# Lecture 11: Pathset Complexity

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**Recall:**

- we fix a connected graph $G$, a strict threshold weighting $\theta$, and the parameter $\varepsilon := n^{-.99}$,
- for $H \subseteq G$, we write $\Delta(H)$ for $\Delta_\theta(H)$ (to simplify notation),
- a *closed subgraph* of $H$ is a union of connected components of $H$,
- an *$H$-pathset* (w.r.t. $G, \theta, \varepsilon$) is a relation $\mathcal{A} \subseteq [n]^{V(H)}$ such that

$$\mu_{V(H_0)}(\mathcal{A}) \leq \varepsilon^{\Delta(H_0)}$$

  for every closed subgraph $H_0$ of $H$.

**Special case of interest / running example:** Ultimately, we wish to prove an $n^{\Omega(\log k)}$ lower bound on the $\mathsf{AC}^0$ formula size of SUB(Path$_k$) (or equivalently SUB(Cycle$_k$)). Later on, we will fix the graph $G = \text{Cycle}_k$ and the threshold weighting $\theta \equiv 1$. This pair has the nice property that every proper subgraph $H \subset G$ satisfies

$$\Delta(H) = |V(H)| - |E(H)| = |\{\text{connected components of } H\}|.$$

## 1 Pathset Formulas and "Upshot Lemma"

We restate the main result of the last lecture ("upshot lemma") in terms of the following notion:s

**Definition 1.** A *pathset formula* (w.r.t. $G, \theta, \varepsilon$) is a rooted binary tree $F$ together with a family of pathsets $\mathcal{A}_{f,H}$ for each $f \in V(F)$ and subgraph $H \subseteq G$ such that

1. if $f$ is a leaf, then $\mathcal{A}_{f,H} = \emptyset$ for all $|E(H)| \geq 2$,
2. if $f$ is a non-leaf with children $f_1$ and $f_2$, then

$$\mathcal{A}_{f,H} \subseteq \bigcup_{H_1, H_2 \,:\, H = H_1 \cup H_2} \mathcal{A}_{f_1,H_1} \bowtie \mathcal{A}_{f_2,H_2}.$$

We view $F$ as "computing" the family of pathsets $\{\mathcal{A}_{\text{root}(F),H}\}_{H \subseteq G}$ (and in particular the $G$-pathset $\mathcal{A}_{\text{root}(F),G}$).

The upshot of the previous lecture was the following lemma:

**Lemma 2** (Upshot of Lecture 10)**.** *Any $\mathsf{AC}^0$ formula (of size $n^{O(\log \log n)}$ and depth $o(\frac{\log n}{\log \log n})$) that solves* SUB($G$) *a.a.s. on $\mathbf{X}_\theta$ induces a pathset formula $F$ of the same size such that $\mathcal{A}_{\text{root}(F),G}$ is 0.99-dense (for sufficiently large $n$).*

We recall the proof of Lemma 2. The pathset formula $F$ is obtained by first converting the $\mathsf{AC}^0$ formula to fan-in 2 and then sampling random $\mathbf{X}_\theta$ and associating each gate $f$ and subgraph $H \subseteq G$ with the $V(H)$-ary relation $\mathcal{A}_{\mathbf{X}_\theta}(f,H) := \{\alpha \in [n]^{V(H)} : \mathrm{ALL}(f{\restriction}R_{\mathbf{X}_\theta,H^{(\alpha)}})\}$. In lecture 10, we showed that

- all such relations are pathsets a.a.s. (with probability $1 - o(1)$),
- the relation $\mathcal{A}_{\mathbf{X}_\theta}(\mathrm{root}(F), G)$ is 0.99-dense with probability $\geq 1/e - o(1)$,
- properties (1) and (2) of Def. 1 hold with probability 1.

The family of relations $\mathcal{A}_{\mathbf{X}_\theta}(f,H)$ therefore constitutes a pathset formula with positive probability over $\mathbf{X}_\theta$ (proving Lemma 2).

## 2 Pathset Complexity

We wish to prove a lower bound on the *pathset formula size* of any dense $G$-pathset. In particular, we aim for an $n^{\Omega(\log k)}$ lower bound in the case where $G = \mathrm{Cycle}_k$ and $\theta \equiv 1$. To this end, we introduce a family of complexity measures on pathsets. These complexity measures are associated with patterns.

For $A \in \mathrm{Pattern}(\subseteq G)$, recall that $G_A = (V_A, E_A)$ denotes the subgraph of $G$ that labels the root of $A$. Recall that $A$ is *atomic* if it is a single node labeled by an single-edge subgraph of $G$ and every non-atomic pattern has the form $C = \langle A, B \rangle$. Also recall notation $A' \preceq A$, which denotes that $A'$ is a *sub-pattern* of $A$.

**Notation 3.** For $A \in \mathrm{Pattern}(\subseteq G)$, let $\mathscr{P}_A$ denote the set of $G_A$-pathsets.

**Definition 4.** For a non-atomic pattern $C = \langle A, B \rangle$ and a pathset $\mathcal{C} \in \mathscr{P}_C$, a *join covering* of $\mathcal{C}$ is an indexed family of pairs $\{(\mathcal{A}_i, \mathcal{B}_i) \in \mathscr{P}_A \times \mathscr{P}_B\}_i$ where $\mathcal{C} \subseteq \bigcup_i \mathcal{A}_i \bowtie \mathcal{B}_i$.

**Definition 5.** *Pathset complexity* is a family of functions $\chi_A : \mathscr{P}_A \to \mathbb{N}$ (one for each pattern $A$) defined by induction on patterns as follows:

- For an atomic pattern $A$ and pathset $\mathcal{A} \in \mathscr{P}_A$,

$$\chi_A(\mathcal{A}) := \begin{cases} 0 & \text{if } \mathcal{A} = \emptyset, \\ 1 & \text{if } \mathcal{A} \neq \emptyset. \end{cases}$$

- For a non-atomic pattern $C = \langle A, B \rangle$ and pathset $\mathcal{C} \in \mathscr{P}_C$,

$$\chi_C(\mathcal{C}) := \min_{\text{join covering } \{(\mathcal{A}_i,\mathcal{B}_i)\}_i \text{ of } \mathcal{C}} \sum_i \max\{\chi_A(\mathcal{A}_i),\ \chi_B(\mathcal{B}_i)\}.$$

**Lemma 6.** *Suppose $\mathcal{A}$ is a $G$-pathset that is computed by a pathset formula of size $s$. Then there exists a pattern $A \in \mathrm{Pattern}(G)$ and a subset $\mathcal{A}' \subseteq \mathcal{A}$ such that*

$$\mu(\mathcal{A}') \geq \frac{\mu(\mathcal{A})}{2^{2^{|E(G)|}}} \quad \textit{and} \quad \chi_A(\mathcal{A}') \leq s \cdot \mathrm{polylog}(s).$$

Time permitting, we will include a proof of Lemma 6 in Lecture 12. However, the idea is simple to describe at a high level. Let $(F, \{\mathcal{A}_{f,H}\}_{f,H})$ be a pathset formula of size $s$ that computes $\mathcal{A}$ (that is, $\mathcal{A}_{\mathrm{root}(F),G} = \mathcal{A}$). To each $\alpha \in \mathcal{A}$, we may associate a pattern $A_\alpha \in \mathrm{Pattern}(G)$ describing the manner in which $\alpha$ is constructed in $F$. In this way, we get a partition of $\mathcal{A}$ into subsets $\mathcal{A}^{(A)}$ for each $A \in \mathrm{Pattern}(G)$. In fact, we need only consider patterns $A$ with the property that $G_B \nsubseteq G_C$ and $G_C \nsubseteq G_B$ for all $\langle B, C\rangle \preceq A$; the number of such patterns is at most $2^{2^{|E(G)|}}$. Hence, for some such pattern $A$, we have $\mu(\mathcal{A}^{(A)}) \geq 2^{-2^{|E(G)|}} \cdot \mu(\mathcal{A})$. A simple inductive argument shows that $\chi_A(\mathcal{A}^{(A)}) \leq s \cdot \mathrm{polylog}(s)$.

Lemmas 2 and 6 have the following immediate corollary:

**Corollary 7.** *If* $\mathrm{SUB}(G)$ *is solvable a.a.s. on* $\mathbf{X}_\theta$ *by* $\mathsf{AC}^0$ *formulas of size* $s$*, then there exists a pattern* $A \in \mathrm{Pattern}(G)$ *and a pathset* $\mathcal{A} \in \mathscr{P}_A$ *such that* $\mu(\mathcal{A}) = \Omega(1)$ *and* $\chi_A(\mathcal{A}) \leq s \cdot \mathrm{polylog}(s)$.

## 3 Three Steps

Given Corollary 7, we achieve our goal of an $n^{\Omega(\log k)}$ lower bound on the $\mathsf{AC}^0$ formula size of $\mathrm{SUB}(\mathrm{Cycle}_k)$ in the following three steps:

**Step 1.** We define a potential function $\Psi : \mathrm{Pattern}(\subseteq G) \to \mathbb{R}_{\geq 0}$. (Similar to $\Delta$, this definition depends on $\theta$; we write $\Psi_\theta(\cdot)$ to make this explicit.)

**Step 2.** In the special case of interest where $G = \mathrm{Cycle}_k$ and $\theta \equiv 1$, we show $\Psi(A) = \Omega(\log k)$ for every $A \in \mathrm{Pattern}(G)$.

**Step 3.** Finally, we show that for every $A \in \mathrm{Pattern}(\subseteq G)$ and pathset $\mathcal{A} \in \mathscr{P}_A$,

$$\chi_A(\mathcal{A}) \geq \varepsilon^{-\Psi(A)} \cdot \mu(\mathcal{A}) = n^{0.99 \cdot \Psi(A)} \cdot \mu(\mathcal{A}).$$

(Corollary 7 and steps 1–3 clearly imply that $\mathrm{SUB}(\mathrm{Cycle}_k)$ has $\mathsf{AC}^0$ formula size $n^{\Omega(\log k)}$.) In the rest of today's lecture, we carry out steps 1 and 2. Next week, we show step 3.

## 4 The Operation $A \ominus B$ on Patterns

**Notation 8.** We write $C = \{A, B\}$ to express that $C$ is either of the patterns $\langle A, B\rangle$ and $\langle B, A\rangle$.

**Definition 9** (The Operation $\ominus$)**.**

- For $S \subseteq V(G)$, we write $\overline{S}$ for the complementary set $V(G) \setminus S$.
- For $H \subseteq G$, we say that $S$ is *$H$-closed* if the induced subgraph of $H$ on $V(H) \cap S$ is a closed subgraph of $H$. In this case, we denote this closed subgraph by $H{\restriction}S$.

  (Note that $S$ is $H$-closed iff $\overline{S}$ is $H$-closed, in which case $H$ is the vertex-disjoint union of $H{\restriction}S$ and $H{\restriction}\overline{S}$. Also note that $S$ is $H$-closed iff $|e \cap S| \neq 1$ for all $e \in E(H)$.)
- For $H, H' \subseteq G$, let $H \ominus H'$ denote the maximum closed subgraph of $H$ that is vertex-disjoint from $H'$.

  (That is, $H \ominus H' = H{\restriction}S$ where $S$ is the union of connected components of $H$ that are vertex-disjoint from $I$.)

- For $A \in \text{Pattern}(\subseteq G)$ and $G_A$-closed $S \subseteq V(G)$, let $A{\restriction}S$ be the pattern obtained from $A$ by "pruning" all leaves labeled by edges $e \in E(G_A)$ such that $e \cap S = \emptyset$. Inductively: if $A$ is atomic and labeled by $e$, then

$$A{\restriction}S := \begin{cases} A & \text{if } e \cap S = \emptyset, \\ \emptyset & \text{if } e \subseteq S. \end{cases}$$

  If $C = \langle A, B\rangle$ and $S$ is $G_C$-closed, then (noting that $S$ is both $G_A$-closed and $G_B$-closed) we have $C{\restriction}S = \langle A{\restriction}S, B{\restriction}S\rangle$.

  (Here we allow the "empty pattern" $\emptyset$ as well as patterns of the form $\langle A, \emptyset\rangle$ and $\langle \emptyset, A\rangle$. W.l.o.g. we may prune away empty branches. Note that $A{\restriction}S$ is not necessarily a sub-pattern of $A$.)

- For $A, B \in \text{Pattern}(\subseteq G)$, let

$$A \ominus B := A{\restriction}V(G_A \ominus G_B).$$

  We refer to the pattern $A \ominus B$ as "$A$ restricted away from $B$". We also let

$$A \ominus \{B_1, B_2\} := A{\restriction}V(G_A \ominus (G_{B_1} \cup G_{B_2})).$$

  (This is consistent with Notation 8, since $A \ominus \{B_1, B_2\} = A \ominus \langle B_1, B_2\rangle = A \ominus \langle B_2, B_1\rangle$.)

# 5 The Potential Function $\Psi : \text{Pattern}(\subseteq G) \to \mathbb{R}_{\geq 0}$

**Definition 10.** We define $\Psi : \text{Pattern}(\subseteq G) \to \mathbb{R}_{\geq 0}$ as the unique pointwise minimal function that satisfies (in)equalities:

- for every atomic pattern $A$,

$$\Psi(A) = \Delta(A),$$

- for every non-atomic pattern $C = \{A, B\}$ and sub-pattern $A' \preceq A$,

$$(\dagger)^C_{A'} \qquad \Psi(C) \geq \Psi(A') + \Delta(B \ominus A') + \Delta(C \ominus \{A', B\}),$$

- for every non-atomic pattern $C = \{A, B\}$ and sub-patterns $A' \preceq A$ and $B' \preceq B$,

$$(\ddagger)^C_{A',B'} \qquad \Psi(C) \geq \frac{1}{2}\Big(\Psi(A') + \Psi(B' \ominus A') + \Delta(C) + \Delta(C \ominus \{A', B'\})\Big).$$

**Obs 1:** $\Psi$ is well-defined, since for any $\Psi_1, \Psi_2$ which satisfy the above, their pointwise minimum $\Psi_0(A) := \min\{\Psi_1(A), \Psi_2(A)\}$ also satisfies the above.

**Obs 2:** For every non-atomic $C = \{A, B\}$, there exist $A' \preceq A$ and $B' \preceq B$ such that at least one of the four inequality $(\dagger)^C_{A'}$, $(\dagger)^C_{B'}$, $(\ddagger)^C_{A',B'}$, $(\ddagger)^C_{B',A'}$ holds with equality.

**Example 11.** Recall from Lecture 8 the recursive-doubling pattern $\mathsf{RD}_{0,k}$ and the maximal-overlapping pattern $\mathsf{MO}_{0,k}$ (both with graph $P_{0,k}$, the path on vertices $0, \dots, k$). We imagine that $P_{0,k}$ is sitting inside a large cycle $G$ with $\theta \equiv 1$, so that $\Delta(P_{a,b}) = 1$ for all $0 \le a < b \le k$.

- We claim that $\Psi(\mathsf{RD}_{0,k}) \ge \frac{1}{2}\log k - O(1)$. This may be shown using inequality $(\dagger)$. To see why, for simplicity assume that 4 divides $k$ and recall that

  $$\mathsf{RD}_{0,k} = \langle \mathsf{RD}_{0,k/2}, \mathsf{RD}_{k/2,k}\rangle = \langle\langle \mathsf{RD}_{0,k/4}, \mathsf{RD}_{k/4,k/2}\rangle, \mathsf{RD}_{k/2,k}\rangle.$$

  We now apply inequality $(\dagger)^C_{A'}$ where $C = \mathsf{RD}_{0,k}$ and $A' = \mathsf{RD}_{0,k/4}$ and $B = \mathsf{RD}_{k/2,k}$:

  $$\Psi(\mathsf{RD}_{0,k}) \ge \Psi(\mathsf{RD}_{0,k/4}) + \underbrace{\Delta(\mathsf{RD}_{k/2,k} \ominus \mathsf{RD}_{0,k/4})}_{=1} + \underbrace{\Delta(\mathsf{RD}_{0,k} \ominus \{\mathsf{RD}_{0,k/4}, \mathsf{RD}_{k/2,k}\})}_{=0} = \Psi(\mathsf{RD}_{0,k/4}) + 1.$$

  This recurrence implies that $\Psi(\mathsf{RD}_{0,k}) \ge \frac{1}{2}\log k - O(1)$.

- We next show that $\Psi(\mathsf{MO}_{0,k}) \ge \frac{1}{2}\log k - O(1)$ using inequality $(\ddagger)$. Recall that

  $$\mathsf{MO}_{0,k} = \langle \mathsf{MO}_{0,k-1}, \mathsf{MO}_{1,k}\rangle.$$

  Assume for simplicity that $k$ is odd. Applying $(\ddagger)^C_{A',B'}$ with $A' = \mathsf{MO}_{0,(k-1)/2} \preceq \mathsf{MO}_{0,k-1}$ and $B' = \mathsf{MO}_{(k+1)/2,k} \preceq \mathsf{MO}_{1,k}$, we have

  $$\Psi(\mathsf{MO}_{0,k}) \ge \frac{1}{2}\begin{pmatrix} \Psi(\mathsf{MO}_{0,(k-1)/2}) + \Psi(\mathsf{MO}_{(k+1)/2,k} \ominus \mathsf{MO}_{0,(k-1)/2}) + \\ \Delta(\mathsf{MO}_{0,k}) + \Delta(\mathsf{MO}_{0,k} \ominus \{\mathsf{MO}_{0,(k-1)/2}, \mathsf{MO}_{(k+1)/2,k}\}) \end{pmatrix} = \Psi(\mathsf{MO}_{0,(k-1)/2}) + \frac{1}{2}.$$

  This recurrence implies that $\Psi(\mathsf{MO}_{0,k}) \ge \frac{1}{2}\log k - O(1)$.

# 6 Proof of $\Psi(A) = \Omega(\log k)$ for all $A \in \mathrm{Pattern}(\mathrm{Cycle}_k)$

To streamline notation, we will write $\Psi_A$ and $\Delta_A$ instead of $\Psi(A)$ and $\Delta(A)$. We now fix $G = \mathrm{Cycle}_k$ and $\theta \equiv 1$. We will show that $\Psi_A = \Omega(\log k)$ for all $A \in \mathrm{Pattern}(G)$.

However, rather than considering $A \in \mathrm{Pattern}(G)$, we will focus on patterns $A \in \mathrm{Pattern}(\subset G)$ (that is, patterns $A$ with $G_A \subset G$). For such patterns, we have $\Delta_A = |\{\text{connected components of } G_A\}|$.

**Notation 12.** For $A \in \mathrm{Pattern}(\subset G)$, let $\ell_A$ denote the length of the longest path in $G_A$ (= the maximum number of edges in a component of $G_A$).

We will show that $\Psi(A) \ge \frac{1}{6}\log \ell_A + \Delta_A$ for all $A \in \mathrm{Pattern}(\subset G)$. It follows that $\Psi(A) = \Omega(\log k)$ for all $A \in \mathrm{Pattern}(G)$. (To see why, note that $\Psi(A') \le \Psi(A)$ for all $A' \preceq A$; next note that $A$ must have a sub-pattern $A'$ with $k/2 \le |E_{A'}| < k$ and that any such $A'$ has $\frac{1}{6}\log \ell_{A'} + \Delta_{A'} > \frac{1}{6}\log k$.)

**Lemma 13.** *For all $C = \langle A, B\rangle \in \mathrm{Pattern}(\subset G)$ and sub-patterns $A' \preceq A$ and $B' \preceq B$,*

$$\Delta_C \le \Delta_{A'} + \Delta_{B' \ominus A'} + \Delta_{C \ominus \{A', B'\}}.$$

*Proof.* Each connected component of $G_C$ contains at least one connected component from at least one of the vertex-disjoint graphs $G_{A'}$, $G_{B'\ominus A'}$ and $G_{C\ominus\{A',B'\}}$. □

**Lemma 14.** *For every $A \in \text{Pattern}(\subset G)$ and $G_A$-closed set $S$, we have $\Psi_A \geq \Psi_{A\restriction S} + \Delta_{A\restriction\overline{S}}$.*

*Proof.* We argue by induction on patterns. The lemma is trivial when $A$ is atomic. For the induction step, consider any non-atomic pattern $C = \langle A, B\rangle$ and assume the lemma holds for all smaller patterns. Let $S$ be any $C$-closed set. Noting that $C\restriction S = \{A\restriction S, B\restriction S\}$ and every sub-pattern of $A\restriction S$ has the form $A'\restriction S$ where $A' \preceq A$ (and similarly for $B\restriction S$), it follows that that at least one the four inequalities

$$(\dagger)^{C\restriction S}_{A'\restriction S}, \qquad (\ddagger)^{C\restriction S}_{A'\restriction S, B'\restriction S}, \qquad (\dagger)^{C\restriction S}_{B'\restriction S}, \qquad (\ddagger)^{C\restriction S}_{B'\restriction S, A'\restriction S}$$

is tight for some $A' \preceq A$ and $B' \preceq B$. Without loss of generality, we consider just the first two possibilities.

First, consider the case that there exists $A' \preceq A$ such that $(\dagger)^{C\restriction S}_{A'\restriction S}$ is tight, that is,

$$\Psi_{C\restriction S} = \Psi_{A'\restriction S} + \Delta_{(B\ominus A')\restriction S} + \Delta_{(C\ominus\{A',B\})\restriction S}. \tag{1}$$

In this case, we have

$$\begin{aligned}
\Psi_C &\geq \Psi_{A'} + \Delta_{B\ominus A'} + \Delta_{C\ominus\{A',B\}} && (\text{by } (\dagger)^{C}_{A'})\\
&\geq \Psi_{A'} + \Delta_{B\ominus A'} + \Delta_{C\ominus\{A',B\}}\\
&\quad + \Delta_{C\restriction\overline{S}} - \Delta_{A'\restriction\overline{S}} - \Delta_{(B\ominus A')\restriction\overline{S}} - \Delta_{(C\ominus\{A',B\})\restriction\overline{S}} && (\text{Lemma 13})\\
&= \Psi_{A'} - \Delta_{A'\restriction\overline{S}} + \Delta_{B\ominus A'\restriction S} + \Delta_{C\ominus\{A',B\}\restriction S} + \Delta_{C\restriction\overline{S}}\\
&\geq \Psi_{A'\restriction S} + \Delta_{B\ominus A'\restriction S} + \Delta_{C\ominus\{A',B\}\restriction S} + \Delta_{C\restriction\overline{S}} && (\text{ind. hyp.})\\
&= \Psi_{C\restriction S} + \Delta_{C\restriction\overline{S}} && (\text{by (1)}).
\end{aligned}$$

Finally, consider the alternative that there exist $A' \preceq A$ and $B' \preceq B$ such that $(\ddagger)^{C\restriction S}_{A'\restriction S, B'\restriction S}$ is tight, that is,

$$\Psi_{C\restriction S} = \tfrac{1}{2}\big(\Psi_{A'\restriction S} + \Psi_{(B'\ominus A')\restriction S} + \Delta_{C\restriction S} + \Delta_{(C\ominus\{A',B'\})\restriction S}\big). \tag{2}$$

In this case, we have

$$\begin{aligned}
\Psi_C &\geq \tfrac{1}{2}\big(\Psi_{A'} + \Psi_{B'\ominus A'} + \Delta_C + \Delta_{C\ominus\{A',B'\}}\big) && (\text{by } (\ddagger)^{C}_{A',B'})\\
&\geq \tfrac{1}{2}\big(\Psi_{A'} + \Psi_{B'\ominus A'} + (\Delta_{C\restriction S} + \Delta_{C\restriction\overline{S}}) + \Delta_{C\ominus\{A',B'\}}\big) +\\
&\quad \tfrac{1}{2}\big(\Delta_{C\restriction\overline{S}} - \Delta_{A'\restriction\overline{S}} - \Delta_{(B'\ominus A')\restriction\overline{S}} - \Delta_{(C\ominus\{A',B'\})\restriction\overline{S}}\big) && (\text{Lemma 13})\\
&= \tfrac{1}{2}\big(\Psi_{A'} - \Delta_{A'\restriction\overline{S}} + \Psi_{B'\ominus A'} - \Delta_{(B'\ominus A')\restriction\overline{S}} + \Delta_{C\restriction S} + \Delta_{(C\ominus\{A',B'\})\restriction S}\big) + \Delta_{C\restriction\overline{S}}\\
&\geq \tfrac{1}{2}\big(\Psi_{A'\restriction S} + \Psi_{(B'\ominus A')\restriction S} + \Delta_{C\restriction S} + \Delta_{(C\ominus\{A',B'\})\restriction S}\big) + \Delta_{C\restriction\overline{S}} && (\text{ind. hyp.})\\
&= \Psi_{C\restriction S} + \Delta_{C\restriction\overline{S}} && (\text{by (2)}).
\end{aligned}$$

Having shown $\Psi_C \geq \Psi_{C\restriction S} + \Delta_{C\restriction\overline{S}}$ in both cases, we are done. □

Finally, we prove our lower bound on $\Psi_A$:

**Theorem 15.** *For every* $A \in \mathrm{Pattern}(\subset G)$, *we have* $\Psi_A \geq \frac{1}{6}\log(\ell_A) + \Delta_A$.

(As remarked earlier, this show that $\Psi_A \geq \frac{1}{6}\log k$ for every $A \in \mathrm{Pattern}(G)$.)

*Proof.* We argue by induction on patterns. The base case where $A$ is atomic is trivial. For the induction step, let $A$ be a non-atomic pattern and assume the lemma holds for all smaller patterns. We will consider a sequence of cases. In each case, after showing that $\Psi_A \geq \frac{1}{6}\log(\ell_A) + \Delta_A$ under a given hypothesis, we will proceed assuming the negation of that hypothesis. The sequences of cases is summarized at the end of the proof.

First, consider the case that $G_A$ is disconnected (i.e. $\Delta_A \geq 2$). Let $S \subseteq V_A$ be the largest component of $G_A$. We have

$$\begin{aligned}
\Psi_A &\geq \Psi_{A\restriction S} + \Delta_{A\restriction\overline{S}} && \text{(Lemma 14)}\\
&\geq \tfrac{1}{6}\log(\ell_{A\restriction S}) + \Delta_{A\restriction S} + \Delta_{A\restriction\overline{S}} && \text{(induction hypothesis)}\\
&= \tfrac{1}{6}\log(\ell_A) + \Delta_A.
\end{aligned}$$

Therefore, we proceed under the assumption that $G_A$ is connected (i.e. $\Delta_A = 1$). Without loss of generality, we assume that $G_A = P_{0,k}$ (i.e. $\ell_A = k$). [We re-use the symbol $k$, which is smaller than the "$k$" of the ambient graph $G = \mathrm{Cycle}_k$.] Our goal is to show that

$$\Psi_A \geq \tfrac{1}{6}\log(k) + 1.$$

Consider the case that there exists a sub-pattern $A' \preceq A$ such that $|E_{A'}| \geq k/8$ and $\Delta_{A'} \geq 2$. Note that $\ell_{A'} \geq |E_{A'}|/\Delta_{A'}$ (i.e. the number of edges in the largest component of $G_{A'}$ is at least the number of edges in $G_{A'}$ divided by the number of components in $G_{A'}$). We have

$$\begin{aligned}
\Psi_A \geq \Psi_{A'} &\geq \tfrac{1}{6}\log(\ell_{A'}) + \Delta_{A'} && \text{(induction hypothesis)}\\
&\geq \tfrac{1}{6}\log(k) - \tfrac{1}{2} - \tfrac{1}{6}\log(\Delta_{A'}) + \Delta_{A'} && (\ell_{A'} \geq |E_{A'}|/\Delta_{A'} \geq k/8\Delta_{A'})\\
&\geq \tfrac{1}{6}\log(k) - \tfrac{1}{2} - \tfrac{1}{6}\log(2) + 2 && (\Delta_{A'} \geq 2)\\
&= \tfrac{1}{6}\log(k) + \tfrac{4}{3}\\
&> \tfrac{1}{6}\log(k) + 1.
\end{aligned}$$

Therefore, we proceed under the following assumption:

$$(\circledast) \qquad \text{for all } A' \preceq A, \text{ if } |E_{A'}| \geq k/8 \text{ then } \Delta_{A'} = 1.$$

Going forward, the following notation will be convenient: for a proper sub-pattern $B \prec A$, let $B^{\uparrow}$ denote the parent of $B$ in $A$, and let $B^{\sim}$ denote the sibling of $B$ in $A$. Note that $B^{\uparrow} = \{B, B^{\sim}\} \preceq A$.

From our assumptions so far (i.e. $G_A = P_{0,k}$ and $(\circledast)$), it follows that there exist proper sub-patterns $B, Z \prec A$ such that

$$v_0 \in V_B, \qquad v_k \in V_Z, \qquad |E_B|, |E_Z| < k/8, \qquad |E_{B^{\uparrow}}|, |E_{Z^{\uparrow}}| \geq k/8.$$

Fix any choice of such $B$ and $Z$. Note that both $G_{B^{\uparrow}}$ and $G_{Z^{\uparrow}}$ are connected by $(\circledast)$. In particular, $G_{B^{\uparrow}}$ is a path of length $|E_{B^{\uparrow}}|$ with initial endpoint $v_0$, and $G_{Z^{\uparrow}}$ is a path of length $|E_{Z^{\uparrow}}|$ with final endpoint $v_k$.

Consider the case that $\ell_{B^\uparrow} < k/2$ and $\ell_{Z^\uparrow} < k/2$. Note that $V_{B^\uparrow}$ and $V_{Z^\uparrow}$ are disjoint and hence $Z^\uparrow \ominus B^\uparrow = Z^\uparrow$. Let $Y$ denote the least common ancestor of $B^\uparrow$ and $Z^\uparrow$ in $A$. We have

$$\begin{aligned}
\Psi_A \geq \Psi_Y &\geq \tfrac{1}{2}\big(\Psi_{B^\uparrow} + \Psi_{Z^\uparrow \ominus B^\uparrow} + \Delta_Y + \Delta_{Y \ominus \{B^\uparrow, Z^\uparrow\}}\big) && \text{(by } (\ddagger)^{Y}_{B^\uparrow, Z^\uparrow}) \\
&\geq \tfrac{1}{2}\big(\Psi_{B^\uparrow} + \Psi_{Z^\uparrow}\big) + \tfrac{1}{2} && (\Delta_Y \geq 1) \\
&\geq \tfrac{1}{2}\big(\tfrac{1}{6}\log(\ell_{B^\uparrow}) + \Delta_{B^\uparrow} + \tfrac{1}{6}\log(\ell_{Z^\uparrow}) + \Delta_{Z^\uparrow}\big) + \tfrac{1}{2} && \text{(ind. hyp.)} \\
&\geq \tfrac{1}{6}\log(k/8) + \tfrac{3}{2} \\
&= \tfrac{1}{6}\log(k) + 1.
\end{aligned}$$

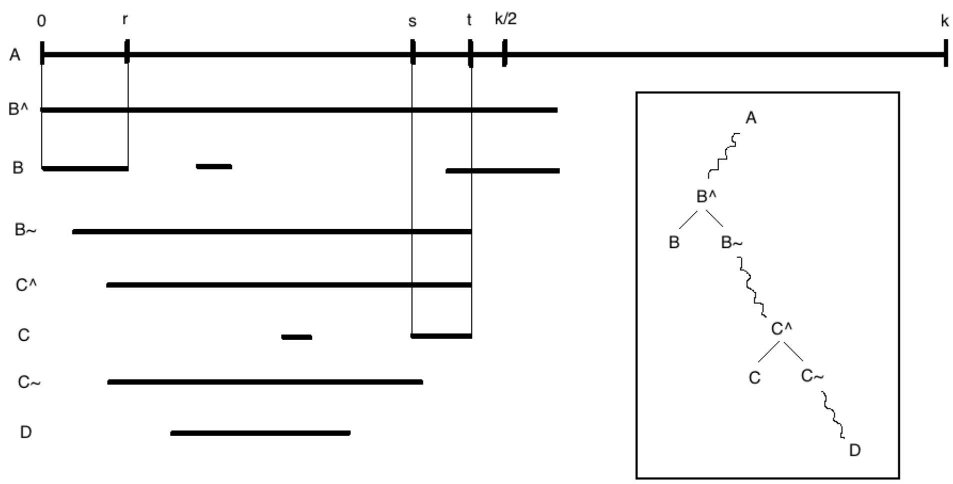
(We remark that this is the only place in the proof where the inequality $(\ddagger)$ is used and the only tight case responsible for the fraction $\frac{1}{6}$.)

Therefore, we proceed under the assumption that $\ell_{B^\uparrow} \geq k/2$ or $\ell_{Z^\uparrow} \geq k/2$. Without loss of generality, we assume that $\ell_{B^\uparrow} \geq k/2$. (We now forget about $Z$ and $Z^\uparrow$.)

Before continuing, let's take stock of the assumptions we have made so far:

$$G_A = P_{0,k}, \quad (\circledast), \quad B \preceq A, \quad v_0 \in V_B, \quad |E_B| < k/8, \quad |E_{B^\uparrow}| = \ell_{B^\uparrow} \geq k/2.$$

We next identify three vertices $v_r, v_s, v_t$ where $0 < r < s < t \leq k$. The following illustration might be helpful for what follows:




We first define $v_r \in V_B$ and $v_t \in V_{B^\sim}$ as follows. Let $\{v_0, \dots, v_r\}$ be the component of $G_B$ containing $v_0$. (That is, the component of $v_0$ in $G_B$ is a path whose initial vertex is $v_0$; let $v_r$ be the final vertex in this path.) Let $v_t$ be the vertex in $V_{B^\sim}$ with maximal index $t$ (i.e. furthest away from $v_0$).

Note that $E_B$ contains edges $\{v_i, v_{i+1}\}$ for all $i \in \{0, \dots, r-1\} \cup \{t, \dots, \lceil k/2 \rceil - 1\}$. (If $t \geq k/2$, then the set $\{t, \dots, \lceil k/2 \rceil - 1\}$ is empty. If $t < k/2$, then since $G_{B^\uparrow} = G_B \cup G_{B^\sim}$ is a path of length $\geq k/2$ and $G_{B^\sim}$ does not contain vertices $v_{t+1}, \dots, v_{\lceil k/2 \rceil}$, it follows that $G_B$ contains all edges between $v_t$ and $v_{\lceil k/2 \rceil}$.) Therefore, $r + (k/2) - t \leq |E_B| < k/8$. It follows that

$$t - r > 3k/8.$$

Next, note that $|E_{B^\sim}| \geq |E_{B^\uparrow}| - |E_B| \geq (k/2) - (k/8) > k/8$. It follows that there exists a proper sub-pattern $C \prec B^\sim$ such that

$$v_t \in V_C, \qquad |E_C| < k/8, \qquad |E_{C^\uparrow}| \geq k/8.$$

Fix any choice of such $C$.

Consider the case that $\ell_{C^\uparrow} < 3k/8$. Since $G_{C^\uparrow}$ is connected (by (⊛)) and $v_t \in V_{C^\uparrow}$ and $t - r > 3k/8$, it follows that $V_{C^\uparrow} \cap \{v_0, \dots, v_r\} = \emptyset$ and hence $\Delta_{B \ominus C^\uparrow} \geq 1$. We have

$$\begin{aligned}
\Psi_A \geq \Psi_{B^\uparrow} &\geq \Psi_{C^\uparrow} + \Delta_{B \ominus C^\uparrow} + \Delta_{B^\uparrow \ominus \{B, C^\uparrow\}} && (\text{by } (\dagger)^{B^\uparrow}_{C^\uparrow}) \\
&\geq \Psi_{C^\uparrow} + 1 \\
&\geq \tfrac{1}{6}\log(\ell_{C^\uparrow}) + \Delta_{C^\uparrow} + 1 && (\text{ind. hyp.}) \\
&\geq \tfrac{1}{6}\log(k/8) + 2 \\
&> \tfrac{1}{6}\log(k) + 1.
\end{aligned}$$

Therefore, we proceed under the assumption that $\ell_{C^\uparrow} \geq 3k/8$. Since $E_{C^\uparrow} = E_C \cup E_{C^\sim}$, we have

$$|E_{C^\sim}| \geq |E_{C^\uparrow}| - |E_C| > (3k/8) - (k/8) = k/4.$$

We now define vertex $v_s \in V_C$. Since $v_t$ is the vertex of $G_{B^\sim}$ with maximal index, it follows that $\{v_t, v_{t+1}\} \notin E_{B^\sim}$ and hence $\{v_t, v_{t+1}\} \notin E_C$ (since $C \prec B^\sim$). Therefore, the component of $G_C$ containing $v_t$ is a path with final vertex $v_t$; let $v_s$ be the initial vertex in this path. That is, $\{v_s, \dots, v_t\}$ is the component of $G_C$ which contains $v_t$.

Recall that $t - r > 3k/8$ and note that $t - s \leq |E_C| < k/8$. Therefore,

$$s - r = (t - r) - (t - s) > (3k/8) - (k/8) = k/4.$$

We now claim that there exists a proper sub-pattern $D \prec C^\sim$ such that

$$k/8 \leq |E_D| < k/4.$$

To see this, note that there exists a chain of sub-patterns $C^\sim = D_0 \succ D_1 \succ \dots \succ D_j$ such that $D_j$ is atomic and $D_i = D_{i-1}^\uparrow$ and $|E_{D_i}| \geq |E_{D_i^\sim}|$ for all $i \in \{1, \dots, j\}$. Since $|E_{D_0}| > k/4$ and $|E_{D_j}| = 1$ and $|E_{D_{i-1}}| = |E_{D_i}| + |E_{D_i^\sim}| \leq 2|E_{D_i}|$, it must be the case that there exists $i \in \{1, \dots, j\}$ such that $k/8 \leq |E_{D_i}| < k/4$.

Since $|E_D| \geq k/8$, (⊛) implies that $G_D$ is connected. Since $|E_D| < k/4$ and $s - r > k/4$, it follows that $V_D$ cannot contain both $v_r$ and $v_s$. We are now down to our final two cases: either $v_r \notin V_D$ or $v_s \notin V_D$.

First, suppose that $v_r \notin V_D$. We have $\Delta_{B \ominus D} \geq 1$ and hence

$$\begin{aligned}
\Psi_A \geq \Psi_{B^\uparrow} &\geq \Psi_D + \Delta_{B \ominus D} + \Delta_{B^\uparrow \ominus \{B, D\}} && (\text{by } (\dagger)^{B^\uparrow}_{D}) \\
&\geq \Psi_D + 1 \\
&\geq \tfrac{1}{6}\log(\ell_D) + \Delta_D + 1 && (\text{ind. hyp.}) \\
&\geq \tfrac{1}{6}\log(k/8) + 2 \\
&> \tfrac{1}{6}\log(k) + 1.
\end{aligned}$$

Finally, we are left with the alternative that $v_s \notin V_D$. In this case $\Delta_{C\ominus D} \geq 1$ and hence (substituting $C$ for $B$ in the above), we have

$$\Psi_A \geq \Psi_{C^\uparrow} \geq \Psi_D + \Delta_{C\ominus D} + \Delta_{C^\uparrow \ominus \{C,D\}} \geq \Psi_D + 1 > \tfrac{1}{6}\log(k) + 1.$$

We have now covered all cases. In summary, we considered cases in the following sequence:

| | | |
|---|---|---|
| (I) | $\Delta_A \geq 2$ | else assume w.l.o.g. $G_A = P_{0,k}$, |
| (II) | $\exists A' \prec A$ with $\Delta_{A'} \geq 2$ and $\ell_{A'} \geq k/8$ | else assume ($\circledast$), |
| (III) | $\lvert E_{B^\uparrow}\rvert < k/2$ and $\lvert E_{Z^\uparrow}\rvert < k/2$ | else assume w.l.o.g. $\lvert E_{B^\uparrow}\rvert \geq k/2$, |
| (IV) | $\lvert E_{C^\uparrow}\rvert < 3k/8$ | else assume $\lvert E_{C^\uparrow}\rvert \geq 3k/8$, |
| (V) | $v_r \notin E_D$ or $v_s \notin E_D$. | |

Since $\Psi_A \geq \frac{1}{6}\log(\ell_A) + \Delta_A$ in each of cases (I)–(V), the proof is complete. □