**ANNOUNCEMENTS**

*Final reminder to fill your course evaluation!*

At end of lecture today, pick up PSET #4 and any unclaimed psets or midterms

No more tutorials. Continue to ask questions on Piazza!

Office hours will continue Thursdays 4-5:30pm on April 5 and 12

Final Exam: Wednesday, April 18, 9am-12pm, EX 200
Marking Scheme

- 25% for each midterm + 35% final exam
- Final exam out of 100, plus 8 extra credit points (max possible score of 108)
- Will count 2nd midterm as +10 points for everyone. Will scale up final exam if necessary to ensure average is at least 70.
- Problem sets (15%): worth 3.25% each, plus 2% · max pset grade
- Extra credit problems: worth up to 3% – exact formula to be determined
**NOT on the Final Exam**

- Won’t ask you to write a deduction. However, you should know the definitions of deduction, (PC) rule and (Q1) axiom. (Appendix not provided.)
- No question about Henkin axioms or details of the proof of the soundness/completeness/compactness theorems.
- You don’t need to memorize the axioms of Robinson Arithmetic $N$.
- You don’t need the memorize the precise definition of Godel numbering ($\text{"}t\text{"}$ and $\text{"}\phi\text{"}$), but you should know the basic idea (for instance, $\text{"}\alpha \lor \beta\text{"} = \langle\text{some number, }\text{"}\alpha\text{"}, \text{"}\beta\text{"}\rangle = 2^{\text{some number}} + 13^{\text{"}\alpha\text{"}} + 15^{\text{"}\beta\text{"}} + 1$).
- Rosser’s Theorem ($N$ is not recursively completable) and 2nd Incompleteness Theorem not on the exam.
ON the Final Exam

• ~50% of exam on Chapters 1–4. One question involving the Compactness Theorem (worth ~10%), relatively easy compared with Midterm 2

• ~50% of exam on Chapter 5 and 6 (the parts we covered in detail)

• Knowledge of Ehrenfeucht-Fraisse Games will help with extra credit problem (8%).

• You should know the statement of Self-Reference Lemma, 1st Incompleteness Theorem and Tarski’s theorem. (I won’t ask you to recite the proof, but it’s good if you understand!)

• Be able to express yourself in first-order logic. (e.g. write a $\Delta$-formula defining PRIME, or write the axioms for equivalence relations, or write a sentence that true in a particular $\mathcal{L}$-structure $\mathfrak{A}$ and false in another $\mathcal{L}$-structure $\mathfrak{B}$)

• Know the key definitions from Chapter 1-3: free variables, substitutions, def. of $\models$ and $\vdash$, statements of soundness/completeness/compactness theorems.
ON the Final Exam

• Know the definitions of isomorphism (page 27) and elementary equivalence.

• Be able to classify terms/formulas/numbers according to our notation: ⌜t⌝, ⌜ϕ⌝, ⟨a₁, . . . , aₖ⟩ are numbers: ⌜a⌝, ⌜ϕ⌝ are terms, Deductionₙ(⌜a⌝, ⌜b⌝) is a formula.

• Know the sequence-coding function ⟨a₁, . . . , aₖ⟩ = ∏ₖᵢ=₁(pᵢ)ᵃᵢ₊₁.

• Know the definition of Σ, Π, Δ-definable and representable and recursive/complete/consistent set of axioms.

• Understand the concept of “construction sequences” as a means of building Δ-formulas.

• Know that Deductionₙ is Δ-definable and Thmₙ is Σ-definable (and know what these sets are, though not necessarily the exact formulas which define them).
Key Results in Chapters 5 and 6

• $N$ proves every $\Sigma$-sentence which is true in $\mathbb{N}$

• $\Delta$-definable $\Rightarrow$ representable $\Rightarrow$ $\Sigma$-definable

• $A$ is representable $\iff$ membership in $A$ is computable by an algorithm (e.g., Turing machine) in finite time.

  You should understand the $\Rightarrow$ direction: Suppose $\varphi(x)$ represents $A \subseteq \mathbb{N}$. To determine whether or not $a \in A$, start enumerating the (infinite) list of deductions-from-$N$ until finding a deduction which shows that $N \vdash \varphi(a)$ or $N \vdash \neg \varphi(a)$.

• If $A$ is recursive (i.e. if $\{\ulcorner \alpha \urcorner : \alpha \in A\}$ is representable), then the set $\text{Thm}_A := \{\ulcorner \varphi \urcorner : A \models \varphi\}$ is $\Sigma$-definable.

• Self-reference lemma, 1st incompleteness theorem, Tarski’s undefinability theorem
Lemma 6.2.2 (Self-Reference Lemma). If $\beta(x)$ is an $\mathcal{L}_{NT}$-formula with only $x$ free, then there is a sentence $\theta$ such that $N \vdash \theta \leftrightarrow \beta(\overline{\theta})$. 
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Theorem 6.3.6 (Gödel’s First Incompleteness Theorem, 1931). Suppose that $A$ is a consistent and recursive set of axioms in the language $\mathcal{L}_{NT}$. Then there is a sentence $\theta$ such that $\mathfrak{N} \models \theta$ but $A \nvdash \theta$.

Idea. If $A \nvdash N$, we’re done. If $A \vdash N$, we consider a sentence $\theta$ such that

$$N \vdash \theta \iff \neg Thm_A(\overline{\theta}).$$

This forces $\mathfrak{N} \models \theta$ and $A \nvdash \theta$ (otherwise we get a contradiction).
Lemma 6.2.2 (Self-Reference Lemma). If $\beta(x)$ is an $\mathcal{L}_{NT}$-formula with only $x$ free, then there is a sentence $\theta$ such that $\mathcal{N} \vdash \theta \leftrightarrow \beta(\overline{\theta^n})$.

Theorem 6.3.6 (Gödel’s First Incompleteness Theorem, 1931). Suppose that $A$ is a consistent and recursive set of axioms in the language $\mathcal{L}_{NT}$. Then there is a sentence $\theta$ such that $\mathcal{N} \models \theta$ but $A \nvdash \theta$.

Idea. If $A \nvdash \mathcal{N}$, we’re done. If $A \vdash \mathcal{N}$, we consider a sentence $\theta$ such that

$$\mathcal{N} \vdash \theta \leftrightarrow \neg \text{Thm}_A(\overline{\theta^n}).$$

This forces $\mathcal{N} \models \theta$ and $A \nvdash \theta$ (otherwise we get a contradiction).

Theorem 6.3.10 (Tarski’s Undefinability Theorem, 1936). The set $\{\overline{\varphi^n} : \mathcal{N} \models \varphi\}$ of Gödel numbers of formulas true in $\mathcal{N}$ is not definable.

Idea. Toward a contradiction, assume $\beta(x)$ defines $\{\overline{\varphi^n} : \mathcal{N} \models \varphi\}$. Consider the sentence $\theta$ such that $\mathcal{N} \vdash \theta \leftrightarrow \neg \beta(\overline{\theta^n})$. Contradiction is immediate, as

$$\mathcal{N} \models \theta \iff \mathcal{N} \models \neg \beta(\overline{\theta^n}) \iff \mathcal{N} \not\models \theta.$$
Theorem 6.4.5 (Rosser’s Theorem). If $A$ is a set of $\mathcal{L}_{NT}$-axioms that is recursive, consistent, and extends $N$, then $A$ is incomplete.

Idea. Consider a sentence $\theta$ such that

$$
N \vdash \theta \iff (\forall x) \left[ \text{Deduction}_A(x, \text{⌜}\theta\text{⌝}) \rightarrow (\exists y < x) \text{Deduction}_A(y, \text{⌜}\neg\theta\text{⌝}) \right].$

Can show that $A \not\vdash \theta$ and $A \not\vdash \neg\theta$. (A contradiction ensues if we assume $A \vdash \theta$ or $A \vdash \neg\theta$.)
Theorem 6.4.5 (Rosser’s Theorem). If $A$ is a set of $\mathcal{L}_{NT}$-axioms that is recursive, consistent, and extends $N$, then $A$ is incomplete.

Idea. Consider a sentence $\theta$ such that

$$N \vdash \theta \iff (\forall x)[Deduction_A(x, \overline{\theta}) \rightarrow (\exists y < x)\ Deduction_A(y, \overline{\neg \theta})].$$

Can show that $A \not\vdash \theta$ and $A \not\vdash \neg \theta$. (A contradiction ensues if we assume $A \vdash \theta$ or $A \vdash \neg \theta$.)

Obs. Either $\mathfrak{M} \models \theta$ or $\mathfrak{M} \models \neg \theta$. In either case, we get a sentence which is true in $\mathfrak{M}$ and not provable from $A$. Thus, Rosser’s Theorem $\Rightarrow$ 1st Incompleteness Theorem.
**Peano Arithmetic**

**Definition.** The axioms of *Peano Arithmetic* (1889), denoted $PA$, are the eleven axioms of Robinson arithmetic together with axioms

$$
Induction_\varphi \equiv \left[ \varphi(0) \land (\forall x)[\varphi(x) \rightarrow \varphi(Sx)] \right] \rightarrow (\forall x)\varphi(x)
$$

for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.
**Peano Arithmetic**

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for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.

- Clearly, $\mathfrak{M} \models PA$ (since $\mathfrak{M} \models \text{Induction}_\varphi$ for each $\varphi(x)$). Therefore, $PA$ is consistent.
**Peano Arithmetic**

**Definition.** The axioms of *Peano Arithmetic* (1889), denoted $PA$, are the eleven axioms of Robinson arithmetic together with axioms

$$Induction_\varphi : \equiv \left[ \varphi(0) \land (\forall x)[\varphi(x) \rightarrow \varphi(Sx)] \right] \rightarrow (\forall x)\varphi(x)$$

for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.

- Clearly, $\mathfrak{N} \models PA$ (since $\mathfrak{N} \models Induction_\varphi$ for each $\varphi(x)$). Therefore, $PA$ is consistent.

- $PA$ is easily seen to be recursive: there is a simple algorithm to decide membership in $\{ \{\alpha\} : \alpha \in PA \}$. By 1st Incompleteness Theorem, there exists a sentence $\theta$ such that $\mathfrak{N} \models \theta$ but $PA \not\vdash \theta$. (In particular, $PA$ is not complete.)
Peano Arithmetic

Definition. The axioms of Peano Arithmetic (1889), denoted $PA$, are the eleven axioms of Robinson arithmetic together with axioms

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for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.

- Clearly, $\mathbb{N} \models PA$ (since $\mathbb{N} \models \text{Induction}_\varphi$ for each $\varphi(x)$). Therefore, $PA$ is consistent.

- $PA$ is easily seen to be recursive: there is a simple algorithm to decide membership in $\{\ulcorner \alpha \urcorner : \alpha \in PA\}$. By 1st Incompleteness Theorem, there exists a sentence $\theta$ such that $\mathbb{N} \models \theta$ but $PA \not\vdash \theta$. (In particular, $PA$ is not complete.)

- Whereas Robinson arithmetic $N$ is very weak (it doesn’t prove $(\forall x)(\forall y)(x + y = y + x)$), Peano arithmetic $PA$ is quite powerful – it proves any result you have seen in MAT315. (It is even claimed that $PA \vdash$ Fermat’s Last Theorem.)
2ND INCOMPLETENESS THEOREM

The sentence $\text{Con}_A$:

Let $A$ be a recursive set of $\mathcal{L}_{NT}$-sentences.

Recall that the set $\text{THM}_A := \{ \ulcorner \varphi \urcorner : A \vdash \varphi \}$ is $\Sigma$-definable. Fix a $\Sigma$-formula $\text{Thm}_A(x)$ which defines $\text{THM}_A$.

Let $\text{Con}_A$ be the sentence

$$\text{Con}_A \equiv \neg \text{Thm}_A(\ulcorner \bot \urcorner).$$

This sentence expresses “$A$ is consistent”: note that $A$ is consistent if, and only if, $\mathfrak{M} \models \text{Con}_A$. 
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \nvdash Con_A$. 
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $\text{PA}$, then $A \nvdash \text{Con}_A$.

- $\text{PA}$ itself is consistent and recursive. Therefore, $\text{PA} \nvdash \text{Con}_{\text{PA}}$. 
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \not\models \text{Con}_A$.

- $PA$ itself is consistent and recursive. Therefore, $PA \not\models \text{Con}_PA$.

- How do you and I know that $PA$ is consistent? We can prove $\mathfrak{M}$ is a model of $\text{Con}_PA$ using the usual axioms of $ZFC$ (Zermelo-Frankl set theory with choice). Therefore, $ZFC \models \text{Con}_PA$ (interpreting the sentence $\text{Con}_PA$ in the language of set theory).

   However, $ZFC \not\models \text{Con}_{ZFC}$. 

**2nd Incompleteness Theorem**

**Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)**

*If* $A$ *is any consistent, recursive set of* $\mathcal{L}_{NT}$*-sentences which extends* $PA$, *then* $A \nvdash Con_A$.

- $PA$ itself is consistent and recursive. Therefore, $PA \vdash Con_{PA}$.

- How do you and I know that $PA$ is consistent? We can prove $\mathcal{M}$ is a model of $Con_{PA}$ using the usual axioms of $ZFC$ (Zermelo-Frankl set theory with choice). Therefore, $ZFC \vdash Con_{PA}$ (interpreting the sentence $Con_{PA}$ in the language of set theory).

However, $ZFC \nvdash Con_{ZFC}$.

- 2nd Incompleteness Theorem answered a question asked by David Hilbert in 1900 by showing that no “sufficiently powerful formal system” (including set theory $ZFC$) can prove its own consistency.
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \nvdash \text{Con}_A$.

- Alternative phrasing of 2nd Incompleteness Theorem: If $A$ is recursive extension of $PA$, then $A$ is consistent $\iff A \nvdash \text{Con}_A$.

(If $A$ is inconsistent, then $A \vdash \text{Con}_A$ since $A$ proves everything.)
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
If $A$ is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends $PA$, then $A \not\vdash Con_A$.

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  (If $A$ is inconsistent, then $A \vdash Con_A$ since $A$ proves everything.)

- Since $PA \not\vdash Con_{PA}$, it follows that $PA \cup \{\neg Con_{PA}\}$ is consistent. (Added in slides: This is because, if we assume that $PA \cup \{\neg Con_{PA}\} \vdash \bot$, then $PA \vdash \neg Con_{PA} \rightarrow \bot$ by the Deduction Theorem; it would then follow that $PA \vdash Con_{PA}$ by the (PC) rule, but this contradicts the fact that $PA \not\vdash Con_{PA}$.) Therefore, there exists a model $\mathcal{M}$ of $PA \cup \{\neg Con_{PA}\}$. (This model looks a lot like $\mathcal{N}$ — for example, addition $+^\mathcal{M}$ is commutative. However, $Th(\mathcal{M}) \neq Th(\mathcal{N})$.)
2nd Incompleteness Theorem

Theorem 6.6.3 (Godel’s 2nd Incompleteness Theorem)
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- Since $PA \not\vdash Con_{PA}$, it follows that $PA \cup \{\neg Con_{PA}\}$ is consistent. Therefore, there exists a model $\mathfrak{M}$ of $PA \cup \{\neg Con_{PA}\}$. (This model looks a lot like $\mathfrak{N}$ — for example, addition $+_{\mathfrak{M}}$ is commutative. However, $Th(\mathfrak{M}) \neq Th(\mathfrak{N})$.)

- Since $PA$ is consistent, why not take $Con_{PA}$ as an additional axiom? Let $PA' := PA \cup \{Con_{PA}\}$. Then $PA' \vdash Con_{PA}$, but $PA' \not\vdash Con_{PA'}$. So we are left with the same problem.
**Hilbert-Bernays Derivability Conditions**

**Lemma.** $PA$ satisfies the following “derivability conditions” for all formulas $\alpha$ and $\beta$:

(D1) If $PA \vdash \alpha$, then $PA \vdash Thm_{PA}(\overline{\alpha})$.

If $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$”.
**Hilbert-Bernays Derivability Conditions**

**Lemma.** \(PA\) satisfies the following “derivability conditions” for all formulas \(\alpha\) and \(\beta\):

(D1) If \(PA \vdash \alpha\), then \(PA \vdash Thm_{PA}(\overline{\overline{\alpha}})\).

If \(PA\) proves \(\alpha\), then \(PA\) proves “\(PA\) proves \(\alpha\)”.

(D2) \(PA \vdash Thm_{PA}(\overline{\overline{\alpha}}) \rightarrow Thm_{PA}(\overline{\overline{Thm_{PA}(\overline{\overline{\alpha}})}})\).

\(PA\) proves “if \(PA\) proves \(\alpha\), then \(PA\) proves “\(PA\) proves \(\alpha\)”.”
Hilbert-Bernays Derivability Conditions

Lemma. $PA$ satisfies the following “derivability conditions” for all formulas $\alpha$ and $\beta$:

(D1) If $PA \vdash \alpha$, then $PA \vdash Thm_{PA}(\overline{\alpha})$.

If $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha$”.

(D2) $PA \vdash Thm_{PA}(\overline{\alpha}) \rightarrow Thm_{PA}(\overline{\text{Thm}_{PA}(\overline{\alpha})})$.

$PA$ proves “if $PA$ proves $\alpha$, then $PA$ proves “$PA$ proves $\alpha””.

(D3) $PA \vdash [Thm_{PA}(\overline{\alpha}) \land Thm_{PA}(\overline{\alpha \rightarrow \beta})] \rightarrow Thm_{PA}(\overline{\beta})$.

$PA$ proves “if $PA$ proves $\alpha$ and $PA$ proves $\alpha \rightarrow \beta$, then $PA$ proves $\beta$”.
**Hilbert-Bernays Derivability Conditions**

**Lemma.** *PA* satisfies the following “derivability conditions” for all formulas \( \alpha \) and \( \beta \):

1. (D1) If \( PA \vdash \alpha \), then \( PA \vdash \text{Thm}_{PA}(\alpha) \).

   If \( PA \) proves \( \alpha \), then \( PA \) proves “\( PA \) proves \( \alpha \)”.

2. (D2) \( \vdash \text{Thm}_{PA}(\alpha) \rightarrow \text{Thm}_{PA}(\text{Thm}_{PA}(\alpha)) \).

   \( PA \) proves “if \( PA \) proves \( \alpha \), then \( PA \) proves “\( PA \) proves \( \alpha \)””.

3. (D3) \( \vdash \text{Thm}_{PA}(\alpha) \wedge \text{Thm}_{PA}(\alpha \rightarrow \beta) \rightarrow \text{Thm}_{PA}(\beta) \).

   \( PA \) proves “if \( PA \) proves \( \alpha \) and \( PA \) proves \( \alpha \rightarrow \beta \), then \( PA \) proves \( \beta \)”.

Moreover, if \( A \) is a recursive extension of \( PA \), then \( A \) satisfies derivability conditions (D1)–(D3) with respect to \( \text{Thm}_A(x) \).
Proof of 2nd Incompleteness Theorem. Let $A$ be a consistent, recursive extension of $PA$. Let $\theta$ be a sentence such that

\[
(*) \quad N \models \theta \leftrightarrow \neg \text{Thm}_A(\overline{\overline{\theta}}).
\]

By proof of 1st Incompleteness Theorem, we know that $A \nvdash \theta$.

CLAIM: $A \vdash Con_A \rightarrow \theta$. (It follows that $A \nvdash Con_A$.)
Proof of 2nd Incompleteness Theorem. Let \( A \) be a consistent, recursive extension of \( PA \). Let \( \theta \) be a sentence such that

\[
(*) \quad N \vdash \theta \iff \neg \text{Thm}_A(\overline{\overline{\theta}}).
\]

By proof of 1st Incompleteness Theorem, we know that \( A \not\vdash \theta \).

CLAIM: \( A \vdash \text{Con}_A \rightarrow \theta \). (It follows that \( A \not\vdash \text{Con}_A \).)

PROOF OF CLAIM: By (\( * \)), we have \( A \vdash \text{Thm}_A(\overline{\overline{\theta}}) \rightarrow \neg \theta \).

(D1): \( A \vdash \text{Thm}_A(\overline{\overline{\text{Thm}_A(\overline{\overline{\theta}})}} \rightarrow \neg \theta) \)

(D2): \( A \vdash \text{Thm}_A(\overline{\overline{\theta}}) \rightarrow \text{Thm}_A(\overline{\overline{\text{Thm}_A(\overline{\overline{\theta}})}}) \).

(D3): \( A \vdash (\overline{\overline{\text{Thm}_A(\overline{\overline{\text{Thm}_A(\overline{\overline{\theta}})}})}} \land \overline{\overline{\text{Thm}_A(\overline{\overline{\text{Thm}_A(\overline{\overline{\theta}})}} \rightarrow \neg \theta)}}) \rightarrow \text{Thm}_A(\overline{\overline{\neg \theta}}) \).

By (PC) rule, \( A \vdash \text{Thm}_A(\overline{\overline{\theta}}) \rightarrow \text{Thm}_A(\overline{\overline{\neg \theta}}) \).

Next step: \( A \vdash \text{Thm}_A(\overline{\overline{\theta}}) \rightarrow \text{Thm}_A(\overline{\overline{\bot}}) \), hence \( A \vdash \text{Con}_A \rightarrow \neg \text{Thm}_A(\overline{\overline{\theta}}) \).

By (\( * \)) and (PC) rule: \( A \vdash \text{Con}_A \rightarrow \theta \). Q.E.D.
Complete, Consistent, Recursive Theories

The 1st Incompleteness Theorem implies that $Th(\mathcal{M})$ has no complete, consistent, recursive axiomatization.

The same is true of any theory that “interprets” $Th(\mathcal{M})$, such as models of ZFC (set theory).
**Complete, Consistent, Recursive Theories**

The 1st Incompleteness Theorem implies that $Th(\mathbb{N})$ has no complete, consistent, recursive axiomatization.

The same is true of any theory that “interprets” $Th(\mathbb{N})$, such as models of $ZFC$ (set theory).

In contrast, there are beautiful examples of complete, consistent, recursive theories:

- $Th(\mathbb{N}, 0, 1, +)$ (Presburger Arithmetic)
- $Th(\mathbb{R}, 0, 1, +, \cdot, <)$ (the theory of real closed fields)
- Euclidean Geometry: the theory $Th(\mathbb{R}^2, \text{Between}, \text{Congruent})$ where

  \[
  \text{Between} := \{(a, b, c) \in (\mathbb{R}^2)^3 : b \in ac\}, \\
  \text{Congruent} := \{(a, b, c, d) \in (\mathbb{R}^2)^4 : |ab| = |cd|\}.
  \]

Here $ab$ denotes the line segment between points $a, b \in \mathbb{R}^2$, and $|ab|$ is the length of $ab$. (See Tarski’s axioms.)