Welcome to MAT135 LEC0501 (Assaf)

What is the integral $\int \frac{1}{\text{cabin}} d\text{cabin}$?

Two challenging integrals from last week:

$$\int \sin(e^t)dt = Si(e^x) + C$$

For $\int \sqrt{\tan(x)}$, substitute $u = \tan(x)$ to get:

$$\int \frac{\sqrt{u}}{u^2+1} = ???$$

This is very hard. Further developments next week.

S7.2 – Integration Methods – Integration by Parts

Assaf Bar-Natan

"Sometimes I lie awake, night after night
Coming apart at the seams
Eager to please, ready to fight
Why do I go to extremes?"

–"Why Do I Go To Extremes", Billy Joel

Jan. 27, 2020

Reading Comprehension

- The differentiation rule that gives us integration by parts is the ______ rule.
- The integration by parts technique tells us that $\int uv'dx = \underline{\hspace{2cm}} - \underline{\hspace{2cm}}$.

Takeaway

When faced with an integral that is a product of functions, try integration by parts. (Sometimes this will work for other functions too)

Leibniz and The Product Rule

Let us seek to obtain others in addition, such as

$$\int t\,dy = \int y\,dx.$$

Now this furnishes us with nothing new; but $\int tw + \int xw = xy$ or $t\,dy + x\,dy = \overline{dxy}$, and $t = \dfrac{dx}{dy}\,y$; hence the latter $= \dfrac{\overline{dxy} - x\,dy}{dy}$. Therefore $\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}$.

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MANUSCRIPT DATED NOV. 21, 1675. 107

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Leibniz used the fundamental theorem and integration by parts to conclude the product rule!

Building Fur – I.B.P Example

A cold snap hits the cats, and Mia's body starts building up her fur at a rate of $f(t)$ pounds per day. If $f(t) = 0.5 * t^2 e^t$, how much hair has she built up after ten days?

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- Does this simplify the question?

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$$\int_0^{10} 0.5t^2 e^{-t} dt = \left[0.5t^2(-e^{-t})\right]_0^{10} + \int_0^{10} te^{-t} dt$$

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Does this simplify the question enough to solve? Evaluating the last integral, and plugging in the bounds gives $\int_0^{10} 0.5t^2e^{-t}dt \approx 0.997$.

Takeaway

Integration by parts is useful when there is a product of functions, and we want one of them to "disappear".

dETAILS Mnemonic

$$\int uv' dx = uv - \int vu' dx$$

Here is a mnemonic for what functions to use for v' (read backwards for what functions to use as u)

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Bonus: find $\int xSi(x)dx$.

Integration by Parts – Functions Given Strangely

Let's say we have two functions, f, and g. g is given as a table of values, and f is given as a formula. Which of the following are easy to estimate?

- $f \cdot g$ evaluated at some point
- $\int_a^b f' g dx$
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Hint: For which of these integrands can you write a table of values?

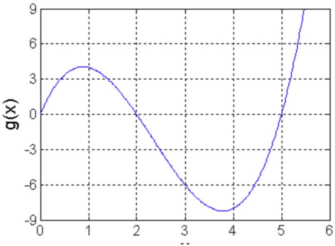
Integration by Parts – Functions Given Strangely

Let's say we have two functions, f, and g. g is given as a table of values, and f is given as a formula.

$$\int_a^b f(x)g'(x) = [fg]_a^b - \int_a^b f'(x)g(x)dx$$

We can now write a table for $f'(x)$, for $g(x)$, and $f'(x)g(x)$, and estimate the integral on the right.

Graphical Estimation



$$\int_0^5 f(x)g'(x) = f(5)g(5) - f(0)g(0) - \int_0^5 g(x)f'(x)dx$$
$$= -\int_0^5 2g(x)$$

Spot The Error

The Calculus Cats find a note on the floor. It reads:

To compute $\int \tan(x)dx$, we integrate by parts.

$$u = \frac{1}{\cos(x)} \qquad v' = \sin(x)$$
$$u' = \tan(x)\sec(x) \qquad v = -\cos(x)$$

so

$$\int \tan(x)dx = \int uv'dx = uv - \int vu'dx = -1 + \int \tan(x)$$

Simplifying, we get $0 = -1$.

The cats are stressed by this, to say the least. Can you help them?

Spot The Error

The Calculus Cats find a note on the floor. It reads:

To compute $\int_{\pi/6}^{\pi/4} \tan(x)dx$, we integrate by parts.

$$u = \frac{1}{\cos(x)} \qquad v' = \sin(x)$$
$$u' = \tan(x)\sec(x) \qquad v = -\cos(x)$$

so

$$\int_{\pi/6}^{\pi/4} \tan(x)dx = \int_{\pi/6}^{\pi/4} uv'dx = uv - \int_{\pi/6}^{\pi/4} vu'dx = -1 + \int_{\pi/6}^{\pi/4} \tan(x)$$

Simplifying, we get $0 = -1$.

The cats are even more stressed by this. Can you help them?

Plans for the Future

For next time:

Integration and Computer Algebra Systems. No reading, but bring a computer, and review what we've done!