Recall:

- we fix a connected graph $G$, a strict threshold weighting $\theta$, and the parameter $\varepsilon := n^{-0.99}$,
- for $H \subseteq G$, we write $\Delta(H)$ for $\Delta(\theta)(H)$ (to simplify notation),
- a closed subgraph of $H$ is a union of connected components of $H$,
- an $H$-pathset (w.r.t. $G, \theta, \varepsilon$) is a relation $\mathcal{A} \subseteq [n]^{V(H)}$ such that
  \[ \mu_{V(H_0)}(\mathcal{A}) \leq \varepsilon^{\Delta(H_0)} \]
  for every closed subgraph $H_0$ of $H$.

Special case of interest / running example: Ultimately, we wish to prove an $n^{\Omega(\log k)}$ lower bound on the $\text{AC}^0$ formula size of $\text{SUB}(\text{Path}_k)$ (or equivalently $\text{SUB}(\text{Cycle}_k)$). Later on, we will fix the graph $G = \text{Cycle}_k$ and the threshold weighting $\theta \equiv 1$. This pair has the nice property that every proper subgraph $H \subset G$ satisfies
\[ \Delta(H) = |V(H)| - |E(H)| = |\{ \text{connected components of } H \}|. \]

1 Pathset Formulas and “Upshot Lemma”

We restate the main result of the last lecture (“upshot lemma”) in terms of the following notion:

**Definition 1.** A *pathset formula* (w.r.t. $G, \theta, \varepsilon$) is a rooted binary tree $F$ together with a family of pathsets $\mathcal{A}_{f,H}$ for each $f \in V(F)$ and subgraph $H \subseteq G$ such that

1. if $f$ is a leaf, then $\mathcal{A}_{f,H} = \emptyset$ for all $|E(H)| \geq 2$,
2. if $f$ is a non-leaf with children $f_1$ and $f_2$, then
   \[ \mathcal{A}_{f,H} \subseteq \bigcup_{H_1,H_2 : H = H_1 \cup H_2} \mathcal{A}_{f_1,H_1} \bowtie \mathcal{A}_{f_2,H_2}. \]

We view $F$ as “computing” the family of pathsets \( \{ \mathcal{A}_{\text{root}(F),H} \}_{H \subseteq G} \) (and in particular the $G$-pathset $\mathcal{A}_{\text{root}(F),G}$).

The upshot of the previous lecture was the following lemma:

**Lemma 2** (Upshot of Lecture 10). Any $\text{AC}^0$ formula (of size $n^{O(\log \log n)}$ and depth $o(\frac{\log n}{\log \log n})$) that solves $\text{SUB}(G)$ a.a.s. on $X_\theta$ induces a pathset formula $F$ of the same size such that $\mathcal{A}_{\text{root}(F),G}$ is 0.99-dense (for sufficiently large $n$).
We recall the proof of Lemma 2. The pathset formula \( F \) is obtained by first converting the \( \text{AC}^0 \) formula to fan-in 2 and then sampling random \( X_\theta \) and associating each gate \( f \) and subgraph \( H \subseteq G \) with the \( V(H) \)-ary relation \( A_{X_\theta}(f, H) := \{ \alpha \in [n]^{V(H)} : \text{ALL}(f \mid R_{X_\theta,H(\omega)}) \} \). In lecture 10, we showed that

- all such relations are pathsets a.a.s. (with probability \( 1 - o(1) \)),
- the relation \( A_{X_\theta}(\text{root}(F), G) \) is 0.99-dense with probability \( \geq 1/e - o(1) \),
- properties (1) and (2) of Def. 1 hold with probability 1.

The family of relations \( A_{X_\theta}(f, H) \) therefore constitutes a pathset formula with positive probability over \( X_\theta \) (proving Lemma 2).

2 Pathset Complexity

We wish to prove a lower bound on the pathset formula size of any dense \( G \)-pathset. In particular, we aim for an \( n^{\Omega(\log k)} \) lower bound in the case where \( G = \text{Cycle}_k \) and \( \theta \equiv 1 \). To this end, we introduce a family of complexity measures on pathsets. These complexity measures are associated with patterns.

For \( A \in \text{Pattern}(\subseteq G) \), recall that \( G_A = (V_A, E_A) \) denotes the subgraph of \( G \) that labels the root of \( A \). Recall that \( A \) is atomic if it is a single node labeled by a single-edge subgraph of \( G \) and every non-atomic pattern has the form \( C = \langle A, B \rangle \). Also recall notation \( A' \preceq A \), which denotes that \( A' \) is a sub-pattern of \( A \).

Notation 3. For \( A \in \text{Pattern}(\subseteq G) \), let \( \mathcal{P}_A \) denote the set of \( G_A \)-pathsets.

Definition 4. For a non-atomic pattern \( C = \langle A, B \rangle \) and a pathset \( C \in \mathcal{P}_C \), a join covering of \( C \) is an indexed family of pairs \( \{(A_i, B_i) \in \mathcal{P}_A \times \mathcal{P}_B \}_i \) where \( C \subseteq \bigcup_i A_i \bowtie B_i \).

Definition 5. Pathset complexity is a family of functions \( \chi_A : \mathcal{P}_A \rightarrow \mathbb{N} \) (one for each pattern \( A \)) defined by induction on patterns as follows:

- For an atomic pattern \( A \) and pathset \( A \in \mathcal{P}_A \),
  \[
  \chi_A(A) := \begin{cases} 
  0 & \text{if } A = \emptyset, \\
  1 & \text{if } A \neq \emptyset.
  \end{cases}
  \]

- For a non-atomic pattern \( C = \langle A, B \rangle \) and pathset \( C \in \mathcal{P}_C \),
  \[
  \chi_C(C) := \min_{\text{join covering } \{(A_i, B_i)\}_i \text{ of } C} \sum_i \max\{\chi_A(A_i), \chi_B(B_i)\}.
  \]

Lemma 6. Suppose \( A \) is a \( G \)-pathset that is computed by a pathset formula of size \( s \). Then there exists a pattern \( A \in \text{Pattern}(G) \) and a subset \( A' \subseteq A \) such that

\[
\mu(A') \geq \frac{\mu(A)}{2^{2|E(G)|}} \quad \text{and} \quad \chi_A(A') \leq s \cdot \text{polylog}(s).
\]
Time permitting, we will include a proof of Lemma 6 in Lecture 12. However, the idea is simple to describe at a high level. Let \( \{f,\{A_{f,H}\}_1\} \) be a pathset formula of size \( s \) that computes \( A \) (that is, \( A_{\text{root}(F),G} = A \)). To each \( \alpha \in A \), we may associate a pattern \( A_\alpha \in \text{Pattern}(G) \) describing the manner in which \( \alpha \) is constructed in \( F \). In this way, we get a partition of \( A \) into subsets \( A^{(a)} \) for each \( A \in \text{Pattern}(G) \). In fact, we need only consider patterns \( A \) with the property that \( G_B \not\subseteq G_C \) and \( G_C \not\subseteq G_B \) for all \( (B,C) \subseteq A \); the number of such patterns is at most \( 2^{2|E(G)|} \). Hence, for some such pattern \( A \), we have \( \mu(A^{(a)}) \geq 2^{-2|E(G)|} \cdot \mu(A) \). A simple inductive argument shows that \( \chi_A(A^{(a)}) \leq s \cdot \text{polylog}(s) \).

Lemmas 2 and 6 have the following immediate corollary:

**Corollary 7.** If \( \text{SUB}(G) \) is solvable a.a.s. on \( X_\theta \) by \( \text{AC}^0 \) formulas of size \( s \), then there exists a pattern \( A \in \text{Pattern}(G) \) and a pathset \( A \in \mathcal{P}_A \) such that \( \mu(A) = \Omega(1) \) and \( \chi_A(A) \leq s \cdot \text{polylog}(s) \).

### 3 Three Steps

Given Corollary 7, we achieve our goal of an \( n^{\Omega(k)} \) lower bound on the \( \text{AC}^0 \) formula size of \( \text{SUB}(\text{Cycle}_k) \) in the following three steps:

**Step 1.** We define a potential function \( \Psi : \text{Pattern}(\subseteq G) \rightarrow \mathbb{R}_{\geq 0} \). (Similar to \( \Delta \), this definition depends on \( \theta \); we write \( \Psi_\theta(\cdot) \) to make this explicit.)

**Step 2.** In the special case of interest where \( G = \text{Cycle}_k \) and \( \theta \equiv 1 \), we show \( \Psi(A) = \Omega(\log k) \) for every \( A \in \text{Pattern}(G) \).

**Step 3.** Finally, we show that for every \( A \in \text{Pattern}(\subseteq G) \) and pathset \( A \in \mathcal{P}_A \),

\[
\chi_A(A) \geq \varepsilon^{-\Psi(A)} \cdot \mu(A) = n^{0.99 \cdot \Psi(A)} \cdot \mu(A).
\]

(Corollary 7 and steps 1–3 clearly imply that \( \text{SUB}(\text{Cycle}_k) \) has \( \text{AC}^0 \) formula size \( n^{\Omega(k)} \).) In the rest of today’s lecture, we carry out steps 1 and 2. Next week, we show step 3.

### 4 The Operation \( A \oplus B \) on Patterns

**Notation 8.** We write \( C = \{A, B\} \) to express that \( C \) is either of the patterns \( \langle A, B \rangle \) and \( \langle B, A \rangle \).

**Definition 9** (The Operation \( \oplus \)).

- For \( S \subseteq V(G) \), we write \( \overline{S} \) for the complementary set \( V(G) \setminus S \).
- For \( H \subseteq G \), we say that \( S \) is \( H \)-closed if the induced subgraph of \( H \) on \( V(H) \cap S \) is a closed subgraph of \( H \). In this case, we denote this closed subgraph by \( H \upharpoonright S \).
  (Note that \( S \) is \( H \)-closed iff \( \overline{S} \) is \( H \)-closed, in which case \( H \) is the vertex-disjoint union of \( H \upharpoonright S \) and \( H \upharpoonright \overline{S} \). Also note that \( S \) is \( H \)-closed iff \( |e \cap S| \neq 1 \) for all \( e \in E(H) \).)
- For \( H, H' \subseteq G \), let \( H \oplus H' \) denote the maximum closed subgraph of \( H \) that is vertex-disjoint from \( H' \).
  (That is, \( H \oplus H' = H \upharpoonright S \) where \( S \) is the union of connected components of \( H \) that are vertex-disjoint from \( I \).

3
For $A \in \text{Pattern}(\subseteq G)$ and $G_A$-closed $S \subseteq V(G)$, let $A|S$ be the pattern obtained from $A$ by “pruning” all leaves labeled by edges $e \in E(G_A)$ such that $e \cap S = \emptyset$. Inductively: if $A$ is atomic and labeled by $e$, then

$$A|S := \begin{cases} A & \text{if } e \cap S = \emptyset, \\ \emptyset & \text{if } e \subseteq S. \end{cases}$$

If $C = \langle A, B \rangle$ and $S$ is $G_C$-closed, then (noting that $S$ is both $G_A$-closed and $G_B$-closed) we have $C|S = \langle A|S, B|S \rangle$.

(Here we allow the “empty pattern” $\emptyset$ as well as patterns of the form $\langle A, \emptyset \rangle$ and $\langle \emptyset, A \rangle$. W.l.o.g. we may prune away empty branches. Note that $A|S$ is not necessarily a sub-pattern of $A$.)

For $A, B \in \text{Pattern}(\subseteq G)$, let

$$A \ominus B := A|V(G_A \ominus G_B).$$

We refer to the pattern $A \ominus B$ as “$A$ restricted away from $B$”. We also let

$$A \ominus \{B_1, B_2\} := A|V(G_A \ominus (G_{B_1} \cup G_{B_2})).$$

(This is consistent with Notation 8, since $A \ominus \{B_1, B_2\} = A \ominus \langle B_1, B_2 \rangle = A \ominus \langle B_2, B_1 \rangle$.)

5 The Potential Function $\Psi : \text{Pattern}(\subseteq G) \to \mathbb{R}_{\geq 0}$

**Definition 10.** We define $\Psi : \text{Pattern}(\subseteq G) \to \mathbb{R}_{\geq 0}$ as the unique pointwise minimal function that satisfies (in)equalities:

- for every atomic pattern $A$,
  $$\Psi(A) = \Delta(A),$$

- for every non-atomic pattern $C = \{A, B\}$ and sub-pattern $A' \preceq A$,
  $$\Psi(C)^C_{A'} \geq \Psi(A') + \Delta(B \ominus A') + \Delta(C \ominus \{A', B\}),$$

- for every non-atomic pattern $C = \{A, B\}$ and sub-patterns $A' \preceq A$ and $B' \preceq B$,
  $$\Psi(C)^C_{A', B'} \geq \frac{1}{2} \left( \Psi(A') + \Psi(B' \ominus A') + \Delta(C) + \Delta(C \ominus \{A', B'\}) \right).$$

**Obs 1:** $\Psi$ is well-defined, since for any $\Psi_1, \Psi_2$ which satisfy the above, their pointwise minimum $\Psi_0(A) := \min\{\Psi_1(A), \Psi_2(A)\}$ also satisfies the above.
Obs 2: For every non-atomic $C = \{A, B\}$, there exist $A' \preceq A$ and $B' \preceq B$ such that at least one of the four inequality $(\dag)_A$, $(\dag)_B$, $(\ddag)_A$, $(\ddag)_B$ holds with equality.

Example 11. Recall from Lecture 8 the recursive-doubling pattern $RD_{0,k}$ and the maximal-overlapping pattern $MO_{0,k}$ (both with graph $P_{0,k}$, the path on vertices $0, \ldots, k$). We imagine that $P_{0,k}$ is sitting inside a large cycle $G$ with $\theta = 1$, so that $\Delta(P_{a,b}) = 1$ for all $0 \leq a < b \leq k$.

- We claim that $\Psi(RD_{0,k}) \geq \frac{1}{2} \log k - O(1)$. This may be shown using inequality $(\dag)$. To see why, for simplicity assume that $4$ divides $k$ and recall that

$$RD_{0,k} = \langle RD_{0,k/2}, RD_{k/2,k/4} \rangle = \langle RD_{0,k/4}, RD_{k/4,k/2}, RD_{k/2,k} \rangle.$$

We now apply inequality $(\dag)_A$ where $C = RD_{0,k}$ and $A' = RD_{0,k/4}$ and $B = RD_{k/2,k}$:

$$\Psi(RD_{0,k}) \geq \Psi(RD_{0,k/4}) + \Delta(RD_{k/2,k} \ominus RD_{0,k/4}) + \Delta(RD_{0,k} \ominus \{RD_{0,k/4}, RD_{k/2,k}\}) = \Psi(RD_{0,k/4}) + 1.$$

This recurrence implies that $\Psi(RD_{0,k}) \geq \frac{1}{2} \log k - O(1)$.

- We next show that $\Psi(MO_{0,k}) \geq \frac{1}{2} \log k - O(1)$ using inequality $(\ddag)$. Recall that

$$MO_{0,k} = \langle MO_{0,k-1}, MO_{1,k} \rangle.$$

Assume for simplicity that $k$ is odd. Applying $(\ddag)_A$ with $A' = MO_{0,(k-1)/2} \preceq MO_{0,k-1}$ and $B' = MO_{(k+1)/2,k} \preceq MO_{1,k}$, we have

$$\Psi(MO_{0,k}) \geq \frac{1}{2} \left( \Psi(MO_{0,(k-1)/2}) + \Psi(MO_{(k+1)/2,k} \ominus MO_{0,(k-1)/2}) + \Delta(MO_{0,k}) + \Delta(MO_{0,k} \ominus \{MO_{0,(k-1)/2}, MO_{(k+1)/2,k}\}) \right) = \Psi(MO_{0,(k-1)/2}) + \frac{1}{2}.$$

This recurrence implies that $\Psi(MO_{0,k}) \geq \frac{1}{2} \log k - O(1)$.

6 Proof of $\Psi(A) = \Omega(\log k)$ for all $A \in \text{Pattern}(\text{Cycle}_k)$

To streamline notation, we will write $\Psi_A$ and $\Delta_A$ instead of $\Psi(A)$ and $\Delta(A)$. We now fix $G = \text{Cycle}_k$ and $\theta = 1$. We will show that $\Psi_A = \Omega(\log k)$ for all $A \in \text{Pattern}(G)$.

However, rather than considering $A \in \text{Pattern}(G)$, we will focus on patterns $A \in \text{Pattern}(\subset G)$ (that is, patterns $A$ with $G_A \subset G$). For such patterns, we have $\Delta_A = |\{\text{connected components of } G_A\}|$.

Notation 12. For $A \in \text{Pattern}(\subset G)$, let $\ell_A$ denote the length of the longest path in $G_A$ (= the maximum number of edges in a component of $G_A$).

We will show that $\Psi(A) \geq \frac{1}{6} \log \ell_A + \Delta_A$ for all $A \in \text{Pattern}(\subset G)$. It follows that $\Psi(A) = \Omega(\log k)$ for all $A \in \text{Pattern}(G)$. (To see why, note that $\Psi(A') \leq \Psi(A)$ for all $A' \preceq A$; next note that $A$ must have a sub-pattern $A'$ with $k/2 \leq |E_{A'}| < k$ and that any such $A'$ has $\frac{1}{6} \log \ell_{A'} + \Delta_{A'} > \frac{1}{6} \log k$.)

Lemma 13. For all $C = \langle A, B \rangle \in \text{Pattern}(\subset G)$ and sub-patterns $A' \preceq A$ and $B' \preceq B$,

$$\Delta_C \leq \Delta_{A'} + \Delta_{B'} \ominus A' + \Delta_C \ominus \{A', B'\}.$$
Proof. Each connected component of $G_C$ contains at least one connected component from at least one of the vertex-disjoint graphs $G_{A'}$, $G_{B'\ominus A'}$ and $G_{C\ominus (A',B')}$. \hfill \square

Lemma 14. For every $A \in \text{Pattern}(\subset G)$ and $G_A$-closed set $S$, we have $\Psi_A \geq \Psi_{A|S} + \Delta_{A|\overline{S}}$.

Proof. We argue by induction on patterns. The lemma is trivial when $A$ is atomic. For the induction step, consider any non-atomic pattern $C = (A,B)$ and assume the lemma holds for all smaller patterns. Let $S$ be any $C$-closed set. Noting that $C|S = \{A|S, B|S\}$ and every sub-pattern of $A|S$ has the form $A'|S$ where $A' \preceq A$ (and similarly for $B|S$), it follows that that at least one of the four inequalities

$$
(\dagger)^{C|S}_{A'|S}, \quad (\dagger)^{C|S}_{A'|S,B'|S}, \quad (\dagger)^{C|S}_{B'|S}, \quad (\dagger)^{C|S}_{B'|S,A'|S}
$$

is tight for some $A' \preceq A$ and $B' \preceq B$. Without loss of generality, we consider just the first two possibilities.

First, consider the case that there exists $A' \preceq A$ such that $(\dagger)^{C|S}_{A'|S}$ is tight, that is,

$$
(1) \quad \Psi_{C|S} = \Psi_{A'|S} + \Delta_{(B\ominus A')|S} + \Delta_{(C\ominus (A',B))|S}.
$$

In this case, we have

$$
\begin{align*}
\Psi_{C} & \geq \Psi_{A'} + \Delta_{B\ominus A'} + \Delta_{C\ominus (A',B)} \\
& \geq \Psi_{A'} + \Delta_{B\ominus A'} + \Delta_{C\ominus (A',B)} \\
& \quad + \Delta_{C|\overline{S}} - \Delta_{A'|\overline{S}} - \Delta_{(B\ominus A')|\overline{S}} - \Delta_{(C\ominus (A',B))|\overline{S}} \\
& = \Psi_{A'} - \Delta_{A'|\overline{S}} + \Delta_{B\ominus A'|S} + \Delta_{C\ominus (A',B)|S} + \Delta_{C|\overline{S}} \\
& \geq \Psi_{A'|S} + \Delta_{B\ominus A'|S} + \Delta_{C\ominus (A',B)|S} + \Delta_{C|\overline{S}} \\
& = \Psi_{C|S} + \Delta_{C|\overline{S}}
\end{align*}
$$

(\text{by (1)}.)

Finally, consider the alternative that there exist $A' \preceq A$ and $B' \preceq B$ such that $(\dagger)^{C|S}_{A'|S,B'|S}$ is tight, that is,

$$
(2) \quad \Psi_{C|S} = \frac{1}{2}(\Psi_{A'|S} + \Psi_{(B'\ominus A')|S} + \Delta_{C|S} + \Delta_{(C\ominus (A',B'))|S}).
$$

In this case, we have

$$
\begin{align*}
\Psi_{C} & \geq \frac{1}{2}(\Psi_{A'} + \Psi_{B'\ominus A'} + \Delta_{C} + \Delta_{C\ominus (A',B')}) \\
& \geq \frac{1}{2}(\Psi_{A'} + \Psi_{B'\ominus A'} + \Delta_{C|S} + \Delta_{C|\overline{S}} + \Delta_{C\ominus (A',B')} + \\
& \quad \frac{1}{2}(\Delta_{C|\overline{S}} - \Delta_{A'|\overline{S}} - \Delta_{(B'\ominus A')|\overline{S}} - \Delta_{(C\ominus (A',B'))|\overline{S}}) \\
& = \frac{1}{2}(\Psi_{A'} - \Delta_{A'|\overline{S}} + \Delta_{B'\ominus A'|S} + \Delta_{C|S} + \Delta_{(C\ominus (A',B'))|S} + \Delta_{C|\overline{S}} \\
& \geq \frac{1}{2}(\Psi_{A'|S} + \Psi_{(B'\ominus A')|S} + \Delta_{C|S} + \Delta_{(C\ominus (A',B'))|S} + \Delta_{C|\overline{S}} \\
& = \Psi_{C|S} + \Delta_{C|\overline{S}}
\end{align*}
$$

(\text{by (2)}.)

Having shown $\Psi_{C} \geq \Psi_{C|S} + \Delta_{C|\overline{S}}$ in both cases, we are done. \hfill \square
Finally, we prove our lower bound on $\Psi_A$:

**Theorem 15.** For every $A \in \text{Pattern}(G)$, we have $\Psi_A \geq \frac{1}{6} \log(\ell_A) + \Delta_A$.

(As remarked earlier, this show that $\Psi_A \geq \frac{1}{6} \log k$ for every $A \in \text{Pattern}(G)$.)

*Proof.* We argue by induction on patterns. The base case where $A$ is atomic is trivial. For the induction step, let $A$ be a non-atomic pattern and assume the lemma holds for all smaller patterns. We will consider a sequence of cases. In each case, after showing that $\Psi_A \geq \frac{1}{6} \log(\ell_A) + \Delta_A$ under a given hypothesis, we will proceed assuming the negation of that hypothesis. The sequences of cases is summarized at the end of the proof.

First, consider the case that $G_A$ is disconnected (i.e. $\Delta_A \geq 2$). Let $S \subseteq V_A$ be the largest component of $G_A$. We have

$$\Psi_A \geq \Psi_A|_S + \Delta_A|_S \quad \text{(Lemma 14)}$$

$$\geq \frac{1}{6} \log(\ell_A|_S) + \Delta_A|_S + \Delta_A|_S \quad \text{(induction hypothesis)}$$

$$= \frac{1}{6} \log(\ell_A) + \Delta_A.$$  

Therefore, we proceed under the assumption that $G_A$ is connected (i.e. $\Delta_A = 1$). Without loss of generality, we assume that $G_A = P_{0,k}$ (i.e. $\ell_A = k$). [We re-use the symbol $k$, which is smaller than the “$k$” of the ambient graph $G = \text{Cycle}_k$.] Our goal is to show that

$$\Psi_A \geq \frac{1}{6} \log(k) + 1.$$  

Consider the case that there exists a sub-pattern $A' \preceq A$ such that $|E_{A'}| \geq k/8$ and $\Delta_{A'} \geq 2$. Note that $\ell_{A'} \geq |E_{A'}|/\Delta_{A'}$ (i.e. the number of edges in the largest component of $G_{A'}$ is at least the number of edges in $G_{A'}$ divided by the number of components in $G_{A'}$). We have

$$\Psi_A \geq \Psi_A \geq \frac{1}{6} \log(\ell_{A'}) + \Delta_{A'} \quad \text{(induction hypothesis)}$$

$$\geq \frac{1}{6} \log(k) - \frac{1}{2} - \frac{1}{6} \log(\Delta_{A'}) + \Delta_{A'} \quad (\ell_{A'} \geq |E_{A'}|/\Delta_{A'} \geq k/8\Delta_{A'})$$

$$\geq \frac{1}{6} \log(k) - \frac{1}{2} - \frac{1}{6} \log(2) + 2 \quad (\Delta_{A'} \geq 2)$$

$$= \frac{1}{6} \log(k) + \frac{4}{3}$$

$$> \frac{1}{6} \log(k) + 1.$$  

Therefore, we proceed under the following assumption:

$$\text{(\@)} \quad \text{for all } A' \preceq A, \text{ if } |E_{A'}| \geq k/8 \text{ then } \Delta_{A'} = 1.$$  

Going forward, the following notation will be convenient: for a proper sub-pattern $B \prec A$, let $B^\uparrow$ denote the parent of $B$ in $A$, and let $B^\downarrow$ denote the sibling of $B$ in $A$. Note that $B^\uparrow = \{B, B^\downarrow\} \preceq A$.

From our assumptions so far (i.e. $G_A = P_{0,k}$ and (\@)), it follows that there exist proper sub-patterns $B, Z \prec A$ such that

$$v_0 \in V_B, \quad v_k \in V_Z, \quad |E_B|, |E_Z| < k/8, \quad |E_{B^\uparrow}|, |E_{Z^\uparrow}| \geq k/8.$$  

Fix any choice of such $B$ and $Z$. Note that both $G_{B^\uparrow}$ and $G_{Z^\uparrow}$ are connected by (\@). In particular, $G_{B^\uparrow}$ is a path of length $|E_{B^\uparrow}|$ with initial endpoint $v_0$, and $G_{Z^\uparrow}$ is a path of length $|E_{Z^\uparrow}|$ with final endpoint $v_k$. 

7
Consider the case that $\ell_B^\uparrow < k/2$ and $\ell_Z^\uparrow < k/2$. Note that $V_{B^\uparrow}$ and $V_{Z^\uparrow}$ are disjoint and hence $Z^\uparrow \ominus B^\uparrow = Z^\uparrow$. Let $Y$ denote the least common ancestor of $B^\uparrow$ and $Z^\uparrow$ in $A$. We have

$$
\Psi_A \geq \Psi_Y \geq \frac{1}{2}(\Psi_{B^\uparrow} + \Psi_{Z^\uparrow \ominus B^\uparrow} + \Delta_Y + \Delta_{Y \ominus (B^\uparrow, Z^\uparrow)}) \quad \text{(by (‡)_{B^\uparrow, Z^\uparrow})}
$$

$$
\geq \frac{1}{2}(\Psi_{B^\uparrow} + \Psi_{Z^\uparrow}) + \frac{1}{2}
$$

$$
\geq \frac{1}{2}\left(\frac{1}{6}\log(\ell_B^\uparrow) + \Delta_B^\uparrow + \frac{1}{6}\log(\ell_Z^\uparrow) + \Delta_Z^\uparrow\right) + \frac{1}{2}
$$

$$(\Delta_Y \geq 1) \quad \text{(ind. hyp.)}
$$

$$
\geq \frac{1}{6}\log(k/8) + \frac{3}{2} \quad \text{and hypothesis.}
$$

(We remark that this is the only place in the proof where the inequality (‡) is used and the only tight case responsible for the fraction $\frac{1}{6}$.)

Therefore, we proceed under the assumption that $\ell_B^\uparrow \geq k/2$ or $\ell_Z^\uparrow \geq k/2$. Without loss of generality, we assume that $\ell_B^\uparrow \geq k/2$. (We now forget about $Z$ and $Z^\uparrow$.)

Before continuing, let’s take stock of the assumptions we have made so far:

$$
G_A = P_{0,k}, \quad (\odot), \quad B \preceq A, \quad v_0 \in V_B, \quad |E_B| < k/8, \quad |E_{B^\uparrow}| = \ell_{B^\uparrow} \geq k/2.
$$

We next identify three vertices $v_r, v_s, v_t$ where $0 < r < s < t \leq k$. The following illustration might be helpful for what follows:

![Illustration of the graph](image)

We first define $v_r \in V_B$ and $v_t \in V_{B^\uparrow}$ as follows. Let $\{v_0, \ldots, v_r\}$ be the component of $G_B$ containing $v_0$. (That is, the component of $v_0$ in $G_B$ is a path whose initial vertex is $v_0$; let $v_r$ be the final vertex in this path.) Let $v_t$ be the vertex in $V_{B^\uparrow}$ with maximal index $t$ (i.e. furthest away from $v_0$).

Note that $E_B$ contains edges $\{v_i, v_{i+1}\}$ for all $i \in \{0, \ldots, r-1\} \cup \{t, \ldots, \lceil k/2 \rceil - 1\}$. (If $t \geq k/2$, then the set $\{t, \ldots, \lceil k/2 \rceil - 1\}$ is empty. If $t < k/2$, then since $G_B^\uparrow = G_B \cup G_{B^-}$ is a path of length $\geq k/2$ and $G_{B^-}$ does not contain vertices $v_{t+1}, \ldots, v_{\lceil k/2 \rceil}$, it follows that $G_B$ contains all edges between $v_t$ and $v_{\lceil k/2 \rceil}$.) Therefore, $r + (k/2) - t \leq |E_B| < k/8$. It follows that

$$
t - r > 3k/8.
$$
Next, note that \(|E_{B^}\geq |E_{B^t}| - |E_B|\geq (k/2) - (k/8) > k/8\). It follows that there exists a proper sub-pattern \(C < B^\sim\) such that

\[ v_t \in V_C, \quad |E_C| < k/8, \quad |E_{C^+}| \geq k/8. \]

Fix any choice of such \(C\).

Consider the case that \(\ell_{C^+} < 3k/8\). Since \(G_{C^+}\) is connected (by (\(\oplus\))) and \(v_t \in V_{C^+}\) and \(t - r > 3k/8\), it follows that \(V_{C^+} \cap \{v_0, \ldots, v_r\} = \emptyset\) and hence \(\Delta_{B \oplus C^+} \geq 1\). We have

\[
\Psi_A \geq \Psi_{B^t} \geq \Psi_{C^+} + \Delta_{B \oplus C^+} + \Delta_{B^+ \oplus \{B, C^+\}} \quad \text{(by (\(\dagger\))\(B^t\))}
\geq \Psi_{C^+} + 1
\geq \frac{1}{6} \log(\ell_{C^+}) + \Delta_{C^+} + 1 \quad \text{(ind. hyp.)}
\geq \frac{1}{6} \log(k/8) + 2
\geq \frac{1}{6} \log(k) + 1.
\]

Therefore, we proceed under the assumption that \(\ell_{C^+} \geq 3k/8\). Since \(E_{C^+} = E_C \cup E_{C^\sim}\), we have

\[ |E_{C^\sim}| \geq |E_{C^+}| - |E_C| > (3k/8) - (k/8) = k/4. \]

We now define vertex \(v_s \in V_C\). Since \(v_t\) is the vertex of \(G_{B^\sim}\) with maximal index, it follows that \(\{v_t, v_{t+1}\} \notin E_{B^\sim}\) and hence \(\{v_t, v_{t+1}\} \notin E_C\) (since \(C < B^\sim\)). Therefore, the component of \(G_C\) containing \(v_t\) is a path with final vertex \(v_t\); let \(v_s\) be the initial vertex in this path. That is, \(\{v_s, \ldots, v_t\}\) is the component of \(G_C\) which contains \(v_t\).

Recall that \(t - r > 3k/8\) and note that \(t - s \leq |E_C| < k/8\). Therefore,

\[ s - r = (t - r) - (t - s) > (3k/8) - (k/8) = k/4. \]

We now claim that there exists a proper sub-pattern \(D < C^\sim\) such that

\[ k/8 \leq |E_D| < k/4. \]

To see this, note that there exists a chain of sub-patterns \(C^\sim = D_0 \succ D_1 \succ \cdots \succ D_j\) such that \(D_j\) is atomic and \(D_t = D^\uparrow_{t-1}\) and \(|E_{D_i}| \geq |E_{D^\sim_i}|\) for all \(i \in \{1, \ldots, j\}\). Since \(|E_{D_0}| > k/4\) and \(|E_{D_j}| = 1\) and \(|E_{D_{i-1}}| = |E_{D_i}| + |E_{D^\sim_i}| \leq 2|E_{D_i}|\), it must be the case that there exists \(i \in \{1, \ldots, j\}\) such that \(k/8 \leq |E_{D_i}| < k/4\).

Since \(|E_D| \geq k/8\), (\(\oplus\)) implies that \(G_D\) is connected. Since \(|E_D| < k/4\) and \(s - r > k/4\), it follows that \(V_D\) cannot contain both \(v_r\) and \(v_s\). We are now down to our final two cases: either \(v_r \notin V_D\) or \(v_s \notin V_D\).

First, suppose that \(v_r \notin V_D\). We have \(\Delta_{B \oplus D} \geq 1\) and hence

\[
\Psi_A \geq \Psi_{B^t} \geq \Psi_D + \Delta_{B \oplus D} + \Delta_{B^+ \oplus \{B, D\}} \quad \text{(by (\(\dagger\))\(B^t\))}
\geq \Psi_D + 1
\geq \frac{1}{6} \log(\ell_D) + \Delta_D + 1 \quad \text{(ind. hyp.)}
\geq \frac{1}{6} \log(k/8) + 2
\geq \frac{1}{6} \log(k) + 1.
\]
Finally, we are left with the alternative that $v_s \notin V_D$. In this case $\Delta_{C \oplus D} \geq 1$ and hence (substituting $C$ for $B$ in the above), we have

$$
\Psi_A \geq \Psi_{C^+} \geq \Psi_D + \Delta_{C \oplus D} + \Delta_{C^\top \ominus (C,D)} \geq \Psi_D + 1 \geq \frac{1}{6} \log(k) + 1.
$$

We have now covered all cases. In summary, we considered cases in the following sequence:

(I) $\Delta_A \geq 2$ \hspace{1cm} \text{else assume w.l.o.g. } G_A = P_{\emptyset,k},

(II) $\exists A' \prec A$ with $\Delta_{A'} \geq 2$ and $\ell_{A'} \geq k/8$ \hspace{1cm} \text{else assume (⊕)},

(III) $|E_{B^+}| < k/2$ and $|E_{Z^+}| < k/2$ \hspace{1cm} \text{else assume w.l.o.g. } |E_{B^+}| \geq k/2,

(IV) $|E_{C^+}| < 3k/8$ \hspace{1cm} \text{else assume } |E_{C^+}| \geq 3k/8,

(V) $v_r \notin E_D$ or $v_s \notin E_D$.

Since $\Psi_A \geq \frac{1}{6} \log(\ell_A) + \Delta_A$ in each of cases (I)–(V), the proof is complete. \qed