Review for mat298

Goals: Show applications of linear algebra to ODEs.

- linear algebra

- ODEs

Did not do: special methods in solving ODEs.

Systems of linear Equations: $Ax = b$

- Existence/uniqueness of solutions
- Solving
- Solution space
- The inverse of a matrix

Example:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 2 & -1 \end{bmatrix} x = \begin{bmatrix} -1 \\ 12 \\ 3 \\ -14 \\ -14 \end{bmatrix}$$

Is it solvable?

$$rref \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 12 \\ -1 & 0 & 0 & 3 \\ 3 & -1 & 0 & -14 \\ 4 & 2 & -1 & -14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrices and linear mappings $Ax = b$

- Definition of linear maps

- The matrix of a Linear map

- Rank, nullity of a matrix

Example: Let L be the map which rotates $\mathbb{R}^3$ around the z-axis by 60 degrees and then reflects everyting at the plane $x = y$. What is the matrix L_A corresponding to this map?

Solution:

$$L_A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

Vector spaces

- Vector space
- Subspaces
- Span
- linear dependence/independence
- basis
- dimension

Example: Is the vector

$$\begin{bmatrix} -1 \\ 12 \\ 3 \\ -14 \\ -14 \end{bmatrix}$$

in the span of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

Eigenvalues and Eigenvectors

- Definitions
- computing them (charachteristic polynomial
- Their significance in ODEs
- Similar matrices
- Normal forms
- Martrix exponentials
  - power series
  - generalized eigenvectors
  - computing matrix exponentials using normal forms

Example: Let L_A be the matrix corresponding to the Linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates everyting by 57 degrees around the line spanned by the vector $(1, 2, 3)$. Find a real eigenvalue and Eigenvector of thios matrix.

Solution: $\lambda = 1$, $v = (1, 2, 3)$

Example: Let

$$A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

Compute e^A.

Solution: Eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -1$. Eigenvectors $v_1 = (2, -1)$, $v_2 = (1, -1)$ Set

$$P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Then

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = B$$

So

$$e^A = Pe^BP^{-1} = \begin{bmatrix} -2e + e^{-1} & -2e + 2e^{-1} \\ e - e^{-1} & e - 2e^{-1} \end{bmatrix}$$

Orthogonality

- Shortest distance of a point in $\mathbb{R}^n$ to a subspace $W \subset \mathbb{R}^n$.

- Gram-Schmidt othonormalization

Linear ODEs

- Linear ODE of one variable $x' = \lambda x$.
  General solution $x(t) = Ce^{\lambda t}$.

- System of linear ODEs, general solution

- the initial value problem
  - solution principle of superposition
  - solution using matrix exponentials

- Qualitative theory of the 2×2-case (classification).

- pplane

Example: Solve the initial value problem

$$X' = AX\,, \qquad X(0) = (1, 2)$$

With

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$X(t) = e^{tA}X(0) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t + 2te^t \\ 2e^t \end{bmatrix}$$

Non-linear ODEs

- one non-linear ODE
  - phasespace, direction field
  - (hyperbolic) equilibria

- system of two non-linear ODEs

- Jacobi matrix

- (hyperbolic) equilibria

- periodic solutions / limit cycles

Example: Find and classify the equilibria of the following system:

$$\begin{aligned} x' =& y - x^3 - 2xy \\ y' =& -x - y^3 \end{aligned}$$

Solution: Equilibria:

$$y - x^3 - 2xy = 0, \qquad -x - y^3 = 0$$

one solution is $(x_0, y_0) = (0, 0)$. To find others, equation 2 gives $x = -y^3$. Then equation 1 is

$$y + y^9 + 2y^4 = 0$$

Real solutions $y_0 = 0$, $y_1 = -1$. Hence $x_0 = 0$, $x_1 = 1$.

Let's classify $(1, -1)$

$$F(x, y) = \begin{bmatrix} y - x^3 - 2xy \\ -x - y^3 \end{bmatrix}$$

Jacobian:

$$dF(x, y) = \begin{bmatrix} -3x^2 - 2y & 1 - 2x \\ -1 & -3y^2 \end{bmatrix}$$

$$dF(1, -1) = \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix}$$

Eigenvalues $\lambda_1 = -3.4$, $\lambda_2 = -0.59$.

$\Longrightarrow$ the point $(1, -1)$ is hyperbolic equilibrium, nodal sink.