CSC2429, MAT1304: Circuit Complexity December 6, 2016

Lecture 12: Pathset Complexity, Continued

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Recall:

- We fix a connected graph G, a strict threshold weighting θ, and the parameter $\varepsilon := n^{-.99}$.
- For $A \in \text{Pattern}(\subseteq G)$, $\mathscr{P}_A$ denotes the set of G_A-pathsets (i.e. relations $\mathcal{A} \subseteq [n]^{V_A}$ that satisfy density constraints $\mu_S(\mathcal{A}) \leq \varepsilon^{\Delta(A \upharpoonright S)}$ for every G_A-closed $S \subseteq V_A$).
- *Pathset complexity* is a family of functions $\{\chi_A : \mathscr{P}_A \to \mathbb{N}\}_{A \in \text{Pattern}(\subseteq G)}$ defined by induction on patterns as follows:
  - for an atomic pattern A and pathset $\mathcal{A} \in \mathscr{P}_A$,
  $$\chi_A(\mathcal{A}) := \begin{cases} 0 & \text{if } \mathcal{A} = \emptyset, \\ 1 & \text{if } \mathcal{A} \neq \emptyset. \end{cases}$$
  - for a non-atomic pattern $C = \langle A, B \rangle$ and pathset $\mathcal{C} \in \mathscr{P}_C$,
  $$\chi_C(\mathcal{C}) := \underbrace{\min_{\{(\mathcal{A}_i, \mathcal{B}_i) \in \mathscr{P}_A \times \mathscr{P}_B\}_i : \mathcal{C} \subseteq \bigcup_i \mathcal{A}_i \bowtie \mathcal{B}_i}}_{\text{“join covering” of } \mathcal{C}} \sum_i \max\{\chi_A(\mathcal{A}_i),\, \chi_B(\mathcal{B}_i)\}.$$
- For $A, B \in \text{Pattern}(\subseteq G)$, recall the definition of $A \ominus B \in \text{Pattern}(G_A \ominus G_B)$. Also recall notation $C = \{A, B\}$, which means $C = \langle A, B \rangle$ or $\langle B, A \rangle$.
- Potential function $\Psi : \text{Pattern}(\subseteq G) \to \mathbb{R}_{\geq 0}$ is the unique pointwise minimal function that satisfies (in)equalities:
  - for every atomic pattern A,
  $$\Psi(A) = \Delta(A),$$
  - for every non-atomic pattern $C = \{A, B\}$ and sub-pattern $A' \preceq A$,
  $$(\dagger)^C_{A'} \qquad \Psi(C) \geq \Psi(A') + \Delta(B \ominus A') + \Delta(C \ominus \{A', B\}),$$
  - for every non-atomic pattern $C = \{A, B\}$ and sub-patterns $A' \preceq A$ and $B' \preceq B$,
  $$(\ddagger)^C_{A',B'} \qquad \Psi(C) \geq \frac{1}{2}\Big(\Psi(A') + \Psi(B' \ominus A') + \Delta(C) + \Delta(C \ominus \{A', B'\})\Big).$$

  Recall that, for every $C = \{A, B\}$, at least one of the inequalities $(\dagger)^C_{A'}$, $(\dagger)^C_{B'}$, $(\ddagger)^C_{A',B'}$, $(\ddagger)^C_{B',A'}$ is tight for some $A' \preceq A$ and $B' \preceq B$.

Today we will complete Step 3 of Lecture 11 by proving the following:

Theorem 1. *(For all G, θ, ε) and for every $A \in \mathrm{Pattern}(\subseteq G)$ and $\mathcal{A} \in \mathscr{P}_A$,*

$$\chi_A(\mathcal{A}) \geq (1/\varepsilon)^{\Psi(A)} \cdot \mu(\mathcal{A}).$$

As explained in Lecture 11, this theorem completes the proof of the average-case $n^{\Omega(\log k)}$ lower bound on the AC^0 formula size of both SUB(Cycle$_k$) and SUB(Path$_k$) (equivalent to the problem DISTANCE-k CONNECTIVITY).

1 Extending χ_A to non-pathset relations $\mathcal{A} \subseteq [n]^{V_A}$

In order to prove Theorem 1, we require an induction hypothesis that speaks about arbitrary V_A-ary relations $\mathcal{A} \subseteq [n]^{V_A}$, not just G_A-pathsets $\mathcal{A} \in \mathscr{P}_A$. To that end, for each $A \in \mathrm{Pattern}(\subseteq G)$, we formally extend the domain of the pathset complexity function χ_A from $\mathscr{P}_A$ to the set of all V_A-ary relations. (This will allow us to state a few nice properties of pathset complexity.)

Definition 2. For $A \in \mathrm{Pattern}(\subseteq G)$ and a non-pathset relation $\mathcal{A} \subseteq [n]^{V_A}$ (such that $\mathcal{A} \notin \mathscr{P}_A$), we define $\chi_A(\mathcal{A})$ as follows:

$$\chi_A(\mathcal{A}) := \min_{\underbrace{\{\mathcal{A}_i \in \mathscr{P}_A\}_i \,:\, \mathcal{A} \subseteq \bigcup_i \mathcal{A}_i}_{\text{covering of } \mathcal{A} \text{ by pathsets}}} \sum_i \chi_A(\mathcal{A}_i).$$

Definition 3. For non-atomic $C = \langle A, B \rangle \in \mathrm{Pattern}(\subseteq G)$, let

$$\mathscr{P}^{\bowtie}_{A,B,C} := \big\{(\mathcal{A}, \mathcal{B}, \mathcal{C}) \in \mathscr{P}_A \times \mathscr{P}_B \times \mathscr{P}_C : \mathcal{C} \subseteq \mathcal{A} \bowtie \mathcal{B}\big\}.$$

The following lemma is established by a straightforward induction on patterns. (The proof is left as an easy exercise.)

Lemma 4. *For every non-atomic $C = \langle A, B \rangle \in \mathrm{Pattern}(\subseteq G)$ and relation $\mathcal{C} \subseteq [n]^{V_C}$, we have*

$$\chi_C(\mathcal{C}) = \min_{\{(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i) \in \mathscr{P}^{\bowtie}_{A,B,C}\}_i \,:\, \mathcal{C} \subseteq \bigcup_i \mathcal{C}_i} \sum_i \max\{\chi_A(\mathcal{A}_i),\, \chi_B(\mathcal{B}_i)\}.$$

From Lemma 4, it follows that $\chi_A(\cdot)$ satisfies the following inequalities (for all $A, B \in \mathrm{Pattern}(\subseteq G)$ and $C = \langle A, B \rangle$):

(**monotonicity**)	$\chi_A(\mathcal{A}') \leq \chi_A(\mathcal{A})$	for all $\mathcal{A}' \subseteq \mathcal{A} \subseteq [n]^{V_A}$,
(**sub-additivity**)	$\chi_A(\bigcup_i \mathcal{A}_i) \leq \sum_i \chi_A(\mathcal{A}_i)$	for all $\{\mathcal{A}_i \subseteq [n]^{V_A}\}_i$,
(**join inequality**)	$\chi_C(\mathcal{C}) \leq \max\{\chi_A(\mathcal{A}),\, \chi_B(\mathcal{B})\}$	for all $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \in \mathscr{P}^{\bowtie}_{A,B,C}$.

Remark 5. In fact, $\{\chi_A : \{V_A\text{-ary relations}\} \to \mathbb{N}\}_{A \in \mathrm{Pattern}(\subseteq G)}$ is the unique pointwise maximal family of functions satisfying these three inequalities, in addition to "base case" equations $\chi_A(\mathcal{A}) = 1$ for each atomic pattern A and non-empty pathset $\mathcal{A} \in \mathscr{P}_A$. This gives an alternative characterization of pathset complexity.

We next establish a few nice structural properties of this expanded version of pathset complexity. This properties (two different forms of monotonicity) give inequalities between $\chi_A(\cdot)$ and $\chi_{A'}(\cdot)$ for sub-patterns $A' \preceq A$, as well as between $\chi_A(\cdot)$ and $\chi_{A\restriction S}(\cdot)$ for G_A-closed sets S.

For $A \in \text{Pattern}(\subseteq G)$ and $\mathcal{A} \subseteq [n]^{V_A}$, it will be helpful to recall the definition of $\mu_S(\mathcal{A})$ and $\mathsf{proj}_S(\mathcal{A})$ and $\mathcal{A}|^z_S$ where $S \subseteq V_A$ and $z \in [n]^{V_A \setminus S}$ (consult the handout on relations on the course website). As a notational shorthand: if B is a pattern such that $V_B \subseteq V_A$, we will write $\mu_B(\mathcal{A})$ and $\mathsf{proj}_B(\mathcal{A})$ and $\mathcal{A}|^z_S$ instead of $\mu_{V_B}(\mathcal{A})$ and $\mathsf{proj}_{V_B}(\mathcal{A})$ and $\mathcal{A}|^z_{V_B}$.

2 χ is monotone under projections to sub-patterns

The next lemma gives a key property of pathset complexity.

Lemma 6 (**Projection Lemma**). *For every $A \in \text{Pattern}(\subseteq G)$ and relation $\mathcal{A} \subseteq [n]^{V_A}$ and sub-pattern $A' \preceq A$, we have*

$$\chi_{A'}(\mathsf{proj}_{A'}(\mathcal{A})) \leq \chi_A(\mathcal{A}).$$

(Remark: $\mathsf{proj}_{A'}(\mathcal{A})$ is not necessarily a $G_{A'}$-pathset, even if $\mathcal{A}$ is a G_A-pathset. Lemma 6 is essentially the reason why we extend the domain of pathset complexity to include all relations.)

Proof. The lemma is trivial when A' is empty or A is atomic. For the remaining case, it suffices to show that $\chi_A(\mathsf{proj}_A(\mathcal{C})) \leq \chi_C(\mathcal{C})$ for every non-atomic pattern $C = \langle A, B \rangle$ and relation $\mathcal{C} \subseteq [n]^{V_C}$. (By induction, it follows that $\chi_{C'}(\mathsf{proj}_{C'}(\mathcal{C})) \leq \chi_C(\mathcal{C})$ for all sub-patterns $C' \preceq C$.) By Lemma 4, there exists $\{(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i) \in \mathscr{P}^{\bowtie}_{A,B,C}\}_i$ such that $\mathcal{C} \subseteq \bigcup_i \mathcal{C}_i$ and $\chi_C(\mathcal{C}) = \sum_i \max\{\chi_A(\mathcal{A}_i), \chi_B(\mathcal{B}_i)\}$. For all i, since $\mathcal{C}_i \subseteq \mathcal{A}_i \bowtie \mathcal{B}_i$, it follows that $\mathsf{proj}_A(\mathcal{C}_i) \subseteq \mathcal{A}_i$. Therefore, $\mathsf{proj}_A(\mathcal{C}) \subseteq \bigcup_i \mathcal{A}_i$. We now have

$$\begin{aligned} \chi_A(\mathsf{proj}_A(\mathcal{C})) &\leq \chi_A\big(\textstyle\bigcup_i \mathcal{A}_i\big) && \text{(monotonicity)} \\ &\leq \textstyle\sum_i \chi_A(\mathcal{A}_i) && \text{(sub-additivity)} \\ &\leq \textstyle\sum_i \max\{\chi_A(\mathcal{A}_i), \chi_B(\mathcal{B}_i)\} \\ &= \chi_C(\mathcal{C}). \end{aligned}$$

□

3 χ is monotone under closed restrictions

Pathset complexity is monotone in a second respect.

Lemma 7 (**Restriction Lemma**). *For every $A \in \text{Pattern}(\subseteq G)$ and relation $\mathcal{A} \subseteq [n]^{V_A}$ and G_A-closed set S and tuple $z \in \Omega^{V_A \setminus S}$, we have*

$$\chi_{A \upharpoonright S}(\mathcal{A}|^z_S) \leq \chi_A(\mathcal{A}).$$

Proof. The lemma is trivial when A is atomic. For the induction step, consider a non-atomic pattern $C = \{A, B\}$ and assume the lemma holds for A and B. Consider a relation $\mathcal{C} \subseteq [n]^{V_C}$ and G_C-closed set $U \subseteq V_C$ and $z \in \Omega^{V_C \setminus S}$. Let

$$S := V_A \cap U, \qquad T := V_B \cap U, \qquad x := z_{V_A \setminus S}, \qquad y := z_{V_B \setminus T}.$$

Note that S is G_A-closed and T is G_B-closed and $C{\upharpoonright}U = \{A{\upharpoonright}S, B{\upharpoonright}T\}$.

By Lemma 4, there exists $\{(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i) \in \mathscr{P}^{\bowtie}_{A,B,C}\}_i$ such that $\mathcal{C} \subseteq \bigcup_i \mathcal{C}_i$ and $\chi_C(\mathcal{C}) = \sum_i \max\{\chi_A(\mathcal{A}_i), \chi_B(\mathcal{B}_i)\}$. For all i, we have $\mathcal{A}_i|^x_S \in \mathscr{P}_{A \upharpoonright S}$ and $\mathcal{B}_i|^y_T \in \mathscr{P}_{B \upharpoonright T}$ and $\mathcal{C}_i|^z_U \in \mathscr{P}_{C \upharpoonright U}$ (this is an easy exercise using the definition of H-pathset). Therefore, $(\mathcal{A}_i|^x_S, \mathcal{B}_i|^y_T, \mathcal{C}_i|^z_U) \in \mathscr{P}^{\bowtie}_{A\restriction S, B\restriction T, C\restriction U}$. Noting that $\mathcal{C}|^z_U \subseteq \bigcup_i \mathcal{C}_i|^z_U$, we have

$$\begin{aligned}
\chi_{C\restriction U}(\mathcal{C}|^z_U) &\leq \chi_{C\restriction U}(\textstyle\bigcup_i \mathcal{C}_i|^z_U) && \text{(monotonicity)}\\
&\leq \textstyle\sum_i \chi_{C\restriction U}(\mathcal{C}_i|^z_U) && \text{(sub-additivity)}\\
&\leq \textstyle\sum_i \max\{\chi_{A\restriction S}(\mathcal{A}_i|^x_S),\ \chi_{B\restriction T}(\mathcal{B}_i|^y_T)\} && \text{(join inequality)}\\
&\leq \textstyle\sum_i \max\{\chi_A(\mathcal{A}_i),\ \chi_B(\mathcal{B}_i)\} && \text{(induction hypothesis)}\\
&= \chi_C(\mathcal{C}).
\end{aligned}$$

□

4 Bounding the density of sub-relation of a join

The final lemma that we require for our main theorem is a restatement of Lemma 7 in the handout on relations (a general bound on the density of a sub-relation of a join of two relations).

Lemma 8. *For all patterns $A' \preceq A$ and $B' \preceq B$ and $C = \{A, B\}$, and for all relations $\mathcal{A} \subseteq [n]^{V_A}$ and $\mathcal{B} \subseteq [n]^{V_B}$ and $\mathcal{C} \subseteq [n]^{V_C}$ such that $\mathcal{C} \subseteq \mathcal{A} \bowtie \mathcal{B}$, we have*

$$\mu(\mathcal{C}) \leq \mu(\mathsf{proj}_{A'}(\mathcal{A})) \cdot \mu_{B' \ominus A'}(\mathsf{proj}_{B'}(\mathcal{B})) \cdot \mu_{C \ominus \{A', B'\}}(\mathcal{C}).$$

5 The main lower bound

We now prove our main result: a lower bound on pathset complexity $\chi_A(\mathcal{A})$ in terms of the potential function $\Psi(A)$ and the density of $\mathcal{A}$. Theorem 9, below, is a restatement of Theorem 1 for general relations $\mathcal{A} \subseteq [n]^{V_A}$ (as opposed to just pathsets $\mathcal{A} \in \mathscr{P}_A$).

Theorem 9 (**Pathset Complexity Lower Bound**). *For every $A \in$ Pattern($\subseteq G$) and relation $\mathcal{A} \subseteq [n]^{V_A}$,*

$$(1/\varepsilon)^{\Psi(A)} \cdot \mu(\mathcal{A}) \leq \chi_A(\mathcal{A}).$$

Proof. We argue by induction on patterns. Consider the base case that A is atomic. Let $\mathcal{A} \subseteq [n]^{V_A}$ be any relation, and let $t = \chi_A(\mathcal{A})$. By Definition 2, there exist pathsets $\mathcal{A}_1, \ldots, \mathcal{A}_t \in \mathscr{P}_A$ such that $\mathcal{A} \subseteq \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_t$. For all i, we have $\mu(\mathcal{A}_i) \leq \varepsilon^{\Delta(A)}$. Therefore, $\mu(\mathcal{A}) \leq \mu(\mathcal{A}_1) \cup \cdots \cup \mu(\mathcal{A}_t) \leq \chi_A(\mathcal{A}) \cdot \varepsilon^{\Delta(A)}$. Since A is atomic, we have $\Psi(A) = \Delta(A)$. We conclude that $(1/\varepsilon)^{\Psi(A)} \cdot \mu(\mathcal{A}) \leq \chi_A(\mathcal{A})$.

For the induction step, consider a non-atomic pattern $C = \langle A, B \rangle$ and assume the theorem holds for all patterns smaller than C.

Claim 10. $(1/\varepsilon)^{\Psi(C)} \cdot \mu(\mathcal{C}) \leq \max\{\chi_A(\mathcal{A}),\ \chi_B(\mathcal{B})\}$ *for all* $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \in \mathscr{P}^{\bowtie}_{A,B,C}$.

Before proving Claim 10, let's see that it suffices to prove the theorem. Let $\mathcal{C}$ be any relation in $\Sigma^{\otimes C}$. By Lemma 4, there exists $\{(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i) \in \mathscr{P}^{\bowtie}_{A,B,C}\}_i$ such that $\mathcal{C} \subseteq \bigcup_i \mathcal{C}_i$ and $\chi_C(\mathcal{C}) = \sum_i \max\{\chi_A(\mathcal{A}_i),\ \chi_B(\mathcal{B}_i)\}$. By monotonicity and sub-additivity of density $\mu(\cdot)$, we have $\mu(\mathcal{C}) \leq \sum_i \mu(\mathcal{C}_i)$. Applying Claim 10 to each $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i)$, we conclude

$$(1/\varepsilon)^{\Psi(C)} \cdot \mu(\mathcal{C}) \leq \sum_i (1/\varepsilon)^{\Psi(C)} \cdot \mu(\mathcal{C}_i) \leq \sum_i \max\{\chi_A(\mathcal{A}_i),\ \chi_B(\mathcal{B}_i)\} = \chi_C(\mathcal{C}).$$

It remains to prove Claim 10. Fix a trio of pathset $(\mathcal{A}, \mathcal{B}, \mathcal{C}) \in \mathscr{P}^{\bowtie}_{A,B,C}$. By definition of $\Psi(C)$, at least one of the inequalities $(\dagger)^C_{A',B}$, $(\ddagger)^C_{A',B'}$, $(\dagger)^C_{B',A}$, $(\ddagger)^C_{B',A'}$ is tight for some $A' \preceq A$ and $B' \preceq B$. Without loss of generality, we consider only the first two possibilities.

Case (†) **(one-sided induction):** Assume there exists a sub-pattern $A' \preceq A$ such that $(\dagger)^{C}_{A'}$ is tight, that is,

$$\Psi(C) = \Psi(A') + \Delta(B \ominus A') + \Delta(C \ominus \{A', B\}). \tag{1}$$

By Lemma 8 (with $B' = B$),

$$\mu(\mathcal{C}) \leq \mu(\mathsf{proj}_{A'}(\mathcal{A})) \cdot \mu_{B \ominus A'}(\mathcal{B}) \cdot \mu_{C \ominus \{A',B\}}(\mathcal{C}).$$

Since $\mathcal{B} \in \mathscr{P}_B$ and $\mathcal{C} \in \mathscr{P}_C$, we have density constraints

$$\mu_{B \ominus A'}(\mathcal{B}) \leq \varepsilon^{\Delta(B \ominus A')},$$
$$\mu_{C \ominus \{A',B\}}(\mathcal{C}) \leq \varepsilon^{\Delta(C \ominus \{A',B\})}.$$

Combining the three above inequalities, we have

$$\mu(\mathcal{C}) \leq \varepsilon^{\Delta(B \ominus A') + \Delta(C \ominus \{A',B\})} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})). \tag{2}$$

By the Projection Lemma, together with the induction hypothesis, we have

$$\begin{aligned}
(1/\varepsilon)^{\Psi(C)} \cdot \mu(\mathcal{C}) &\overset{(1)}{=} (1/\varepsilon)^{\Psi(A') + \Delta(B \ominus A') + \Delta(C \ominus \{A',B\})} \cdot \mu(\mathcal{C}) \\
&\overset{(2)}{\leq} (1/\varepsilon)^{\Psi(A')} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})) \\
&\leq \chi_{A'}(\mathsf{proj}_{A'}(\mathcal{A})) && \text{(ind. hyp.)} \\
&\leq \chi_A(\mathcal{A}) && \text{(Lemma 6),} \\
&\leq \max\{\chi_A(\mathcal{A}),\, \chi_B(\mathcal{B})\}.
\end{aligned}$$

Therefore, Claim 10 holds in this case.

Case (‡) **(balanced induction):** Assume that there exist sub-patterns $A' \preceq A$ and $B' \preceq B$ such that $(\ddagger)^{C}_{A',B'}$ is tight, that is,

$$\Psi(C) = \frac{1}{2}\Big(\Psi(A') + \Psi(B' \ominus A') + \Delta(C) + \Delta(C \ominus \{A', B'\})\Big). \tag{3}$$

By Lemma 8,

$$\mu(\mathcal{C}) \leq \mu(\mathsf{proj}_{A'}(\mathcal{A})) \cdot \mu_{B' \ominus A'}(\mathsf{proj}_{B'}(\mathcal{B})) \cdot \mu_{C \ominus \{A',B'\}}(\mathcal{C}). \tag{4}$$

By definition of $\mu_{B' \ominus A'}(\cdot)$, there exists $z \in \Omega^{V_{B'} \setminus V_{B' \ominus A'}}$ such that

$$\mu_{B' \ominus A'}(\mathsf{proj}_{B'}(\mathcal{B})) = \mu(\mathsf{proj}_{B'}(\mathcal{B})|^{z}_{B' \ominus A'}). \tag{5}$$

Since $\mathcal{C} \in \mathscr{P}_C$, we have density constraints

$$\mu(\mathcal{C}) \leq \varepsilon^{\Delta(C)}, \tag{6}$$
$$\mu_{C \ominus \{A',B'\}}(\mathcal{C}) \leq \varepsilon^{\Delta(C \ominus \{A',B'\})}. \tag{7}$$

Taking the geometric mean of inequalities (4) and (6), we have

$$\mu(\mathcal{C}) \overset{(4),(6)}{\leq} \sqrt{\varepsilon^{\Delta(C)} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})) \cdot \mu_{B' \ominus A'}(\mathsf{proj}_{B'}(\mathcal{B})) \cdot \mu_{C \ominus \{A',B'\}}(\mathcal{C})}$$

$$\overset{(5),(7)}{\leq} \sqrt{\varepsilon^{\Delta(C)+\Delta(C \ominus \{A',B'\})} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})) \cdot \mu(\mathsf{proj}_{B'}(\mathcal{B})|^{z}_{B' \ominus A'})}. \tag{8}$$

Using both the Projection Lemma and the Restriction Lemma, together with the induction hypothesis, we have

$$\begin{aligned} (1/\varepsilon)^{\Psi(A')} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})) &\leq \chi_{A'}(\mathsf{proj}_{A'}(\mathcal{A})) && \text{(ind. hyp.)} \\ &\leq \chi_A(\mathcal{A}) && \text{(Lemma 6)} \end{aligned} \tag{9}$$

and

$$\begin{aligned} (1/\varepsilon)^{\Psi(B' \ominus A')} \cdot \mu(\mathsf{proj}_{B'}(\mathcal{B})|^{z}_{B' \ominus A'}) &\leq \chi_{B' \ominus A'}(\mathsf{proj}_{B'}(\mathcal{B})|^{z}_{B' \ominus A'}) && \text{(ind. hyp.)} \\ &\leq \chi_{B'}(\mathsf{proj}_{B'}(\mathcal{B})) && \text{(Lemma 7)} \\ &\leq \chi_B(\mathcal{B}) && \text{(Lemma 6).} \end{aligned} \tag{10}$$

We now have

$$\begin{aligned} (1/\varepsilon)^{\Psi(C)} \cdot \mu(\mathcal{C}) &\overset{(3)}{=} \sqrt{(1/\varepsilon)^{\Psi(A')+\Psi(B' \ominus A')+\Delta(C)+\Delta(C \ominus \{A',B'\})}} \cdot \mu(\mathcal{C}) \\ &\overset{(8)}{\leq} \sqrt{(1/\varepsilon)^{\Psi(A')+\Psi(B' \ominus A')} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})) \cdot \mu(\mathsf{proj}_{B'}(\mathcal{B})|^{z}_{B' \ominus A'})} \\ &\leq \max\left\{(1/\varepsilon)^{\Psi(A')} \cdot \mu(\mathsf{proj}_{A'}(\mathcal{A})),\ (1/\varepsilon)^{\Psi(B' \ominus A')} \cdot \mu(\mathsf{proj}_{B'}(\mathcal{B})|^{z}_{B' \ominus A'})\right\} \\ &\overset{(9),(10)}{\leq} \max\{\chi_A(\mathcal{A}),\ \chi_B(\mathcal{B})\}. \end{aligned}$$

This completes the proof of Claim 10 and thus of Theorem 9. □