CSC2429, MAT1304: Circuit Complexity November 22, 2016

Handout on relations (projection, restriction, join, density)

Instructor: Benjamin Rossman

1 Relations

We consider relations over a fixed universe $[n]$ $(= \{1, \dots, n\})$. Rather than k-tuples and k-ary relations over $[n]$ (i.e. elements and subsets of $[n]^k$ where $k \in \mathbb{N}$), we will speak of tuples and relations over $[n]$ indexed by finite sets. Below, let V, W, S, T represent arbitrarily finite sets.

Definition 1 (V-Tuples)**.**

- A *V-tuple* is an element $x \in [n]^V$. For $v \in V$, the v-th coordinate of x is denoted $x_v \in [n]$.
- There is a unique $\emptyset$-ary tuple, which we denote by $\circ$. That is, $[n]^\emptyset = \{\circ\}$.
- For $x \in [n]^V$ and $S \subseteq V$, we write $x_S \in [n]^S$ for the restriction of x to coordinates in S.
- For $y \in [n]^S$ and $z \in [n]^{V \setminus S}$, we write yz for the V-tuple $x \in [n]^V$ with $x_S = y$ and $x_{V \setminus S} = z$. (This concatenation operation is unordered, that is, $yz = zy$.)

Definition 2 (V-ary Relations, Projection, Restriction)**.**

- A *V-ary relation* is a set $\mathcal{A} \subseteq [n]^V$ of V-tuples.
- For $\mathcal{A} \subseteq [n]^V$ and $S \subseteq V$, the *S-projection of* $\mathcal{A}$ is the S-ary relation
$$\mathsf{proj}_S(\mathcal{A}) := \{x_S : x \in \mathcal{A}\}.$$
- For $\mathcal{A} \subseteq [n]^V$ and $S \subseteq V$ and $z \in [n]^{V \setminus S}$, the *$S$-restriction of* $\mathcal{A}$ *at* z is the S-ary relation
$$\mathcal{A}|_S^z := \{y \in [n]^S : yz \in \mathcal{A}\}.$$
Note that $\mathcal{A}|_S^z \subseteq \mathsf{proj}_S(\mathcal{A})$.

Definition 3 (Join)**.**

- The *join* of relations $\mathcal{A} \subseteq [n]^V$ and $\mathcal{B} \subseteq [n]^W$ is the $V \cup W$-ary relation defined by
$$\mathcal{A} \bowtie \mathcal{B} := \{x \in [n]^{V \cup W} : x_V \in \mathcal{A} \text{ and } x_W \in \mathcal{B}\}.$$

The join operation $\bowtie$ is a hybrid of intersection $\cap$ and product $\times$. If $V = W$, then we have $\mathcal{A} \bowtie \mathcal{B} = \mathcal{A} \cap \mathcal{B}$. On the opposite extreme, if $V \cap W = \emptyset$, then we have $\mathcal{A} \bowtie \mathcal{B} = \mathcal{A} \times \mathcal{B}$ (i.e. the set $\{xy : x \in \mathcal{A} \text{ and } y \in \mathcal{B}\}$). As an operation on relations, note that $\bowtie$ is associative, commutative and idempotent (i.e. $\mathcal{A} \bowtie (\mathcal{B} \bowtie \mathcal{C}) = (\mathcal{A} \bowtie \mathcal{B}) \bowtie \mathcal{C}$ and $\mathcal{A} \bowtie \mathcal{B} = \mathcal{B} \bowtie \mathcal{A}$ and $\mathcal{A} \bowtie \mathcal{A} = \mathcal{A}$). Also note that $\emptyset$-ary relation $\{\circ\}$ is an identity under $\bowtie$ (i.e. $\mathcal{A} \bowtie \{\circ\} = \mathcal{A}$).

Exercise 4. Suppose $\mathcal{A} \subseteq [n]^V$ and $\mathcal{B} \subseteq [n]^W$ and $\mathcal{C} \subseteq [n]^{V \cup W}$ such that $\mathcal{C} \subseteq \mathcal{A} \bowtie \mathcal{B}$.

i. Show that $\mathsf{proj}_V(\mathcal{C}) \subseteq \mathcal{A}$.

ii. For $S \subseteq V$, show that $\mathsf{proj}_S(\mathcal{C}) \subseteq \mathsf{proj}_S(\mathcal{A})$.

2 Density

Definition 5 (Density, Projection Density, Restriction Density)**.**

- The *density* of a relation $\mathcal{A} \subseteq [n]^V$, denoted $\mu(\mathcal{A})$, is defined by

$$\mu(\mathcal{A}) := \frac{|\mathcal{A}|}{n^{|V|}} \quad \Big(= \mathop{\mathbb{P}}_{x \in [n]^V}[\ x \in \mathcal{A}\ ]\Big).$$

- For $S \subseteq V$, the S*-projection density* of $\mathcal{A}$ is the quantity $\mu(\mathsf{proj}_S(\mathcal{A}))$. Note that

$$\mu(\mathsf{proj}_S(\mathcal{A})) = \mathop{\mathbb{P}}_{y \in [n]^S}\Big[\ \bigvee_{z \in [n]^{V \setminus S}} yz \in \mathcal{A}\ \Big].$$

- The S*-restriction density* of $\mathcal{A}$, denoted $\mu_S(\mathcal{A})$, is defined by

$$\mu_S(\mathcal{A}) := \max_{z \in [n]^{V \setminus S}} \mu(\mathcal{A}|_S^z) \quad \Big(= \max_{z \in [n]^{V \setminus S}} \mathop{\mathbb{P}}_{y \in [n]^V}[\ yz \in \mathcal{A}\ ]\Big).$$

  Note that $\mu_S(\mathcal{A}) \le \mu(\mathsf{proj}_S(\mathcal{A}))$ and $\mu_V(\mathcal{A}) = \mu(\mathcal{A})$ and $\mu_\emptyset(\mathcal{A}) = 1_{[\mathcal{A} \neq \emptyset]}$.

Exercise 6. Let $\mathcal{A} \subseteq [n]^V$ and $S \subseteq V$.

iii. Show that $\mu(\mathcal{A}) \le \mu(\mathsf{proj}_S(\mathcal{A})) \cdot \mu_{V \setminus S}(\mathcal{A})$.

iv. Show that inequality (iii) cannot be strengthened to $\mu(\mathcal{A}) \le \mu(\mathsf{proj}_S(\mathcal{A})) \cdot \mu(\mathsf{proj}_{V \setminus S}(\mathcal{A}))$. That is, find an example of $\mathcal{A}$ and S where this inequality is false.

v. For $S'' \subseteq S' \subseteq S$, show that $\mu_{S''}(\mathsf{proj}_S(\mathcal{A})) \le \mu_{S''}(\mathsf{proj}_{S'}(\mathcal{A}))$.

3 Bounding the density of a sub-relation of a join

The following lemma will play a role in our lower bound on the AC^0 formula size of SUB(Path$_k$).

Lemma 7. *Suppose $\mathcal{A} \subseteq [n]^V$ and $\mathcal{B} \subseteq [n]^W$ and $\mathcal{C} \subseteq [n]^{V \cup W}$ such that $\mathcal{C} \subseteq \mathcal{A} \bowtie \mathcal{B}$. Then for all $S \subseteq V$ and $T \subseteq W$, we have*

$$\mu(\mathcal{C}) \le \mu(\mathsf{proj}_S(\mathcal{A})) \cdot \mu_{T \setminus S}(\mathsf{proj}_T(\mathcal{B})) \cdot \mu_{(V \cup W) \setminus (S \cup T)}(\mathcal{C}).$$

Proof. Lemma 7 is mainly derived by two applications of inequality (iii). First we apply (iii) with respect to $\mathcal{C}$ and $S \cup T$:

$$\mu(\mathcal{C}) \le \mu(\mathsf{proj}_{S \cup T}(\mathcal{C})) \cdot \mu_{(V \cup W) \setminus (S \cup T)}(\mathcal{C}).$$

Next we apply (iii) with respect to $\mathsf{proj}_{S \cup T}(\mathcal{C})$ and S:

$$\mu(\mathsf{proj}_{S \cup T}(\mathcal{C})) \le \mu(\mathsf{proj}_S(\mathcal{C})) \cdot \mu_{T \setminus S}(\mathsf{proj}_{S \cup T}(\mathcal{C})).$$

Now observe that

$$\mu(\mathsf{proj}_S(\mathcal{C})) \overset{\text{(ii)}}{\le} \mu(\mathsf{proj}_S(\mathcal{A})),$$

$$\mu_{T \setminus S}(\mathsf{proj}_{S \cup T}(\mathcal{C})) \overset{\text{(v)}}{\le} \mu_{T \setminus S}(\mathsf{proj}_T(\mathcal{C})) \overset{\text{(ii)}}{\le} \mu_{T \setminus S}(\mathsf{proj}_T(\mathcal{B})).$$

Combining the four above inequalities finishes the proof. $\square$