SOME MORE "HAND EXERCISES" FOR CHAPTERS 5,8,10

(1) Which of the following subsets of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) are subspaces? Why? In parts a)-f) draw the corresponding subsets of $\mathbb{R}^2$.
   (a) $\{(x_1, x_2) \mid x_1 + x_2 = 0\}$
   (b) $\{(x_1, x_2) \mid x_1 + x_2 = 1\}$
   (c) $\{(x_1, x_2) \mid x_1 = 0,\ x_2 \geq 0\}$
   (d) $\{(x_1, x_2) \mid x_1 + x_2 = 0,\ \text{and } x_1 - x_2 = 0\}$
   (e) $\{(x_1, x_2) \mid x_1 + x_2^2 = 0\}$
   (f) $\{(x_1, x_2) \mid x_1 = 3\lambda,\ x_2 = -2\lambda,\ \lambda \in \mathbb{R}\}$
   (g) $\{(x_1, x_2, x3) \mid 4x_1 - x_2 + 7x_3 = 0\}$
   (h) $\{(x_1, x_2, x3) \mid x_1 + 2x_3 = 0,\ \text{and} x_2^2 = 0\}$

(2) Let P denote the vector space of all polynomials in one variable. Which of the following sets are subspaces?
   (a) $\{p \mid p(1) = p(2)\}$
   (b) $\{p \mid p(1) = p(2) + 1\}$
   (c) $\{p \mid p(0) = 1\}$
   (d) $\{p \mid p(1) = 0\}$
   (e) $\{p \mid p(1) \leq 0\}$
   (f) $\{p \mid p \text{ has degree exactly } 3\}$
   (g) $\{p \mid p \text{ has degree } \leq 3\}$
   (h) $\{p \mid p''(1) = 0\}$
   (i) $\{p \mid p''(1) = -1\}$
   (j) $\{p \mid \int_0^1 p(t)dt = 0\}$
   (k) $\{p \mid \int_0^1 p(t)dt = 7\}$

(3) Are the following sets of vectors linearly independent? If yes, do they form a basis of $\mathbb{R}^3$?

$$a)\quad \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \qquad b)\quad \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\4\\9\end{bmatrix}, \begin{bmatrix}1\\8\\27\end{bmatrix}$$

(4) Find four vectors in $\mathbb{R}^3$ (which are necessarily linearly dependent) such that each three of them are linearly independent.

(5) Find bases for the subspaces $W = \{(x_1, x_2) \mid 2x_1 + 3x_2 = 0\} \subset \mathbb{R}^2$ and $V = \{(x_1, x_2, x_3) \mid x_1 - x_2 + 2x_3 = 0\} \subset \mathbb{R}^3$.

(6) Show that the functions $\cos(x)$ and $f(x) = x^2$ are linearly independent.

(7) Write the vector v as a linear combination of the vectors b_1, b_2, and b_3.

$$v = \begin{bmatrix}3\\0\\0\end{bmatrix}, \quad b_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad b_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad b_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

(8) Find a minimal spanning set for the following subspace of $\mathbb{R}^4$:

$$span \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(9) Find the null-spaces of the following matrices:

$$a) \quad \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & -2 & 3 & -1 \\ 1 & -2 & 3 & -1 \end{bmatrix}, \qquad b) \quad \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

(10) Calculate the inverse and the determinant of the following matrices:

$$A = \begin{bmatrix} 0 & 1 & -4 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 1 & -15 & \\ 2 & 0 & 0 & -1 \\ -3 & -2 & 1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

(11) **This one is slightly tricky!**
Calculate the following determinant:

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

Hint: Substract - starting with the last column- from each column x_1 times the left neighbor. This does not change the value of the determinant. Then use the recursive formula for the determinant and proceed by induction...

Compare your result with exercise 3 b).

(12) Find the eigenvalues and eigenvectors of the following matrices

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -3 & -2 & 3 \\ -2 & -2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} i & -1 & 2+i \\ 1 & 0 & -3 \\ -2+i & 3 & 3i \end{bmatrix}$$

(13) Find a matrix whose characteristic polynomial is given by $p(\lambda) = (2-\lambda)^3(4-\lambda)^5$.

(14) Let A be an $n \times n$-matrix and put $A_1 = A - \alpha I_n$. Show that λ is an eigenvalue of A if and only if $\lambda - \alpha$ is an eigenvalue of A_1. How do the eigenvectors compare?

(15) Use Gram-Schmidt orthogonalization to find an orthogonal basis for the subspace of exercise 8

(16) Consider the vectorspace spanned by the polynomials $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$. An analogue of the dot-product for this space is given by

$$f \cdot g = \int_0^1 f(t)g(t)dt\,.$$

Use Gram-Schmidt to transform the basis $\{p_0, p_1, p_2\}$ to an orthonormal basis.