Welcome to MAT135 LEC0101 (Assaf)

As you walk in, look up the lyrics to your favourite song. Are there any math-y words in the lyrics?

S3.3 – We are all **products** of our environments

Assaf Bar-Natan

" Can't seem to get no rest?
Put our product to the test
You'll feel just fine "

–" The Big Bright Green Pleasure Machine ", Simon and Garfunkel

Oct. 18, 2019

Interpreting Leibniz

In Nov. 11, 1675, Leibniz wrote the following in his notebook (translated from Latin):

> Let us now examine whether $dx\,dy$ is the same thing as $d\overline{xy}$, and whether dx/dy is the same thing as $d\dfrac{x}{y}$; it may be seen that if $y = z^2 + bz$, and $x = cz + d$; then
>
> $$dy = z^2 + 2\beta z + \beta^2, + bz + b\beta, - z^2 - bz,$$
>
> and this becomes $dy = 2z + b\beta$.
>
> In the same way $dx = +c\beta$, and hence
>
> $$dx\,dy = \overline{2z+b}\,c\beta^2.$$
>
> But you get the same thing if you work out $d\overline{xy}$ in a straightforward manner. For in each of the several factors there is a separate destruction, the one not influencing the other; and it is the same thing in the case of divisors.

Desciphering Leibniz

Some hints:

- $\overline{xy}$ is notation for (xy)
- Putting a d in front of a quantity means "find the instantaneous rate of change"
- β is an *infinitessimal* quantity, a concept that was used in early calculus. Feel free to ignore it.

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Leibniz' Example

Hence it appears that it is incorrect to say that $dv\,d\psi$ is the same thing as $dv\psi$, or that $\frac{dz}{d\psi} = d\frac{v}{\psi}$; although just above I stated that this was the case, and it appeared to be proved. This is a difficult point. But now I see how this is to be settled.

If we have v and ψ, and they form some quantity, say $\phi = v\psi$ or v/ψ, and if the values of v and ψ are expressed as rationals in terms of some one thing, for instance, in terms of the abscissa x, then the calculus will always show that the same difference is produced, and that $d\phi$ is the same as $dv\,d\psi$ or $dv/d\psi$. But now I see the former can never happen, nor can it come to the latter by separation of parts; for example,

$$x+\beta, \frown x+\beta,, -, x, x, \text{ becomes } 2\beta x,$$

which is quite a different thing from

$$x+\beta, -x,, \frown x+\beta, -x \text{ which gives } \beta^2.$$

Hence it must be concluded that $dv\psi$ is not the same as $dv\,d\psi$, and

Leibniz' Correction

Let us seek to obtain others in addition, such as

$$\int t\,dy = \int y\,dx.$$

Now this furnishes us with nothing new; but $\int tw + \int xw = xy$ or $t\,dy + x\,dy = \overline{dxy}$, and $t = \dfrac{dx}{dy}\,y$; hence the latter $= \dfrac{\overline{dxy} - x\,\overline{dy}}{dy}$.

Therefore $\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}$.

Now this is a really noteworthy theorem and a general one for all curves. But nothing new can be deduced from it, because we had already obtained it.

However, from another principle we shall obtain a new theorem; for it is known that the sum of every $BP = BC^2/2$; that is to say, $BP = \dfrac{a^2}{t-x}$, $t = \dfrac{\beta y}{w} = \dfrac{\overline{dx}}{dy}\,y$, and therefore

$$BP = \frac{a^2\,dy}{\overline{dx}\,y - \overline{dy}\,x} = \frac{\overline{dy^2}}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

We therefore have two equations in which dx occurs, namely

Quotient Rule and Constant Rule

The quotient rule tells us:

$$\frac{d}{dx}\left(\frac{f(x)}{c}\right) = \frac{c\frac{d}{dx}f(x) - f(x)\frac{d}{dx}c}{c^2}$$

and the constant multiple rule tells us:

$$\frac{d}{dx}\left(\frac{f(x)}{c}\right) = \frac{1}{c}\frac{d}{dx}f(x)$$

Do these rules agree? Justify your response in a short paragraph. Be sure to include formulas!

Plans for the Future

For Monday:

WeBWork 3.4 and read section 3.4