RESEARCH STATEMENT

ASSAF BAR-NATAN

My research is in low-dimensional topology, where I study surfaces of both finite and infinite-type and objects constructed from them. For finite-type surfaces, I am interested in understanding Teichmüller space – the space of all hyperbolic metrics on the surface. I study this space by looking at geodesics in the Thurston metric.

I study infinite-type surfaces by looking at combinatorial graphs that are built from them, such as the grand arc graph [6], and by examining the actions of mapping class groups on these objects.

My research has two main focuses. The first is to answer questions posed by Papadopoulos–Théret [38] about geodesics in the Thurston metric. The second is to follow the work of Bavard-Walker, Patel-Miller-Patel and Fanoni–Ghaswala–McLeay [8, 7, 1, 16] to study δ-hyperbolic metric spaces coming from infinite-type surfaces with the hopes of reproducing a Nielsen-Thurston type classification.

1. Teichmüller theory and the Thurston metric

For any surface S with $\chi(S) < 0$, Teichmüller space, $\mathcal{T}(S)$ is the space of all hyperbolic structures on S up to homotopy isotopic to the identity. I study Teichmüller space by endowing it with a metric called the Thurston metric, which Thurston showed can be described in two different ways [43]:

$$d_{\mathrm{Th}}(X,Y) = \log\left(\inf_{\varphi : X \to Y} L_\varphi\right) = \sup_{\alpha \in \pi_1(S)} \log\left(\frac{l_\alpha(Y)}{l_\alpha(X)}\right)$$

where φ is Lipschitz, and in an appropriate homotopy class, and $l_\alpha(X)$ is the length of α on the hyperbolic structure X. The infimum is realized by an *optimal map*, and the above supremum is realized when passing to chain-recurrent laminations, which are Hausdorff limits of simple closed curves. The intersection of all such laminations that are limits of curves approaching the supremum is a nonempty lamination that depends only on X and Y. This lamination is denoted by $\Lambda(X,Y)$ and is called the *maximally-stretched lamination*.

The space $(\mathcal{T}(S_g), d_{Th})$ is a geodesic metric space, and when $\Lambda(X,Y)$ is maximal with respect to inclusion, a construction of Thurston shows that there exists a unique geodesic between X and Y, called a *stretch path* [43]. Generic geodesics in the Thurston metric are not unique [29], and for $X, Y \in \mathcal{T}(S)$, we can define the *geodesic envelope* as the collection of all points in Teichmüller space that appear along geodesic paths from X to Y.

There are many open questions about the geodesic envelope [15, 38]:

Question 1. *(Problem V in* [38]*) Describe an arbitrary geodesic in the Thurston metric. In particular, is any geodesic a limit of a concatenations of stretch paths?*

Question 2. *(Problem 2.2 in* [41]*) Given $X, Y \in \mathcal{T}(S)$, describe the set* $\mathrm{Env}(X,Y) = \cup G$*, where G denotes a Thurston geodesic connecting X to Y.*

In recent work of Huang-Oshika-Papadopoulos [23], infinitesimal properties of the Thurston metric are studied, and a rich stratification structure is shown to exist on the unit tangent sphere at any point in $\mathcal{T}(S)$.

In my Ph.D. thesis [4], I study the shape and width of the geodesic envelope, in a similar way to [23] studying the tangent sphere. I prove that, while geodesic paths aren't necessarily unique, there are specific surfaces for which any two geodesic paths from X to Y stay close together in Teichmüller space:

Theorem 1 (A. Bar-Natan, 2022). *If $S = S_{1,1}$ or $S = S_{0,4}$ then there exists a constant c such that for any two geodesics in $\mathcal{T}(S)$, $g_1(t)$ and $g_2(t)$ from X to Y, we have $d_{Th}(g_1(t), g_2(t)) \leq c$.*

The difficulty in proving this theorem lies in understanding the shape of the geodesic envelope. A useful tool to study the envelope is the *infinitesimal envelope*, defined to be the set of all 1-jets of geodesics from X to Y. I show that one can understand the infinitessimal envelope using stretch paths, and then apply this to the geodesic envelope as a whole.

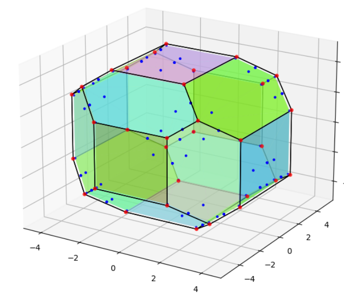

FIGURE 1. The Infinitesimal geodesic envelope in $\mathcal{T}(S_2)$

Theorem 2 (A. Bar-Natan, 2022). *If v is a vector in the infinitesimal envelope, then v is a convex combination of 1-jets of stretch paths. Moreover, 1-jets of stretch paths corresponding to maximal chain-recurrent laminations are extremal in the infinitesimal envelope.*

To prove Theorem 1, I show that the geodesic envelope is an intersection of two cone-offs of the infinitesimal envelope as in [15]. I then use coarse estimates of the Thurston metric coming from [15], and estimate the lengths of *earthquake paths* to yield the result. A key step in the proof is the following lemma:

Lemma 1 (A. Bar-Natan, 2022). *Let α be a simple closed curve in S, then there exists some $C = C(S)$ such that for any $X \in \mathcal{T}(S)$,*

$$d_{Th}(X, Eq_{\alpha,t}(X)) \leq \log(e^{l_\alpha(X)/2}t) + C$$

Where $Eq_{\alpha,t}$ is the path starting at X and earthquaking along α.

I will use this research project as a jumping off point to answer further questions about the combinatorial structure of the infinitesimal envelope and about geodesic envelopes in general:

Research Goal. *In what sense are geodesic envelopes foliated by stretch paths? Is there a combinatorial structure of a convex body on the full geodesic envelope that comes from the infinitesimal picture?*

By using the machinery in [29] and adapting their methods of "sufficiently horizontal" foliations to this context, I want to prove a theorem similar to Theorem 1 when $\Lambda(X, Y)$ is a lamination that is "close to" filling.

Research Goal. *Assume that there exists an $\varepsilon > 0$ such that an ε-neighbourhood of $\Lambda(X, Y)$ is filling. Does there exist a bound on the width geodesic envelope from X to Y depending only on the underlying surface and on ε?*

2. Infinite-type surfaces

If S is a surface, we define the *mapping class group of S*, denoted $\mathrm{Map}(S)$, as the group of homeomorphisms of S up to homotopy. $\mathrm{Map}(S)$ is a finitely-generated [30] discrete group that has been extensively studied in the 1900s by Dehn, Nielsen, and Thurston [13, 44]. The celebrated Nielsen-Thurston classification [44] says that any mapping class is one of the following:

- *Elliptic*, meaning that it has finite order in $\mathrm{Map}(S)$, or
- *Reducible*, meaning that it fixes a collection of homotopy classes of simple closed curves, or
- *Pseudo-Anosov*, meaning that it preserves two transverse singular foliations on the surface.

In other words, a mapping class either repeats itself like a reflection, leaves a part of the surface untouched, or jumbles up the surface in a very precise and tractable way.

A powerful tool used to study the mapping class group is the *curve graph* (c.f. [31, 32, 40, 17]). Originally defined by Harvey [20], the curve graph of a surface S, denoted $\mathcal{C}(S)$, is the graph whose vertices are isotopy classes of simple closed curves, and whose edges correspond to the existence of disjoint representatives. The curve graph and the mapping class group are intimitely related by a natural action of $\mathrm{Map}(S)$ on $\mathcal{C}(S)$ by setting $[f] \circ [\alpha] = [f(\alpha)]$. The curve graph has important properties as a metric space: it is δ-hyperbolic [22, 32], infinite-diameter [40], and its automorphism group is the mapping class group of the surface [10]. Moreover, it can detect the Nielsen Thurston classification: $g \in \mathrm{Map}(S)$ is pseudo-Anosov if and only if the action of g on $\mathcal{C}(S)$ is *loxodromic*, meaning that, up to bounded error, g acts as a translation along some axis in $\mathcal{C}(S)$.

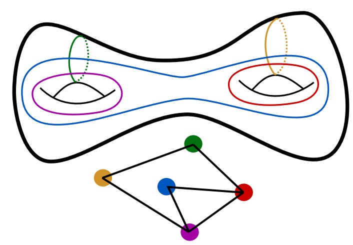

Figure 2. A small part of the curve graph

My research focuses on trying to find infinite-type analogues of the Nielsen Thurston classification.

2.1. **The Grand Arc Graph:**

An *infinite-type surface* is a surface whose fundamental group is not finitely generated.

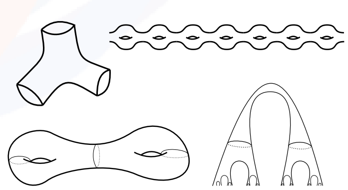

Figure 3. Some finite and infinite-type surfaces

When Σ is infinite-type, $\mathcal{C}(\Sigma)$ has diameter 2, and is therefore not as useful as in the finite-type case. In AIM Workshop on infinite-type surfaces, the following question was posed:

Question 3 (AIM Workshop Problem 2.1)**.** *What combinatorial objects are "good" analogues of the curve complex, either uniformly for all infinite-type surfaces or for some class of infinite-type surfaces?*

My work with Verberne generalizes past works [8, 7, 16] to define the grand arc graph, $\mathcal{G}(\Sigma)$, and in [6] we prove:

Theorem 3 (A. Bar-Natan, Y. Verberne, 2021)**.** *Let Σ be any surface with at least 3, and finitely-many self-similar equivalence classes of maximal ends.*

Then $\mathcal{G}(\Sigma)$ is a nonempty, connected, infinite-diameter δ-hyperbolic metric space. Moreover, the constant δ is independent of Σ.

2.2. **Describing the Boundary:** For finite-type surfaces, by studying the *visual boundary* of the curve graph [26], one can distill a new proof of the celebrated Nielsen-Thurston classification [9]. Thus, I hope to:

Research Goal. *Use the grand arc graph and its visual boundary, to create a Nielsen-Thurston type classification for mapping class groups of infinite-type surfaces.*

A big step in the project is to describe the visual boundary of the grand arc graph and the induced action of the mapping class group. Verberne and I already prove [6]:

Theorem 4 (A. Bar-Natan, Y. Verberne, 2021)**.** *The action of the mapping class group is not continous on $\mathcal{G}(\Sigma)$, nevertheless, it is continuous on the visual boundary of $\mathcal{G}(\Sigma)$*

The next step is to identify when mapping classes are loxodromic, elliptic (ie, fixing a point in $\mathcal{G}(\Sigma)$), or parabolic (ie, fixing a point on $\partial\mathcal{G}(\Sigma)$).

Some examples of elliptic and loxodromic homeomorphisms already exist [6, 2], but it is still unknown if any parabolic homeomorphisms exist.

Research Goal. *Do any mapping classes act parabolically on the visual boundary of the grand arc graph?*

In the Nielsen Thurston classification, pseudo-Anosov mapping classes are characterized by their loxodromic action on the curve graph. We show that a similar picture happens for infinite-type surfaces [6]:

Theorem 5 (A. Bar-Natan, Y. Verberne, 2021)**.** *If Σ is an infinite-type surface with nonempty grand arc graph, and if $W \subset \Sigma$ is a finite-type witness for the grand arc graph, and $\varphi \in \mathrm{Map}(W)$ is a pseudo-Anosov mapping class, then $\bar{\varphi} \in \mathrm{Map}(\Sigma)$ defined by taking φ on W and id on W^c acts loxodromically on the grand arc graph.*

Witnesses are subsurfaces that intersect every grand arc, and have been used to study many arc and curve graphs [40].

2.3. **Constructing Loxodromic Actions:** These "pseudo-Anosov" mapping classes are all compactly supported, and in order to obtain a clearer picture of a possible Nielsen-Thurston classification, we would like to find non-compactly supported infinite-type mapping classes that act loxodromically on arc complexes.

For finite-type surfaces, Penner's construction of pseudo-Anosov mapping classes [39] says that if α and β are a pair of curves of distance ≥ 2 in the curve graph, then the composition of *Dehn twists* around α and β is a pseudo-Anosov mapping class. Dehn twists act elliptically on the curve graph, and in the infinite-type setting, they are joined by *handle shifts* as simple examples of mapping classes. A handle shift is a mapping class which is supported on a "periodic" strip of a surface, and acts by shifting everything in the strip in a $\mathbb{Z}$-action.

In a recent paper [2], Abbott, Miller, and Patel use handle-shifts to construct infinte-type mapping classes that act loxodromically on the *relative arc graph*. Their construction uses three handle shifts, and could generalize to the grand arc graph with some care.

Research Goal. *Generalize the constructions of* [2] *to the grand arc graph, and use them to construct infinite-type loxodromic actions on the grand arc graph.*

3. Previous Research and Publications

My previous research was centered around low-dimensional topology, metric geometry, and geometric group theory.

3.1. Flip Graphs of Infinite-type Surfaces. The triangulation graph, $T(S)$ of a surface, is the graph whose nodes are triangulations with vertices at punctures of S, and whose edges are flip moves (see figure 4).

When S is a finite-type surface, $T(S)$ is an infinite-diameter connected metric space [21, 19], and the mapping class group, $\mathrm{Map}(S)$ acts naturally on it. For almost all surfaces, $\mathrm{Map}(S)$ and $T(S)$ are quasi-isometric, and $\mathcal{MF}(S) = T(S)/\mathrm{Map}(S)$ has finite diameter [14]. Thus, $T(S)$ provides a combinatorial analogue of Teichmüller space when S is finite-type, and is an interesting object of study.

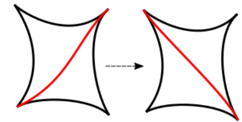

Figure 4. An edge in the flip graph corresponds to a flip move.

When a surface Σ is of infinite-type, single flips prove to be too restrictive, so we redefine edges in $T(\Sigma)$ to allow for infinitely-many simultaneous flips in disjoint quadrilaterals. We call this new graph the *flip graph*, and denote it by $\mathcal{F}(\Sigma)$. The flip graph has uncountably-many connected components [18], coming from orbits of the mapping class group. Nevertheless, we may check Ivanov's metaconjecture [24], which does hold for $T(S)$ for almost all finite-type surfaces [10].

Question 4 (Ivanov's Metaconjecture)**.** *Every object naturally associated to a surface S and having a sufficiently rich structure has $\mathrm{Map}(S)$ as its groups of automorphisms.*

Throughout my time at Canada/USA Mathcamp, I worked with a group of high-schoolers, A. Goel, B. Halstead, P. Hamrick, S. Shenoy, and R. Verma to study this generalization, and answered it in the following:

Theorem 6 (A. Bar-Natan, A. Goel, B. Halstead, P. Hamrick, S. Shenoy, and R. Verma, 2022)**.** *If Σ is an infinite-type surface, then there exist automorphisms of $\mathcal{F}(\Sigma)$ which are not induced by any mapping class of Σ.*

3.2. The gerrymandering jumble. In political redistricting, the compactness of a district is used as a quantitative proxy for its fairness. Several well-established, yet competing, notions of geographic compactness are commonly used to evaluate the shapes of regions, including the *Polsby-Popper score* (isoperimetric ratio), the *convex hull score* (ratio of area to convex hull area), and the *Reock score* (ratio of area to area of minimal bounding circle). These scores are used to compare two or more districts or plans, and each of these scores can be computed on the sphere or in $\mathbb{R}^2$ after passing through a map projection.

As a basis for legal frameworks, these scores were thought to be resilient against differing map projections from S^2 to the plane [11, 28, 12]. However, in a paper, joint with L. Najt and Z. Schutzman, we prove that this is not the case:

Theorem 7. *[A. Bar-Natan, L. Najt, Z. Schutzman, 2020] For any of the above compactness scores, and any map projection $\varphi : U \to \mathbb{R}^2$ defined on $U \subset S^2$, there exist regions A and B such that A is more compact than B on the sphere, but $\varphi(B)$ is more compact than $\varphi(A)$.*

3.3. Medium-scale Ricci curvature for croups. In the 2000s, Ollivier defined a notion of metric Ricci curvature at finite scales for graphs and other non-manifold geometries [34, 35, 36, 37] by offering a geometric interpretation of classical Ricci curvature as follows:

curvature measures the extent to which corresponding points on spheres are "closer together" or "farther apart" than the centers of the spheres. Inspired by this definition, together with M. Duchin and R. Kropholler [5], we propose a new notion of curvature on a finitely-generated group, denoted by $\kappa : G \to \mathbb{R}_{\geq 0}$.

We then show various properties of this curvature, and that this curvature can detect some flatness properties of the group G:

Theorem 8 (A. Bar-Natan, M. Duchin, R. Kropholler, 2020)**.** *Let G be any group.*

- *If G is generated by a symmetric set S for which $\kappa(s) = 0$ for all $s \in S$. Then G is virtually abelian.*
- *If G is virtually abelian with a finite-index free abelian subgroup H, then there exists a generating set S for which $\kappa(h) = 0$ for any $h \in H$.*

These results have been studied further in various contexts, ranging from further extensions of our work on conjugation curvature on hyperbolic groups [25], to lamplighter and Houghton's group [27], to growth rates of groups [33] and to solvable Baumslag-Solitar groups [42].

3.4. **2-systems of arcs on a punctured disk.** In the field of combinatorial topology, we are often interested in maximal families of curves or arcs (always considered up to homotopy) that have specific properties. In my masters' thesis, I proved the following theorem [3] about families of arcs intersecting twice:

Theorem 9 (A. Bar-Natan, 2020)**.** *Let D_n be the n-punctured disk. Let $\mathcal{A}$ be a family of essential simple arcs with endpoints on ∂D_n which are pairwise nonhomotopic and pairwise intersect at most twice. Then $|\mathcal{A}| \leq \binom{n+1}{3}$.*

References

[1] C. R. Abbott, N. Miller, and P. Patel. Infinite-type loxodromic isometries of the relative arc graph, 2021.

[2] Carolyn Abbott, Nicholas Miller, and Priyam Patel. Infinite-type loxodromic isometries of the relative arc graph. 09 2021.

[3] A. Bar-Natan. Arcs on punctured disks intersecting at most twice with endpoints on the boundary. *Groups, Geometry, and Dynamics*, 14:1309 —- 1332, 2020.

[4] A. Bar-Natan. Geodesic envelopes in the thurston metric on teichmuller space. *in preparation*, 2022.

[5] A. Bar-Natan, M. Duchin, and R. Kropholler. Conjugation curvature for cayley graphs. *Journal of Topology and Analysis*, pages 1–21, 11 2020.

[6] A. Bar-Natan and Y. Verberne. The grand arc graph. *in preparation*, 2021.

[7] J. Bavard and A. Walker. The Gromov boundary of the ray graph, 2018.

[8] J. Bavard and A. Walker. Two simultaneous actions of big mapping class groups, 2018.

[9] M. Bestvina and M. Handel. Train-tracks for surface homeomorphisms. *Topology*, 34(1):109–140, 1995.

[10] T. E. Brendle and D. Margalit. Normal subgroups of mapping class groups and the metaconjecture of ivanov. *Journal of the American Mathematical Society*, 2019.

[11] Jowei Chen. Common Cause, et al., vs. Robert A. Rucho, League Of Women Voters of North Carolina vs. Robert A. Rucho, deposition of Jowei Chen. *In The United States District Court For The Middle District Of North Carolina*, page 167, 2017.

[12] M. Chikina, A. Frieze, and W. Pegden. Assessing significance in a Markov chain without mixing. *Proceedings of the National Academy of Sciences*, 114(11):2860–2864, feb 2017.

[13] Max Dehn. Die gruppe der abbildungsklassen: Das arithmetische feld auf flächen. *Acta Math. (in German)*, 69:135–206, 1938.

[14] V. Disarlo and H. Parlier. The geometry of flip graphs and mapping class groups. *Trans. Amer. Math. Soc. 372*, 372:3809–3844, 2019.

[15] D. Dumas, A. Lenzhen, R. Kasra, and J. Tao. Coarse and fine geometry of the thurston metric. *Forum of Mathematics, Sigma*, 8:e28, 2020.
[16] F. Fanoni, T. Ghaswala, and A. McLeay. Homeomorphic subsurfaces and the omnipresent arcs, 2020.
[17] B. Farb and D. Margalit. *A primer on mapping class groups*. Princeton University Press, 2012.
[18] A. Fossas and H. Parlier. Flip graphs for infinite type surfaces, 2020.
[19] J. L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Annals of Mathematics*, 121(2):215–249, 1985.
[20] W. J. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State University of New York, Stony Brook, N. Y. 1978), volume 97 of Annals of Mathematics Studies*, pages 245–251. Princeton University Press, Princeton N. J., 1981.
[21] A. Hatcher. On triangulations of surfaces. *Topology and its Applications*, 40(2):189–194, 1991.
[22] S. Hensel, P. Przytycki, and R. C. H. Webb. 1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs. *Journal of the European Mathematical Society (JEMS)*, 17(4):755–762, 2015.
[23] Y. Huang, K. Ohshika, and A. Papadopoulos. The infinitesimal and global thurston geometry of teichmüller space, 2021.
[24] N. V. Ivanov. Fifteen problems about the mapping class groups. *arXiv: Geometric Topology*, 2006.
[25] A. Keisling. Medium-scale ricci curvature for hyperbolic groups. 2020.
[26] Erica Klarreich. The boundary at infinity of the curve complex and the relative teichmüller space. *Groups, Geometry, and Dynamics*, 2022.
[27] R. Kropholler and B. Mallery. Medium-scale curvature at larger radii in finitely generated groups. 04 2020.
[28] League of Women Voters of Pennsylvania, *et al.* Petitioners' brief in support of proposed remedial plans, 2 2018.
[29] A. Lenzhen, K. Rafi, and J. Tao. The shadow of a thurston geodesic to the curve graph. *Journal of Topology*, 8:1085–1118, 2015.
[30] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Annals of Mathematics*, 76:531, 1962.
[31] H. Masur and Y. Minsky. Geometry of the complex of curves I: Hyperbolicity. *Inventiones mathematicae*, 138(1):103–149, 1999.
[32] H. Masur and Y. Minsky. Geometry of the complex of curves II: Hierarchical structure. *Geometric and Functional Analysis*, 10(4):902–974, 2000.
[33] T. Nguyen and S. Wang. Cheeger-gromoll splitting theorem for groups, 2020.
[34] Y. Ollivier. Ricci curvature of metric spaces. *C. R. Math. Acad. Sci. Paris*, 347(11):643–646, 2007.
[35] Y. Ollivier. Ricci curvature of markov chains on metric spaces. *J. Funct. Anal.*, 256(3):810–864, 2009.
[36] Y. Ollivier. A survey of ricci curvature for metric spaces and markov chains. *Probabilistic Approach to Geometry*, 57, 01 2010.
[37] Y. Ollivier. A visual introduction to riemannian curvatures and some discrete generalizations. *Analysis and Geometry of Metric Measure Spaces: Lecture Notes of the 50th Séminaire de Mathématiques Supérieures (SMS), Montréal*, 2013.
[38] A. Papadopoulos and G. Théret. On teichmueller's metric and thurston's asymmetric metric on teichmueller space. a. papadopoulos. handbook of teichmüller theory. *European Mathematical Society Publishing House*, 1, 11, 2007.
[39] Robert C. Penner. A construction of pseudo-anosov homeomorphisms. *Trans. Amer. Math. Soc.*, 310(1), 1988.
[40] S. Schleimer. Notes on the curve complex. `http://homepages.warwick.ac.uk/~masgar/math.html`.
[41] W. Su. Problems on thurston metric. *AIM Workshop on "Lipschitz Metric on Teichmüller space*, 2014.
[42] J. Taback and A. Walker. Conjugation curvature in solvable baumslag-solitar groups. *Journal of Topology and Analysis*, mar 2021.
[43] W. P. Thurston. Minimal stretch maps between hyperbolic surfaces, preprint. 1986.
[44] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin of the American Mathematical Society*, 19:417–431, 1988.