EXAMPLES OF BUILDINGS

1. SPHERICAL BUILDINGS


1.1. An example. Here’s an example of a 1-dimensional spherical building, drawn in three different ways:

This building is a graph with 14 vertices, half black and half white (in the picture on the right, one of the black vertices is at infinity). Each vertex has degree 3, and the graph is bipartite – each edge connects a black and a white vertex. (In a building, these top-dimensional simplices are called chambers.)

The graph is made up of hexagons (later, apartments) with vertices colored alternatingly black and white. If you count carefully, you can find 28 hexagons. In fact, each pair of edges is contained in at least one hexagon, so the diameter of the graph is 3. Furthermore, the graph has girth 6 – there are no cycles of length < 6.

The graph has a lot of symmetries – in the center and left figures, one can see obvious symmetries of order 3 and 7, and one can construct assorted symmetries of the graph on the left that fix the edge on the top (in fact, there are 8). In fact, there are isomorphisms that take any vertex to any vertex, and edge to any edge, and any hexagon to any hexagon.

How do we construct graphs like this? In this case, this is a building based on the vector space \((\mathbb{Z}/2)^3\). Each black vertex corresponds to a 1-dimensional subspace, each white vertex corresponds to a 2-dimensional subspace, and edges correspond to one subspace containing another. Given a basis of \((\mathbb{Z}/2)^3\), we get a hexagon, with three black vertices corresponding to the basis vectors and three white vertices corresponding to the spaces spanned by pairs of basis vectors. The symmetries come from the action of \(\text{SL}_3(\mathbb{Z}/2)\) on the vector space.

In these notes, we’ll define a spherical building as a simplicial complex equipped with a system of apartments which satisfies certain axioms. In this example, we constructed a building based on an algebraic structure – it is a remarkable result of Tits that all buildings of dimension at least 2 are based on some algebraic structure.

1.2. Finite reflection groups. In order to define spherical buildings, we need to first need to define the simplest spherical buildings, which are triangulations of the sphere arising from finite reflection groups.

Suppose that \(W\) is a finite reflection group of rank \(n\), that is, a finite group of isometries of the sphere \(S^{n-1}\) which is generated by reflections. We think of \(S^{n-1}\) as the unit sphere in \(\mathbb{R}^n\), and we will alternately think of \(W\) as a group of isometries of \(S^{n-1}\) or as a group of isometries of \(\mathbb{R}^n\) that fix the origin. If \(P\) is a plane through the origin in \(\mathbb{R}^n\), we denote the reflection through \(P\) by \(s_P\). Consider the set

\[
\mathcal{H} = \{ P \mid s_P \in W \}
\]
of planes which are the fixed point of some reflection in $W$. Then $W$ permutes the set $\mathcal{H}$, since if $P \in \mathcal{H}$ and $w \in W$, then $wPs^{-1} = s_wP$, so $wP \in \mathcal{H}$.

Example 1. The simplest example is the dihedral group $D_{2m}$, which consists of the symmetry group of the regular $m$-gon. Every finite reflection group of rank 2 is isomorphic to $D_{2m}$ for some $m$. In this case, $\mathcal{H}$ consists of $m$ lines, each forming an angle of $\pi/m$ with the adjacent lines. If $s$ and $t$ are the reflections in two neighboring lines, one can check that $st$ is a rotation by angle $2\pi/m$ and $s$ and $t$ generate $D_{2m}$; in fact,

$$D_{2m} = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle.$$

The planes in $\mathcal{H}$ decompose $S^{n-1}$ into a CW-complex $\Sigma$, which we call a Coxeter complex. Call the top-dimensional cells chambers. If $C_0, \ldots, C_n$ are a sequence of chambers such that $C_i$ and $C_{i+1}$ are adjacent across a codimension 1 face, we call the sequence a gallery of length $n$, and we define the combinatorial distance $d(C, C')$ to be the minimum length of a gallery which starts with $C$ and ends with $C'$. If $C_0, \ldots, C_n$ is a gallery with $n = d(C_0, C_n)$, we call it a minimal gallery. If $C$ and $C'$ are chambers, then $d(C, C')$ is equal to the number of hyperplanes in $\mathcal{H}$ which separate $C$ and $C'$.

Let $C$ be a chamber and let $S \subset \mathcal{H}$ be the set of reflections through planes containing the walls of $C$. Let $\{P_1, \ldots, P_d\}$ be the set of planes bounding $C$, so $S = \{s_1, \ldots, s_d\}$ is the corresponding set of reflections.

Then

**Lemma 1.** For all $i \neq j$, there is some natural number $k_{ij}$ such that the angle between $P_i$ and $P_j$ is $\pi/k_{ij}$.

**Proof.** Any two distinct planes through the origin intersect in a subspace $V = P_i \cap P_j$ of codimension 2, so the subgroup $\langle s_i, s_j \rangle$ generated by $s_i$ and $s_j$ acts by reflections on $\mathbb{R}^n/V \cong \mathbb{R}^2$. It is straightforward to see that $\langle s_i, s_j \rangle$ is finite if and only if the angle between $P_i$ and $P_j$ is of the form $a\pi/b$ for $a/b$ in lowest terms. In this case, it has order $2b$, and it contains reflections through the $b$ planes obtained by rotating $P_i$ around $V$ by angles $\pi/b, 2\pi/b, \ldots, \pi$. Since $P_i$ and $P_j$ both bound $C$, there is no plane in $\mathcal{H}$ “between” $P_i$ and $P_j$, so the angle between them must be $\pi/b$ for some natural number $b$. \quad \Box

In particular, for all $i, j$, the angle between $P_i$ and $P_j$ is acute. Some linear algebra then implies that $C$ is a simplex (see [AB08, Prop. 1.37]), so $d = n$.

We can use these angles to present $W$. If $P_i$ and $P_j$ form an angle of $\pi/k_{ij}$, then $(s_is_j)^{k_{ij}} = 1$. The following theorem shows that this gives a presentation of $W$:

**Theorem 1** (see [AB08, Thm. 1.69]).

1. $W = \langle S \mid s_i^2 = 1, (s_is_j)^{k_{ij}} = 1 \rangle$
2. $W$ acts simply transitively on the set of chambers. That is, if $C, C'$ are chambers, there is a unique $w \in W$ such that $wC = C'$.
3. The graph whose vertices are the chambers of $\Sigma$, with an edge between each pair of adjacent chambers, is a Cayley graph for $W$.

**Proof.** If $n = 2$, then $W = D_{2m}$ for some $m$ and the properties are easy to verify. We assume that $n > 2$.

The basic idea of the theorem is that there is a bijection between galleries starting at $C$ and words. For any $g \in S$, $gC$ is adjacent to $C$. If $g_1, \ldots, g_k \in S$ and $w = g_1 \ldots g_k$, let $w_i = g_1 \ldots g_i$. Then $w_i+1C = w_ig_{i+1}C$ is adjacent to $w_iC$, so

$$C, w_1C, \ldots, w_kC$$

is a gallery. Conversely, given a gallery $C = C_0, \ldots, C_k$, we can construct a unique word $w$ of length $k$ such that $w_1C = C_1$.

In particular, $W$ acts transitively on the set of chambers. This lets us show that $S$ generates $W$. Suppose that $P \in \mathcal{H}$. Then there is some $C'$ adjacent to $P$, and there is an element $w \in \langle S \rangle$ such that $wC = C'$. In particular, there is a plane $Q \in \mathcal{H}$ adjacent to $C$ such that $wQ = P$, so $wsw^{-1} = s_P$. Since $W$ is generated by reflections, this means that $S$ generates $W$.

Clearly, $W$ satisfies the relations $s_i^2 = 1, (s_is_j)^{k_{ij}} = 1$; we claim that these relations are enough to present $W$. Suppose that $w$ and $w'$ are two words representing the same group element $g$. Then if $C' = gC$, we can use the correspondence above to find two galleries $C = C_0, \ldots, C_k = C'$ and $C = C_0', \ldots, C_k' = C''$ from $C$ to $C'$. Draw two curves $\gamma$ and $\gamma'$ through these galleries; since these are curves in $S^{n-1}$, they are homotopic. Let $h : [0, 1]^2 \to S^n$ be a homotopy from $\gamma$ to $\gamma'$. By transversality, we may assume that $h$ doesn't intersect
any cell of $\Sigma$ of codimension $> 2$. Then, in the same way that a gallery corresponds to a word, the homotopy $h$ corresponds to a van Kampen diagram for $w^{-1}w'$. 

That is, we can decompose $[0,1]^2$ into cells, each of which is the preimage of a chamber of $\Sigma$. Given each cell $D_i$, we construct a curve $\lambda_i : [0,1] \to [0,1]^2$ which connects $(0,0)$ to a point in $D_i$; by transversality, we can ensure that $h \circ \lambda_i$ avoids any codimension 2 cells. We can then construct a gallery by taking the chambers that $h \circ \lambda_i$ passes through in order, and label $D_i$ by the corresponding word by a relation of the form $(s_is_j)^{m_{ij}}$. By taking the dual skeleton of $[0,1]^2$, we get a van Kampen diagram reducing $w$ to $w'$. 

Therefore,

$$W = \langle S \mid s_i^2 = 1, (s_is_j)^{k_{ij}} = 1 \rangle.$$ 

Furthermore, if $g, g' \in W$ are such that $gC = gC'$, the same argument applies, so $g = g'$. The second part of the lemma follows – there is a bijection between $W$ and the set of chambers given by $g \mapsto gC$, and any two adjacent chambers differ by a generator of $W$. 

In fact, the closure $\bar{C}$ is a fundamental domain for $\Sigma$ [AB08, Thm. 1.104]. We can use this fact to color the vertices of $\Sigma$; if $v$ is a vertex, there is a unique vertex of $C$ which is in the orbit of $\sigma$ under $W$. (This is why the vertices in the example can be colored black and white.) This coloring is invariant under the action of $W$. We can associate the vertices of $C$ with the set $S$; since $C$ is a simplex, each vertex of $C$ is disjoint from exactly one of the hyperplanes $P_i$ bounding $C$, so this gives a bijection between the set of $W$-orbits of vertices and $S$. 

Likewise, we can color the hyperplanes of $H$ in a $W$-invariant way; for each $H \in H$, there is a $g \in W$ such that $gH = P_i$, and $i$ is independent of the choice of $g$.

1.3. **Defining buildings.** We roughly follow the definition in Chapter 4 of [AB08]. A spherical building $\Delta$ is a simplicial complex which can be equipped with a set of subcomplexes called apartments which satisfy three axioms:

1. Each apartment is the Coxeter complex of a finite reflection group.
2. For each pair of simplices $A, B \subset \Delta$, there is an apartment containing $A$ and $B$.
3. For any two apartments $\Sigma$ and $\Sigma'$, there is an isomorphism $\Sigma \to \Sigma'$ which fixes $\Sigma \cap \Sigma'$ pointwise.

Frequently, one also assumes that $\Delta$ is thick – that every codimension 1 simplex is part of at least three apartments. 

In particular, all the apartments are isomorphic to the Coxeter complex of some finite reflection group $W$; we say that $\Delta$ is of type $W$. It’s useful to try to check these axioms by hand for the example in Section 1.1. For instance, the drawing on the left shows that for every simplex $\sigma$ in the building, there is at least one hexagon containing $\sigma$ and the top edge.

As before, we call the top-dimensional simplices of $\Delta$ chambers. The following properties follow from the axioms:

- The link of any face of a building is a building.
- For every apartment $\Sigma$ and every chamber $C \subset \Sigma$, there is a retraction $\rho_{C, \Sigma} : \Delta \to \Sigma$ which restricts to an isomorphism on each apartment of $\Delta$ which contains $C$.
- Every apartment is convex with respect to the combinatorial distance. That is, any minimal gallery between two chambers of $\Sigma$ is contained in $\Sigma$.

1.4. **Examples.** The only rank-1 finite reflection group is the isometry group of the 0-sphere; i.e., the group of order two which acts on a pair of points by swapping them. Therefore, a rank-1 spherical building is a collection of points. By the axioms, any pair of points forms an apartment.

Any connected bipartite graph with diameter $m$, girth $2m$, and in which every vertex is part of at least two edges is a rank-2 building with an apartment system consisting of all of the $2m$-gons ([AB08, Prop. 4.44]). In fact, every spherical building of rank 2 is a graph like this.

As we go to higher dimensions, though, buildings start becoming more rigid. Here are some constructions:

- The Coxeter complex of any finite reflection group is a building with one apartment.
• Given a $k$-dimensional vector space $V$ over a field, we can construct a building $\Sigma_V$ whose vertex set is the set of proper nonempty subspaces of $V$. The vertices $V_0, \ldots, V_n \subset V$ span a simplex if the $V_i$ form a flag; that is,

$$V_1 \subset \cdots \subset V_n.$$  

Then the largest dimension of a simplex is $k - 2$. If $x_1, \ldots, x_k \in V$ is a basis of $V$, then any permutation $\sigma$ of the $x_i$’s corresponds to a flag

$$\langle x_{\sigma_1} \rangle \subset \langle x_{\sigma_2} \rangle \subset \cdots \subset \langle x_{\sigma_k} \rangle$$

and thus to a $k - 2$-dimensional simplex. The subcomplex formed by these simplices is homeomorphic to a sphere; in fact, it is the barycentric subdivision of the surface of the $(k - 1)$-simplex with vertices $\langle x_1 \rangle, \ldots, \langle x_n \rangle$. These spheres form the apartments of $\Sigma_V$. It is an exercise in linear algebra to prove that $\Sigma_V$ satisfies the required axioms. This is known as the flag complex of the set of subspaces of $V$.

This construction generalizes the example in Section 1.1. That example is $\Sigma_V$ when $V = (\mathbb{Z}/2)^3$; it is a rank-2 building of type $D_6$, the symmetry group of the triangle. In general, this construction produces buildings modeled on the symmetry group of the $n$-simplex.

• The same example can be constructed in a different way. We work with $V = \mathbb{R}^n$ for definiteness. Consider the group $\text{SL}_n(\mathbb{R})$. The parabolic subgroups are the proper subgroups conjugate to a subgroup containing the upper triangular matrices. Any parabolic subgroup of $\text{SL}_n(\mathbb{R})$ is conjugate to a group of upper block-triangular matrices; for example:

$$P_{3,1} = \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
0 & 0 & 0 & *
\end{pmatrix}.$$  

Inclusion gives a partial ordering on the set of parabolic subgroups, and one can define a simplicial complex whose set of simplices is the set $P$ of parabolic subgroups. If $P, P' \in P$, then $P$ is a face of $P'$ if $P' \subset P$. For example, the subgroup $B$ of upper-triangular matrices is a minimal parabolic subgroup and thus a maximal simplex. The simplex has dimension $n - 2$ and its $n - 1$ vertices correspond to the $n - 1$ maximal parabolic subgroups containing $B$. Intersections of these maximal parabolics correspond to the $2^{n-1} - 1$ faces of the $(n - 2)$-simplex.

Constructing apartments in this building takes some effort. Let $N$ be the group of matrices that keep the set

$$\{ \langle e_1 \rangle, \ldots, \langle e_n \rangle \}$$

invariant (we call these monomial matrices) and let $T = B \cap N$ be the group of diagonal matrices. ($T$ is a maximal torus in $\text{SL}_n(\mathbb{R})$ and $N$ is the normalizer of $T$.) Each matrix in $N$ is the product of a permutation matrix and a diagonal matrix, so the quotient $W = N/T$ is isomorphic to the permutation group $S_n$. We call $W$ the Weyl group of $\text{SL}_n(\mathbb{R})$. The group $\text{SL}_n(\mathbb{R})$ acts on $P$ by conjugation, and one can check explicitly that the orbit of $B$ under $W$ forms a subcomplex isomorphic to the Coxeter complex of $S_n$. This is an apartment, and we can form an apartment system by taking the orbit of this apartment under $\text{SL}_n(\mathbb{R})$.

We can construct an isomorphism between this building and the one constructed in the previous example by associating parabolic subgroups of $\text{SL}_n(\mathbb{R})$ with the flags in $\mathbb{R}^n$ that they stabilize. Each upper block-triangular subgroup stabilizes some subset of the spaces in the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle,$$

and since any other parabolic subgroup is a conjugate of an upper block-triangular matrix, every parabolic subgroup stabilizes some flag. Conversely, the stabilizer of any flag is a parabolic subgroup.

This is a particular case of constructing a building from a $BN$-pair – in fact, given a group with subgroups $B$ and $N$ satisfying certain properties, one can construct a building.

• Buildings modeled on other Coxeter complexes can be constructed through similar means. For example, if

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$
is the standard symplectic form on \( \mathbb{R}^{2n} \), then one can construct the flag complex of the set of isotropic subspaces of \( \mathbb{R}^{2n} \) (subspaces on which the form vanishes.) This is a building, and apartments in this building are modeled on the barycentric subdivision of the \( n \)-dimensional octahedron.

Just like the SL example, we can view this building in terms of parabolic subgroups too. In this case, these are subgroups of \( \text{Sp}_{2n}(\mathbb{R}) \), the symplectic group over \( \mathbb{R} \). We let \( P \) be the set of upper-triangular matrices in \( \text{Sp}_{2n}(\mathbb{R}) \) and we let \( N \) be the set of symplectic monomial matrices. That is, \( N \) is the set of matrices that fixes the set

\[
\{ (x_1, \ldots, x_n), (y_1, \ldots, y_n) \}.
\]

One can check that each matrix in \( N \) consists of a product of a permutation matrix sending each pair \( \{x_i, y_i\} \) to a pair \( \{x_j, y_j\} \) and a diagonal matrix.

Then \( T = B \cap N \) is a group of diagonal matrices and \( W = N/T \) is isomorphic to the set of permutations of \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) that send each pair \( \{x_i, y_i\} \) to a pair \( \{x_j, y_j\} \). This is the symmetry group of the \( n \)-dimensional octahedron. The \( n \)-dimensional octahedron is the convex hull of the points \( \pm e_i \), where the \( e_i \) are coordinate vectors; if we associate \( x_i \) with \( e_i \) and \( y_i \) with \( -e_i \), the isomorphism is clear. Then, just as before, each conjugate of \( B \) corresponds to a chamber of a building, and the orbit of \( B \) under \( W \) is an apartment isomorphic to a Coxeter complex modeled on \( W \).

In higher ranks, Tits proved a rigidity theorem which states (roughly) that every thick irreducible spherical building comes from an algebraic construction. For example, every thick spherical building of type \( A_n \) (with Coxeter complex corresponding to the isometry group of the \( n \)-simplex) is the flag complex of a vector space over a division ring.

2. SYMMETRIC SPACES

Reference: Ji, “Buildings and their applications in geometry and topology” ([Ji06])

I’m interested in buildings because of their uses to study symmetric spaces — recall that a symmetric space is a homogeneous space with an involution centered at each point which acts on the tangent space by sending each vector \( v \) to \( -v \).

Quick review: Any symmetric space is the quotient of a Lie group by a maximal compact subgroup. A non-compact non-flat irreducible symmetric space is a quotient of a simple Lie group by a maximal compact subgroup, and has a non-positively curved metric. One standard example is

\[
\mathcal{H}^n = \text{Isom}(\mathcal{H}^n)/\text{Stab}_{\text{Isom}(\mathcal{H}^n)}(p) = \text{SO}(n, 1)/\text{SO}(n-1)
\]

where \( \text{Stab}_{\text{Isom}(\mathcal{H}^n)}(p) \) is the stabilizer of a point in hyperbolic space and \( \text{SO}(n, 1) \) is the set of linear maps of \( \mathbb{R}^{n+1} \) which preserve the quadratic form

\[
dx_1^2 + \cdots + dx_n^2 - dx_0^2.
\]

If \( G \) is a semisimple algebraic group over a field \( k \) (for example, a product of simple groups), it has a BN-pair. This lets us construct a building (the Tits building) as in the examples of \( \text{SL}_n(\mathbb{R}) \) and \( \text{Sp}_{2n}(\mathbb{R}) \) above; the simplices of the building consist of parabolic subgroups, ordered by reverse inclusion. This can be described algebraically (\( B \) is the Borel subgroup, which is the maximal connected solvable subgroup, and \( N \) is the normalizer of a maximal torus).

Let’s look at this more geometrically.

Suppose that \( X \) is the symmetric space corresponding to \( G(\mathbb{R}) \). Then \( X \) is a CAT(0) space and we can consider its Gromov boundary \( \partial_X \), consisting of the set of equivalence classes of geodesic rays. Isometries of \( X \) act on the Gromov boundary, and the parabolic subgroups consist of the stabilizers of points in that boundary. Furthermore, if \( P \subset P' \) are parabolic subgroups, then \( X^P \cap X^{P'} \). In fact, we can decompose the Gromov boundary into sets of fixed points of parabolic subgroups such that there is a natural bijection between the set of points of the Gromov boundary and the set of points in the Tits building. Note, though, that this is just a bijection, not a homeomorphism; this will become clear in the examples.

2.1. Examples.

- The most familiar symmetric space is hyperbolic space. The rank of the Tits building is the same as the rank of the symmetric space; hyperbolic space has rank 1, so the Tits building is a collection of
points, with an apartment for every pair of points. In particular, the topology on the Tits building is the discrete topology.

We can write a connected component of the isometry group of $H^2 \times H^2$ as $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$, and the stabilizer of a point at infinity is conjugate to the upper triangular matrices; call this subgroup $B$.

- A slightly more complicated example comes from taking the product of two hyperbolic spaces. We can think of $\partial \infty(H^2 \times H^2)$ as the join of $S^1$ with $S^1$. A unit-speed geodesic ray is characterized by a direction in each factor and the ratio of the speeds in each factor, except when one of those speeds is zero.

We can write a connected component of the isometry group of $H^2 \times H^2$ as $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. A generic geodesic ray $\gamma$ has nonzero speed in each factor, and the stabilizer $P = \text{Stab}(\gamma)$ of $\gamma$ is conjugate to $B \times B$, where $B$ is the group of upper-triangular matrices. In fact, this is the stabilizer of many points at infinity, corresponding to rays with the same directions as $\gamma$, but different speeds. These form a 1-simplex $\sigma$ bounded by rays of the form $p_1 \times p_1$ and $p_2 \times p_2$, where $p_1, p_2 \in H^2$.

In turn, $\gamma_1 \times p_1$ is stabilized by a parabolic subgroup conjugate to $B \times \text{SL}_2(\mathbb{R})$ and $p_1 \times p_2$ is stabilized by a conjugate of $\text{SL}_2(\mathbb{R}) \times B$. These subgroups form a 1-simplex in the Tits building. Furthermore, $F = \gamma_1 \times \gamma_2$ is a flat in $H^2 \times H^2$ which is invariant under the action of a maximal torus $T \subset \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. (That is, the product of 1-parameter families of hyperbolic translations along each geodesic.) Then $T$ stabilizes $\gamma$. Indeed, the fixed point set of $T$ is $\partial \infty F$.

If $N$ is the normalizer of $T$, it acts on $\partial \infty F$ or on $T$ (by conjugation). Since $T$ acts trivially, we consider $W = N/T$. This is the Weyl group of $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$, and it is a reflection group which acts on $T$ or $\partial \infty F$. In fact, the orbit of $\sigma$ is a Coxeter complex. Translates of this Coxeter complex form the apartments of the Tits building, and each apartment of the Tits building is the boundary of some flat in $H^2 \times H^2$.

- This picture generalizes to other groups. Try doing the same thing for $\text{SL}_n(\mathbb{R})/\text{SO}(n)$, for example.

### 3. Euclidean buildings

Suppose that a reflection group $G$ acts properly discontinuously and cocompactly on $\mathbb{R}^n$. The same techniques that we used to construct a Coxeter complex for spherical Coxeter groups apply to euclidean Coxeter groups, so we can construct a Coxeter complex for $G$. A euclidean building $\Delta$ is a simplicial complex which can be equipped with a set of subcomplexes called apartments which satisfy three axioms:

1. Each apartment is a euclidean Coxeter complex.
2. For each pair of simplices $A, B \subset \Delta$, there is an apartment containing $A$ and $B$.
3. For any two apartments $\Sigma$ and $\Sigma'$, there is an isomorphism $\Sigma \to \Sigma'$ which fixes $\Sigma \cap \Sigma'$ pointwise.

(These are exactly the axioms for a spherical building, except with the spherical apartments replaced by euclidean ones.)

#### 3.1. Examples.

- The easiest examples are infinite trees and products of trees; any two edges in a tree are part of a geodesic, and any two geodesics intersect in an interval.

Before we construct other examples, we should consider trees in detail. Consider the group $G = \text{SL}_2(\mathbb{Q}_p)$, where $\mathbb{Q}_p$ is the $p$-adic numbers. Then $G$ acts on the infinite $p$-ary tree.

A lattice in $\mathbb{Q}_p^2$ is a finitely-generated $\mathbb{Z}_p$-module whose span over $\mathbb{Q}_p$ is all of $\mathbb{Q}_p^2$. If $\Lambda, \Lambda'$ are lattices, we say that $\Lambda$ and $\Lambda'$ are similar if $\Lambda = p^k \Lambda'$ for some $k$. Let $\mathcal{L}$ be the set of similarity classes of lattices in $\mathbb{Q}_p^2$. We will construct a graph $\Delta$ whose vertex set is $\mathcal{L}$. If $\Lambda$ and $\Lambda'$ are lattices, we draw an edge from $\Lambda$ to $\Lambda'$ if, for some lattices $\Gamma$ and $\Gamma'$ equivalent to $\Lambda$ and $\Lambda'$,

\[ \Gamma \subset \Gamma' \subset \rho^{-1} \Gamma. \]

This is a symmetric relation; if (*) holds, then

\[ p\Gamma' \subset \Gamma \subset \Gamma'. \]

Note that (*) is equivalent to $[\Gamma' : \Gamma] = p$.

We claim that the result is the infinite $p$-ary tree. Suppose $\Gamma$ is a lattice. By applying an element of $\text{GL}_2(\mathbb{Q}_p)$, we may assume that $\Gamma = \mathbb{Z}_p^2$, so if

\[ p\Gamma \subset \Gamma' \subset \Gamma, \]

then
then $\Gamma'/p\Gamma \subseteq \Gamma/p\Gamma = (\mathbb{Z}/p)^2$, and each neighbor of $\Gamma$ corresponds to a proper nontrivial subgroup of $(\mathbb{Z}/p)^2$. There are $p + 1$ such subgroups, so each vertex of $\Delta$ has degree $p + 1$.

Furthermore, $\Delta$ is connected. Suppose $\Lambda$ is a lattice; given a basis $v_1, v_2 \in \mathbb{Q}_p^2$ of $\Lambda$, we can write

$$\Lambda = (v_1, v_2)\mathbb{Z}_p^2,$$

where $(v_1, v_2)$ is a $2 \times 2$ matrix with columns equal to $v_1$ and $v_2$. By applying column operations with coefficients in $\mathbb{Z}_p$, we can write

$$\Lambda = \begin{pmatrix} p^a z_1 & 0 \\ p^b z_2 & p^c z_3 \end{pmatrix} \mathbb{Z}_p^2,$$

with $z_1, z_2, z_3$ units in $\mathbb{Z}_p$. If $k = \min\{a, b, c\}$, then $\Lambda \subset p^k \mathbb{Z}_p^2$. Then $p^k \mathbb{Z}_p^2/\Lambda$ is a finite abelian group of prime power order, and we can construct a composition series

$$p^k \mathbb{Z}_p^2 = \Lambda_0 \triangleright \cdots \triangleright \Lambda_d = \Lambda.$$

Each quotient group in this series is $\mathbb{Z}/p$, so the $\Lambda_i$ provide a path in $\Delta$ from $\mathbb{Z}_p^2$ to $\Lambda$. Similar arguments show that there is a unique path from $\mathbb{Z}_p^2$ to $\Lambda$ with no backtracking, so $\Delta$ is a tree (known as the Bruhat-Tits tree of $G$).

Given a basis $\{v_1, v_2\}$ of $\mathbb{Q}_p^2$, we can construct a sequence of lattices $\langle v_1, p^k v_2 \rangle \sim \langle p^k v_1, v_2 \rangle$; these form an apartment in $\Delta$. These form an apartment system for $\Delta$; a linear algebra argument implies that any two simplices are contained in a common apartment. (See [Lub10] for more details on this tree and some of its applications.)

- The advantage of the above construction is that it generalizes to produce a Bruhat-Tits building $\Delta$ of rank $k - 1$ for $\operatorname{SL}_k(\mathbb{Q}_p)$. As for the Bruhat-Tits tree, we take the vertex set of our building to be the set of lattices in $\mathbb{Q}_p^k$ up to similarity. If

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots \subseteq \Gamma_d \subseteq p^{-1} \Gamma_0,$$

then we add a simplex spanned by the corresponding vertices.

If $\{v_1, \ldots, v_k\}$ is a basis of $\mathbb{Q}_p^k$, then lattices of the form

$$\langle p^{a_1} v_1, \ldots, p^{a_k} v_k \rangle$$

form an apartment in $\Delta$. With some work, one can show that these apartments form an apartment system for $\Delta$.

4. Brief notes

Each of these could be expanded to fill several lectures; I only have space here to sketch some further topics:

- Root groups and discrete valuations: Why does $\operatorname{SL}_2(\mathbb{R})$ only act on a spherical building, but $\operatorname{SL}_2(\mathbb{Q}_p)$ acts on a euclidean one? One reason why is that $\mathbb{Q}_p$ has a discrete valuation.

Consider the action of $\operatorname{SL}_2(\mathbb{Q}_p)$ on the $p$-ary tree $T_p$. Suppose $\gamma$ is a ray in $T_p$ and consider the group of isometries $f$ such that $d(f(\gamma(t)), \gamma(t)) \to 0$ as $t \to \infty$. If $\gamma$ is the ray in $T_p$ consisting of lattices

$$\begin{pmatrix} 1 & 0 \\ 0 & p^t \end{pmatrix} \mathbb{Z}_p^2$$

for $t > 0$, this is the group $N$ of matrices of the form

$$\begin{pmatrix} z_1 & q \\ 0 & z_2 \end{pmatrix}$$

where $z_1, z_2 \in \mathbb{Z}_p$ and $q \in \mathbb{Q}_p$. This is a root group, and it has a discrete valuation, namely, $|q|$. This valuation corresponds to the amount of $\gamma$ that is fixed under the action of the group element — if $|q|$ is large, then the group element moves a larger portion of $\gamma$. In particular, you can’t multiply many small elements of $N$ to get a big element.

On the other hand, we could try to do the same thing for $\operatorname{SL}_2(\mathbb{R})$; we can define $\gamma$ to be the imaginary axis in the upper half-plane model and consider the group $N_\mathbb{R}$ of isometries $f$ such that $d(f(\gamma(t)), \gamma(t)) \to 0$ as $t \to \infty$. In this case, it’s a group of unipotent matrices isomorphic to $\mathbb{R}$. The amount of $\gamma$ that moves by distance $\geq 1$ under some element of $N_\mathbb{R}$ is governed by the log of the
coefficients. Now, though, we can combine many small elements of $N$ to get a big element; we can’t interpret an element of $N\mathbb{R}$ as stabilizing some ray.

- **Asymptotic cones:** On the other hand, log is almost a discrete valuation. That is, for every $x, y \in \mathbb{R}$,
  \[ \log |x + y| \leq \max\{\log |x|, \log |y|\} + \log 2, \]
  rather than the usual inequality
  \[ \eta(x + y) \leq \max\{\eta(x), \eta(y)\} \]
  for a discrete valuation. So one might expect that an appropriate scaling limit of $\mathcal{H}^2$ should have the structure of a building. In fact, this is true; Kleiner and Leeb proved that the asymptotic cone of a symmetric space of non-compact type is a building (for an appropriate generalization of the definition of building.) They used this to prove rigidity results for symmetric spaces and buildings.

- **Restriction of scalars and $S$-arithmetic groups:** We saw in class that Sol is a horosphere of a product of hyperbolic planes and that $\text{SL}_2(\mathbb{Z}[\sqrt{2}])$ is an arithmetic group with subgroups that are lattices in Sol. Similar arguments can be used to find actions of $S$-arithmetic groups on products of buildings; for instance, $\text{SL}_2(\mathbb{Z}[1/2])$ acts on $\mathcal{H}^2 \times T^2$ (with actions given by maps $\text{SL}_2(\mathbb{Z}[1/2]) \to \text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{Z}[1/2]) \to \text{SL}_2(\mathbb{Q})$).

**References**

