, we may run the similar recursion which only depends on the datum of γ . We recover in the totally ramified case the deformed master symmetric function $\widehat{\mathbf{f}}_{\bar{p},\bar{q}}$ from Section 5, and in general have proved Theorem 1.12 from the introduction, namely

Theorem 6.19. *Let $\gamma \in \mathfrak{g}(F)$ be compact and elliptic. Then the master symmetric function admits a canonical t -deformation, namely*

$$\widehat{\mathbf{f}}_\gamma = \sum_\lambda \widetilde{\Gamma}_\lambda(\gamma) \widetilde{H}_\lambda$$

where \widetilde{H}_λ are the modified Macdonald polynomials. In particular, the Shalika germs $\Gamma_\lambda(\gamma)$ admit a canonical t -deformation.

6.2. The formula for Shalika germs. In this section, we will state and prove the main formula for Shalika germs. Let

$$\mathbf{f}_{\bar{p},\bar{q}} = \sum_{\lambda \vdash n} \Gamma_\lambda(\gamma) \widetilde{h}_\lambda$$

be the Shalika expansion of the master symmetric function for γ elliptic. The cabling process passing from $\gamma^{<i}$ to γ with new Newton exponents (p, q) (above, the correspondence is $p = e, q = d$) expands $\mathbf{f}_{\bar{p},\bar{q}}$ in the $\{e_\lambda\}$, replacing each e_λ by $E_{q,p,\lambda}$. On the level of Shalika expansions, denote the transition matrix between the bases $\{h_\lambda\}_{\lambda \vdash n}$ and $\{h_{\lambda'}\}_{\lambda' \vdash n/p}$ by $M = \{M_{\lambda,\lambda'}\}$. Denote also $n' = n/p$.

Theorem 6.20. *The matrix M has a combinatorial description as follows:*

$$(6.4) \quad M_{\lambda,\lambda'} = \left(c_{\lambda'} \sum_{\mu \vdash n'} \frac{|S_\lambda \cap C_\mu|}{b_\mu \lambda!} \prod_{i=1}^{\ell(\mu)} \left(\sum_{\alpha \models p\mu_i} \text{wt}(\alpha)_{q/p} \widetilde{h}_\alpha \right) \right) \Big|_{\widetilde{h}_\lambda}$$

In particular, it is possible to interpret this sum combinatorially as follows: For each λ' , take the partitions μ which refine it. Next, consider the compositions the parts of e_μ and attach to each composition the respective weight. Finally, sum up the result (weighted with the appropriate prefactors above).

Proof. We need to compute the slope " q/p plethysm" of the functions $\widetilde{h}_{\lambda'}$, i.e. expand them in the p_μ and replace each p_k by $P_{kq, kp}$ and bring the result back to the basis \widetilde{h}_λ . In order to do this, we note that by the *untransformed* complete homogeneous symmetric functions satisfy

$$h_\lambda = \sum_\mu \frac{|S_\lambda \cap C_\mu|}{\lambda!} p_\mu$$

where $|S_\lambda \cap C_\mu|$ is the number of permutations simultaneously lying in the Young/parabolic subgroup $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ and the conjugacy class C_μ of permutations with cycle type μ .

Since

$$\widetilde{h}_{\lambda'} = c_{\lambda'} h_{\lambda'}[X/(1-q)],$$

we get

$$\widetilde{h}_{\lambda'} = c_{\lambda'} \sum_{\mu \vdash n'} \frac{|S_{\lambda'} \cap C_\mu|}{\lambda'! b_\mu} p_\mu$$

where $b_\mu = \prod_i (1 - q^{\mu_i})$ is the principal specialization of p_μ and $c_{\lambda'}$ is as before.

Now, $\varphi_{q/p}(p_\mu) = \prod_{i=1}^{\ell(\mu)} P_{\mu_i q, \mu_i p} =: P_{\mu q, \mu p}$ which at $t = 1$ we know how to write down using Proposition 5.17. Using the prescription of Lemma 3.4 and the fact that $\tilde{h}_\lambda \tilde{h}_\mu = \tilde{h}_{\lambda+\mu}$,

$$P_{\mu q, \mu p} = \prod_i \left(\sum_{\alpha = \mu_i p} \text{wt}_{q/p}(\alpha) \tilde{h}_\alpha \right)$$

is a linear combination of $\tilde{h}_\lambda, \lambda \vdash N$ where the coefficient of each \tilde{h}_λ is the sum of the weights associated to all compositions of the rows of μ adding up to λ after sorting. \square

Remark 6.21. One may view Theorem 6.20 as giving a combinatorial expression for the " λ' -colored" master symmetric functions of torus knots at $a = 0, t = 1$.

Remark 6.22. There is a "slope zero" analog of this Theorem, where F'/F is an unramified extension, say of residue degree r . By Proposition 6.15 this amounts to applying the Adams operation τ_r . In this case, we only need to take $\text{wt}(\alpha)_{0/r}$ to be defined as in Eq. (5.11) but with $S_{0/r}(i) := 0$ for all i .

6.2.1. *Graphs.* In this section, we conjecture a different combinatorial approach to the renormalized Shalika germs.

We start with a lemma. Let $\lambda, \mu \vdash n \geq 1$. Let $\mathcal{G}(\lambda)$ be the set of directed graphs (loops allowed) with vertex set the boxes of the Ferrers diagram of λ , labeled with $\{1, \dots, n\}$ and edges only between boxes in the same row. Further, require each vertex to have in- and outdegree 1. Let $\mathcal{G}(\lambda, \mu) \subset \mathcal{G}(\lambda)$ be the subset of graphs whose connected components sort to give the partition μ . Note that μ is necessarily a *refinement* of λ .

Lemma 6.23. *There is a natural bijection $S_\lambda \cap C_\mu \leftrightarrow \mathcal{G}(\lambda, \mu)$*

Proof. Writing a cycle decomposition for elements on the left gives rise to a graph by drawing the boxes labeled $1, \dots, n$ and adding edges $a_i \rightarrow a_{i+1}$ for each cycle $(a_1 \dots a_k)$. The converse is clear. \square

Next, we note that by deleting at least one edge from each cycle of a graph $\mathbf{G} \in \mathcal{G}(\lambda)$, we get a composition of n , by remembering the ordering on the original boxes of λ . This composition naturally refines λ . Accordingly for $\mathbf{G} \in \mathcal{G}(\lambda)$, we say $\alpha \vDash n$ refines \mathbf{G} if we can obtain the composition α by deleting edges from \mathbf{G} . Finally, for $e \geq 1$, let $e\mathbf{G}$ be the graph obtained by e -dilating each cycle in \mathbf{G} .

In order to only have one kind of combinatorial object, we may further associate to each $\mathbf{G}' \in \mathcal{G}(\lambda', \mu)$ and a composition $\alpha \leq e\mathbf{G}'$ exactly $\prod_{i=1}^{\ell(\mu)} \mu_i$ different graphs by cyclic permutation of vertices in \mathbf{G}' . It is easy to see these graphs $\mathbf{G} \leq m\mathbf{G}'$ are the ones coming exactly from $m\mathbf{G}'$ by removal of one or more edges so that the resulting composition is α .

Next, define the weight of a graph to be

$$\text{wt}(\mathbf{G})_{M/N} = q^{\sum_{v \in \mathbf{G}} \text{coarm}(v) S_{M/N}(v)}$$

where coarm is the i -coordinate of the vertex minus 1, counting from the start of the chain v belongs to.

Conjecture 6.24. *In the renormalized basis $h_\lambda[\frac{X}{1-q}] = \tilde{h}_\lambda/c_\lambda$, the transition matrix of Shalika germs is given by*

$$(6.5) \quad M'_{\lambda, \lambda'} = \frac{c_\lambda}{c_{\lambda'}} M_{\lambda, \lambda'} = (-1)^{N-\ell(\lambda)} \frac{1}{\lambda!} \sum_{\mathbf{G}' \in \mathcal{G}(\lambda)} (-1)^{n-\ell(\alpha(\mathbf{G}'))} \sum_{\substack{\mathbf{G} \leq p\mathbf{G}' \\ \text{sort}(\mathbf{G})=\lambda}} \text{wt}(\mathbf{G}_{q/p})$$

This is a purely combinatorial conjecture, which we expect to be verifiable by direct comparison of Eqs. (6.4) (6.5).

While it may not seem obvious from this formula, from the easily checked fact that the symmetric functions $c_\lambda e_\lambda$ expand with $\mathbb{Z}[q]$ -coefficients in the basis \tilde{h}_λ it is also clear that this latter transition matrix has entries in $\mathbb{Z}[q]$.

Proposition 6.25. *The renormalized Shalika germs $c_\lambda \Gamma_\lambda(\gamma)$ are integral, i.e. $c_\lambda \Gamma_\lambda(\gamma) \in \mathbb{Z}[q]$.*

In effect, Eq. (6.5) gives a conjectural combinatorial interpretation for these integral polynomials.

Remark 6.26. Eq. (6.5) was conjectured in a slightly different form by the second author in 2018, based on extensive computer experiments, a slightly different algorithm based on [67], and an expectation for (6.5) when $q \rightarrow 1$.

Remark 6.27. The renormalized Shalika germs are not in $\mathbb{N}[q]$ in general, even up to an overall sign. In particular it is easy to find examples for which $c_\lambda \Gamma_\lambda(\gamma)$ has both positive and negative integer coefficients.

6.3. The formulas for orbital integrals. In this section, we give a combinatorial formulation of the orbital integrals and comment on the non-elliptic case.

Definition 6.28. For $\gamma \in \mathfrak{m} \subset \mathfrak{g}$, where \mathfrak{m} is the Lie algebra of a Levi subgroup conjugate to $L(\mu)$, let $\gamma_1, \dots, \gamma_{\ell(\mu)}$ we define the master symmetric function to be

$$\mathbf{f}_\gamma = \prod_{i=1}^{\ell(\mu)} \mathbf{f}_{\gamma_i}$$

Remark 6.29. This is only a definition at $t = 1$. For the equivalued, deformed case the relevant symmetric functions are defined in [7]. In the DAHA-version, the superpolynomials (in general) are defined in [13, Section 4.2.], but as far as the authors are aware, this has not been explored on the level of the elliptic Hall algebra.

Theorem 6.30. *Let γ be compact and regular semisimple, and let $\mathbf{1}_\lambda$ be the characteristic function of the standard parahoric \mathbf{P}_λ associated to $\lambda \vdash n$. Then*

$$I_\gamma(\mathbf{1}_\lambda) = q^{\dim \mathrm{Sp}_\gamma} \langle \mathbf{f}_\gamma, e_\lambda \rangle$$

where we pair using the Hall inner product and \mathbf{f}_γ is as above.

Proof. Assume first γ is elliptic. From Theorem 2.19, we have

$$I_\gamma(\mathbf{1}_\lambda) = \sum_{\mu} \Gamma_{\mu}^{St}(\gamma) \mathrm{St}_{\mu,c}(\mathbf{1}_\lambda)$$

and by Theorem 6.13 plus Definition 6.10 we have

$$\sum_{\mu} \Gamma_{\mu}^{St}(\gamma) h_{\mu} = \sum_{\mu} \Gamma_{\mu}(\gamma) \tilde{h}_{\mu} = \sum_{\mu} \sigma_{\mu}(\gamma) e_{\mu} = \mathbf{f}_{\gamma}$$

The result then follows from Propositions 3.13, 3.15 and Proposition 2.9.

For general γ , suppose γ belongs to a Levi of type μ , WLOG to the standard one and has blocks $\gamma_1, \dots, \gamma_{\ell(\mu)}$. Then by Proposition 2.43

$$I_{\gamma}^G(\mathbf{1}_{\lambda}) = |\det(\mathrm{ad}(\gamma))|_{\mathrm{Lie}(G)/\mathrm{Lie}(M)}|^{1/2} I_{\gamma}^M(\mathrm{Res}_{\mathfrak{m}}^{\mathfrak{g}}(\mathbf{1}_{\lambda}))$$

Let us write $\text{Res}_\mu = \text{Res}_M^G$. By [70, Lemme IV 3.] and Lemma 2.40, we get

$$\text{Res}_\mu(\mathbf{1}_\lambda) = \sum_{m \in \mathbf{M}(\lambda, \mu)} \otimes_{j=1}^{\ell(\mu)} \mathbf{1}_{m.,j}$$

where $\mathbf{1}_{m.,j}$ is the characteristic function of the corresponding standard parahoric and $\mathbf{M}(\lambda, \mu)$ is as in Definition 3.2. It is clear that this implies

$$I_\gamma^M(\text{Res}_\mu(\mathbf{1}_\lambda)) = \sum_{m \in \mathbf{M}(\lambda, \mu)} \prod_{j=1}^{\ell(\mu)} \langle e_{m.,j}, \mathbf{f}_{\gamma_j} \rangle$$

On the other hand, the first displayed equation on [70, p. 883] implies that we may write the RHS of the above equation as

$$\langle e_\lambda, \prod \mathbf{f}_{\gamma_i} \rangle$$

Comparing to Definition 6.28 and the dimension formula for affine Springer fibers we are done. \square

In particular, given the Newton pairs of an elliptic γ , we compute $I_\gamma(\mathbf{1}_\lambda)$ by forming the master symmetric function \mathbf{f}_γ recursively using Dyck paths, and then expand it in the elementary symmetric functions. In this expansion, we give the coefficient of e_μ the weight $\langle e_\lambda, e_\mu \rangle$ and sum the result up (and finally invert the powers of q appearing). If $\lambda = (n)$ the inner product is always one, so this is just summing up the coefficients in the expansion.

6.4. Examples.

Example 6.31. Let $\gamma = u^7 + u^6 \in \mathfrak{gl}_4(F)$, following Example 1.6. Then on the second step of our induction $n = 4, n' = 2$. Suppose we want to compute the entry $M_{211,2}$ of our transition matrix.

We compute

$$c_2 = (q-1)(q^2-1), \quad b_2 = (1-q^2), \quad b_{11} = (1-q)^2, \quad z_2 = z_{11} = 2$$

and therefore $\tilde{h}_2 = \frac{q+1}{2}p_{11} + \frac{1-q}{2}p_2$. Now since $(p_2, q_2) = (3, 2)$ we must apply the slope $3/2$ plethysm and replace $p_{11} \mapsto P_{3,2}^2, p_{(2)} \mapsto P_{6,4}$.

Now by formula (5.8) we have $P_{3,2} = \frac{1}{1-q}\tilde{h}_{11} - \frac{q}{1-q}\tilde{h}_2$ so

$$P_{3,2}^2 = \frac{1}{(1-q)^2}\tilde{h}_{1111} - \frac{2q}{(1-q)^2}\tilde{h}_{211} + \frac{q^2}{(1-q)^2}\tilde{h}_{22}$$

There are 8 compositions of 4, and we compute

$$S_{3/2}(1) = 2, S_{3/2}(2) = 1, S_{3/2}(3) = 2, S_{3/2}(4) = 1$$

Plugging this in to Eq. (5.11) gives

$$\begin{aligned} \text{wt}(2+1+1)_{3/2} &= \frac{-q(1+q^2)}{(1-q)^2(1-q^2)}, \quad \text{wt}(1+2+1)_{3/2} = \frac{-2q^2}{(1-q)^2(1-q^2)}, \\ \text{wt}(1+1+2)_{3/2} &= \frac{-q(1+q)}{(1-q)^3}, \end{aligned}$$

so that the coefficient of \tilde{H}_{211} in $P_{6,4}$ is

$$-q(1+q^2)(1-q)^2(1-q^2) + \frac{-2q^2}{(1-q)^2(1-q^2)} + \frac{-q(1+q)}{(1-q)^3} = \frac{-2q^2-2q}{(q-1)^3}$$

Taken together, we get

$$\frac{-2q(q+1)}{2(1-q)^2} + \frac{(1-q)(-2q^2-2q)}{2(q-1)^3} = 0$$

One verifies in Sage that the slope $3/2$ plethysm of \tilde{h}_2 has vanishing coefficient for \tilde{h}_{211} . Let us write down the master symmetric function.

$$\begin{aligned} \mathbf{f}_{(1,2),(3,2)} &= \varphi_{3/2}(\varphi_{1/2}(1)) = \varphi_{3/2}(e_2) = \\ &= (q^2 + q + 1)e_{1,1,1,1} + (q^5 + 2q^4 + 4q^3 + 2q^2 + 2q)e_{2,1,1} + \\ &\quad (q^6 + q^4 + q^2)e_{2,2} + (q^7 + q^6 + 2q^5 + q^4)e_{3,1} + q^8 e_4 \end{aligned}$$

Indeed, there are 23 Dyck paths in a 6×4 rectangle with these horizontal steps and area statistics. The weight polynomial of the spherical affine Springer fiber is

$$q^{\dim \text{Sp}_\gamma} \langle \mathbf{f}_{(1,2),(3,2)}, e_4 \rangle |_{q \rightarrow q^{-1}} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 4q^6 + 3q^7 + q^8$$

and that of the Iwahori affine Springer fiber is

$$q^{\dim \text{Sp}_\gamma} \langle \mathbf{f}_{(1,2),(3,2)}, e_{1111} \rangle |_{q \rightarrow q^{-1}} = 1 + 4q + 10q^2 + 20q^3 + 34q^4 + 48q^5 + 54q^6 + 48q^7 + 24q^8$$

Note that the first one is just the sum of the coefficients of the various e_λ up to normalization. It agrees up to $q \mapsto q^{-1}$ with the computation in [12, Eq. (3.1)] – it seems that there is a typo in their paper, repeating one from Piontkowski's work [59].

Example 6.32. The simplest elliptic case with three Puiseux pairs appears in [12, Eq. (3.8.)] as well as in [59] as an example where previous methods fail. This example corresponds to the plane curve singularity $\mathbb{C}[[t^8, t^{12} + t^{14} + t^{15}]]$, so we have $(p_1, q_1) = (p_2, q_2) = (2, 1)$, $(p_3, q_3) = (2, 3)$. The dimension of the ASF is 42 in this case. Using Sage, we compute

$$\begin{aligned} &q^{42} \langle \mathbf{f}_\gamma, e_8 \rangle |_{q \rightarrow q^{-1}} = \\ &q^{42} + 7q^{41} + 24q^{40} + 56q^{39} + 104q^{38} + 166q^{37} + 236q^{36} + 306q^{35} + 370q^{34} + \\ &424q^{33} + 465q^{32} + 492q^{31} + 507q^{30} + 510q^{29} + 504q^{28} + 488q^{27} + 466q^{26} + \\ &437q^{25} + 406q^{24} + 370q^{23} + 335q^{22} + 298q^{21} + 264q^{20} + 230q^{19} + 199q^{18} + \\ &168q^{17} + 143q^{16} + 118q^{15} + 97q^{14} + 78q^{13} + 63q^{12} + 48q^{11} + 38q^{10} + \\ &28q^9 + 21q^8 + 15q^7 + 11q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + q + 1 \end{aligned}$$

which by Theorem 1.10 is the weight polynomial of the compactified Jacobian in this case. We refer the reader to the attached computer program for computing the Shalika germs and other data in this case.

Example 6.33. Let $G = GL_4$ and $\gamma = u^6$. This is an element whose characteristic polynomial is $x^4 - t^6$, so that the link is a $(6, 4)$ -torus link. The element γ is conjugate to one in a Levi isomorphic to $GL_2 \times GL_2$, and on each of the blocks we have an equivalued element of valuation $3/2$. We compute the master symmetric function to be the product of the two factors in this case, namely $\mathbf{f}_\gamma = (e_{11} + qe_2)^2$. The Shalika expansion of \mathbf{f}_γ reads

$$\mathbf{f}_\gamma = \left(\frac{1}{q^2 - 2q + 1} \right) \tilde{h}_{1111} + \left(\frac{-2q}{q^2 - 2q + 1} \right) \tilde{h}_{211} + \left(\frac{q^2}{q^2 - 2q + 1} \right) \tilde{h}_{22}$$

Theorem 6.30 gives that $I_\gamma(\mathbf{1}_{(4)}) = q^8 + 2q^7 + q^6$ and $I_\gamma(\mathbf{1}_{(1^4)}) = 24q^8 + 24q^7 + 6q^6$. Note that up to $q \leftrightarrow t$, the first result agrees with the numerator of [35, Example 1.3.] at $a = 0, q = 1$.

Example 6.34. Let us work out an unramified example. Suppose $k = \mathbb{F}_q$ and $a \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$. Let

$$\gamma = \begin{pmatrix} 0 & at \\ t & 0 \end{pmatrix}$$

Then γ splits over a degree two unramified extension of F . By Hilbert's Theorem 90, stable conjugacy in GL_n is rational conjugacy, so by [72, (3.5.4)] we should have

$$I_\gamma^{GL_n}(\mathbf{1}_{\mathfrak{g}(\mathcal{O})}) = \text{SO}_\gamma(\mathbf{1}_{\mathfrak{st}_n(\mathcal{O})}) = q + 2$$

where SO_γ is the stable orbital integral in SL_n , defined as in [72, Section 3.5.3].

Indeed, we are in a situation where $1 + \gamma \sim 1 + tX$ with X generating \mathbb{F}_{q^2} over \mathbb{F}_q , and by Theorem 6.16 and Proposition 2.9

$$\mathbf{f}_\gamma = -\nabla_{t=1}\tau_2(e_1) = e_{11} + 2qe_2$$

By Theorem 6.30 we get

$$I_\gamma(\mathbf{1}_{(2)}) = q\langle \mathbf{f}_\gamma, e_2 \rangle|_{q \rightarrow q^{-1}} = q + 2$$

as desired.

7. APPLICATIONS

7.1. Affine Springer fibers. Let $G = GL_n/\mathcal{K}$ where $\mathcal{K} = k((t))$ as before, with $k = \mathbb{F}_q$ now. Appropriately modifying the definition of Sp_γ below to account for mixed characteristic F , we get similar results but leave these for the interested reader. Suppose $\mathbf{P} \subset G(\mathcal{K})$ is a parahoric subgroup. Let $\text{Fl}_{\mathbf{P}} = G(\mathcal{K})/\mathbf{P}$ be the corresponding partial affine flag variety.

Definition 7.1. The affine Springer fiber is the reduced ind-subscheme of $\text{Fl}_{\mathbf{P}}$ defined by

$$\text{Sp}_\gamma^{\mathbf{P}}(k) = \{g\mathbf{P} \mid \text{Ad}(g^{-1})\gamma \in \text{Lie}(\mathbf{P})\}$$

Let T_γ be the centralizer of γ . Then it acts naturally on $\text{Sp}_\gamma^{\mathbf{P}}$, and in particular gives rise to an action of $T_\gamma(\mathcal{K})/T_\gamma(\mathcal{O}) =: \Lambda_\gamma$ on $\text{Sp}_\gamma^{\mathbf{P}}$.

Unraveling the definitions, it is not hard to prove

Proposition 7.2. *When γ is totally ramified,*

$$|\text{Sp}_\gamma^{\mathbf{P}}(k)/\Lambda_\gamma(k)| = I_\gamma(\mathbf{1}_{\mathbf{P}})$$

where the Haar measure on the right is normalized as in Definition 2.1.

Combined with Theorem 6.30 this implies

Corollary 7.3. *If \mathbf{P} is of type λ ,*

$$|\text{Sp}_\gamma^{\mathbf{P}}(k)/\Lambda_\gamma(k)| = q^{\dim \text{Sp}_\gamma} \langle \mathbf{f}_\gamma, e_\lambda \rangle|_{q \rightarrow q^{-1}}$$

which is a polynomial in q with nonnegative coefficients.

We note that for more general γ , the orbital integral also seems to be a polynomial in q with nonnegative coefficients. Let us also note the following application to components of affine Springer fibers.

Theorem 7.4. *Let $\gamma \in G$ be compact, tamely ramified and regular semisimple. Then the number of irreducible components of $\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda$ is always at most the order of the Weyl group of GL_n , and always divides this number.*

Proof. By [68, Eq. (4.5)], the number of irreducible components is the coefficient of the leading term in q of the integral of $\mathbf{1}_{11\dots 1}$ along the orbit of γ . By Theorem 6.30, this orbital integral can be computed using \mathbf{f}_γ by pairing it with $e_{11\dots 1}$. On the other hand, \mathbf{f}_γ is formed by multiplying the master symmetric functions for the blocks of γ . Suppose for a moment γ is totally ramified. By Lemma 7.5 the smallest power in the Dyck germs is 1, it appears with coefficient one, and it appears for the least dominant partition. Since pairing with e_λ does not introduce powers of q , the highest power of q appearing in

$$q^{\dim \mathrm{Sp}_\gamma} \langle \mathbf{f}_\gamma, e_{11\dots 1} \rangle|_{q \rightarrow q^{-1}}$$

is $q^{\dim \mathrm{Sp}_\gamma}$ and it appears with coefficient $\langle e_\lambda, e_{11\dots 1} \rangle = \frac{|W|}{|W_\lambda|}$ where λ is the smallest partition in the dominance order appearing in the Dyck expansion of \mathbf{f}_γ . This proves the claim in the totally ramified case.

For the general case where the construction \mathbf{f}_γ involves the operator τ_f , cf. Proposition 6.15, note that the plethysm τ_f is designed so that for any homogeneous symmetric function \mathbf{f} of degree $n' = n/f$, $\lambda \vdash n$, we have

$$\langle e_\lambda, \tau_f(\mathbf{f}) \rangle = \begin{cases} \langle e_{\lambda/f}, \mathbf{f} \rangle, & \text{if } \lambda \text{ is divisible by } f \\ 0, & \text{if } \lambda \text{ is not divisible by } f \end{cases}$$

In particular, we may reduce these cases to the computation above. \square

Lemma 7.5. *Let γ be elliptic and totally ramified. Then the smallest power of q appearing in the coefficients (i.e. Dyck germs) of $\mathbf{f}_{\bar{p}, \bar{q}} = \sum_\lambda \sigma_\lambda(\gamma) e_\lambda$ is $1 = q^0$ and it only appears in front of the smallest partition in dominance order for which $\sigma_\lambda(\gamma) \neq 0$. In addition, it appears with coefficient 1.*

Proof. We will prove this by induction. For one Puiseux pair it is clear, as there is always a Dyck path with area 0. Suppose it holds for Newton pairs $(p_1, q_1), \dots, (p_{i-1}, q_{i-1})$. Then by the recursive construction of $\mathbf{f}_{\bar{p}, \bar{q}}$, the "least dominant" horizontal steps appearing in all the possible concatenations of Dyck paths when applying the plethysm φ_{q_i/p_i} appear from those smallest in the dominance order before applying the plethysm. This combined with the facts that there is always a unique Dyck path with area 0 and

$$\varphi_{q_i/p_i}(e_\lambda) = \prod_{j=1}^{\ell(\lambda)} \left(\sum_{\pi \in \mathbb{D}_{\lambda_j q_i, \lambda_j p_i}} q^{\mathrm{area}(\pi)} e_\pi \right)$$

we are by induction done. \square

Remark 7.6. Let m/n be the minimal root valuation of γ . When γ is totally ramified and elliptic, the above shows that the minimal partition appearing in the e_λ -expansion is formed from the horizontal steps of the maximal staircase partition fitting under a line of slope m/n , i.e. the one with parts $\lfloor \frac{(m-k)n}{m} \rfloor, k = 1, \dots, m$. For example, when $m/n = 3/7$ this gives the partition $4 + 2(+0)$, and the corresponding Dyck path in the 3×7 rectangle has horizontal steps $3, 2, 2$. In particular when $m/n \geq 1$ the horizontal steps give the one-column partition. It is easy to extend this to γ non-elliptic by multiplying the corresponding e_λ together. This gives

another (slightly more general) proof of a Theorem of Z. Yun in type A, which states that the *minimal reduction type* of γ determines the number of components in the Iwahori affine Springer fiber.

Remark 7.7. Theorem 7.4 proves [68, Conjecture 8.7.] in type A. From the main result of [68], there are always exactly $n!$ components when the depth is > 1 . In fact the last statement is true for depth ≥ 1 because any depth-1 element either differs from a depth > 1 element by a central element, or is contained in a Levi subalgebra in which case we can reduce the assertion to the Levi case as in the proof of Theorem 6.30.

Theorem 7.4 has the following interesting corollary about the W -representation given by $H^*(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda)$. Let us assume that γ is totally ramified, and note that the top degree part of the cohomology is always pure. A well-known argument using finite Springer theory tells us there is a graded isomorphism of W -representations for the affine Springer action:

$$H^*(\mathrm{Sp}_\gamma^{\mathbf{P}^\lambda}/\Lambda) \cong H^*(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda)^{W_\lambda}$$

In particular, knowing the dimensions of the top degree cohomologies of each $\mathrm{Sp}_\gamma^{\mathbf{P}^\lambda}/\Lambda$ tells us exactly all the dimensions of the W_λ -invariants of the representation on top degree cohomology of $\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda$. Recall that using the Hall inner product, this is the same as knowing the inner products of the Frobenius character with h_λ . Since the h_λ are a basis of the ring of symmetric functions, this uniquely determines the representation. A similar argument shows that assuming purity, \mathbf{f}_γ actually determines the Frobenius character of $H^*(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda)$ in the elliptic case. In fact, since $\omega(h_\lambda) = e_\lambda$ and the standard involution on symmetric functions is an isometry for the Hall inner product, the Frobenius character will simply be $\omega(\mathbf{f}_\gamma)$.

More precisely, we get

Theorem 7.8. *The $W = S_n$ -representation on $H^{\mathrm{top}}(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda)$ has Frobenius character $h_\nu = \omega(e_\nu)$, where ν is the smallest partition in dominance order appearing in the e_λ -expansion of \mathbf{f}_γ . In particular, when γ has depth ≥ 1 , this is the regular representation by above.*

If γ is further elliptic and totally ramified and we assume the purity conjecture, $\omega(\mathbf{f}_\gamma)$ is the Frobenius character of $H^(\mathrm{Sp}_\gamma^{\mathbf{I}}/\Lambda)$.*

Remark 7.9. Note that this proves [27, Conjecture 7.17.] in type A.

Suppose for a moment γ is a split element, i.e. lies in some split maximal torus. In [11], Zongbin Chen proves that the generating function (summing over elements of varying root valuation data) for the number of points on a so called *fundamental domain* of Sp_γ is rational, and that the number of points only depends on the root valuation datum. This is further related to the "weighted" Shalika expansion of Arthur, indeed the rationality is proved using homogeneity properties of these functions. See [11] for more details. We have not compared our techniques with the weighted Arthur-Shalika expansion, but it would be interesting to see how Chen's results could be combined with ours to yield stronger rationality results.

7.2. Compactified Jacobians. In this section, we apply Theorem 1.10 to show that the point-counts of compactified Jacobians of rational, unibranch plane curves are polynomials in q .

Let us recall some relevant material from [44]. Let C be a reduced, projective and geometrically connected curve over the residue field k , with only planar singularities. Suppose for simplicity that the normalization of C is rational. Let $\overline{\text{Pic}}(C)$ be the compactified Picard scheme of C . It is the moduli space whose closed points parametrize torsion-free rank one sheaves on C . For each $c \in \text{Sing}(C)$ fix an isomorphism $\widehat{\mathcal{O}}_{C,c} \cong k[[x, y]]/f$ and let Sp_c be the affine Springer fiber associated to $\gamma_c := \gamma_f \in \mathfrak{gl}_{\deg_x f}$ where γ_f is the companion matrix of f . Let Λ_c be the lattice part of the centralizer of γ_f and $\Lambda = \text{Pic}(C)/\text{Jac}(C)$. Fix a section $\Lambda \rightarrow \text{Pic}(C)$ of the quotient map.

From [44, Proposition 2.3.1.] we have

Proposition 7.10. *There is a universal homeomorphism*

$$\prod_{c \in \text{Sing}(C)} \text{Sp}_c/\Lambda_c \rightarrow \overline{\text{Pic}}(C)/\Lambda$$

If k is a finite field, we have

Corollary 7.11. *Let k'/k be a finite extension. Then*

$$\left| \prod_{c \in \text{Sing}(C)} \text{Sp}_c(k')/\Lambda_c \right| = |\overline{\text{Pic}}(C)(k')/\Lambda|$$

Combined with Corollary 7.3, we have

Theorem 7.12. *The number of points on $\overline{\text{Pic}}(C)$ is a polynomial in $q = |k|$. In addition, it is a polynomial with nonnegative integer coefficients.*

A standard spreading-out argument, combined with [37, Theorem 1] and the previous Theorem gives

Corollary 7.13. *Let $k = \mathbb{C}$. Then $X = \overline{\text{Pic}}(C)$ is strongly polynomial-count in the sense of Katz [37], and the E -polynomial*

$$E_X(x, y) := \sum_{p, q} e_{p, q} x^p y^q$$

is given by the weight polynomial of $\overline{\text{Pic}}(C)$ as $E_X(x, y) = P_X(xy)$, defined by

$$P_X(q) = \sum_{i, j} (-1)^i q^j \dim \text{gr}_W^j H^i(\overline{\text{Pic}}(C))$$

Yet another corollary of Corollary 7.3 together with Corollary 7.11 and Definition 5.5 is a virtual version of [12, Conjecture 2.4.(iii)], which compares Betti numbers of Jacobian factors with superpolynomials at $q = 1$ (or $t = 1$).

Proposition 7.14. *For unibranch C , the weight polynomial of $\overline{\text{Jac}}(C)$ is given by the superpolynomial at $a = 0, q = 1$, with t replaced by q .*

7.3. Orbital integrals. Let us finally comment on possible other applications of our results, as the explicit computation of orbital integrals bears on many problems in number theory and automorphic forms.

For example, in [65] Shin and Templier prove an equidistribution theorem for "families" of automorphic L-functions (for any G). Their main result [65, Theorem 1.3.] rests on an explicit, residue-characteristic independent bound for the size of orbital integrals derived by Kottwitz from the Shalika germ expansion. For $G = GL_n$, our methods should be applicable to give sharper bounds and as they remark,

possible improvements on their analytic results. It would be interesting to see more analytic applications of our results.

In his Beyond Endoscopy -proposal [43], Langlands computes global orbital integrals for $GL(2)$ using "elementary" methods. In the thesis of Espinosa Lara [45], which builds on work of Altug [1], the corresponding local orbital integrals are computed and compared via a product formula to Langlands' results. In Altug's work analysis of orbital integrals is used to "isolate" the contribution of the trivial representation to a certain trace formula Langlands introduces.

A priori, as suspected by Arthur in [2], it should be possible to use an explicit computation of the local orbital integrals (which is where our results come in) to have similar results for GL_n . It would be interesting to see how the possible application to Beyond Endoscopy plays out.

8. HILBERT SCHEMES OF POINTS

In this section, we give a conjectural geometric expression for the Shalika germs of γ in terms of the Hilbert scheme of points on \mathbb{A}^2 . Here we work over a field K , which is algebraically closed of characteristic zero.

Let $\text{Hilb}^n(\mathbb{A}^2)$ be the Hilbert scheme of n points on \mathbb{A}^2 , see e.g. [31]. There is a natural action of \mathbb{G}_m^2 on it given by scaling the coordinates on \mathbb{A}^2 .

Proposition 8.1 ([31]). *The direct sum of the equivariant K -theory groups of $\text{Hilb}^n(\mathbb{A}^2)$, $n \geq 0$ is naturally isomorphic to \mathcal{F} :*

$$K(\text{Hilb}) := \left(\bigoplus_{n \geq 0} K^{\mathbb{G}_m^2}(\text{Hilb}^n(\mathbb{A}^2)) \right) \otimes_{\mathbb{C}[q^\pm, t^\pm]} \mathbb{C}(q, t) \cong \mathcal{F} \cong \text{Sym}_{q, t}$$

The fixed point basis on the left corresponds to the basis $|\lambda\rangle = \tilde{H}_\lambda$ on the right.

Proposition 8.2. *Under the isomorphism of Proposition 8.1, the action of \mathcal{E} on the Fock space \mathcal{F} is realized on $K(\text{Hilb})$ by certain geometric correspondences.*

Recall from [27] that to each (conjugacy class of) γ we may associate a quasi-coherent sheaf

$$\mathcal{F}_\gamma \in \text{QCoh}^{\mathbb{G}_m}(\text{Hilb}(T^*\mathbb{G}_m))$$

using a \mathbb{Z} -algebra construction.

We now sketch an extension of this construction along the lines of [24] to give a sheaf

$$\mathcal{F}_\gamma \in \text{QCoh}^{\mathbb{G}_m \times \mathbb{G}_m}(\text{Hilb}(\mathbb{A}^2))$$

in which the other grading records the "number of points" grading on the Hilbert scheme of points of the spectral curve/a generalized affine Springer fiber associated to the companion matrix of γ as in [24] (this is just the intersection of the positive part of the affine Grassmannian with the affine Springer fiber for γ conjugated to a specific form).

Namely, let γ be the companion matrix of a polynomial $f \in \mathcal{K}[x]$ and let $\chi(t) = \text{diag}(t^{n-1}, t^{n-2}, \dots, t, 1)$. Then we have

Lemma 8.3. *For any k , the matrix*

$$\chi^{-k} t^k \gamma \chi^k$$

is the companion matrix of $f(t^k x)$.

We denote by C_k the germ of the plane curve singularity

$$\{\text{char}(t^k \chi^{-k} \gamma \chi^k) = 0\}$$

Now we will use the \mathbb{Z} -algebra construction in [27, Section 5] when we take our flavor symmetry to be constructed using this cocharacter of G . Namely, we let η_k be the action on $\text{Ad}(\mathcal{K}) + V(\mathcal{K})$ sending $\gamma \mapsto t^k \chi^{-k} \gamma \chi^k$ and $v \mapsto v$, so that $(1, 0, \dots, 0)^t \mapsto (1, \dots, 0)^t$. Since this is just a twisted form of the cocharacter $\gamma \mapsto t^k \gamma$, the \mathbb{Z} -algebra we get should be (but we have not checked carefully) the Gordon-Stafford \mathbb{Z} -algebra. Assuming this is the case, the construction of [27, Section 7] yields a sheaf \mathcal{F}_γ on $\text{Hilb}^n(\mathbb{A}^2)$ such that by the main theorem of [24] the global sections of $\mathcal{O}(k) \otimes \mathcal{F}_\gamma$ are given by the Borel-Moore homologies of Hilbert schemes of points on C_k :

$$H^0(\mathcal{O}(k) \otimes \mathcal{F}_\gamma) = H_*(\text{Hilb}^\bullet(C_k))$$

If all of the above works out, one hopes to compare the constructions of [62] and [54] to our results as follows. Recall from above the convolution action of the EHA on the K -theory $K(\text{Hilb})$ (see Proposition 8.2). Similar to the construction of the full master symmetric function of Definition 5.3, one constructs a K -class $[\mathcal{G}_\gamma]$ of a complex of coherent sheaves from the datum of γ , with

$$[\mathcal{G}_\gamma] = \widehat{\mathbf{f}}_\gamma = \sum_{\lambda \vdash n} \widetilde{\Gamma}_\lambda(\gamma) \widetilde{H}_\lambda$$

Remark 8.4. Similarly, one may think of the passage from γ^\leftarrow to γ by addition of the largest depth part as an "action" by the EHA on the constructible side but we have not made this precise.

Note that the sheaf \mathcal{F}_γ is \mathbb{G}_m^2 -equivariant, so we may write its class in localized equivariant K -theory as the sum of fixed point classes. Recall from Proposition 8.1 that the fixed points are indexed by $\lambda \vdash n$ and correspond to \widetilde{H}_λ in the Fock space. Now writing

$$[\mathcal{F}_\gamma] = \sum_{\lambda \vdash n} \widehat{\Gamma}_\lambda(\gamma) \widetilde{H}_\lambda$$

inside $K(\text{Hilb})$ gives us coefficients $\widehat{\Gamma}_\lambda(\gamma) \in \mathbb{Q}(q, t)$.

We should emphasize that we do not know whether $\widehat{\Gamma}_\lambda = \widetilde{\Gamma}_\lambda$. But according to [27, Conjecture 1.9.], the sheaf \mathcal{F}_γ or at least its K -theory class $[\mathcal{F}_\gamma]$ agrees with the one constructed using shuffle algebra techniques, i.e. the one denoted $[\mathcal{G}_\gamma]$ above. In other words, the K -class of $[\mathcal{F}_\gamma]$ is presumably the full master symmetric function from Definition 5.3. This would also imply the following conjecture.

Conjecture 8.5. *The coefficients $\widehat{\Gamma}_\lambda(\gamma)$ limit to the Shalika germs $\Gamma_\lambda(\gamma)$ of γ as $t \rightarrow 1$. In particular, they can be thought of as a natural t -deformation of the Shalika germs of γ and*

$$[\mathcal{F}_\gamma] \xrightarrow{t \rightarrow 1} \mathbf{f}_{\vec{p}, \vec{q}}$$

Remark 8.6. When γ is homogeneous i.e. its characteristic polynomial is quasi-homogeneous with the Puiseux pair (m, n) these coefficients appear, up to multiplication by a combinatorial factor, at the end of [58, Section 5] under the name $g_{m/n}$ and some values for them are computed using explicit combinatorics of the Hilbert schemes on the spectral curves. One can check that these coefficients limit to the Shalika germs as $t \rightarrow 1$.

Our conjecture applies to *any* compact regular semisimple element. For example, when γ is the split unramified element from [25], one knows only the leading Shalika germ is nonvanishing, see Proposition 2.42. However, the above suggests

$$[\mathcal{F}_\gamma] = \nabla p_1^n$$

whose expansion in the modified Macdonald polynomials is quite nontrivial (but limits as $t \rightarrow 1$ to the Shalika expansion).

From the point of view of harmonic analysis, this t -deformation seems fascinating. If one further had a version of the Shalika germ expansion enhanced with this second variable, one could then try to mimic the strategy of Waldspurger's recursion to say that the symmetric function attached to the constructible side is obtained from an action of the EHA. There is a combinatorial candidate already, coming from the construction of the master symmetric function using Theorem 4.8. We leave these explorations for future work.

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