**ANNOUNCEMENTS**

*Final reminder to fill your course evaluation!*

At end of lecture today, pick up PSET #4 and any unclaimed psets or midterms

No more tutorials. Continue to ask questions on Piazza!

Office hours will continue Thursdays 4-5:30pm on April 5 and 12

Final Exam: Wednesday, April 18, 9am-12pm, EX 200
Marking Scheme

- 25% for each midterm + 35% final exam
- Final exam out of 100, plus 8 extra credit points (max possible score of 108)
- Will count 2nd midterm as +10 points for everyone. Will scale up final exam if necessary to ensure average is at least 70.
- Problem sets (15%): worth 3.25% each, plus 2% · max pset grade
- Extra credit problems: worth up to 3% – exact formula to be determined
NOT on the Final Exam

- Won’t ask you to write a deduction. However, you should know the definitions of deduction, (PC) rule and (Q1) axiom. (Appendix not provided.)
- No question about Henkin axioms or details of the proof of the soundness/completeness/compactness theorems.
- You don’t need to memorize the axioms of Robinson Arithmetic $N$.
- You don’t need the memorize the precise definition of Godel numbering ($\vdash t$ and $\vdash \varphi$), but you should know the basic idea (for instance, $\vdash \alpha \lor \beta = \langle \text{some number}, \vdash \alpha, \vdash \beta \rangle = 2^{\text{some number} + 1} \cdot 3^{\vdash \alpha} + 15^{\vdash \beta} + 1$).
- Rosser’s Theorem ($N$ is not recursively completable) and 2nd Incompleteness Theorem not on the exam.
ON the Final Exam

• \( \sim 50\% \) of exam on Chapters 1–4. One question involving the Compactness Theorem (worth \( \sim 10\% \)), relatively easy compared with Midterm 2

• \( \sim 50\% \) of exam on Chapter 5 and 6 (the parts we covered in detail)

• Knowledge of Ehrenfeucht-Fraisse Games will help with extra credit problem (8%).

• You should know the statement of Self-Reference Lemma, 1st Incompleteness Theorem and Tarski’s theorem. (I won’t ask you to recite the proof, but it’s good if you understand!)

• Be able to express yourself in first-order logic. (e.g. write a \( \Delta \)-formula defining PRIME, or write the axioms for equivalence relations, or write a sentence that true in a particular \( L \)-structure \( \mathfrak{A} \) and false in another \( L \)-structure \( \mathfrak{B} \))

• Know the key definitions from Chapter 1-3: free variables, substitutions, def. of \( \models \) and \( \vdash \), statements of soundness/completeness/compactness theorems.
ON the Final Exam

• Know the definitions of isomorphism (page 27) and elementary equivalence.

• Be able to classify terms/formulas/numbers according to our notation: \( \llcorner t \lrcorner \), \( \llcorner \varphi \lrcorner \), \( \langle a_1, \ldots, a_k \rangle \) are numbers: \( \overline{a} \), \( \overline{\varphi} \) are terms, \( \text{Deduction}_N(\overline{a}, \overline{b}) \) is a formula.

• Know the sequence-coding function \( \langle a_1, \ldots, a_k \rangle = \prod_{i=1}^{k} (p_i)^{a_i+1} \).

• Know the definition of \( \Sigma \), \( \Pi \), \( \Delta \)-definable and representable and recursive/complete/consistent set of axioms.

• Understand the concept of “construction sequences” as a means of building \( \Delta \)-formulas.

• Know that \( \text{Deduction}_N \) is \( \Delta \)-definable and \( \text{Thm}_N \) is \( \Sigma \)-definable (and know what these sets are, though not necessarily the exact formulas which define them).
Key Results in Chapters 5 and 6

- $N$ proves every sentence which is true in $\mathbb{N}$
- $\Delta$-definable $\Rightarrow$ representable $\Rightarrow$ $\Sigma$-definable
- $A$ is representable $\iff$ membership in $A$ is computable by an algorithm (e.g., Turing machine) in finite time.
  You should understand the $\Rightarrow$ direction: Suppose $\varphi(x)$ represents $A \subseteq \mathbb{N}$. To determine whether or not $a \in A$, start enumerating the (infinite) list of deductions-from-$N$ until finding a deduction which shows that $N \vdash \varphi(a)$ or $N \vdash \neg \varphi(a)$.

- If $A$ is recursive (i.e. if $\{\ulcorner \alpha \urcorner : \alpha \in A\}$ is representable), then the set $\text{Thm}_A := \{\ulcorner \varphi \urcorner : A \models \varphi\}$ is $\Sigma$-definable.

- Self-reference lemma, 1st incompleteness theorem, Tarski’s undefinability theorem
Lemma 6.2.2 (Self-Reference Lemma). If $\beta(x)$ is an $\mathcal{L}_{NT}$-formula with only $x$ free, then there is a sentence $\theta$ such that $\mathcal{N} \vdash \theta \iff \beta(\overline{\theta})$. 
Lemma 6.2.2 (Self-Reference Lemma). If $\beta(x)$ is an $L_{NT}$-formula with only $x$ free, then there is a sentence $\theta$ such that $N \vdash \theta \leftrightarrow \beta(\ulcorner \theta \urcorner)$.

Theorem 6.3.6 (Gödel’s First Incompleteness Theorem, 1931). Suppose that $A$ is a consistent and recursive set of axioms in the language $L_{NT}$. Then there is a sentence $\theta$ such that $\mathfrak{M} \models \theta$ but $A \nvdash \theta$.

Idea. If $A \nvdash N$, we’re done. If $A \vdash N$, we consider a sentence $\theta$ such that

$$N \vdash \theta \leftrightarrow \neg \text{Thm}_A(\ulcorner \theta \urcorner).$$

This forces $\mathfrak{M} \models \theta$ and $A \nvdash \theta$ (otherwise we get a contradiction).
Lemma 6.2.2 (Self-Reference Lemma). If $\beta(x)$ is an $L_{NT}$-formula with only $x$ free, then there is a sentence $\theta$ such that $N \vdash \theta \leftrightarrow \beta(\overline{\theta})$.

Theorem 6.3.6 (Gödel’s First Incompleteness Theorem, 1931). Suppose that $A$ is a consistent and recursive set of axioms in the language $L_{NT}$. Then there is a sentence $\theta$ such that $\mathfrak{N} \models \theta$ but $A \not\vdash \theta$.

Idea. If $A \not\vdash N$, we’re done. If $A \vdash N$, we consider a sentence $\theta$ such that

$$N \vdash \theta \leftrightarrow \neg \text{Thm}_A(\overline{\theta}).$$

This forces $\mathfrak{N} \models \theta$ and $A \not\vdash \theta$ (otherwise we get a contradiction).

Theorem 6.3.10 (Tarski’s Undefinability Theorem, 1936). The set $\{\overline{\varphi} : \mathfrak{N} \models \varphi\}$ of Gödel numbers of formulas true in $\mathfrak{N}$ is not definable.

Idea. Toward a contradiction, assume $\beta(x)$ defines $\{\overline{\varphi} : \mathfrak{N} \models \varphi\}$. Consider the sentence $\theta$ such that $N \vdash \theta \leftrightarrow \neg \beta(\overline{\theta})$. Contradiction is immediate, as

$$\mathfrak{N} \models \theta \iff \mathfrak{N} \models \neg \beta(\overline{\theta}) \iff \mathfrak{N} \not\models \theta.$$

Theorem 6.4.5 (Rosser’s Theorem). If $A$ is a set of $\mathcal{L}_{NT}$-axioms that is recursive, consistent, and extends $N$, then $A$ is incomplete.

Idea. Consider a sentence $\theta$ such that

$$N \vdash \theta \iff (\forall x)[\text{Deduction}_A(x, \overline{\theta}) \rightarrow (\exists y < x)\text{Deduction}_A(y, \overline{\neg \theta})].$$

Can show that $A \not\vdash \theta$ and $A \not\vdash \neg \theta$. (A contradiction ensues if we assume $A \vdash \theta$ or $A \vdash \neg \theta$.)
Theorem 6.4.5 (Rosser’s Theorem). If $A$ is a set of $\mathcal{L}_{NT}$-axioms that is recursive, consistent, and extends $N$, then $A$ is incomplete.

Idea. Consider a sentence $\theta$ such that

$$N \vdash \theta \iff (\forall x)[Deduction_A(x, \overline{\theta}) \rightarrow (\exists y < x)Deduction_A(y, \overline{\neg \theta})].$$

Can show that $A \nvdash \theta$ and $A \nvdash \neg \theta$. (A contradiction ensues if we assume $A \vdash \theta$ or $A \vdash \neg \theta$.)

Obs. Either $\mathfrak{M} \models \theta$ or $\mathfrak{M} \models \neg \theta$. In either case, we get a sentence which is true in $\mathfrak{M}$ and not provable from $A$. Thus, Rosser’s Theorem $\Rightarrow$ 1st Incompleteness Theorem.
Peano Arithmetic

**Definition.** The axioms of *Peano Arithmetic* (1889), denoted $PA$, are the eleven axioms of Robinson arithmetic together with axioms

\[
\text{Induction}_\varphi \equiv \left[ \varphi(0) \land (\forall x)[\varphi(x) \rightarrow \varphi(Sx)] \right] \rightarrow (\forall x)\varphi(x)
\]

for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.
Peano Arithmetic

Definition. The axioms of **Peano Arithmetic** (1889), denoted \( PA \), are the eleven axioms of Robinson arithmetic together with axioms

\[
\text{Induction}_\varphi \equiv \left[ \varphi(0) \land (\forall x)[\varphi(x) \rightarrow \varphi(Sx)] \right] \rightarrow (\forall x)\varphi(x)
\]

for each \( \mathcal{L}_{NT} \)-formula \( \varphi(x) \) with one free variable.

- Clearly, \( \mathfrak{M} \models PA \) (since \( \mathfrak{M} \models \text{Induction}_\varphi \) for each \( \varphi(x) \)). Therefore, \( PA \) is consistent.
**Peano Arithmetic**

**Definition.** The axioms of *Peano Arithmetic* (1889), denoted $\textit{PA}$, are the eleven axioms of Robinson arithmetic together with axioms

$$\text{Induction}_\varphi : \equiv \left[ \varphi(0) \land (\forall x)[\varphi(x) \to \varphi(Sx)] \right] \to (\forall x)\varphi(x)$$

for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.

- Clearly, $\mathfrak{N} \models \textit{PA}$ (since $\mathfrak{N} \models \text{Induction}_\varphi$ for each $\varphi(x)$). Therefore, $\textit{PA}$ is consistent.

- $\textit{PA}$ is easily seen to be recursive: there is a simple algorithm to decide membership in $\{\neg \alpha \downarrow : \alpha \in \textit{PA}\}$. By 1st Incompleteness Theorem, there exists a sentence $\theta$ such that $\mathfrak{N} \models \theta$ but $\textit{PA} \nvdash \theta$. (In particular, $\textit{PA}$ is not complete.)
Peano Arithmetic

Definition. The axioms of *Peano Arithmetic* (1889), denoted \( PA \), are the eleven axioms of Robinson arithmetic together with axioms

\[
\text{Induction}_\varphi \equiv \left[ \varphi(0) \land (\forall x)[\varphi(x) \to \varphi(Sx)] \right] \to (\forall x)\varphi(x)
\]

for each \( \mathcal{L}_{NT} \)-formula \( \varphi(x) \) with one free variable.

- Clearly, \( \mathbb{N} \models PA \) (since \( \mathbb{N} \models \text{Induction}_\varphi \) for each \( \varphi(x) \)). Therefore, \( PA \) is consistent.

- \( PA \) is easily seen to be recursive: there is a simple algorithm to decide membership in \( \{ \ulcorner \alpha \urcorner : \alpha \in PA \} \). By 1st Incompleteness Theorem, there exists a sentence \( \theta \) such that \( \mathbb{N} \models \theta \) but \( PA \not\vdash \theta \). (In particular, \( PA \) is not complete.)

- Whereas Robinson arithmetic \( N \) is very weak (it doesn’t prove \((\forall x)(\forall y)(x + y = y + x)\)), Peano arithmetic \( PA \) is quite powerful – it proves any result you have seen in MAT315. (It is even claimed that \( PA \vdash \) Fermat’s Last Theorem.)
2nd Incompleteness Theorem

The sentence $Con_A$:

Let $A$ be a recursive set of $\mathcal{L}_{NT}$-sentences.

Recall that the set $Thm_A := \{ \ulcorner \varphi \urcorner : A \vdash \varphi \}$ is $\Sigma$-definable. Fix a $\Sigma$-formula $Thm_A(x)$ which defines $Thm_A$.

Let $Con_A$ be the sentence

$$Con_A \equiv \neg Thm_A(\ulcorner \bot \urcorner).$$

This sentence expresses “$A$ is consistent”: note that $A$ is consistent if, and only if, $\mathcal{N} \models Con_A$. 
Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \nvdash Con_A$. 
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $L_{NT}$-sentences which extends $PA$, then $A \nvdash Con_A$.

- $PA$ itself is consistent and recursive. Therefore, $PA \nvdash Con_{PA}$. 
Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \nvdash Con_A$.

- $PA$ itself is consistent and recursive. Therefore, $PA \nvdash Con_{PA}$.
- How do you and I know that $PA$ is consistent? We can prove $\mathcal{M}$ is a model of $Con_{PA}$ using the usual axioms of $ZFC$. Therefore, $ZFC \vdash Con_{PA}$ (interpreting the sentence $Con_{PA}$ in the language of set theory).
  However, $ZFC \nvdash Con_{ZFC}$. 
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If \( A \) is any consistent, recursive set of \( \mathcal{L}_{NT} \)-sentences which extends \( PA \), then \( A \nvdash \text{Con}_A \).

- \( PA \) itself is consistent and recursive. Therefore, \( PA \nvdash \text{Con}_{PA} \).
- How do you and I know that \( PA \) is consistent? We can prove \( \mathfrak{M} \) is a model of \( \text{Con}_{PA} \) using the usual axioms of \( ZFC \). Therefore, \( ZFC \vdash \text{Con}_{PA} \) (interpreting the sentence \( \text{Con}_{PA} \) in the language of set theory).
  However, \( ZFC \nvdash \text{Con}_{ZFC} \).
- 2nd Incompleteness Theorem answered a question asked by David Hilbert in 1900 by showing that no “sufficiently powerful formal system” (including set theory \( ZFC \)) can prove its own consistency.
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \not\vdash Con_A$.

- Alternative phrasing of 2nd Incompleteness Theorem: If $A$ is recursive extension of $PA$, then $A$ is consistent $\iff A \not\vdash Con_A$.
  (If $A$ is inconsistent, then $A \vdash Con_A$ since $A$ proves everything.)
2ND INCOMPLETENESS THEOREM

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \not\vdash \text{Con}_A$.

- Alternative phrasing of 2nd Incompleteness Theorem: If $A$ is recursive extension of $PA$, then $A$ is consistent $\iff A \not\vdash \text{Con}_A$.
  (If $A$ is inconsistent, then $A \vdash \text{Con}_A$ since $A$ proves everything.)

- Since $PA$ is consistent and $PA \not\vdash \neg \text{Con}_{PA}$, it follows that $PA \cup \{\neg \text{Con}_{PA}\}$ is consistent. Therefore, there exists a model $\mathfrak{M}$ of $PA \cup \{\neg \text{Con}_{PA}\}$. (This model looks a lot like $\mathfrak{N}$ — for example, addition $+^\mathfrak{M}$ is commutative. However, $Th(\mathfrak{M}) \neq Th(\mathfrak{N})$.)
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)

If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \nvdash Con_A$.

- Alternative phrasing of 2nd Incompleteness Theorem: If $A$ is recursive extension of $PA$, then $A$ is consistent $\iff A \nvdash Con_A$.

  (If $A$ is inconsistent, then $A \vdash Con_A$ since $A$ proves everything.)

- Since $PA$ is consistent and $PA \nvdash \neg Con_{PA}$, it follows that $PA \cup \{\neg Con_{PA}\}$ is consistent. Therefore, there exists a model $\mathfrak{M}$ of $PA \cup \{\neg Con_{PA}\}$. (This model looks a lot like $\mathfrak{M}$ — for example, addition $+_{\mathfrak{M}}$ is commutative. However, $Th(\mathfrak{M}) \neq Th(\mathfrak{N})$.)

- Since $PA$ is consistent, why not take $Con_{PA}$ as an additional axiom? Let $PA' := PA \cup \{Con_{PA}\}$. Then $PA' \vdash Con_{PA}$, but $PA' \nvdash Con_{PA'}$. So we are left with the same problem.
**Hilbert-Bernays Derivability Conditions**

**Lemma.** $PA$ satisfies the following “derivability conditions” for all formulas $\alpha$ and $\beta$:

(D1) If $PA \vdash \alpha$, then $PA \vdash Thm_{PA}(\overline{\overline{\alpha}})$.

If $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$”. 

Hilbert-Bernays Derivability Conditions

Lemma. $PA$ satisfies the following “derivability conditions” for all formulas $\alpha$ and $\beta$:

(D1) If $PA \vdash \alpha$, then $PA \vdash Thm_{PA}(\overline{\overline{\alpha} \top})$.

If $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$”.

(D2) $PA \vdash Thm_{PA} (\overline{\overline{\alpha} \top}) \rightarrow Thm_{PA}(\overline{\overline{Thm_{PA}(\overline{\overline{\alpha} \top})} \top})$.

$PA$ proves “if $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$’”.
Hilbert-Bernays Derivability Conditions

**Lemma.** $PA$ satisfies the following “derivability conditions” for all formulas $\alpha$ and $\beta$:

(D1) If $PA \vdash \alpha$, then $PA \vdash Thm_{PA}(\overline{\alpha})$.

If $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$”.

(D2) $PA \vdash Thm_{PA}(\overline{\alpha}) \rightarrow Thm_{PA}(\overline{Thm_{PA}(\overline{\alpha})})$.

$PA$ proves “if $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$””.

(D3) $PA \vdash [Thm_{PA}(\overline{\alpha}) \land Thm_{PA}(\overline{\alpha \rightarrow \beta})] \rightarrow Thm_{PA}(\overline{\beta})$.

$PA$ proves “if $PA$ proves $\alpha$ and $PA$ proves $\alpha \rightarrow \beta$, then $PA$ proves $\beta$”.
Hilbert-Bernays Derivability Conditions

Lemma. PA satisfies the following “derivability conditions” for all formulas \( \alpha \) and \( \beta \):

(D1) If \( PA \vdash \alpha \), then \( PA \vdash Thm_{PA}(\overline{\alpha}) \).

If \( PA \) proves \( \alpha \), then \( PA \) proves “\( PA \) proves \( \alpha \)”.

(D2) \( PA \vdash Thm_{PA}(\overline{\alpha}) \rightarrow Thm_{PA}(\overline{Thm_{PA}(\overline{\alpha})}) \).

\( PA \) proves “if \( PA \) proves \( \alpha \), then \( PA \) proves “\( PA \) proves \( \alpha \)””.

(D3) \( PA \vdash [Thm_{PA}(\overline{\alpha}) \land Thm_{PA}(\overline{\alpha \rightarrow \beta})] \rightarrow Thm_{PA}(\overline{\beta}) \).

\( PA \) proves “if \( PA \) proves \( \alpha \) and \( PA \) proves \( \alpha \rightarrow \beta \), then \( PA \) proves \( \beta \)”.

Moreover, if \( A \) is a recursive extension of \( PA \), then \( A \) satisfies derivability conditions (D1)–(D3) with respect to \( Thm_{A}(x) \).
Proof of 2nd Incompleteness Theorem. Let $A$ be a consistent, recursive extension of $PA$. Let $\theta$ be a sentence such that

\[(*) \quad N \vdash \theta \leftrightarrow \neg \text{Thm}_A(\overline{\theta}).\]

By proof of 1st Incompleteness Theorem, we know that $A \not\vdash \theta$.

CLAIM: $A \vdash \text{Con}_A \rightarrow \theta$. (It follows that $A \not\vdash \text{Con}_A$.)
**Proof of 2nd Incompleteness Theorem.** Let $A$ be a consistent, recursive extension of $PA$. Let $\theta$ be a sentence such that

\[(\ast) \quad N \vdash \theta \iff \neg \text{Thm}_A(\overline{\theta}).\]

By proof of 1st Incompleteness Theorem, we know that $A \nvdash \theta$.

CLAIM: $A \vdash \text{Con}_A \rightarrow \theta$. (It follows that $A \nvdash \text{Con}_A$.)

PROOF OF CLAIM: By $(\ast)$, we have $A \vdash \text{Thm}_A(\overline{\theta}) \rightarrow \neg \theta$.

(D1): $A \vdash \text{Thm}_A(\overline{\text{Thm}_A(\overline{\theta})} \rightarrow \neg \theta)$

(D2): $A \vdash \text{Thm}_A(\overline{\theta}) \rightarrow \text{Thm}_A(\overline{\text{Thm}_A(\overline{\theta})} \rightarrow \neg \theta)$.

(D3): $A \vdash (\text{Thm}_A(\overline{\text{Thm}_A(\overline{\theta})}) \land \text{Thm}_A(\overline{\text{Thm}_A(\overline{\theta})} \rightarrow \neg \theta)) \rightarrow \text{Thm}_A(\overline{\neg \theta})$.

By (PC) rule, $A \vdash \text{Thm}_A(\overline{\theta}) \rightarrow \text{Thm}_A(\overline{\neg \theta})$.

Next step: $A \vdash \text{Thm}_A(\overline{\theta}) \rightarrow \text{Thm}_A(\overline{\bot})$, hence $A \vdash \text{Con}_A \rightarrow \neg \text{Thm}_A(\overline{\theta})$

By $(\ast)$ and (PC) rule: $A \vdash \text{Con}_A \rightarrow \theta$. Q.E.D.
**Complete, Consistent, Recursive Theories**

The 1st Incompleteness Theorem implies that $Th(\mathbb{N})$ has no complete, consistent, recursive axiomatization.

The same is true of any theory that “interprets” $Th(\mathbb{N})$, such as models of $ZFC$ (set theory).
**Complete, Consistent, Recursive Theories**

The 1st Incompleteness Theorem implies that $Th(\mathbb{N})$ has no complete, consistent, recursive axiomatization.

The same is true of any theory capable of “interpreting” Robinson arithmetic $N$, such as $ZFC$ (set theory).

In contrast, there are beautiful examples of complete, consistent, recursive theories:

- $Th(\mathbb{N}, 0, 1, +)$ (Presburger Arithmetic)
- $Th(\mathbb{R}, 0, 1, +, \cdot, <)$ (the theory of real closed fields)
- Euclidean Geometry: the theory $Th(\mathbb{R}^2, \text{Between}, \text{Congruent})$ where
  \[
  \text{Between} := \{(a, b, c) \in (\mathbb{R}^2)^3 : b \in ac\},
  \]
  \[
  \text{Congruent} := \{(a, b, c, d) \in (\mathbb{R}^2)^4 : |ab| = |cd|\}.
  \]
  Here $ab$ denotes the line segment between points $a, b \in \mathbb{R}^2$, and $|ab|$ is the length of $ab$. (See Tarski’s axioms.)