GEODESIC ENVELOPES IN TEICHMÜLLER SPACE EQUIPPED WITH THE THURSTON METRIC

by

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ABSTRACT

Geodesic Envelopes in Teichmüller Space Equipped with the Thurston Metric

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The Thurston metric on Teichmüller space, first introduced by W. P. Thurston is an asymmetric metric on Teichmüller space defined by $d_{Th}(X, Y) = \frac{1}{2} \log \sup_{\alpha} \frac{l_{\alpha}(Y)}{l_{\alpha}(X)}$. This metric is geodesic, but geodesics are far from unique. In this thesis, we show that in the once-punctured torus, and in the four-times punctured sphere, geodesics stay a uniformly-bounded distance from each other. In other words, we show that the *width* of the *geodesic envelope*, $E(X, Y)$ between any pair of points $X, Y \in \mathcal{T}(S)$ (where $S = S_{1,1}$ or $S = S_{0,4}$) is bounded uniformly. To do this, we first identify extremal geodesics in $Env(X, Y)$, and show that these correspond to *stretch vectors*, proving a conjecture from [HOP21]. We then compute Fenchel-Nielsen twisting along these paths, and use these computations, along with estimates on earthquake path lengths, to prove the main theorem.

To my wife and family,
who encouraged me to succeed.

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CONTENTS

1 THE THURSTON METRIC

1.1 TEICHMULLER SPACE & THURSTON METRIC

Throughout this thesis, we let S be an orientable surface with no boundary components, such that $\chi(S) < 0$. In particular, this means that S can be endowed with a hyperbolic metric. It is our goal to study the space of such metrics, up to some equivalence. One possibility is to just consider all hyperbolic surfaces up to isometry. This object is called **moduli space**, but in this thesis, the main object will be its universal cover, Teichmüller space.

The reason we work with the universal cover is because we don't just want to distinguish metrics on S, but we also want to identify specific lengths of specific curves on S. To do this, we need to define a marking:

Definition 1.1. A **marking** on S is a homeomorphism $f : S \to X$, where X is a surface endowed with a hyperbolic metric.

Two markings $f : S \to X$ and $g : S \to Y$ are equivalent if fg^{-1} is homotopic to an isometry.

Definition 1.2. We define the **Teichmüller space** of S, denoted by $\mathcal{T}(S)$, to be the space of all markings on S up to equivalence.

We think of points in Teichmüller space as a pair consisting of a metric space X, together with a marking map φ.

If γ is a curve, arc, or arc segment on S, then we define $l_\gamma : \mathcal{T}(S) \to \mathbb{R}_{\geq 0}$ by sending X to $l_\gamma(X)$, the **length** of the geodesic representative of γ on X relative to its endpoints.

Definition 1.3. We define the **Thurston metric** $d_{Th} : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}_{\geq 0}$ by:

$$d_{Th}(X, Y) = \sup_{\alpha} \log \left(\frac{l_\alpha(Y)}{l_\alpha(X)} \right)$$

where the supremum ranges over all simple closed curves α contained in S.

Lemma 1.4. *For any $X, Y \in \mathcal{T}(S)$, $d_{Th}(X, Y)$ is equal to:*

- $d_{Th}(X, Y) = \sup_\alpha \log\left(\frac{l_\alpha(Y)}{l_\alpha(X)}\right)$ *(as above)*
- $L(X, Y) = \inf_{f\sim id} \log(L_f)$*, where* $f : X \to Y$ *is a Lipschitz map homotopic to the identity, and* L_f *is the Lipschitz constant of* f*.*
- $D(X, Y) = \inf_{f\sim id} \sup_{p\in X} \log(\|Df_p\|)$*, where* $f : X \to Y$ *is a homeomorphism that is once differentiable almost everywhere.*

Proof. In [Thu86], Thurston shows that $L(X, Y) = d_{Th}(X, Y)$. It suffices to show that $d_{Th}(X, Y) \leq D(X, Y) \leq L(X, Y)$. The first inequality follows because $\frac{l_\alpha(Y)}{l_\alpha(X)}$ is always bounded above by $\sup_p \|Df_p\|$. The latter inequality is more subtle. Trivially, if in the definition of $L(X, Y)$, f is taken to be differentiable, then the inequality follows by the fact that Lipschitz constants give upper bounds for derivatives. In fact, Thurston [Thu86] explicitly constructs a map f that realizes the infimum in $L(X, Y)$, and this map is a homeomorphism that is differentiable almost everywhere. Thus, the inequality holds. □

Claim. *For any* $X, Y, Z \in \mathcal{T}(S)$*, the Thurston metric satisfies:*

- $d_{Th}(X, Y) \geq 0$
- $d_{Th}(X, Y) = 0$ *if and only if* $X = Y$
- $d_{Th}(X, Z) \leq d_{Th}(X, Y) + d_{Th}(Y, Z)$

Proof. We use the third characterization of d_{Th} in Lemma 1.4.

- If $d_{Th}(X, Y) < 0$, then that would imply the existence of a map $f : X \to Y$ such that $\|Df_p\| < 1$ for all p. In particular, this means that f is not area-preserving, which cannot happen, as $\text{Area}(X) = \text{Area}(Y) = -2\pi\chi(S)$.
- If $d_{Th}(X, Y) = 0$, then there would be a homeomorphism $f : X \to Y$ homotopic to the identity with $\|Df_p\| = 1$ for all p. In particular, f must be an isometry, and X and Y are equivalent.
- This follows immediately from the chain rule.

□

The functions $d_{Th} = L = D$ define an asymmetric complete geodesic metric on $\mathcal{T}(S)$ [Thu86]. By this, we mean that:

Theorem 1.5. *For any* $X, Y, Z \in \mathcal{T}(S)$*,*

- $d_{Th}(X, Y) \geq 0$ *for all* X, Y*, with equality if and only if* $X = Y$

- $d_{Th}(X,Y) \leq d_{Th}(X,Z) + d_{Th}(Z,Y)$
- *There exists a path* $\gamma : [0, d_{Th}(X,Y)] \to \mathcal{T}(S)$ *such that* $\gamma(0) = X$, $\gamma(d_{Th}(X,Y)) = Y$*, and for any* $s \leq t$, $d_{Th}(\gamma(s), \gamma(t)) = t - s$.

For any simple closed curve c, and any $v \in T_X\mathcal{T}(S)$, we define $D_v \log l_c = \frac{d}{dt}|_{t=0} \log(l_c(\alpha(t)))$, where $\alpha(t)$ is some germ whose 1-jet is equal to v. This family of linear functionals induces the Thurston norm on $T_X\mathcal{T}(S)$:

$$\|v\|_{Th} = \sup_c d \log_c(v)$$

The Thurston metric is induced by this norm on the tangent bundle [Thu86], and hence is a Finsler metric.

We define the **unit norm sphere** at $X \in \mathcal{T}(S)$ by:

$$\boldsymbol{S}_X = \{v \in T_X\mathcal{T}(S) : \|v\|_{Th} = 1\}$$

Since the Thurston metric is induced by the Thurston norm, we can think of $\boldsymbol{S}_X$ as the set of tangent vectors which arise from 1-jets of C^1 geodesics starting at X.

More details on $d_v \log_\alpha$ and the characterization of the unit sphere can be found in Section 2.1 and in [Thu86, HOP21, DLRT20].

Throughout this paper, we will work with two different co-ordinate systems for Teichmüller space: Fenchel-Nielsen co-ordinates, and shearing co-ordinates. We review them in this section, following [Mar22] for Fenchel-Nielsen co-ordinates, and following [BBFS09] for shearing co-ordinates. For a more generalized overview of shearing co-ordinates in the case of a filling lamination that is not an ideal triangulation, we refer the reader to [Thé14].

1.2 SHEARING CO-ORDINATES & LAMINATIONS

Throughout this section, let (X, φ) be a point in $\mathcal{T}(S)$. Let $\tilde{X}$ be the universal cover of X, which we will identify with $\mathbb{H}^2$. A **geodesic lamination** λ on X is a closed subset of X which can we decomposed as a disjoint union of (possibly bi-infinite) geodesics. If (Y, ψ) is a different marking on S, then $\psi\varphi^{-1}(\lambda)$ gives a closed collection of disjoint arcs on Y, which we can turn into geodesics by an ambient isotopy, and hence, we can think of λ as a lamination on Y. In this manner, λ can be thought of as a lamination on the underlying surface S without specifying a metric.

A geodesic lamination is called **complete** if its complementary components are triangles. A lamination λ can be lifted to a lamination $\tilde{\lambda}$ on $\tilde{X}$, so λ is complete if and only if $\tilde{\lambda}$ is a triangulation of $\mathbb{H}^2$. A lamination λ

is called **chain-recurrent** if there exists a sequence of multicurves which converge to λ in the Hausdorff topology. An alternate characterization of chain-recurrence is the following:

Definition 1.6. A geodesic lamination λ is called **chain-recurrent** if for any arc segment I contained in λ, and for any $\varepsilon > 0$, there exists a geodesic simple closed curve α in S and an arc segment $J \subset \alpha$ such that $d_H(I, J) < \varepsilon$, where d_H is the Hausdorff distance.

In this sense, the space of chain-recurrent laminations is the closure of the space of multicurves, equipped with Hausdorff convergence [DLRT20].

A **measured lamination** is a lamination λ together with a measure, μ_λ, defined on all arc segments intersecting λ transversely whose endpoints lie on λ. We also require μ to be invariant under homotopy that moves the endpoints of arc segments along leaves of λ.

As an example, consider a **weighted multicurve** – a disjoint union of simple closed curves γ_n with nonzero weights a_n. For any arc segment I, $\mu(I) = \sum_n a_n i(\gamma_n, I)$.

Definition 1.7. The space of projective measured laminations, $\mathcal{PML}(S)$, is the space of all measured laminations up to scaling the measure by a positive real number.

We topologize $\mathcal{PML}(S)$ in the following manner: We say that λ_n converge to λ if for every arc segment I in S, $\mu_{\lambda_n}(I)$ converges to $\mu_\lambda(I)$.

The following is useful for working with $\mathcal{PML}(S)$, and appears in Chapter 1 of [PH92]:

Proposition 1.8. *$\mathcal{PML}(S)$ is compact, and moreover, is the compactification of the space of weighted multicurves with total weight 1.*

We next explicitly describe shearing co-ordinates for a class of laminations that will be of interest later in this thesis.

Let $\lambda = \cup_{i=1}^{9g-9} \lambda_i$ be a complete geodesic lamination consisting of $3g-3$ leaves which are simple closed curves and $6g-6$ bi-infinite leaves. Denote by $\mathcal{C}$ the closed leaves of λ. The leaves of λ give an ideal triangulation of S, and $\mathcal{C}$ form a pair-of-pants decomposition of S.

We wish to define a family of functions $S_{\lambda_i} : \mathcal{T}(S) \to \mathbb{R}$, which will give a co-ordinate system on $\mathcal{T}(S)$. To do this, we first define **shearing between triangles**.

Let Δ_1 and Δ_2 be two triangles in $S \setminus \lambda$, and let $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ be lifts of Δ_1 and Δ_2 to the universal cover $\tilde{X} = \mathbb{H}^2$. Choose some geodesic γ separating the interiors of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$, such that γ is the geodesic between a vertex v_1 of $\tilde{\Delta}_1$ and to a vertex v_2 of $\tilde{\Delta}_2$ (if Δ_1 and Δ_2 are adjacent triangles on S,

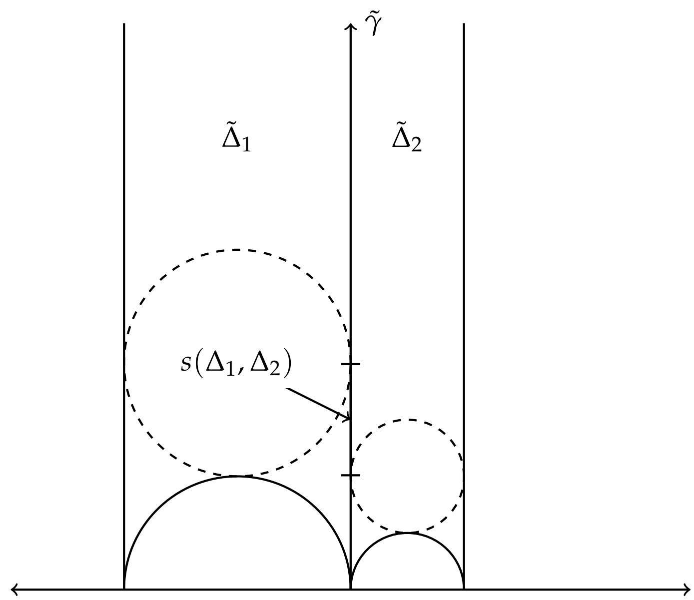

Figure 1.1: The set-up for computing shearing co-ordinates between two adjacent triangles.

choose adjacent lifts, and let γ be their intersection). We orient γ so that the interior of Δ_1 lies to the left of γ.

An ideal triangle in $\mathbb{H}^2$ has a well-defined incenter, incircle, and three distinguished medians along its edges, given by the intersection of the incircle with the edges. For each triangle $\tilde{\Delta}_i$, we define m^i as the median of $\tilde{\Delta}_i$ lying on the edge separating γ from $int(\tilde{\Delta}_i)$.

Let φ_i be the orientation-preserving parabolic isometry of $\mathbb{H}^2$ fixing v_i and sending γ to the edge of $\tilde{\Delta}_i$ containing m^i. We set $q_i = \varphi^{-1}(m^i)$, and define $s(\Delta_1, \Delta_2)$ to be the signed distance between q_1 and q_2, where the sign is inherited from the orientation of γ.

In [BBFS09], it is proven that $s(\Delta_1, \Delta_2)$ is symmetric and does not depend on the geodesic γ. However, $s(\Delta_1, \Delta_2)$ as defined above may depend on the choice of lifts of Δ_1 and Δ_2, unless Δ_1 and Δ_2 are adjacent.

We are now ready to define $s_{\lambda_i}(X) = s(\Delta_1, \Delta_2)$, where Δ_i are chosen as follows:

- If λ_i is an infinite leaf of λ, take Δ_1 and Δ_2 are the (possibly coinciding) triangles adjacent to λ_i

- If $\lambda_i \in \mathcal{C}$, we pick Δ_1 and Δ_2 to be two triangles which asymptote to λ_i from both sides of λ_i.

Lemma 1.9. *Let $c \in \mathcal{C}$, and let $\lambda_c \subset \lambda$ be the sublamination of λ consisting of leaves converging to c.*

Then $s_c(X)$ is well-defined up to $\mathbb{Z}[\{s_{\lambda_c}\}]$.

Proof. This follows from Lemma 3.1 of [BBFS09]. □

Choosing prescribed lifts for every triangle in $S \setminus \lambda$, we get a well-defined **shearing co-ordinates** map $\mathcal{T}(S_g) \to \mathbb{R}^{9g-9}$ given by:

$$X \to \{s_l\}_{l \subset \lambda \text{ is a leaf}}$$

The shearing co-ordinate map does not give an isomorphism, as for any λ and X, the collection $\{s_{\lambda_i}(X)\}$ is linearly dependent (see Lemma 3.2 in [BBFS09]). Choosing a linearly independent subset of these gives a homeomorphism from $\mathcal{T}(S_g)$ to $\mathbb{R}^{6g-6}$ [BBFS09].

1.3 TWISTING CO-ORDINATES

Let $\mathcal{C}$ be a pair-of-pants decomposition of a surface S. In the same way as in our definition for the shearing co-ordinates, it will be convenient to choose prescribed lifts of every $c \in \mathcal{C}$ in $\tilde{X}$. For any closed leaf $c \in \mathcal{C}$, let $P_1(c)$ and $P_2(c)$ be the (possibly same) pairs of pants adjacent to c. Let $\tilde{c}$ be a lift of c, and choose lifts, $\tilde{P}_i(c)$ of $P_i(c)$ which are adjacent to $\tilde{c}$ on either side of $\tilde{c}$.

Let $c \neq c_1$ and $c \neq c_2$ be cuffs of $P_1(c)$ and $P_2(c)$ respectively, and let $\tilde{c}_i$ be the lifts of c_i bounding $\tilde{P}_i(c)$.

Let η_i be the unique simple geodesic segment intersecting $\tilde{c}_i$ with start point on $\tilde{c}$ and end point on $\tilde{c}_i$, which is normal to both $\tilde{c}$ and $\tilde{c}_i$. Let $p_i(c)$ denote the start-point of η_i.

We define the **twist co-ordinate relative to** c, $\tau_c(X)$ to be the signed distance between $p_1(c)$ and $p_2(c)$, so that the sign is *positive* if we turn *left* to get from p_1 to p_2. Choosing different lifts of c will result in twist co-ordinates that differ by an integer multiple of $l_c(X)$.

Additionally, it can be shown that choosing different curves c_i in the pair-of pants will change p_i by a Gaussian-integer multiple of l_c, and hence, $\tau_c(X)$ is well defined up to an integer multiple of $l_c(X)$.

The map $FN : \mathcal{T}(S) \to \mathbb{R}^{6g-6}$ given by $X \to \{l_c(X), \tau_c(X)\}_{c \in \mathcal{C}}$ is a homeomorphism [Mar22], so we call this map the **Fenchel-Nielsen co-ordinate** system on $\mathcal{T}(S)$.

1.4 STRETCH LAMINATIONS, CHAIN-RECURRENCE, AND THURSTON GEODESICS

Definition 1.10. Let I be a possibly infinite closed interval. A **forward** (resp. **backwards**) geodesic is a map $\gamma : I \to M$ satisfying $d_{Th}(\gamma(s), \gamma(t)) = t - s$ (resp. $d_{Th}(\gamma(t), \gamma(s)) = t - s$) for all $t \geq s$.

If $I = [a, b]$ is finite, we say that γ starts at $\gamma(a)$ and ends at $\gamma(b)$.

Throughout this thesis, a "geodesic" means a forward geodesic, unless otherwise stated.

Given $X, Y \in \mathcal{T}(S)$, we define

$$\mathrm{Out}(X) = \{Z \in \mathcal{T}(S) : Z \text{ lies on a forward geodesic starting at } X\}$$
$$\mathrm{In}(Y) = \{Z \in \mathcal{T}(S) : Z \text{ lies on a backwards geodesic starting at } Y\}$$

For $X, Y \in \mathcal{T}(S)$, we define the **geodesic envelope**, $\mathrm{Env}(X, Y) = \mathrm{Out}(X) \cap \mathrm{In}(Y)$.

Given $X, Y \in \mathcal{T}(S)$, we can consider a sequence of multicurves α_i such that $d_{Th}(X, Y) = \lim_{i\to\infty} \frac{l_{\alpha_i}(Y)}{l_{\alpha_i}(X)}$. The space of geodesic laminations equipped with the Hausdorff topology is compact [BZ05], so up to subsequence, α_i converges to some geodesic lamination λ. It turns out [Thu86] that the union of all Hausdorff limits of subsequences of $\{\alpha_i\}_n$ is itself a geodesic lamination, call it $\lambda_{\{\alpha_i\}}$. Moreover, if α'_i is another sequence whose length ratio converges to $d_{Th}(X, Y)$ then any Hausdorff limit of $\{\alpha'_i\}$ is disjoint from $\lambda_{\{\alpha_i\}}$.

Thus, it makes sense to define the **maximally stretched lamination**, $\Lambda(X, Y)$ as the union of all Hausdorff limits of sequences of multicurves whose length ratio converges to $d_{Th}(X, Y)$.

Given $X_0 \in \mathcal{T}(S)$, and any completion of a maximal chain-recurrent lamination λ, there exists an analytic 1-parameter family of metrics $X_t = \mathrm{Stretch}(X_0, \lambda, t) \subset \mathcal{T}(S)$ with the following properties:

- $l_\lambda(X_t) = e^t l_\lambda(X_0)$
- For $0 \leq s \leq t$, $d_{Th}(X_s, X_t) = t - s$
- $\Lambda(X_s, X_t) = \lambda$

This family of metric is called the **Thurston Stretch Path** associated to X_0 and λ. In particular, when $\Lambda(X, Y)$ is maximal amongst all chain-recurrent laminations, there is a unique geodesic from X to Y, and the points along it are precisely the points in $\mathrm{Env}(X, Y)$ [DLRT20].

Remark 1.11. Let X_t be some smooth 1-parameter family of surfaces in $\mathcal{T}(S)$.

The maps $s_\lambda : X_t \to \mathbb{R}^{9g-9}$ and $FN : X_t \to \mathbb{R}^{6g-6}$ are only well-defined up to the choices of lifts in their construction. However, the maps $\dot{s}_\lambda : \mathcal{T}(S) \to \mathbb{R}^{9g-9}$ and $\dot{FN} : X_t \to \mathbb{R}^{6g-6}$ defined by postcomposition of s_λ and FN by differentiation with respect to t are well defined.

Let λ be some chain-recurrent lamination, and let $X, Y \in \mathcal{T}(S)$. We define

$$\mathrm{Out}(X, \lambda) = \{Z \in \mathrm{Out}(X) : \lambda \subset \Lambda(X, Z)\}$$
$$\mathrm{In}(Y, \lambda) = \{Z \in \mathrm{In}(Y) : \lambda \subset \Lambda(Y, Z)\}$$

By this definition, $\mathrm{Env}(X, Y) = \mathrm{Out}(X, \Lambda(X, Y)) \cap \mathrm{In}(Y, \Lambda(X, Y))$. The geodesic envelope can be thought of as a 1-parameter family of cross-sections, where at any time $t > 0$, $\mathrm{Env}_t(X, Y)$ consists of all points of distance t from X lying along geodesics from X to Y.

Definition 1.12. Let $X, Y \in \mathcal{T}(S)$. We define the **width** of the envelope $\mathrm{Env}(X, Y)$ as:

$$\begin{aligned} w(X, Y) &= \sup_{g_1, g_2 \mathcal{G}(X,Y)} \sup_{t \in [0, d_{Th}(X,Y)]} d_{Th}(g_1(t), g_2(t)) \\ &= \sup_t \mathrm{Diam}(\mathrm{Env}_t(X, Y)) \end{aligned}$$

Organization of this Thesis

Chapter 2 is all about infinitesimal envelopes in $\mathcal{T}(S)$. We will begin Chapter 2 by explicitly computing the Fenchel-Nielsen co-ordinates from given shearing co-ordinates with respect to a completion of a pair-of-pants decomposition of S. This allows us to explicitly compute the infinitesimal envelope for certain points in $\mathcal{T}(S)$. We then shift our focus to the more general study of the infinitesimal envelope by looking at stretch vectors in the envelope. We prove:

Theorem 1.13. *Let SV_X be the set of 1-jets of stretch paths starting at X corresponding to maximal chain-recurrent laminations, and let $\boldsymbol{S}_X$ by the unit tangent sphere at X. Then SV_X is precisely the set of extreme points in $\boldsymbol{S}_X$*

In Chapter 3, we begin by computing estimates for the Thurston distance between a point X and its earthquake along a simple closed curve α, in terms of $l_\alpha(X)$. We prove:

Proposition 1.14. *Let α be a simple closed curve on S, and let $X \in \mathcal{T}(S)$. Then there exists some uniform constant C such that*

$$d_{Th}(X, Eq_{\alpha,t}(X)) \leq \log(e^{l_\alpha/2} t) + C$$

We then use this result, combined with the computations in Chapter 2 to show that when S is of sufficiently low complexity, the geodesic envelopes in $\mathcal{T}(S)$ have uniformly bounded width. We show:

Theorem 1.15. *Let S be the once-punctured torus or the four-times punctured sphere. There exists some $B > 0$ such that for any $X, Y \in \mathcal{T}(S)$, $w(X, Y) < B$.*

2 THE INFINITESIMAL ENVELOPE

The goal of this section is to understand the infinitesimal structure of geodesic envelopes in the Thurston metric.

We begin the section by proving Theorem 1.13, which will be one of the main ingredients in the arguments employed in Chapter 3 to prove Theorem 1.15

We continue by examining $\mathrm{Env}_0(X, Y)$ when $\Lambda(X, Y)$ contains a pair-of-pants decomposition of S. In this case, the envelope width is determined by twist parameters along geodesics. This will then allow us to estimate the maximal and minimal twisting within $\mathrm{Env}(X, Y)$, which we will use in Chapter 3.

2.1 STRETCH VECTORS IN THE ENVELOPE

In this section, we study the set of tangent vectors in $T_X\mathcal{T}(S)$ which arise from 1-jets of C^1 geodesics starting at some point $X \in \mathcal{T}(S)$. This set, called the **unit norm sphere**, and denoted by $\boldsymbol{S}_X$ was shown in [HOP21] to have a combinatorial structure of a convex body with a convex stratification whose faces come from chain-recurrent laminations in S. If $v \in \boldsymbol{S}_X$ is a 1-jet of a stretch path $\mathrm{Stretch}(X, \lambda, t)$, we call it a **stretch vector with respect to** λ, and denote it by $v = v_\lambda(X)$.

The following lemma appears in various parts of [Thu86] and [HOP21]. We adapt it to our language:

Lemma 2.1. *For any $X \in \mathcal{T}(S)$, $\boldsymbol{S}_X$ is a topological sphere around $0 \in T_X\mathcal{T}(S)$.*

Moreover, $\boldsymbol{S}_X$ is an infinite union of convex sets, called **faces**, which glue together in a combinatorial way. In [HOP21], it is shown that there is a one-to-one correspondence with topological chain-recurrent laminations and faces of $\boldsymbol{S}_X$. We review their definitions and theorems later in this section.

We further analyze the structure of $\boldsymbol{S}_X$, and prove Theorem 1.13, answering Conjecture 1.12 of [HOP21] in the affirmative:

Theorem. *The set of stretch vectors in $\boldsymbol{S}_X$ with respect to completions of maximal chain-recurrent laminations is precisely the set of extreme points in $\boldsymbol{S}_X$*

This result not only allows us to characterize faces in $\boldsymbol{S}_X$ using stretch paths, but also allows us to find extreme points in the **infinitesimal envelope** from X to Y. These extreme points can later be used to give upper and lower bounds on the twisting width between *any* two geodesics from X to Y, not just stretch lines starting at X, as computed in Chapter 3.

Throughout this section, if Λ is some lamination, we fix the following notation:

- We denote by Λ^{CR} the largest sublamination of Λ which is chain-recurrent (see Chapter 1).

- If Λ is chain-recurrent, we denote by $CR(\Lambda)$ to be the set of chain-recurrent laminations containing Λ that are maximal with respect to inclusion. Abusing notation, we write $CR(\emptyset)$ to denote the set of all chain-recurrent laminations that are maximal with respect to inclusion.

- If Λ is chain-recurrent or is empty, we define:

$$MCR(\Lambda) = \{\mu \in CR(\Lambda) : \forall \nu \in CR(\Lambda), \mu \subset \nu \Rightarrow \mu = \nu\}$$

- If Λ is chain-recurrent or empty, we denote by $M(\Lambda)$ to be the set of completions of laminations in $MCR(\Lambda)$.

- If Λ is chain-recurrent or empty, and $X \in \mathcal{T}(S)$, we define $SV_X(\Lambda) = \{v_\mu(X) : \mu \in M(\Lambda)\}$. We abuse notation and write $SV_X = SV_X(\emptyset) = \{v_\mu(X) : \mu \in M(\emptyset)\}$

Using this notation, Theorem Theorem 1.13 says that the set of extreme points of $\boldsymbol{S}_X$ is precisely $SV_X(\emptyset)$. We prove some lemmas and state some facts about the above sets and spaces.

Fact 2.2. If Λ is chain-recurrent or empty, $CR(\Lambda)$ is compact in the Hausdorff topology.

Proof. The space of all geodesic laminations on a surface is compact [BZ05], so it suffices to show that the set of chain-recurrent laminations containing Λ is closed. Note that containment of Λ is a closed condition, since laminations are closed subsets of S. Let Λ_n be a converging sequence of chain-recurrent laminations. If α_n^k is a sequence of simple closed multicurves converging in the Hausdorff topology to Λ_n, then α_n^n is a sequence of simple closed curves converging to Λ. □

The following lemma follows from the proof of Theorem 8.5 in [Thu86], and also appears as Corollary 2.3 in [DLRT20] and in the discussion following Remark 2.9 in [HOP21].

Lemma 2.3. *If Λ is chain-recurrent or empty, and if $\lambda \in MCR(\Lambda)$, then for any two completions λ_1 and λ_2 of λ, we have that for any $X \in \mathcal{T}(S)$, and $t \geq 0$,* $\mathrm{Stretch}(X, \lambda_1, t) = \mathrm{Stretch}(X, \lambda_2, t)$*, and in particular,* $v_{\lambda_1}(X) = v_{\lambda_2}(X)$

The case for $\mathrm{Env}_0(X, Y)$

Let $X, Y \in \mathcal{T}(S)$ be two points in Teichmuller space, and let $\mathcal{G}(X, Y)$ denote the set of all geodesics parametrized by arc length from X to Y. By definition, $\cup\mathcal{G}(X, Y) = \mathrm{Env}(X, Y)$. If $g \in \mathcal{G}(X, Y)$, we define the 1**-jet** of g by $v_g(X) = \frac{d}{dt}|_{t=0} g(t)$. Note that $v_g(X) \in T_X\mathcal{T}(S)$ is a unit tangent vector.

Definition 2.4. The **infinitesimal envelope**, $\mathrm{Env}_0(X, Y) \subset T_X\mathcal{T}(S)$ is defined by:

$$\mathrm{Env}_0(X, Y) = \{v_g(X) : g \in \mathcal{G}(X, Y)\}$$

For any $X, Y \in \mathcal{T}(S)$, we denote by $\Lambda(X, Y)$ the maximally stretched lamination between X and Y. In this subsection, we prove the following:

Theorem 2.5. *For any $X, Y \in \mathcal{T}(S)$,* $\mathrm{Env}_0(X, Y)$ *is the convex hull of: $SV_X(\Lambda(X, Y))$. Moreover, $SV_X(\Lambda(X, Y))$ is precisely the set of extremal vectors in* $\mathrm{Env}_0(X, Y)$.

Example 2.6. As an example for Theorem 2.5, consider the genus 2 surface, S_2, and let $\lambda = \alpha \cup \beta \cup \gamma$ be a pair-of-pants decomposition consiting of non-separating curves. Let $X \in \mathcal{T}(S_2)$ be arbitrary. Working in Fenchel-Nielsen co-ordinates with respect to α, β, γ, $\mathrm{Out}(X, \lambda)$ can be thought of as a 3-dimensional cone lying in $\mathbb{R}^6$. In particular, every Z in $\mathrm{Out}(X, \lambda)$ of distance t from X has the same length co-ordinates, and the only interesting co-ordinates are the twists.

If $Y \in \mathrm{Out}(X, \lambda)$, we can consider the 3-dimensional projection of $\mathrm{Env}_0(X, Y)$ to the tangent subspace of $T_X\mathcal{T}(S_2)$ spanned by the directions corresponding to twist co-ordinates. Theorem 2.5 then says that the extremal vectors in this projection must be stretch vectors with respect to laminations in $M(\lambda)$.

Using the formulas developed in Section 2.2, we can plot all 1-jets of stretch paths emenating from X and maximally-stretching λ. This is Figure 2.1.

The red dots in the above picture are stretch vectors corresponding to the 32 chain-recurrent completions of λ. The rest of the points are stretch paths corresponding to non chain-recurrent laminations. The eight red vertices at the "corners" of the projected infinitesimal envelope correspond to the completions Λ of λ that have the property that for any pair of curves in λ, there exists a leaf $l \subset \Lambda$ asymptotic to both. There are precisely 8

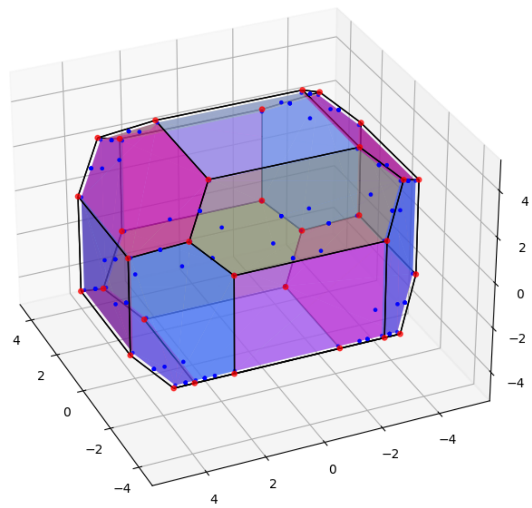

Figure 2.1: The projection of all stretch vectors emenating from X and maximally stretching λ. Combinatorially, this is a chamfered cube.

such laminations, corresponding to the 2^3 possible directions in which leaves can asymptotically twist around α, β, and γ. that have leaves

We will split the proof of Theorem 2.5 up into three main lemmas:

Lemma 2.7. *For any $X, Y \in \mathcal{T}(S)$, if $\Lambda(X, Y)$ has finitely-many completions, then $Env_0(X, Y)$ is the convex hull of $SV_X(\Lambda(X, Y))$*

We will use this lemma to show the general case:

Lemma 2.8. *For any $X, Y \in \mathcal{T}(S)$, $Env_0(X, Y)$ is the convex hull of: $SV_X(\Lambda(X, Y))$*

Finally, we show:

Lemma 2.9. *Let $\lambda \in M(\Lambda(X, Y))$, and let $v_\lambda \in SV_X(\Lambda(X, Y))$ be a convex combination: $v_\lambda = \sum_i a_i v_{\lambda_i}$, where $\lambda_i \in M(\Lambda(X, Y))$. Then $v_{\lambda_i} = v$ for all i.*

Before we prove the lemmas, we will prove a helpful technical lemma about oriented foliations and their 1-jets. When we say *oriented foliation*, we mean a foliation on some manufold M such that every leaf carries with it an orientation, and that these orientations vary continuously along any path transverse to the foliation.

Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be smooth oriented foliations defined on some open domain with smooth (possibly empty) boundary $0 \in U \subset \mathbb{R}^k$. For each oriented foliation, we can construct a vector field of unit-length vectors tangent to the foliation. Such a vecor field can be constructed by taking

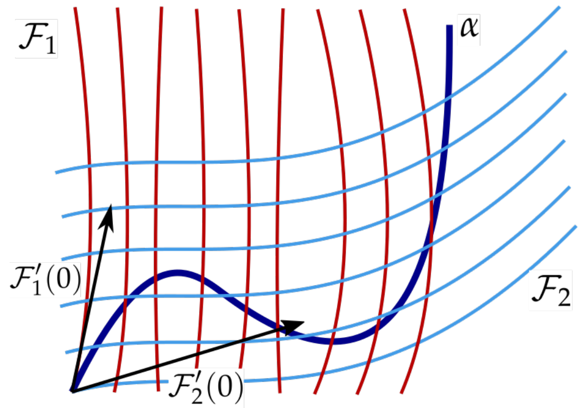

Figure 2.2: The picture of the set-up in 2.10

the associated line field of the foliation and assigning a direction at every point using the orientation of the foliation. We will refer to the flow along the vector field associated to the foliation by the 'flow along the foliation'.

Let $U^\varepsilon = \{y \in U : d(y, \partial y) > \varepsilon\}$ and let $f_i : [0, \varepsilon) \times U^\varepsilon \to \mathbb{R}^k$ be defined by setting $f_i(t, x)$ to be the flow for time t along $\mathcal{F}_i$ starting from x. Taking ε sufficiently small, we can always ensure that $0 \in U^\varepsilon$, and that all of the flows f_i are defined at $x = 0$.

We prove:

Lemma 2.10. *Let $\alpha : [0, T) \to U$ be a path differentiable at 0, and such that $\alpha(0) = 0$. Assume that for every $t < T$, there exist $t_1, \ldots, t_n \geq 0$ and $i_1, \ldots, i_n$ such that $t = \sum_i t_i$ and $\alpha(t) = f_{i_1}(t_1, f_{i_2}(t_2, (\ldots, (f_{i_n}(t_n, 0)))))$. Furthermore, assume that the non-negative span, $\operatorname{Span}_{\geq 0}(\{f_i\}_i)$ is a closed subset of $\mathbb{R}^N$. Then $\alpha'(0) \in \operatorname{Span}_{\geq 0}(\{\mathcal{F}'_i(0)\}_i) = \operatorname{Span}_{\geq 0}(\{\frac{\partial}{\partial t} f_i(0,0)\}_i)$*

Proof. In order to estimate $\alpha'(0)$, we first note that as t gets really small, so must the t_i's. Thus, for small t, we can take a first-order expansion of $\alpha(t) = f_{i_1}(t_1, f_{i_2}(t_2, (\ldots, (f_{i_n}(t_n, 0)))))$ when t_1 is small to get:

$$\begin{aligned}\alpha(t) &= f_{i_1}(0, f_{i_2}(t_2, (\ldots, (f_{i_n}(t_n, 0))))) \\ &\qquad + t_1 \frac{\partial f_{i_1}}{\partial t}(0, f_{i_2}(t_2, (\ldots, (f_{i_n}(t_n, 0))))) + o(t_1^2) \\ &= f_{i_2}(t_2, (\ldots, (f_{i_n}(t_n, 0)))) + t_1 \frac{\partial f_{i_1}}{\partial t}(0, f_{i_2}(t_2, (\ldots, (f_{i_n}(t_n, 0))))) + o(t_1^2)\end{aligned}$$

Where we understand $\frac{\partial f_{i_1}}{\partial t}$ to be the derivative of f_{i_1} with respect to the first co-ordinate.

Expanding the $f_{i_2}(t_2,(\dots,(f_{i_n}(t_n,0))))$ in terms of t_2, we get:

$$\alpha(t) = f_{i_3}(t_3,(\dots,(f_{i_n}(t_n,0)))) + t_2\frac{\partial f_{i_2}}{\partial t}(0, f_{i_3}(t_3,(\dots,(f_{i_n}(t_n,0))))) \tag{2.1}$$

$$+ t_1\frac{\partial f_{i_1}}{\partial t}(0, f_{i_2}(t_2,(\dots,(f_{i_n}(t_n,0))))) + o(t^2) \tag{2.2}$$

If we continue to expand in this manner, we obtain:

$$\alpha(t) = f_{i_n}(t_n,0) + \sum_{j=1}^{n-1} t_j\frac{\partial f_{i_j}}{\partial t}(0, f_{i_{j+1}}(t_{j+1},(\dots,(f_{i_n}(t_n,0))))) + o(t^2) \tag{2.3}$$

$$= t_n\frac{\partial f_{i_n}}{\partial t}(0,0) + \sum_{j=1}^{n-1} t_j\frac{\partial f_{i_j}}{\partial t}(0, f_{i_{j+1}}(t_{j+1},(\dots,(f_{i_n}(t_n,0))))) + o(t^2) \tag{2.4}$$

$$\tag{2.5}$$

We denote $v_j = \frac{\partial}{\partial t}f_j(0,0)$, and notice that $f_j(t_j,0) = t_jv_j + o(t^2)$. Thus,

$$\begin{aligned}
f_{i_{n-1}}(t_{n-1}, f_{i_n}(t_n,0)) &= f_{i_{n-1}}(t_{n-1}, t_nv_{i_n} + o(t^2)) \\
&= f_{i_{n-1}}(t_{n-1},0) + D_0f_{i_{n-1}}(t_{n-1},0)\cdot(t_nv_{i_n}) + o(t^2)) \\
&= t_{i_{n-1}}v_{n-1} + t_nD_0f_{i_{n-1}}(t_{n-1},0)\cdot v_{i_n} + o(t^2) \\
&= t_{i_{n-1}}v_{n-1} + t_nD_0(t_{n-1}v_{i_{n-1}} + o(t^2))\cdot v_{i_n} + o(t^2) \\
&= t_{n-1}v_{i_{n-1}} + t_nt_{n-1}v_{i_{n-1}} + o(t^2) \\
&= t_{n-1}v_{i_{n-1}} + o(t^2)
\end{aligned}$$

Where we used the fact that all functions are smooth and hence have bounded derivatives in a neighbourhood of 0. Note that we can continue this computation to replace the composition terms in 2.3 with simpler, linear terms:

$$\alpha'(0) = t_nv_{i_n} + \sum_{j=1}^{n-1} t_j\frac{\partial f_{i_j}}{\partial t}(0, t_{j+1}v_{i_{j+1}} + o(t^2)) + o(t^2) \tag{2.6}$$

$$= t_nv_n + \sum_{j=1}^{n-1} t_j\left(v_{i_j} + D_0f_{i_j}(0,0)\cdot(t_{j+1}v_{i_{j+1}} + o(t^2))\right) + o(t) \tag{2.7}$$

$$= \sum_{j=1}^{n} t_jv_{i_j} + o(t^2) \tag{2.8}$$

Since $W = \text{Span}_{\geq 0}(\{v_i\}_i)$ is closed, we also get that for any $t > 0$, we have that $d(\alpha(t), W) = o(t^2)$. This means that $\alpha'(0)$ is in W. □

Lemma 2.11. *Assume that $\Lambda(X,Y)$ has finitely-many completions, and let $v \in \mathrm{Env}_0(X,Y)$. Then v is a convex combination of vectors in $SV_X(\Lambda(X,Y))$.*

Proof. Let $v(t) : [0,1] \to \mathcal{T}(S)$ be a geodesic path parametrized by arc length whose 1-jet at 0 is v. Since $v \in \mathrm{Env}_0(X,Y)$, we can freely assume that $v(t)$ lies in $\mathrm{Env}(X,Y)$ for any sufficiently small t.

Working in co-ordinates $\varphi : \mathcal{T}(S_g) \to \mathbb{R}^{6g-6}$, for each completion λ of $\Lambda(X,Y)$, we get a foliation of $\mathbb{R}^{6g-6}$ defined by the Thurston stretch line corresponding to λ. This foliation also comes with a natural *flow direction* by taking the forward corresponding to the stretch path defined by λ.

Note that when λ contains a pair-of-pants decomposition, shearing co-ordinates are smoothly related to Fenchel-Nielsen co-ordinates on $\mathcal{T}(S_g)$ by the computations done in Section 2.2. More generally, by Theorem A of [Bon01] and computations in [Gen15], we have that length functions of simple closed curves are smooth in the shearing co-ordinates. Since Teichmuller space is locally parametrized by length functions of $6g-6$ simple closed curves, it follows that we can think of shearing co-ordinates not just as topological co-ordinates on $\mathcal{T}(S_g)$, but as smooth co-ordinates as well. Stretch lines are smooth in shearing co-ordinates, since they can be realized as rays starting at the origin in the shearing co-ordinates corresponding to the lamination defining the stretch path [Thu86]. Thus, stretch lines are also smooth in Fenchel-Nielsen co-ordinates. In particular, it follows that if λ is a completion of a maximal chain-recurrent lamination containing $\Lambda(X,Y)$, we get a smooth oriented foliation (which we will call $\mathcal{F}_\lambda$) of $\mathbb{R}^{6g-6}$ under any co-ordinates on Teichmüller space.

By Thurston's construction of geodesics using concatenation of stretch lines, it follows that for each t, $v(t)$ can be expressed as a concatenation of flows along the foliations $\mathcal{F}_\lambda$. Moreover, a careful reading of Theorem 8.5 of [Thu86] actually says that we can choose these laminations to lie in $M(\Lambda(X,Y))$.

By Lemma 2.10, using the fact that $SV_X(\Lambda(X,Y))$ is finite and hence $\mathrm{Span}_{\geq 0}(SV_X(\Lambda(X,Y)))$ is closed, we get that $v \in \mathrm{Span}_{\geq 0}(SV_X(\Lambda(X,Y)))$ and we write $v = \sum_{i=1}^N a_i v_{\lambda_i}$, where $\lambda_i \in M(\Lambda(X,Y))$.

Next, we show that v is a convex linear combination of $SV_X(\Lambda(X,Y))$. We write $v = \sum_i a_i v_{\lambda_i}$, and consider the path:

$$\beta(t) = f_1(ka_1 t, f_2(ka_2 t, (\dots, (f_n(ka_n t, 0)))))$$

Where k is chosen such that $k \sum a_i = 1$, and where we denote $f_i(t,X) = \mathrm{Stretch}(X, \lambda_i, t)$. Note that β is a length-parametrized geodesic, since $\Lambda(X,Y)$ is maximally stretched along it. By the same computations in Lemma 2.10, we get that $\beta'(0) = kv$. In particular, since v and $\beta'(0)$ are unit-length vectors, it follows that $k = 1$, and $\sum_i a_i = 1$. $\square$

We now treat the cases when $\Lambda(X, Y)$ has infinitely many completions.

Lemma 2.12. *If Λ is chain-recurrent or empty, then* $\mathrm{Span}_{\geq 0}(SV_X(\Lambda))$ *is closed.*

Proof. We will show that for $\lambda_n \in MCR(\Lambda)$, if $v_{\lambda_n} \in SV_X(\Lambda)$ converge to some v, then v is in the convex hull of $SV_X(\Lambda)$. By 2.2, up to subsequence, there exists some $\lambda \in CR(\Lambda)$ such that $\lambda_n \to \lambda$ in the Hausdorff topology.

By maximality of λ_n, $S \setminus \lambda_n$ has at most finitely-many completions to a triangulation, and since λ is a Hausdorff limit of λ_n, it shares this property. Let $\lambda^1, \ldots, \lambda^k \in M(\lambda)$ be the finitely-many completions of maximal chain-recurrent laminations containing λ.

Since the complementary components of λ_n stabilize close to the complementary components of λ, it follows that for any sufficiently large n, there exist completions of $\lambda_n^i \in M(\lambda_n)$ which Hausdorff converge to λ^i. We can do this, for example, by adding in leaves into the complementary components of λ_n^i which converge to the added leaves of λ^i.

For any t, we consider the family $S_n(t) = \mathrm{Stretch}(X, \lambda_n^i, t)$. Since $S_n(t)$ are smooth geodesics, it follows that they have uniformly bounded first derivatives. By the Arzela-Ascoli theorem, up to subsequence, $S_n(t)$ converge to a continuous path, $S_\infty(t)$ in $\mathcal{T}(S)$, starting at X. Moreover, S_∞ is differentiable at 0, and has derivative equal to v. Note that because λ_n^i converge to λ^i, and $\lambda \subset \lambda^i$, it follows that $\lambda \subset \Lambda(X, S_\infty(t))$, and so $\Lambda(X, S_\infty(t))$ has finitely-many completions.

By 2.7, we have that $\mathrm{Env}_0(X, S_\infty(t))$ is the convex hull of $SV_X(\Lambda(X, S_\infty(t)))$, which is closed. Since this is true for all t, a similar Taylor series argument as in the previous lemmas above shows that, in fact, $v = \frac{d}{dt}|_{t=0} S_\infty(t)$ lies in the convex hull of $SV_X(\Lambda(X, S_\infty(t)))$, as desired. □

The proof of Lemma 2.8 follows in the same way as the proof of Lemma 2.7, where we now use Lemma 2.12 to meet the conditions of Lemma 2.10.

Remark 2.13. The reason we split Lemma 2.8 into two lemmas is a bit of a subtlety. In the argument above, we showed that the non-negative span of $SV_X(\Lambda(X, Y))$ is closed. A priori, the entire proof of Lemma 2.12 could have been skipped if we knew that $SV_X(\Lambda(X, Y))$ was closed.

To argue this, let v_{λ_n} be a converging sequence of stretch vectors converging to some v. One should show that the Hausdorff limit of $\lambda_n \in MCR(\Lambda(X, Y))$ is a lamination $\lambda \in MCR(\Lambda(X, Y))$, and then use the fact that stretch vectors change continuously in their defining laminations to get that $v = v_\lambda$. It is easy to see that λ must be chain-recurrent, and we must show that it is maximal amongst all chain-recurrent laminations,

Showing that $MCR(\Lambda(X, Y))$ is closed is non-trivial, and may in fact be false. It would follow from Remark 2.9 of [HOP21], that says that the complementary components of a maximal chain-recurrent lamination consists of ideal triangles, once-punctured monogons, or once-punctured bigons.

The argument is that Remark 2.9 of [HOP21] tells us that the complementary components of λ must also be triangles or once-punctured monogons and bigons, and hence λ is maximal. However, a proof of the remark is not furnished anywhere in literature, and it is not clear if it is even true. It is possible that λ_n each have a complementary component that is an ideal square, and that in the limit, λ has an ideal square as a complementary component, and one of the diagonal leaves can be added to λ preserving chain-recurrence.

It remains to show that $SV_X(\Lambda(X,Y))$ consists of extremal vectors in $\mathrm{Env}_0(X,Y)$. This follows immediately from the proof of Theorem 5.2 in [Thu86]. We provide the main argument, but refer the reader to details in [Thu86]. We prove Lemma 2.9:

Lemma. *Let Λ be chain-recurrent or empty, and let $\lambda \in M(\Lambda)$, and let $v_\lambda \in SV_X(\Lambda)$ be a convex combination: $v_\lambda = \sum_i a_i v_{\lambda_i}$, where $\lambda_i \in M(\Lambda)$. Then $v_{\lambda_i} = v_\lambda$ for all i.*

Proof. For a lamination μ, denote by μ^{CR} the maximal chain-recurrent sublamination of μ. To prove the lemma, suffices to show that $\lambda_i^{CR} = \lambda^{CR}$.

Suppose not, and let α_n be a sequence of simple closed multi-curves Hausdorff-converging to λ^{CR}. Denoting the curve complex of S by $\mathcal{C}(S)$, and fixing $X \in \mathcal{T}(S)$, we define a map from $T\mathcal{T}(S) \times \mathcal{C}(S)$ by sending v, α to $D_v \log l_\alpha$. This map is linear in the first agument, and is continuous in the second argument, where we think of $\mathcal{C}(S)$ as a space endowed with the Hausdorff topology on X.

By Thurston's construction of $\mathrm{Stretch}(X, \lambda, t)$, it follows that if α_n Hausdorff converge to λ^{CR}, then $\lim_{n\to\infty} D_{v_\lambda} \log(l_{\alpha_n}) = 1$. We argue that for any i, $\lim_{n\to\infty} D_{v_{\lambda_i}} \log(l_{\alpha_n}) < 1$, and the contradiction follows.

Indeed, if $\lim_{n\to\infty} D_{v_{\lambda_i}} \log(l_{\alpha_n}) = c \geq 1$, then there would exist some weights w_n on α_n and some measured lamination λ^m whose support contains λ^{CR}, for which α_n converge to λ^m in measure. Moreover, for this measured lamination, we would have $D_{v_{\lambda_i}} \log(l_{\lambda^m}) = c$. This means that λ^{CR} is maximally stretched along v_{λ_i}, a contradiction. □

Proving Theorem 1.13

We now prove Theorem 1.13:

Proof. Let v be an extremal point in $\boldsymbol{S}_X$. v must lie in $\mathrm{Env}_0(X,Y)$ for some $Y \in \mathcal{T}(S)$. In particular, it must be extremal in $\mathrm{Env}_0(X,Y)$, and hence, by

Theorem 2.5, $v \in SV_X$. Conversely, by Lemma 2.9, any stretch vector must be extremal. □

2.2 COMPUTING TWISTING FROM SHEARING

We compute the twist parameters along Thurston geodesics defined by laminations that are completions of pair of pants decompositions of S. A large portion of this work was independently done in [HOP21], but we leave it here for the sake of exposition. We begin by reviewing some notation that will help us later on.

Introduction and Notation

Let $\mathcal{C} = \{c_1, \ldots, c_N\}$ be a pair-of-pants decomposition of S, and let λ be a chain-recurrent completion of $\mathcal{C}$ to a triangulation. Let $X \in \mathcal{T}(S)$, and let X_t be the Thurston geodesic $\mathrm{Stretch}(X, \lambda, t)$. To compute the twisting along X_t, we must compute $\tau_{c_i}(t)$ as a function of s_λ. This can be done from the definition, but depends on the topological type of λ. Let $c \in \mathcal{C}$ be some curve, and $\tilde{c}$ a lift of it. Let P_1, P_2 be its adjacent pairs-of-pants, with lifts $\tilde{P}_i$ adjacent to $\tilde{c}$.

Let $p_1(t)$ and $q_2(t)$ be defined as in Chapter 1 be the distinguished points used to compute $s_c(X_t)$ and $\tau_c(X_t)$, where we choose lifts carefully so that p_i and q_i lie on the same lift $\tilde{c}$ of c.

We define $\Delta_{P_i,c,\lambda}(t)$ as the signed distance from $q_i(t)$ to $p_i(t)$, where the sign is positive if to get from $q_i(t)$ to $p_i(t)$, one has to turn left from the perspective of $\tilde{P}_i$. The functions $\Delta_{P_i,c,\lambda}(t)$ are actually intrinsic to P_i, and are well-defined up to a choice of the lifts of all of the curves in $\mathcal{C}$. In particular, $\Delta_{P_1,c,\lambda}(t) + \Delta_{P_2,c,\lambda}(t) = s_c(X_t) - \tau_c(X_t)$, so we devote the rest of this section to computing $\Delta_{P,c,\lambda}(t)$, where P is a pair of pants in $S \smallsetminus \mathcal{C}$, $c \in \mathcal{C}$ and λ is a triangulation of P.

Definition 2.14. A **geodesic triangulation** of a hyperbolic pair of pants P is a decomposition of P into two ideal triangles, whose edges are glued together.

Lemma 2.15. *Let P be a hyperbolic pair of pants, then there are exactly 32 geodesic triangulations of P up to boundary-preserving isomorphism of P.*

This fact is stated in the beginning of section 3.3 of [PT07], and follows from the discussion in section 2.6 of [PH92].

In fact, let P be a pair of pants with cuff curves γ_1, γ_2, and γ_3. The topological type of a geodesic triangulation of P is precisely characterized by the following information

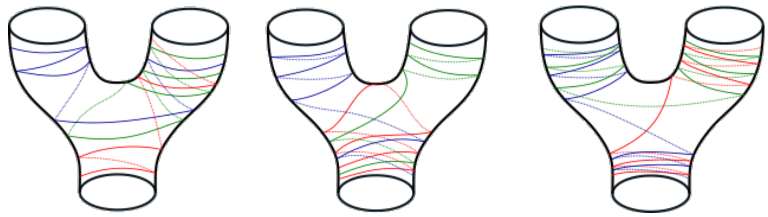

Figure 2.3: Asymmetric, 2-symmetric, and 3-symmetric laminations λ. The curve γ is the bottom one

- The twist direction at every cuff γ_i of P (there are two possibilities per cuff, so eight options in total)
- The number of leaves converging to $\gamma_1, \gamma_2, \gamma_3$, with possibilities $(2,2,2), (1,1,4), (1,4,1)$ and $(4,1,1)$.

Definition 2.16. If γ is a cuff curve of P, and λ is a triangulation of P, then we say that

- λ is 3-symmetric if λ has precisely two leaves converging to γ
- λ is 2-symmetric around γ if λ has precisely four leaves converging to γ
- λ is asymmetric around γ if λ has precisely one leaf converging to γ

We compute $\Delta_{c,P,\lambda}$ in each of these cases. We will first fix some notation in all of the following computations.

Let γ_i be the three boundary components of P, and we will always assume that $c = \gamma_1$. Let γ_{ij} be the leaf of λ which is asymptotic to both γ_i and γ_j. For ease of notation later on, we define $l_i = l_{\gamma_i}(X_0)$, and $s_i = s_{\gamma_i}(X_0)$, $s_{ji} = s_{ij} = s_{\gamma_{ij}}(X_0)$. Note that in the different possibilities for λ, not all of the variables s_{ij} have meaning. For example, if λ is 2-symmetric, γ_{23} is replaced by γ_{22}.

λ is 3-symmetric around c

In this subsection, we compute $\Delta_{P,c,\lambda}$, where λ is a 3-symmetric triangulation which twists left at every cuff. We will compute $\Delta_{P,c,\lambda}(X_0)$ in terms of the shearing co-ordinates of X_0.

We fix an identification of $\tilde{S}$ with the upper-half plane such that $\tilde{\gamma}_1$ is the line $\{(0,t) : t \in \mathbb{R}\}$, oriented so that P lies to the left of this line.

We know that γ_{13} is asymptotic to γ_1, and twists left, meaning that there is a lift of it of the form $\{(x,t) : t \in \mathbb{R}\}$, for some $x < 0$. Similarly, γ_{12} has

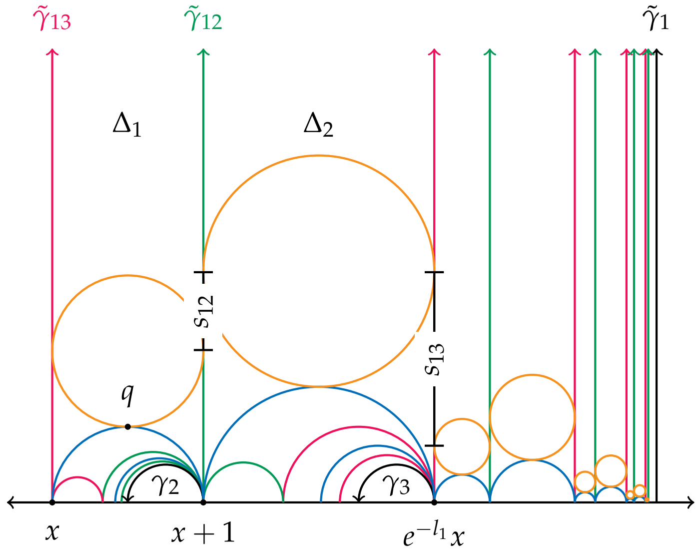

Figure 2.4: Lifting a 3-symmetric lamination

a lift of the form $\{(y,t) : t \in \mathbb{R}\}$, oriented so that $(0,0)$ lies to the right of them.

Up to rescaling the identification above, we can assume that $y = x+1$. This will allow us to explicitly compute x later on. In our choice of specified lifts for the shearing co-ordinates, we choose these lifts of γ_{ij} with this orientation. This choice will not matter after differentiating, as a different choice of lifts will only change the answer by an additive constant.

The deck action of $[\gamma_1]^{-1}$ on $\mathbb{H}^2$ sends the vertical lines $\{(a,t) : t \in \mathbb{R}\}$ to $\{(e^{-l_1}a, t) : t \in \mathbb{R}\}$, and in particular, must send one lift of γ_{ij} to another one. Drawing two lifts of γ_{23}, one from $(x,0)$ to $(x+1,0)$ and one from $(x+1,0)$ to $(e^{-l_1}x, 0)$, we obtain lifts of the complementary triangles of λ.

In particular, one can readily see from this picture that $s_{12} + s_{13} = -l_1$, and in general, $|s_{ij} + s_{jk}| = l_j$, where the sign is positive if γ_{ij} and γ_{jk} twist to the right at γ_j, and is negative otherwise. This is a special case of Lemma 3.2 in [BBFS09]

By definition, the shearing co-ordinates of γ_{12} is the difference between the medians along $\tilde{\gamma}_{12}$ of the two triangles drawn (Δ_1 and Δ_2). This gives us the (Euclidean) radius of the incircle on the right, which tells us that $e^{-l_1}x - (x+1) = e^{s_{12}}$.

In particular,

$$x = \frac{1+e^{s_{12}}}{e^{-l_1}-1}$$

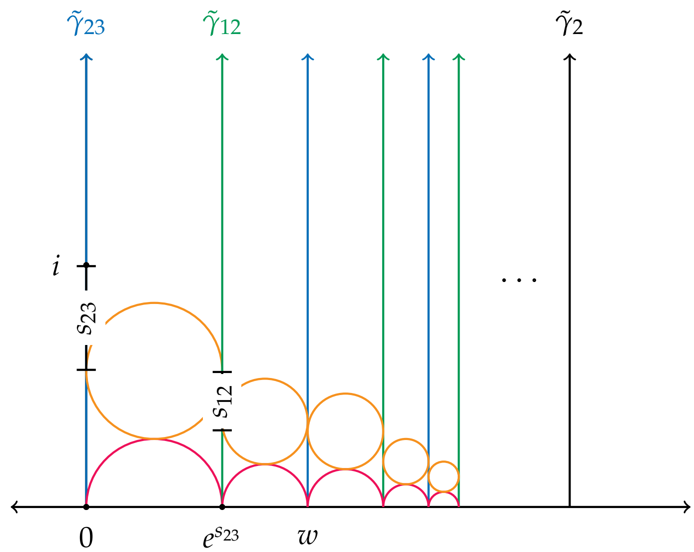

Figure 2.5: The lift of the 3-symmetric lamination after applying the isometry φ

The next step is to find the endpoints of $\tilde{\gamma}_2$ in terms of x. Since γ_{12} is asymptotic to γ_2, it follows that one of the endpoints of γ_2 is $x+1$. The second endpoint can be computed by taking the limit $[\gamma_2]^n \circ x$, where $[\gamma_2]$ is the deck transformation corresponding to γ_2. Call this second endpoint p^*. We now wish to compute p^* in terms of x and the shearing co-ordinates.

To do this, we apply the isometry $\varphi(z) = \frac{x-z}{z-(x+1)}$, which will send the lift of γ_2 at $[p^*, x+1]$ to an arc $[\varphi(p^*), \infty]$. This isometry also sends the arc $[x, x+1]$ to the arc $[0, \infty]$, and we can now use shearing co-ordinates to find that its first translate under $[\gamma_2]$ must lie on $[w, \infty]$ Where $w = e^{s_{23}} + e^{s_{23}} e^{s_{12}}$ (see picture):

Since the translation length of the aciton of $[\gamma_2]$ is precisely l_2, it follows that we must have that $\varphi(p^*) - w = e^{l_2}(\varphi(p^*))$, or, solving for $\varphi(p^*)$ and plugging in $\varphi^{-1}(z) = x + \frac{z}{z+1}$ yields:

$$p^* = x + \left(\frac{e^{s_{23}} + e^{-l_2}}{e^{s_{23}} + 1} \right)$$

All that remains is to compute the intersection point of $[0, \infty]$ and the orthogeodesic between $[p^*, x+1]$ and $[0, \infty]$. In other words, we must find the point p as described at the start of this section.

Under our lift, the incircle of Δ_1 intersects $\tilde{\gamma}_{13}$ at i, meaning that the horocycle connecting this intersection point to $\tilde{\gamma}_1$ must also intersect $\tilde{\gamma}_1$ at i. Thus, $\Delta_{P,c,\lambda}(X_0) = \log(-ip)$.

Claim. *Let C_1 be a circle whose center lies on the x-axis intersects the x-axis at $(a, 0)$ and $(b, 0)$ with $0 < a < b$. Let C_2 be a circle centered at 0 intersecting C_1 at right angles. Then C_1 has radius $\sqrt{ab}$*

Proof. The triangle $(0, 0), (0, \frac{a+b}{2}), C_1 \cap C_2$ is a right triangle with hypothenuse $\frac{a+b}{2}$, and other whose other edges have lengths equal to $\frac{b-a}{2}$ and to the radius of C_2. The claim follows. □

By the above claim, it follows that $p = i\sqrt{(x+1)p^*}$, and since $q(t)$ is located at i by the choice of normalization, we get that:

$$\Delta_{P,c,\lambda}(X_0) = \frac{1}{2}\log\left((x+1)\left(x + \frac{e^{s_{23}} + e^{-l_2}}{e^{s_{23}} + 1}\right)\right) \tag{2.9}$$

where $x = \frac{1+e^{s_{12}}}{e^{-l_1}-1}$, and s_{ij} are the shearing co-ordinates of X_0, as described before.

Recall that these shearing co-ordinates $\{s_{ij}\}$ satisfy the system of equations $s_{ij} + s_{jk} = -l_j$, as we assumed that the arcs γ_{ij} all twist left around the cuffs. In particular, by solving this system of equations, we can compute $\Delta_{P,c,\lambda}(X_0)$ as an explicit function of $l_{\gamma_1}, l_{\gamma_2}$, and l_{γ_3}.

A similar computation also holds when the twist directions of λ around the γ_i's are arbitrary. However, in this case, every appearance of l_i is replaced with $-l_i$ if λ twists to the right around γ_i, and the twisting is in the opposite direction if λ twists to thr right around γ_1.

For example, if λ twists to the right around γ_1, then we choose a normalization and lifts of γ_{13} and γ_{12} that are vertical lines intersecting the x-axis at x and $x+1$. The next lift of γ_{13} is then the vertical line at $e^{l_1}x$, and not at $e^{-l_1}x$. The computation of x, p^*, and therefore $\Delta_{\mathcal{P},c,\lambda}$ follow in the same way, but with l_1 replaced with $-l_1$, and the signed distance from p to q picking up a negative sign.

If λ twists to the right around γ_2, then in Figure 2.4, $\tilde{\gamma}_2$ would be drawn starting at $x+1$, and going to the right of $\tilde{\gamma}_{12}$. After passing through the same mobius transformation, we arrive at a reflected picture in place of Figure 2.5, where the deck transformation pushing lifts of γ_{23} towards $\tilde{\gamma}_2$ acts in the opposite direction. Thus, we would get $\varphi(p^*) - w = e^{-l_2}\varphi(p^*)$, and the rest follows.

Similar computations can be done for all permutations of twisting directions of λ around the γ_i's, yielding:

$$\varepsilon_1 \frac{1}{2}\log\left((x+1)\left(x + \frac{e^{s_{23}} + e^{-\varepsilon_2 l_2}}{e^{s_{23}} + 1}\right)\right) \tag{2.10}$$

where

- $\varepsilon_i = 1$ if λ twists to the left around l_i, and $\varepsilon_i = -1$ otherwise.

- $x = x(t) = \frac{1+e^{s_{12}}}{e^{-\varepsilon_1 l_1}-1}$,
- s_{ij} are the shearing co-ordinates, which satisfy $s_{ij} = \frac{1}{2}(\varepsilon_k l_k - \varepsilon_i l_i - \varepsilon_j l_j)$ for $k \neq i \neq j \neq k$

The formula above for $\Delta_{P,\gamma_1,\lambda}$ does not appear to be symmetric under replacing the labels of γ_2 and γ_3, and indeed there is no reason for it to be, as we made the choice to compute p using γ_2. However, one may verify that choosing to compute p using γ_3 is analogous to replacing x with $x+1$, and replacing $x+1$ with $e^{-l_1}x$, and rescaling by a factor of $e^{-l_1}x - (x+1)$. This yields a difference of $l_1/2$ in the computation, as expected.

λ is 2-symmetric around c

In this section, we compute $\Delta_{P,c,\lambda}$, where λ is a 2-symmetric or asymmetric triangulation with respect to c, which twists left at every cuff. We will compute $\Delta_{P,c,\lambda}(X_0)$ in terms of the shearing co-ordinates of X_0.

Let γ_i be the three boundary components of P, and label $c = \gamma_1$. Let γ_{ij} be the leaf of λ which is asymptotic to both γ_i and γ_j. Note that not all combinations of i and j yield a viable γ_{ij}, as in the case that λ is 2-symmetric, γ_{23} does not exist as no leaf is aysmptotic to both γ_2 and γ_3. In this case,

We define $l_i = l_{\gamma_i}(X_0)$, and $s_i = s_{\gamma_i}(X_0)$, $s_{ij} = s_{\gamma_{ij}}(X_0)$.

We follow a similar computation as in the section above to get that:

$$\Delta_{P,c,\lambda}(X_0) = \varepsilon_1 \frac{1}{2} \log\Big((x+1)(x+e^{-\varepsilon_2 l_2})\Big) \tag{2.11}$$

where

$$x = \frac{1 + e^{s_{12}} + e^{s_{12}+s_{11}} + e^{s_{12}+s_{11}+s_{13}}}{e^{-\varepsilon_1 l_1} - 1}$$

and

- $\varepsilon_i = 1$ if λ twists to the left around l_i, and $\varepsilon_i = -1$ otherwise.
- s_{ij} are the shearing co-ordinates, which satisfy:

$$\begin{aligned} s_{11} &= \frac{1}{2}(-\varepsilon_1 l_1 + \varepsilon_2 l_2 + \varepsilon_3 l_3) \\ s_{12} &= -\varepsilon_2 l_2 \\ s_{13} &= -\varepsilon_3 l_3 \end{aligned}$$

λ is asymmetric around c

In this section, we compute $\Delta_{P,c,\lambda}$, where λ is a 2-symmetric triangulation with respect to c, which twists left at every cuff.

Let γ_i be the three boundary components of P, and label $c = \gamma_1$ as before. Let γ_{ij} be defined as before, and assume that λ is 2-symmetric around γ_2. When computing $\Delta_{P,c,\lambda}$, we will use an orthogeodesic from $\tilde{\gamma}_2$ to $\tilde{\gamma}_1$. We define l_i, s_{ij} as before, noting that in this case, the only co-ordinates are s_{12}, s_{23}, and s_{22}.

We follow a similar computation as in the section above to get that,

$$\Delta_{P,c,\lambda}(X_0) = \varepsilon_1 \frac{1}{2} \log\left((x+1) \left(x + \frac{e^{s_{22}} + e^{s_{22}+s_{23}} + e^{2s_{22}+s_{23}} + e^{-\varepsilon_2 l_2}}{e^{s_{22}} + e^{s_{22}+s_{23}} + e^{2s_{22}+s_{23}} + 1} \right) \right) \tag{2.12}$$

Where

- $\varepsilon_i = 1$ if λ twists to the left around l_i, and $\varepsilon_i = -1$ otherwise.
- $x = \frac{1}{e^{-\varepsilon_1 l_1} - 1}$
- s_{ij} are the shearing co-ordinates, which satisfy:

$$\begin{aligned} s_{22} &= \frac{1}{2}(-\varepsilon_2 l_2 + \varepsilon_1 l_1 + \varepsilon_3 l_3) \\ s_{12} &= -\varepsilon_1 l_1 \\ s_{23} &= -\varepsilon_3 l_3 \end{aligned}$$

Twist Widths Between Stretch Paths

We have seen how twisting co-ordinates change along stretch paths defined by any lamination which is a completion of a pair-of-pants decomposition of a surface S. We now end this subsection with an explicit formula for the twisting along stretch paths.

Consider the foliation of $\mathcal{T}(S)$ given by stretch paths corresponding to λ. For any $Y \in \mathcal{T}(S)$, we can define the **negative-time stretch path** from Y by setting $\mathrm{Stretch}(Y, \lambda, -t)$ to be the unique point $X \in \mathcal{T}(S)$ such that $\mathrm{Stretch}(X, \lambda, t) = Y$, and $d_{Th}(\mathrm{Stretch}(Y, \lambda, -t), Y) = t$. Denote $Y^{\lambda}_{-t} = \mathrm{Stretch}(Y, \lambda, -t)$. For $X \in \mathcal{T}(S)$, we denote $X^{\lambda}_t = \mathrm{Stretch}(X, \lambda, t)$,

Fix a pair of pants decomposition $\mathcal{P}$, and some curve $c \in \mathcal{P}$. Let P_1 and P_2 be the (possibly non-distinct) pairs-of-pants adjacent to c. We employ the shorthand: $\Delta^i_{\lambda} = \Delta_{P_i,c,\lambda}$, where $\Delta_{P_i,c,\lambda}$ was defined in Section 2.2

Note that for any $X \in \mathcal{T}(S)$, we have:

$$\tau_c(X) = s_c(X) - \Delta_\lambda^1 - \Delta_\lambda^2 \tag{2.13}$$

Lemma 2.17. *For any $X, Y \in \mathcal{T}(S)$, we have:*

$$\tau_c(X_t^\lambda) = \tau_c(X)e^t + \Delta_\lambda^1(0)e^t + \Delta_\lambda^2(0)e^t - \Delta_\lambda^1(t) - \Delta_\lambda^2(t)$$
$$\tau_c(Y_{-t}^\lambda) = \tau_c(Y)e^{-t} + \Delta_\lambda^1(0)e^{-t} + \Delta_\lambda^2(0)e^{-t} - \Delta_\lambda^1(-t) - \Delta_\lambda^2(-t)$$

Where $\Delta_\lambda^i(s)$ is Δ_λ^i, where every shearing co-ordinate is multiplied by a factor of e^s for positive or negative s.

Proof. Note that $s_c(X_t^\lambda) = s_c(X)e^t$, and similarly $s_c(Y_{-t}^\lambda) = s_c(Y)e^{-t}$. This is because c is contained in λ, and in the shearing co-ordinates (s_λ) on Teichmüller space, stretch lines are exponential scalings.

For any t, and plugging in $s_c(X_t) = e^t s_c(X)$ to Equation 2.13, we have $\tau_c(X_t) = s_c(X)e^t - \Delta_\lambda^1(t) - \Delta_\lambda^2(t)$. To find $s_c(X)$, we set $t = 0$ and rearrange, giving us $s_c(X) = \tau_c(X) + \Delta_\lambda^1(0) + \Delta_\lambda^2(0)$. We then get:

$$\tau_c(X_t^\lambda) = \tau_c(X)e^t + \Delta_\lambda^1(0)e^t + \Delta_\lambda^2(0)e^t - \Delta_\lambda^1(t) - \Delta_\lambda^2(t)$$

A similar computation follows for Y_{-t}^λ. □

If ν is another completion of $\mathcal{P}$, we define the **twist width** $\Delta\tau_c(\lambda, \nu, t)$ by $\tau_c(X_t^\lambda) - \tau_c(X_t^\nu)$, and write:

$$\Delta\tau_c(\lambda, \nu, t) = e^t\Delta_\lambda^1(0) - \Delta_\lambda^1(t) \tag{2.14}$$
$$+ e^t\Delta_\lambda^2(0) - \Delta_\lambda^2(t) \tag{2.15}$$
$$- \left(e^t\Delta_\nu^1(0) - \Delta_\nu^1(t)\right) \tag{2.16}$$
$$- \left(e^t\Delta_\nu^2(0) - \Delta_\nu^2(t)\right) \tag{2.17}$$

In order to estimate the twist width, it suffices to estimate each of the terms above separately.

Observe that by the computations in the previous section, $\Delta_\lambda^i(t)$ is always of the form $\frac{1}{2}\log(R_{\mathcal{P},\lambda^i}(t))$, where $R^i_{\mathcal{P},\lambda}(t)$ is some rational polynomial over the set $\{e^{l_\alpha(t)}\}_{\alpha\in\mathcal{P}}$. Thus, if $\alpha \in \mathcal{P}$, then we have that $l_\alpha(t) = l_\alpha(X_0)e^t$, and so, for any real k, we compute:

$$
\begin{aligned}
&= \frac{1}{2}e^t \log(e^{kl_\alpha(X_0)}R^i_{\mathcal{P},\lambda}(0)) - \frac{1}{2}\log(e^{kl_\alpha(t)}R^i_{\mathcal{P},\lambda}(t)) \\
&= \frac{1}{2}e^t \left(kl_\alpha(X_0) + \log(R^i_{\mathcal{P},\lambda}(0))\right) - \frac{1}{2}\left(kl_\alpha(t) + \log(R^i_{\mathcal{P},\lambda}(t))\right) \\
&= \frac{1}{2}e^t \log(R^i_{\mathcal{P},\lambda}(0)) - \frac{1}{2}e^t \log(R^i_{\mathcal{P},\lambda}(0)) \\
&= e^t\Delta^i_\lambda(0) - \Delta^i_\lambda(t)
\end{aligned}
$$

Where t can also hold negative values. Thus, we are left with:

Fact 2.18. When computing twist widths, we are free to mutliply and divide $R^i_{\mathcal{P},\lambda}(t)$ by any factor of the form $e^{l_\alpha(t)}$, where $\alpha \in \mathcal{P}$.

3 BOUNDED WIDTH

In this chapter, we prove Theorem 1.15. To prove this theorem, we first use Theorem 1.13 to show that it suffices to compute the distance between points in the envelope that lie on stretch paths. We then examine the lengths of simple closed curves in the maximally-stretched lamination. We use twisting computations from Section 2.2 to find upper and lower boundes of the twisting around it. When the curve is long, we use an earthquake around it to bound the distance in the envelope. When the curve is short, we resort to other estimates of the Thurston distance.

Throughout this chapter, we will use coarse estimate notation as in [DLRT20]. We introduce it here.

Definition 3.1. Given two quantities (or functions), A and B, we write $A \overset{+}{\prec} B$ if there exists a constant C independent of A and B such that $A \leq B + C$.

Similarly, we write $A \overset{*}{\prec} B$ if there exists a constant C independent of A and B such that $A \leq BC$. We write $A \overset{+}{\asymp} B$ (resp. $A \overset{*}{\asymp} B$) if $A \overset{+}{\prec} B$ and $B \overset{+}{\prec} A$ (resp. $A \overset{*}{\prec} B$ and $B \overset{*}{\prec} A$)

Remark 3.2. The relations $\overset{*}{\prec}$, $\overset{*}{\asymp}$, $\overset{+}{\prec}$, and $\overset{+}{\asymp}$ *do not* behave as regular equalities and inequalities. For example, if $A \overset{+}{\asymp} B$ and $C \overset{+}{\asymp} D$, it is not necessarily true that $\frac{A}{C} \overset{+}{\asymp} \frac{B}{D}$.

3.1 TWISTING AND EARTHQUAKES

In this section, we prove the following, which is of interest of its own right:

Proposition. *Let α be a simple closed curve on S, and let $X \in \mathcal{T}(S)$. Then:*

$$d_{Th}(X, Eq_{\alpha,t}(X)) \overset{+}{\prec} \log(e^{l_\alpha/2}t)$$

The following lemma will be useful in the proof of Proposition 1.14:

Lemma 3.3. *Let $\varepsilon > 0$, and $X \in \mathcal{T}(S)$ be given. Let α, β be geodesic arc segments, and assume that β is contained in the ε-thick part of X, $X_{\geq\varepsilon}$. Then*

$$i(\alpha, \beta) \leq \frac{4l_\alpha(X)l_\beta(X)}{\varepsilon^2}$$

Lemma 3.3 has appeared in literature in the setting of flat structures [Raf07], and in various other contexts for extremal lengths [Min93]. We prove it here in the setting of hyperbolic lengths and geometric intersection numbers. In particular, we get:

Corollary 3.4. *If α, β are any closed curves contained in $X_{\geq\varepsilon}$, then $i(\alpha,\beta) \overset{*}{\prec}_{\varepsilon} l_\alpha(X) l_\beta(X)$*

We prove Lemma 3.3:

Proof. Let w be the subarc of α of length $\varepsilon/2$ which maximizes $|w \cap \beta|$. Since β is contained $X_{\geq\varepsilon}$, it follows that every arc segment of $\beta \setminus (\beta \cap w)$ must be of length at least $\varepsilon/2$, otherwise we could find an essential loop of length ε in $X_{\geq\varepsilon}$ (α and β are geodesic segments and hence in minimal position). Denote $|w \cap \beta| = N$, and by the above observation, we get that $N \leq \frac{l_\beta}{\varepsilon/2}$. Additionally, since w was chosen to maximize $|w \cap \beta|$, it follows that $\frac{l_\alpha}{\varepsilon/2} \leq N$, and the proof follows by stringing together these inequalities. $\square$

To prove Proposition 1.14, we will use the coarse estimates of the Thurston metric from [LRT14] using a short marking:

Definition 3.5. Let $X \in \mathcal{T}(S_{g,n})$. A **short marking** on X is a collection of $N = 3g - g + n$ simple closed curves, $\{\beta_i, \beta_i'\}_{i=1}^{N}$ obtained as follows:

- Define β_1 to be the shortest curve on X. Then inducively, define β_i to be the shortest curve on X disjoint from β_j for all $j < i$.
- Define β_i' to be the shortest curve which intersects β_i transversely.

We call β_i and β_i' **duals** to each other, and we denote $\bar{\beta}_i = \beta_i'$ and $\bar{\beta}_i' = \beta_i$.

We can estimate intersection numbers of curves with a short marking in a similar way to Proposition 3.1 of [LRT14]. By gaining control of the number of intersections between a curve α and the curves in the short marking, we can gain control of the length of each curve in the short marking after earthquaking along α. We prove:

Lemma 3.6. *Let μ_X be a short marking on X, and let α be a simple closed curve. For any $\beta \in \mu_X$, we have:*

$$i(\alpha,\beta)/l_\beta(X) \overset{*}{\prec} e^{l_\alpha/2}$$

Proof. We split the proof into two cases, depending on the length of β. We denote $l_\beta = l_\beta(X)$, and always assume that $i(\alpha,\beta) \geq 1$, otherwise the lemma follows trivially.

$l_\beta < \frac{1}{e}$: In this case, β has a collar of radius $2\sinh^{-1}(1/\sinh(l_\beta/2))$. By estimating $\sinh(x) \leq 2x$ for $x < 1$, and applying the inequality $\sinh^{-1}(x) \geq \log(x)$, we obtain a lower-bound on the width of the collar around β: $2\log(1/l_\beta)$.

For every intersection of α and β, α must enter and exit the collar of β, meaning that $i(\alpha,\beta) \leq \frac{l_\alpha}{2\log(1/l_\beta)}$ Since α intersects β, it follows in particular that we must have $l_\alpha \geq 2\log(1/l_\beta)$, or, $l_\beta \geq e^{-l_\alpha/2}$. Thus,

$$\frac{i(\alpha,\beta)}{l_\beta} \leq \frac{l_\alpha}{l_\beta 2\log(1/l_\beta)}$$

The function $x\log(1/x)$ is increasing on $[0,\frac{1}{e}]$, meaning that $2l_\beta\log(1/l_\beta) \geq e^{-l_\alpha/2}l_\alpha l_\alpha$, and plugging this in, we get the appropriate estimate.

$\frac{1}{e} \leq l_\beta$: By Propositon 3.1 of [LRT14], there exists some C such that $l_\alpha \geq C\sum_{\beta\in\mu_X} i(\alpha,\beta)l_{\bar\beta}$. In particular, we get $i(\alpha,\beta) \leq C\frac{l_\alpha}{l_{\bar\beta}}$, so $\frac{i(\alpha,\beta)}{l_\beta} \leq \frac{l_\alpha}{l_\beta l_{\bar\beta}}$.

Let $k(S) < \frac{1}{e}$ be such that if $l_{\bar\beta} \leq k(S)$, then $\bar\beta$ has a collar of width at least $B(S)$, where $B(S)$ is the Bers constant of S. In particular, if $l_{\bar\beta} \leq k(S)$, then this means that $\bar\beta$ is a curve in the short marking that was a part of a short pair-of-pants decomposition.

If $l_{\bar\beta} \geq k(S)$, then $\frac{i(\alpha,\beta)}{l_\beta} \leq C\frac{e}{K(S)}l_\alpha \overset{*}{\prec} e^{l_\alpha/2}$.

We now assume that $\frac{1}{e} \leq l_\beta$ and that $l_{\bar\beta} \leq k(S)$. In this case, $\bar\beta$ is part of a short pair-of-pants decomposition, and we can use the collar estimates from before to get that $\bar\beta$ has a collar of width at least $2\log(1/l_{\bar\beta})$, so $l_\beta \geq 2\log(1/l_{\bar\beta})$. If α intersects $\bar\beta$, then we must have $l_\alpha \geq 2\log(1/l_{\bar\beta})$, and, in a similar fashion to the case where $l_\beta < \frac{1}{e}$, we get:

$$\frac{i(\alpha,\beta)}{l_\beta} \leq \frac{l_\alpha}{l_\beta l_{\bar\beta}} \leq \frac{l_\alpha}{l_{\bar\beta}2\log(1/l_{\bar\beta})} \leq e^{l_\alpha/2}$$

We are left with the case that α does not intersect $\bar\beta$. Let $U(\bar\beta)$ be a collar around $\bar\beta$ whose boundary components have length at least $\frac{1}{e}$. By construction of the dual, β must be entirely contained in $X_{\geq\frac{1}{e}} \cup U(\bar\beta)$. Let $\beta_1 = \beta \cap X_{\geq\frac{1}{e}}$. Note that since α does not intersect $\bar\beta$, it follows that α is disjoint from $U(\bar\beta)$, and so $i(\alpha,\beta) = i(\alpha,\beta_1)$. By Lemma 3.3, $i(\alpha,\beta_1) \overset{*}{\prec} l_\alpha l_{\beta_1} \overset{*}{\prec} l_\alpha l_\beta$. Thus, $\frac{i(\alpha,\beta)}{l_\beta} \overset{*}{\prec} e^{l_\alpha/2}$. $\square$

Theorem 3.7. *(Theorem E in [LRT14]) For any $X,Y \in \mathcal{T}(S_{g,n})$, we have:*

$$d_{Th}(X,Y) \overset{+}{\asymp} \max_{\beta\in\mu_X} \log\frac{l_\beta(Y)}{l_\beta(X)}$$

We now prove Proposition 1.14:

Proof. Let μ_X be a short marking on X. For any $\beta \in \mu_X$, if β is disjoint from α, then $l_\beta(Eq_{\alpha,t}(X)) = l_\beta(X)$, and so $\log \frac{l_\beta(Eq_{\alpha,t}(X))}{l_\beta(X)} = 0$. Otherwise, we can bound the length $l_\beta(Eq_{\alpha,t}(X))$ by:

$$l_\beta(Eq_{\alpha,t}(X)) \leq l_\beta + ti(\alpha, \beta)$$

In particular, by Theorem 3.7,

$$d_{Th}(X, Eq_{\alpha,t}(X)) \stackrel{+}{\asymp} \max_{\beta \in \mu_X} \log \left(1 + t\frac{i(\alpha,\beta)}{l_\beta(X)}\right)$$

Applying Proposition 1.14, we get that $\frac{i(\alpha,\beta)}{l_\beta} \stackrel{*}{\prec} e^{l_\alpha/2}$, meaning that $d_{Th}(X, Eq_{\alpha,t}(X)) \stackrel{+}{\prec} \log(e^{l_\alpha/2}t)$, as desired. □

3.2 BOUNDED ENVELOPES IN $\mathcal{T}(S)$

In this section, we prove that geodesic envelopes have uniformly bounded width in $\mathcal{T}(S_{1,1})$ and in $\mathcal{T}(S_{0,4})$. We first show:

Lemma 3.8. *Let S be the once-punctured torus or the four-times punctured sphere. There exists some $B = B(S) \in \mathbb{R}$ such that for any $X, Y \in \mathcal{T}(S)$, we have that if $\Lambda(X,Y)$ is a simple closed curve, then $w(X,Y) < B$.*

From the lemma, we immediately obtain:

Theorem. *Let S be the once-punctured torus or the four-times punctured sphere. There exists some $B \in \mathbb{R}$ such that for any $X, Y \in \mathcal{T}(S)$, $w(X,Y) < B$.*

Proof. Let $X, Y \in \mathcal{T}(S)$ be arbitrary, and let $\Lambda(X,Y)$ be the maximally-stretched lamination from X to Y. For the four-times punctured sphere, by Proposition 26 of [BZ05], $\Lambda(X,Y)$ is either maximal, in which case there is a unique geodesic from X to Y, or $\Lambda(X,Y)$ contains a simple closed curve, α. For the once-punctured torus, the same result holds by Theorem 1.1 of [DLRT20]. Now, notice that geodesic from X to Y must be contained in $\mathrm{Out}(X,\alpha) \cap In(Y,\alpha)$, which, by Lemma 3.8, has width bounded by B. □

The rest of this section is dedicated to proving Lemma 3.8. We first fix some notation. Fix $Y \in \mathcal{T}(S)$ and a simple closed curve α.

- Denote by λ^L and λ^R be the two maximal chain-recurrent laminations containing α, twisting to the left and right (respectively) around α (see Figure 3.1)
- If $S = S_{1,1}$, define $l_0 = l_\alpha(Y)/2$, if $S = S_{0,4}$, define $l_0 = l_\alpha(Y)/4$. Define $l_s = l_0e^s$ for any real s.

- For $Y \in \mathcal{T}(S)$, we define Y^L_{-t} as the unique point in $\mathcal{T}(S_{0,4})$ such that $\Lambda(Y^L_{-t}, Y) = \lambda^L$, and $d_{Th}(Y^L_{-t}, Y) = t$. Define Y^R_{-t} in the same manner.
- Let P_1, P_2 be the two (possibly non-distinct) pairs of pants on opposite sides of α. We define $\Delta^L_i(t) = \Delta_{P_i,\alpha,\lambda^L}(t)$ and $\Delta^R_i(t) = \Delta_{P_i,\alpha,\lambda^R}(t)$ (see Section 2.2)

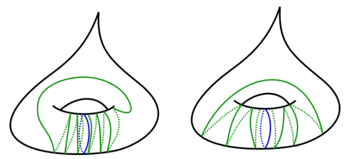

Figure 3.1: Definition of λ^R (right) and λ^L (left). The blue curve is α

Employing the above notation, and following the computations in Lemma 2.17, we compute the relative twisting $\mathrm{Twist}_\alpha(Y^L_{-t}, Y^R_{-t})$ in either case of S:

Lemma 3.9. *Let $\alpha, l_0, l_s, Y^{\pm}_{-t} \in \mathcal{T}(S)$ be defined as above, then:*

$$\mathrm{Twist}_\alpha(Y^+_{-t}, Y^-_{-t}) = 4e^{-t}\log\coth(l_0/2) - 4\log\coth\left(\frac{e^{-t}l_0}{2}\right) \tag{3.1}$$

Proof. We first treat the case where $S = S_{0,4}$. In this case, we use the computations from Lemma 2.17 to get:

$$\begin{aligned}\mathrm{Twist}_\alpha(Y^L_{-t}, Y^R_{-t}) &= e^{-t}(\Delta^L_1(0) + \Delta^L_2(0)) + e^{-t}(\Delta^R_1(0) + \Delta^R_2(0))\\ &\quad -(\Delta^L_1(-t) + \Delta^L_2(-t)) - (\Delta^R_1(-t) + \Delta^R_2(-t))\end{aligned}$$

Using Equation 2.11 with $l_2 = l_3 = 0$, and $s_{11} = -2l_s$, we get $x = 2\frac{1+e^{-l_s}}{e^{-2l_s}-1}$, and so:

$$\begin{aligned}\Delta^L_1 = \Delta^L_2 &= \frac{1}{2}\log((x+1)^2)\\ &= \log\left(\frac{e^{-2l_s} + 1 + 2e^{-l_s}}{1 - e^{-2l_s}}\right)\\ &= \log\coth(l_s)\end{aligned}$$

where in the last line we used Fact 2.18. Similarly, and $\Delta^R_1 = \Delta^R_2 = -\log(\coth(l_s))$, and the result follows.

Next, we consider the case where $S = S_{1,1}$. Using Equation 2.10 with $l_2 = l_1 = 2l_s$ and $l_3 = 0$, and getting $s_{12} = -2l_s$, and $s_{23} = \frac{1}{2}(l_1 - l_2 - l_3) = 0$. This gives $x = \frac{1+e^{-2l_s}}{e^{-2l_s}-1}$, and we compute:

$$x + 1 = \frac{2e^{-2l_s}}{e^{-2l_s} - 1}$$
$$x + \frac{e^{s_{23}} + e^{-2l_s}}{e^{s_{23}} + 1} = \frac{1}{2}\frac{(1 + e^{-2l_s})^2}{e^{-2l_s} - 1}$$

Thus, after applying Fact 2.18, we get:

$$\Delta_1^L = \log\left(\coth(l_s)\right)$$

Δ_2^L, and Δ_i^R are similarly computed, giving the result. □

Remark 3.10. Parts of the above lemma also follows from formulas 18 and 19 in [DLRT20], with slight modifications to treat negative t. However, it is a reaffirming sanity check that the computations preformed in Chapter 2 agree with those done in [DLRT20]

Lemma 3.11. *Let S be the once-punctured torus or the four-times punctured sphere, and let α be a simple closed curve in S. Then there exists a constant ε_0 such that for any $X \in \mathcal{T}(S)$, if $l_\alpha(X) < \varepsilon_0$, then α is the systole of X.*

Proof. Any pair of distinct simple closed curves on S must intersect. Therefore, if α is sufficiently short, then by the collar lemma, any other simple closed curve in S must be at least the length of the Bers constant of S. In particular, this implies that α is the shortest curve on S. □

We henceforth write ε_S to be the constant from Lemma 3.11.

Lemma 3.12. *Let $S = S_{1,1}$ or $S = S_{0,4}$, and let $X \in \mathcal{T}(S)$. Let $Y \in \mathcal{T}(S)$ be obtained from X by twisting along a simple closed curve α Let $\beta = \bar{\alpha}$ be the dual curve of α as in Definition 3.5. If $l_\alpha(X) < \varepsilon_S$, then*

The Thin Part of $\mathcal{T}(S)$

In this subsection, we show that $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is uniformly bounded if $l_0 e^{-t}$ is small.

Lemma 3.13. *Let $\varepsilon = \min(\varepsilon_{S_{0,4}}, \varepsilon_{S_{1,1}}, \log(2))$, and assume that $l_0 \leq \varepsilon$. Furthermore, assume that $Y \in \mathcal{T}(S)$ for $S = S_{1,1}$ or $S = S_{0,4}$. Then $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is bounded uniformly.*

Proof. Using Theorem E of [DLRT20], it suffices to check that $\frac{l_\beta(Y^R_{-t})}{l_\beta(Y^L_{-t})}$ is uniformly bounded, where β is the dual curve to α. Recall that $l_{-t} = l_0 e^{-t} \overset{+}{\asymp} l_\alpha(Y^L_{-t}) = l_\alpha(Y^R_{-t})$. By the collar lemma, and Proposition 3.1 of [LRT14], we know that $l_\beta(Y^L_{-t}) \overset{*}{\asymp} \log(1/l_{-t}) \overset{*}{\succ} \log(1/\varepsilon)$. Since Y^R_{-t} is obtained from Y^L_{-t} by twisting along α for time $\mathrm{Twist}_\alpha(Y^L_{-t}, Y^R_{-t})$, we can estimate the length ratio of β by:

$$\begin{aligned}\frac{l_\beta(Y^R_{-t})}{l_\beta(Y^L_{-t})} &\leq \frac{l_\beta(Y^L_{-t}) + l_\alpha(Y^L_{-t})|\mathrm{Twist}_\alpha(Y^R_{-t}, Y^L_{-t})|}{l_\beta(Y^L_{-t})} \\ &\overset{*}{\prec} 1 + \frac{l_0 e^{-t}}{\log(1/\varepsilon)} 4\left(e^{-t}\log\coth(l_0) + \log\coth(l_0 e^{-t})\right)\end{aligned}$$

Where we used the fact that $l_\alpha(Y^L_{-t}) \leq 4 l_0 e^s$.

We remark that by our choice of ε, we have that $\frac{1}{l_0 e^{-t}} \geq \log\coth(l_0 e^{-t}) > 0$, meaning that:

$$\frac{l_\beta(Y^R_{-t})}{l_\beta(Y^L_{-t})} \overset{*}{\prec} 1 + 4\frac{l_0 e^{-t}}{\log(1/\varepsilon)}\left(\frac{e^{-t}}{l_0} + \frac{e^t}{l_0}\right) = 1 + \frac{1}{\log(1/\varepsilon)}(e^{-2t} + 1)$$

Which is uniformly bounded in l_0 and in t. □

Lemma 3.14. *Assume that $\varepsilon \leq l_0 \leq 2$, and that $l_0 e^{-t} < \varepsilon$ from the previous lemma. Then $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is bounded uniformly.*

Proof. As in the previous lemma, we must estimate

$$\frac{l_\beta(Y^R_{-t})}{l_\beta(Y^L_{-t})} \overset{*}{\prec} 1 + \frac{l_0 e^{-t}}{\log(1/\varepsilon)} 4\left(e^{-t}\log\coth(l_0) + \log\coth(l_0 e^{-t})\right)$$

The $\log\coth(l_0)$ term is bounded uniformly, so it suffices to observe that $l_0 e^{-t}\log\coth(l_0 e^{-t})$ is bounded uniformly. □

Lemma 3.15. *Assume that $2 \geq l_0$, and that $l_0 e^{-t} < \varepsilon$ from the previous lemmas. Then $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is bounded uniformly.*

Proof. Let $t_0 = \log(l_0)$, and set $s = t - t_0$. Note that $t_0 \geq \log(2) > 0$ and that $s \in (-\log(\varepsilon), \infty)$ is bounded from below. With this notation, and the same argumens as before, we have:

$$\frac{l_\beta(Y^R_{-t})}{l_\beta(Y^L_{-t})} \overset{*}{\prec} 1 + 4\frac{e^{-s}}{s}\|\left(e^{-s-t_0}\log\coth(e^{t_0}) + \log\coth(e^{-s})\right)\|$$

As t_0 ranges from 0 to ∞, we have that $\frac{1}{e^{t_0}} > \log\coth(e^{t_0}) > 0$, and similarly $\frac{1}{e^{-s}} > \log\coth(e^{-s}) > 0$, giving:

$$\frac{l_\beta(Y^R_{-t})}{l_\beta(Y^L_{-t})} \overset{*}{\prec} 1 + 4\frac{e^{-s}}{s}\left(e^{-s} + e^s)\right)$$

Which is bounded uniformly in s, since s is bounded from below by $-\log(\varepsilon)$. $\square$

Together, the previous three lemmas tell us:

Lemma 3.16. *Let $\varepsilon = \min(\varepsilon_{S_{1,1}}, \varepsilon_{S_{0,4}}, \log(2))$, and assume that $l_0 e^{-t} \leq \varepsilon$. Then $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is bounded uniformly.*

The Thick Part of $\mathcal{T}(S)$

In this subsection, we show that $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is uniformly bounded when $l_0 e^{-t}$ is large.

Lemma 3.17. *In the setting of the above lemmas, if $1 < l_0 e^{-t}$, then $d_{Th}(Y^L_{-t}, Y^R_{-t})$ is uniformly bounded.*

Proof. We first treat the case when $S = S_{1,1}$ is the once-punctured torus. We recall that $l_\alpha(Y^L_{-t}) = l_\alpha(Y^R_{-t}) = 2l_0$, and by Proposition 1.14, it suffices to verify that for $l_0 e^{-t} > 1$, we have that

$$\begin{aligned} e^{l_0 e^{-t}} \mathrm{Twist}_\alpha(Y^L_{-t}, Y^R_{-t}) &= 4e^{l_0 e^{-t}}\left(e^{-t}\log\coth(l_0) - \log\coth(l_0 e^{-t})\right) \\ &\leq 4e^{l_0 e^{-t}} e^{-t}\log\coth(l_0) + 4e^{l_0 e^{-t}}\log\coth(l_0 e^{-t}) \end{aligned}$$

is bounded uniformly. Note that $1 < \coth(l_0) < \coth(l_0 e^{-t})$, so it suffices to observe that $e^{2l_0 e^{-t}}\log\coth(l_0 e^{-t})$ is bounded uniformly in t and l_0, as long as $l_0 e^{-t} > 1$.

The case when $S = S_{0,4}$ is similar; we must verify that $e^{2l_0 e^{-t}}\mathrm{Twist}_\alpha(Y^L_{-t}, Y^R_{-t})$ is uniformly bounded. $\square$

Remark 3.18. It is a striking coincidence that the twisting along a simple closed curve is decaying at least as fast as $e^{-2l_0 e^{-t}}$ in the case of the once-punctured torus and the four-times punctured sphere. In higher complexity surfaces, this is not generally the case. In fact, the factor of $\frac{1}{2}$ is crucial in Proposition 1.14, since $e^{(2+\varepsilon)l_0 e^{-t}}\log\coth(l_0 e^{-t})$ is not uniformly bounded for any $\varepsilon > 0$.

In fact, similar computations can be performed to estimate $d_{Th}(Y^R_{-t}, Y^L_{-t})$, giving:

Lemma 3.19. *$d_{Th}(Y^L_{-t}, Y^R_{-t})$ and $d_{Th}(Y^R_{-t}, Y^L_{-t})$ are uniformly bounded.*

Proof. We've seen the cases for when $l_0 e^{-t} < \varepsilon$ and when $l_0 e^{-t} > 1$. then let Y^R_0 (resp. Y^L_0) be the point on Y^R_{-t} (resp. Y^L_0) for which $l_\alpha(Y^L_0) = 1$.

Let t be such that $\varepsilon \leq l_0 e^{-t} \leq 1$, by the triangle inequality, $d_{Th}(Y^R_{-t}, Y^L_{-t}) \leq 2(1-\varepsilon) + d_{Th}(Y^R_0, Y^L_0)$ which is uniformly bounded. A similar argument can be shown for $d_{Th}(Y^L_{-t}, Y^R_{-t})$ □

Geodesic Envelopes

Before proving Lemma 3.8, we first analyze the large-scale structure of $Env(X,Y)$ for $X, Y \in \mathcal{T}(S)$, where $S = S_{1,1}$ or $S = S_{0,4}$. Let α be a simple closed curve in S, and let Y, Y^L_{-t} and Y^R_{-t} be defined as before

Lemma 3.20. *The set $In(Y,\alpha)$ is a closed set. Moreover, in Fenchel-Nielsen co-ordinates, we can write:*

$$In(Y,\alpha) = \{X \in \mathcal{T}(S) : \tau_\alpha(Y^L_{-t}) \leq \tau_\alpha(X) \leq \tau_\alpha(Y^R_{-t}), l_\alpha(X) = l_\alpha(Y)e^{-t} \textit{for some } t\}$$

Where $\tau_\alpha(Y^R_{-t})$ and $\tau_\alpha(Y^L_{-t})$ are functions of only $l_\alpha(Y)$ and t.

Proof. Denote the second set by $A(Y)$. Note that Y^R_{-t} and Y^L_{-t} are two smooth paths in $\mathcal{T}(S)$, which intersect at Y. In particular, they split $\mathcal{T}(S)$ into four complementary components (this is because stretch paths do not accumulate inside $\mathcal{T}(S)$), one of which is $A(Y)$.

If $X \in A(Y)$, then consider Stretch(X, λ^L, t), and note that this path must Y^R_{-t} at X'. The contatenation of Stretch(X, λ^L, t) from X to X', and the path Y^R_{-t} from X' to Y shows that $X \in In(Y,\alpha)$.

Conversely, assume that $X \in In(Y,\alpha)$ but not in $A(Y)$. Without loss of generality, assume that $\tau_\alpha(X) > \tau_\alpha(Y^R_{-t})$. Let $\gamma(t)$ be the path from X to Y which maximally stretches α along it. By Theorem 2.5, it follows that $\tau_\alpha(\gamma(t)) \geq \tau_\alpha(\text{Stretch}(X, \lambda^R, t))$ for any t, and so $\gamma(t)$ cannot reach Y (by construction of Y^R_{-t}). □

Corollary 3.21. *Let $S = S_{1,1}$ or $S = S_{0,4}$. For any $X, Y \in \mathcal{T}(S)$, we have:*

- *$Env(X,Y)$ is a unique geodesic from X to Y or,*
- *$Env(X,Y)$ is a combinatorial quadrilateral whose edges are stretch paths*

Proof. If we are not in the first case, then $\Lambda(X,Y)$ must be a simple closed curve, α. Notice that $Env(X,Y) = In(Y,\alpha) \cap \text{Out}(X,\alpha)$. From Theorem 2.5, we see that Out$(\Lambda(X,Y))$ is a cone starting at X and bounded by $\text{Stretch}(X, \lambda^L, t)$ and $\text{Stretch}(X, \lambda^R, t)$. The intersection of two cones in $\mathcal{T}(S)$ is a combinatorial quadrilateral. $\square$

This characterization allows us to prove Lemma 3.8

Theorem. *Let $X, Y \in \mathcal{T}(S)$ be such that $\Lambda(X, Y) = \alpha$. For any $Z, Z' \in Env(X, Y)$, if $l_\alpha(Z) = l_\alpha(Y)$, then $d_{Th}(Z, Z')$ is bounded uniformly. In particular, $w(X, Y)$ is uniformly bounded.*

Proof. We must have that $\text{Twist}_\alpha(Z, Z') \leq \text{Twist}_\alpha(Y^L_{-t}, Y^R_{-t})$, so using the exact same computations as in the previous subsection, $d_{Th}(Z, Z') \leq d_{Th}(Y^L_{-t}, Y^R_{-t})$. The theorem then follows from Lemma 3.19 $\square$

BIBLIOGRAPHY

[BBFS09] M. Bestvina, K. Bromberg, K. Fujiwara, and J. Souto, *Shearing coordinates and convexity of length functions on teichmüller space*, American Journal of Mathematics **135** (2009).

[Bon01] F. Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. **5** (2001).

[BZ05] F. Bonahon and X. Zhu, *The metric space of geodesic laminations on a surface ii: small surfaces*, 05 2005, pp. 509–547.

[DLRT20] D. Dumas, A. Lenzhen, K. Rafi, and J. Tao, *Coarse and fine geometry of the thurston metric*, Forum of Mathematics, Sigma **8** (2020), e28.

[Gen15] M. Gendulphe, *DERIVATIVES OF LENGTH FUNCTIONS AND SHEARING COORDINATES ON TEICHMÜLLER SPACES*, working paper or preprint, June 2015.

[HOP21] Y. Huang, K. Ohshika, and A. Papadopoulos, *The infinitesimal and global thurston geometry of teichmüller space*, 2021.

[LRT14] A. Lenzhen, K. Rafi, and J. Tao, *The shadow of a thurston geodesic to the curve graph*, Journal of Topology **8** (2014).

[Mar22] B. Martelli, *An introduction to geometric topology*, Bruno Martelli, 2022.

[Min93] Y. Minsky, *Teichmüller geodesics and ends of hyperbolic 3-manifolds*, Topology **32** (1993), 625–647.

[PH92] R.C. Penner and J. Harer, *Combinatorics of train tracks*, Annals of Math. Studies 125, Princeton University Press, 1992.

[PT07] A. Papadopoulos and G. Théret, *On teichmueller's metric and thurston's asymmetric metric on teichmueller apace. handbook of teichmüller theory*, European Mathematical Society Publishing House **1, 11** (2007).

[Raf07] K. Rafi, *Thick-thin decomposition for quadratic differentials*, Mathematical Research Letters **14** (2007), 333–341.

[Thé14] G. Théret, *Convexity of length functions and thurston's shear coordinates*.

[Thu86] W. P. Thurston, *Minimal stretch maps between hyperbolic surfaces, preprint*.

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