Next Steps (Section 5.8)

Δ-definability of sets

\[
\text{Terms} := \{ \llbracket t \rrbracket : \text{terms } t \} = \{ a \in \mathbb{N} : a = \llbracket t \rrbracket \text{ for some term } t \},
\]
\[
\text{Formulas} := \{ \llbracket \varphi \rrbracket : \text{formulas } \varphi \} = \{ a \in \mathbb{N} : a = \llbracket \varphi \rrbracket \text{ for some formula } \varphi \}.
\]
**Δ-Definition of Terms** = \{\lceil t \rceil : t \text{ is a term}\}  

<table>
<thead>
<tr>
<th>\lceil \neg \alpha \rceil = \langle 1, \lceil \alpha \rceil \rangle</th>
<th>\lceil t_1 t_2 \rceil = \langle 7, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle</th>
<th>\lceil + t_1 t_2 \rceil = \langle 13, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle</th>
<th>\lceil &lt; t_1 t_2 \rceil = \langle 19, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>\lceil \alpha \lor \beta \rceil = \langle 3, \lceil \alpha \rceil, \lceil \beta \rceil \rangle</td>
<td>\lceil 0 \rceil = \langle 9 \rangle</td>
<td>\lceil \cdot t_1 t_2 \rceil = \langle 15, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle</td>
<td>\lceil v_i \rceil = \langle 2i \rangle</td>
</tr>
<tr>
<td>\lceil \forall v_i (\alpha) \rceil = \langle 5, \lceil v_i \rceil, \lceil \alpha \rceil \rangle</td>
<td>\lceil St \rceil = \langle 11, \lceil t \rceil \rangle</td>
<td>\lceil Et_1 t_2 \rceil = \langle 17, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle</td>
<td></td>
</tr>
</tbody>
</table>
**Δ-Definition of Terms** = \{\[t\]: \(t\) is a term\}

\[
\begin{align*}
\text{⌜¬\(α\)⌟} &= \langle 1, \text{⌜α⌟} \rangle \\
\text{⌜=\(t_1t_2\)⌟} &= \langle 7, \text{⌜t_1⌟, ⌜t_2⌟} \rangle \\
\text{⌜+\(t_1t_2\)⌟} &= \langle 13, \text{⌜t_1⌟, ⌜t_2⌟} \rangle \\
\text{⌜<\(t_1t_2\)⌟} &= \langle 19, \text{⌜t_1⌟, ⌜t_2⌟} \rangle \\
\text{⌜(α ∨ β)⌟} &= \langle 3, \text{⌜α⌟, ⌜β⌟} \rangle \\
\text{⌜0⌟} &= \langle 9 \rangle \\
\text{⌜·\(t_1t_2\)⌟} &= \langle 15, \text{⌜t_1⌟, ⌜t_2⌟} \rangle \\
\text{⌜v_i⌟} &= \langle 2i \rangle \\
\text{⌜(∀v_i)(α)⌟} &= \langle 5, \text{⌜v_i⌟, ⌜α⌟} \rangle \\
\text{⌜St⌟} &= \langle 11, \text{⌜t⌟} \rangle \\
\text{⌜Et_1t_2⌟} &= \langle 17, \text{⌜t_1⌟, ⌜t_2⌟} \rangle
\end{align*}
\]

Recall the inductive definition of an \(\mathcal{L}_{NT}\)-term \(t\): it is either

- a variable symbol \(v_i\),
- \(St_1\) where \(t_1\) is term,
- the constant symbol 0,
- \(+t_1t_2\) or \(·t_1t_2\) or \(Et_1t_2\) where \(t_1, t_2\) are terms.
△-Definition of Terms = \{\lceil t \rceil : \text{t is a term}\}

\begin{array}{|l|}
\hline
\lceil \neg \alpha \rceil = \langle 1, \lceil \alpha \rceil \rangle & \lceil t_1 \circ t_2 \rceil = \langle 7, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle & \lceil +t_1 t_2 \rceil = \langle 13, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle & \lceil \prec t_1 t_2 \rceil = \langle 19, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\lceil (\alpha \lor \beta) \rceil = \langle 3, \lceil \alpha \rceil, \lceil \beta \rceil \rangle & \lceil 0 \rceil = \langle 9 \rangle & \lceil \cdot t_1 t_2 \rceil = \langle 15, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle & \lceil v_i \rceil = \langle 2i \rangle \\
\lceil (\forall v_i)(\alpha) \rceil = \langle 5, \lceil v_i \rceil, \lceil \alpha \rceil \rangle & \lceil St \rceil = \langle 11, \lceil t^\rceil \rangle & \lceil Et_1 t_2 \rceil = \langle 17, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\hline
\end{array}

Recall the inductive definition of an \(L_{NT}\)-term \(t\): it is either

- a variable symbol \(v_i\),
- \(St_1\) where \(t_1\) is term,
- the constant symbol \(0\),
- \(+t_1 t_2\) or \(\cdot t_1 t_2\) or \(Et_1 t_2\) where \(t_1, t_2\) are terms.

Let’s start with \(\Delta\)-definition of

\[\text{Variables} := \{\lceil v_i \rceil : i = 1, 2, \ldots\}\]  

(= \{2^{2i+1} : i = 1, 2, \ldots\}).

by the formula

\[\text{Variable}(x) \equiv (\exists y < x)[\text{Even}(y) \land (0 < y) \land (x = 2^{Sy})].\]
\(\Delta\)-Definition of Terms = \(\{ \ulcorner t \urcorner : t \text{ is a term} \}\)

\[
\begin{aligned}
\ulcorner \lnot \alpha \urcorner &= \langle 1, \ulcorner \alpha \urcorner \rangle & \ulcorner =t_1t_2 \urcorner &= \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \ulcorner +t_1t_2 \urcorner &= \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \ulcorner <t_1t_2 \urcorner &= \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner (\alpha \lor \beta) \urcorner &= \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle & \ulcorner 0 \urcorner &= \langle 9 \rangle & \ulcorner \cdot t_1t_2 \urcorner &= \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \ulcorner v_i \urcorner &= \langle 2i \rangle \\
\ulcorner (\forall v_i)(\alpha) \urcorner &= \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle & \ulcorner St \urcorner &= \langle 11, \ulcorner t \urcorner \rangle & \ulcorner Et_1t_2 \urcorner &= \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle
\end{aligned}
\]

Recall the inductive definition of an \(L_{NT}\)-term \(t\): it is either

- a variable symbol \(v_i\),
- \(St_1\) where \(t_1\) is term,
- the constant symbol 0,
- \(+t_1t_2\) or \(\cdot t_1t_2\) or \(Et_1t_2\) where \(t_1, t_2\) are terms.

We would like to write:

\[
Term(x) \equiv Variable(x) \lor x = 2^{10} \lor (\exists y < x)[Term(y) \land x = \overset{2^{12} \cdot 3^{Sy}}{< 11, y>}] \lor \ldots \lor \text{“}x \text{ is } +t_1t_2 \text{ or } \cdot t_1t_2 \text{ or } Et_1t_2\text{”}
\]

However, there is a problem with this “\(\Delta\)-formula”.
Δ-Definition of Terms = \{ \mathcal{T}^- : t \text{ is a term} \}

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>\neg \alpha^- = \langle 1, \alpha^- \rangle</td>
<td>\alpha^-</td>
</tr>
<tr>
<td>\alpha^- t_1 t_2^- = \langle 7, \alpha^- t_1^- t_2^- \rangle</td>
<td>\alpha^- t_1^- t_2^-</td>
</tr>
<tr>
<td>\alpha^- + t_1 t_2^- = \langle 13, \alpha^- t_1^- t_2^- \rangle</td>
<td>\alpha^- + t_1 t_2^-</td>
</tr>
<tr>
<td>\alpha^- \times t_1 t_2^- = \langle 19, \alpha^- t_1^- t_2^- \rangle</td>
<td>\alpha^- \times t_1 t_2^-</td>
</tr>
<tr>
<td>\neg \alpha^- = \langle 9 \rangle</td>
<td>\neg \alpha^-</td>
</tr>
<tr>
<td>\alpha^- 0^- = \langle 15, \alpha^- t_1^- t_2^- \rangle</td>
<td>\alpha^- 0^-</td>
</tr>
<tr>
<td>\alpha^- v_i^- = \langle 2i \rangle</td>
<td>\alpha^- v_i^-</td>
</tr>
<tr>
<td>\alpha^- \varpi t_1 t_2^- = \langle 17, \alpha^- t_1^- t_2^- \rangle</td>
<td>\alpha^- \varpi t_1 t_2^-</td>
</tr>
<tr>
<td>\alpha^- \eta t^- = \langle 3, \alpha^- \eta \rangle</td>
<td>\alpha^- \eta t^-</td>
</tr>
<tr>
<td>\alpha^- \alpha^- t^- = \langle 5, \alpha^- \eta \rangle</td>
<td>\alpha^- \alpha^- t^-</td>
</tr>
<tr>
<td>\alpha^- \xi t^- = \langle 11, \alpha^- \eta \rangle</td>
<td>\alpha^- \xi t^-</td>
</tr>
<tr>
<td>\alpha^- \omega t^- = \langle 12, \alpha^- \eta \rangle</td>
<td>\alpha^- \omega t^-</td>
</tr>
<tr>
<td>\alpha^- \chi t^- = \langle 13, \alpha^- \eta \rangle</td>
<td>\alpha^- \chi t^-</td>
</tr>
<tr>
<td>\alpha^- \delta t^- = \langle 17, \alpha^- \eta \rangle</td>
<td>\alpha^- \delta t^-</td>
</tr>
</tbody>
</table>

Recall the inductive definition of an \textbf{L}_{NT}-term \textit{t}: it is either

- \textbullet a variable symbol \textit{v}_i,
- \textbullet \textit{St}_1 \text{ where } \textit{t}_1 \text{ is term},
- \textbullet the constant symbol 0,
- \textbullet \alpha^- + \textit{t}_1 \textit{t}_2 \text{ or } \alpha^- \times \textit{t}_1 \textit{t}_2 \text{ or } \alpha^- \varpi \textit{t}_1 \textit{t}_2 \text{ where } \textit{t}_1, \textit{t}_2 \text{ are terms.}

We would like to write:

\[
\text{Term}(x) \equiv \text{Variable}(x) \lor x = 2^{10} \lor (\exists y < x)[\text{Term}(y) \land x = 2^{12} \cdot 3^y] \\
\lor \ldots \\
\lor \text{"x is } + t_1 t_2 \text{ or } \times t_1 t_2 \text{ or } \varpi t_1 t_2"
\]

This is a not legitimate formula of first-order logic! Note the circular use of the subformula Term(y).
\textbf{Definition of Terms} = \{ \text{"t"} : t \text{ is a term} \}

\textbf{Definition.} A \textit{term construction sequence} for a term \( t \) is a finite sequence of terms \( (t_1, \ldots, t_\ell) \) such that \( t_\ell \equiv t \) and, for each \( k \in \{1, \ldots, \ell\} \), the term \( t_k \) is either

- a variable symbol,
- the constant symbol 0,
- \( S_{t_j} \) for some \( j < k \), or
- \(+t_i t_j \) or \( \cdot t_i t_j \) or \( E_{t_i t_j} \) for some \( i, j < k \).
\textbf{Definition of Terms} = \{\texttt{''}t\texttt{''} : t \text{ is a term}\}

\textbf{Definition.} A \textit{term construction sequence} for a term $t$ is a finite sequence of terms $(t_1, \ldots, t_{\ell})$ such that $t_{\ell} \equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term $t_k$ is either

- a variable symbol,
- the constant symbol 0,
- $St_j$ for some $j < k$, or
- $+t_i t_j$ or $t_i t_j$ or $Et_i t_j$ for some $i, j < k$.

\textbf{Example.} $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$. 
**Definition.** A *term construction sequence* for a term $t$ is a finite sequence of terms $(t_1, \ldots, t_\ell)$ such that $t_\ell \equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term $t_k$ is either

- a variable symbol,
- the constant symbol 0,
- $St_j$ for some $j < k$, or
- $+t_it_j$ or $\cdot t_it_j$ or $Et_it_j$ for some $i, j < k$.

**Example.** $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$.

**Lemma.** Every term $t$ has a term construction sequence of length at most the number of symbols in $t$.

(Easy proof by induction.)
**Δ-Definition of Terms** = \{ \overline{t} : t \text{ is a term} \}

**Definition.** A term construction sequence for a term \( t \) is a finite sequence of terms \( (t_1, \ldots, t_\ell) \) such that \( t_\ell :\equiv t \) and, for each \( k \in \{1, \ldots, \ell\} \), the term \( t_k \) is either

- a variable symbol,
- the constant symbol \( 0 \),
- \( St_j \) for some \( j < k \), or
- \( +t_it_j \) or \( \cdot t_it_j \) or \( Et_it_j \) for some \( i, j < k \).

**Key to defining Terms:** We will write a \( Δ \)-formula defining the set

\[
\text{TermConSeq} = \{(c, a) : c = \langle \overline{t_1}, \ldots, \overline{t_\ell} \rangle \text{ and } a = \overline{t_\ell} \text{ where } (t_1, \ldots, t_\ell) \text{ is a term construction sequence}\}.
\]
**Δ-Definition of Terms** = \{⌜t⌝ : t is a term\}

\[\text{TermConSeq}(c,a) \equiv \]
\[\text{Codenumber}(c) \land (\exists \ell < c) \left[ \text{Length}(c,\ell) \land \text{IthElement}(a,\ell,c) \land \right.\]
\[(\forall k \leq \ell)(\exists e_k < c) \left[ \text{IthElement}(e_k,k,c) \land \right.\]
\[\left( \text{Variable}(e_k) \lor e_k = 2^{10} \right) \]
\[\lor (\exists j < k)(\exists e_j < c)[\text{IthElement}(e_j,j,c) \land e_k = 2^{12} \cdot 3^{Se_j}] \]
\[\lor \ldots \]

**Key to defining Terms:** We will write a Δ-formula defining the set

\[\text{TermConSeq} = \{(c,a) : c = \langle \text{⌜}t_1\text{⌝}, \ldots, \text{⌜}t_\ell\text{⌝}\rangle \text{ and } a = \text{⌜}t_\ell\text{⌝}\text{ where } \langle t_1,\ldots,t_\ell\rangle\text{ is a term construction sequence}\}\].
\Delta\text{-Definition of Terms} = \{ t \downarrow : t \text{ is a term} \}

Now there is an obvious way to define Term(a):

\[
\text{Term}(a) \equiv (\exists c) \text{TermConSeq}(c, a).
\]

To make this a \(\Delta\)-formula, we need an upper bound on \(c\) as a function of \(a\).
**Δ-Definition of Terms** = \{\text{⟦} t \text{⟧} : t \text{ is a term} \}

Now there is an obvious way to define \( \text{Term}(a) \):

\[
\text{Term}(a) \equiv (\exists c) \text{TermConSeq}(c, a).
\]

To make this a Δ-formula, we need an upper bound on \( c \) as a function of \( a \).

Suppose \( a = \text{⟦} t \text{⟧} \). Another easy lemma by induction: The number of symbols in \( t \) is at most \( a \). Therefore, there exists a term construction sequence \((t_1, \ldots, t_\ell)\) for \( t \) with length \( \leq a \). We may assume that each \( t_k \) is a subterm of \( t \), so that \( \text{⟦} t_k \text{⟧} \leq \text{⟦} t \text{⟧} = a \) for all \( k \in \{1, \ldots, \ell\} \).

Let \( c := \langle \text{⟦} t_1 \text{⟧}, \ldots, \text{⟦} t_\ell \text{⟧} \rangle \). We have

\[
c = 2^{t_1+1}3^{t_2+1} \cdots (p_\ell)^{t_\ell+1} \leq (p_\ell)^{t_1+\cdots+t_\ell+\ell} \leq (p_\ell)^{la+\ell} \leq (p_a)^{a^2+a}.
\]

Easy fact: The \( a \)th prime number \( p_a \) is at most \( a^a \). (In fact, \( p_a \leq 2a^2 \) using the Prime Number Theorem: \( a(\log a + \log \log a - 1) < p_a < a(\log a + \log \log a) \) for all \( a \geq 6 \).) We conclude that \( c \leq a^{a(a^2+a)} \leq a^{2a^3} \).
\textbf{\textit{\Delta}-Definition of Terms} = \{"t" : t is a term\}

Now there is an obvious way to define Term(a):

\[\text{Term}(a) \equiv (\exists c) \text{TermConSeq}(c, a).\]

To make this a \textit{\Delta}-formula, we need an upper bound on \(c\) as a function of \(a\). We may therefore take

\[\text{Term}(a) \equiv (\exists c \leq E_a \cdot SS_0 \cdot EaSSS_0) \text{TermConSeq}(c, a).\]
**Construction Sequences for Other Recursive Definitions**

In a similar way, using the notion of a *formula construction sequence*, we get a $\Delta$-definition of the set

\[
\text{FORMULAS} = \{ \lbrack \varphi \rbrack : \varphi \text{ is a formula} \}.
\]

**Definition.** A *formula construction sequence* for a formula $\varphi$ is a finite sequence of terms $(\varphi_1, \ldots, \varphi_\ell)$ such that $\varphi_\ell :\equiv \varphi$ and, for each $k \in \{1, \ldots, \ell\}$, the term $\varphi_k$ is either

- $= t_1 t_2$ for some terms $t_1$ and $t_2$
- $< t_1 t_2$ for some terms $t_1$ and $t_2$
- $\neg \varphi_j$ for some $j < k$
- $(\varphi_i \lor \varphi_j)$ for some $i, j < k$
- $(\forall x)(\varphi_i)$ for some $i < k$ and $x \in \text{Vars}$
This idea is very general: using an appropriate notion of construction sequence, we get a $\Delta$-definition of any recursively defined set or function.

For example, recall the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

defined by $f(1) = f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$. We can define the function $f(n)$ in terms of the set of codes of construction sequences

$$\text{FIBONACCI CONSEQ} = \{ \langle f(1), f(2), \ldots, f(n) \rangle : n = 1, 2, \ldots \}.$$
Next Steps (Sections 5.11–5.12)

The following are $\Delta$-definable:

**LogicalAxiom** := \{ $\Gamma \varphi \vdash : \varphi$ is a logical axiom \}

**RuleOfInference** := \{ $(\langle \Gamma \gamma_1 \vdash, \ldots, \Gamma \gamma_n \vdash \rangle, \Gamma \varphi \vdash) : (\{\gamma_1, \ldots, \gamma_n\}, \varphi)$ is a rule of inference \}

**Axiom**$_N$ := \{ $\Gamma N_1 \vdash, \ldots, \Gamma N_{11} \vdash$ \}

**Deduction**$_N$ := \{ $(\langle \Gamma \delta_1 \vdash, \ldots, \Gamma \delta_1 \vdash \rangle, \Gamma \varphi \vdash) : (\delta_1, \ldots, \delta_n)$ is a deduction from $N$ of $\varphi$ \}.

Important $\Delta$-definable functions:

$\text{Num}(a) := \Gamma a \vdash,$

$\text{TermSub}(\Gamma u \vdash, \Gamma x \vdash, \Gamma t \vdash) := \Gamma u^x \vdash,$

$\text{Sub}(\Gamma \varphi \vdash, \Gamma x \vdash, \Gamma t \vdash) := \Gamma \varphi^x \vdash.$