Announcements:

Problem Set #3 posted on website. Due date: Tuesday, March 6

Midterm #2 will take place in class on Tuesday, March 13

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Completeness Theorem. If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

Follows from:

Model Existence Theorem. Every consistent set of sentences has a model.

(If $\Sigma \nvdash \bot$, then there exists a model $\mathfrak{A} \models \Sigma$.)

Proof of the Model Existence Theorem

Let $\mathcal{L}$ be a countable language, and let Σ be a *consistent* set of $\mathcal{L}$-sentences.

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We construct a language $\mathcal{L}' = \mathcal{L} \cup \{\text{countably many new constant symbols}\}$ and a set of $\mathcal{L}'$-sentences $\widehat{\Sigma}$ such that

(1) $\Sigma \subseteq \widehat{\Sigma}$,

(2) $\widehat{\Sigma}$ is consistent,

(3) $\widehat{\Sigma}$ contains a Henkin axiom $(\exists x\theta) \to \theta^x_c$ for each $\mathcal{L}'$-sentence $\exists x\theta$.

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We then complete $\widehat{\Sigma}$ to a maximal consistent set of $\mathcal{L}'$-sentences Σ' (by adding either φ or $\neg\varphi$ one-at-a-time for each $\mathcal{L}'$-sentence φ). In addition to properties (1)–(3), Σ' has the property:

(4) For every $\mathcal{L}'$-sentence φ,

$$\varphi \in \Sigma' \iff \Sigma' \vdash \varphi \iff \neg\varphi \notin \Sigma' \iff \Sigma' \nvdash \neg\varphi.$$

The $\mathcal{L}'$-Structure $\mathfrak{A}$:

We define an equivalence relation $\sim$ on $\{$variable-free $\mathcal{L}'$-terms$\}$ by

$$t \sim u \overset{\text{def}}{\iff} (t = u) \in \Sigma' \iff \Sigma' \vdash (t = u).$$

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CLAIM: $\sim$ is symmetric.

PROOF: Assume $t \sim u$. Then $\Sigma' \vdash (t = u)$.

We know $\vdash (\forall x)(\forall y)[(x = y) \to (y = x)]$ (from Chapter 2).

Therefore, $\vdash (t = u) \to (u = t)$ by (Q1) axiom and (PC) rule.

Therefore, $\Sigma' \vdash (u = t)$ by (PC) rule; hence $u \sim t$.

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Let $\mathfrak{A}$ be the following structure:

- $A := \{[t]_\sim : t \text{ is a variable-free } \mathcal{L}'\text{-term}\}$ (universe of $\mathfrak{A}$)
- $c^{\mathfrak{A}} := [c]_\sim$ for each constant symbol c
- $f^{\mathfrak{A}}([t_1]_\sim, \ldots, [t_n]_\sim) := [f t_1 \ldots t_n]_\sim$ for each n-ary f
- $([t_1]_\sim, \ldots, [t_n]_\sim) \in R^{\mathfrak{A}} \iff^{\text{def}} R t_1 \ldots t_n \in \Sigma'$ for each n-ary R

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- $f^{\mathfrak{A}}([t_1]_\sim, \ldots, [t_n]_\sim) := [ft_1 \ldots t_n]_\sim$ for each n-ary f
- $([t_1]_\sim, \ldots, [t_n]_\sim) \in R^{\mathfrak{A}} \iff^{\text{def}} Rt_1 \ldots t_n \in \Sigma'$ for each n-ary R

NOTE: To show that $f^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$ are well-defined, we must show that if $t_i \sim u_i$ for $i = 1, \ldots, n$, then

- $[ft_1 \ldots t_n]_\sim = [fu_1 \ldots u_n]_\sim$ (that is, $(ft_1 \ldots t_n = fu_1 \ldots u_n) \in \Sigma'$) and
- $Rt_1 \ldots t_n \in \Sigma' \iff Ru_1 \ldots u_n \in \Sigma'$.

Proposition 3.2.6. For every $\mathcal{L}'$-sentence φ, we have

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NOTE: This finishes the proof of the Model Existence Theorem (and hence also the Completeness Theorem):

- As a consequence of Prop 3.2.6, $\mathfrak{A} \models \Sigma'$.
- Since $\Sigma \subset \widehat{\Sigma} \subset \Sigma'$, it follows that $\mathfrak{A} \models \Sigma$.
- Therefore, Σ has an $\mathcal{L}'$-structure model.
  (Recall that Σ is a set of $\mathcal{L}$-sentences.)
- To show that Σ has an $\mathcal{L}$-structure model, simply ignore the extra constant symbols in $\mathfrak{A}$.

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PROOF. We argue by induction on the complexity[1] of φ.

As usual, we consider five cases:

1. $\varphi :\equiv Rt_1 \dots t_n$ (base case)
2. $\varphi :\equiv t_1 = t_2$ (base case)
3. $\varphi :\equiv \neg\alpha$ (easy induction step)
4. $\varphi :\equiv \alpha \vee \beta$ (easy induction step)
5. $\varphi :\equiv \forall x\alpha$ (non-trivial induction step)

[1] As defined by $2 \cdot \#(\forall \text{ symbols in } \varphi) + \#(\neg \text{ symbols in } \varphi) + \#(\vee \text{ symbols in } \varphi)$.

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CASE 1. $\varphi :\equiv Rt_1 \dots t_n$

We have

$$\varphi \in \Sigma' \iff ([t_1]_\sim, \dots, [t_n]_\sim) \in R^{\mathfrak{A}} \iff \mathfrak{A} \models \varphi.$$

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CASE 2. $\varphi :\equiv t_1 = t_2$

We use the fact (shown by induction) that $\overline{s}(t) = [t]_\sim$ for every variable-free term t and $s : \mathit{Vars} \to A$.

It follows:

$$\begin{aligned}
\mathfrak{A} \models \varphi &\iff \mathfrak{A} \models \varphi[s] \text{ for every } s : \mathit{Vars} \to A \\
&\iff \overline{s}(t_1) = \overline{s}(t_2) \text{ for every } s : \mathit{Vars} \to A \\
&\iff [t_1]_\sim = [t_2]_\sim \text{ (since } \overline{s}(t) = [t]_\sim) \\
&\iff (t_1 = t_2) \in \Sigma' \\
&\iff \varphi \in \Sigma'.
\end{aligned}$$

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CASE 3. $\varphi :\equiv \neg\alpha$

$$\begin{aligned}
\mathfrak{A} \models \varphi &\iff \mathfrak{A} \not\models \alpha \\
&\iff \alpha \notin \Sigma' \text{ (induction hypothesis applied to } \alpha) \\
&\iff \neg\alpha \in \Sigma' \text{ (since } \alpha \in \Sigma' \iff \neg\alpha \notin \Sigma') \\
&\iff \varphi \in \Sigma'.
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CASE 4. $\varphi :\equiv \alpha \vee \beta$

Another trivial induction step.

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CASE 5. $\varphi :\equiv \forall x \alpha$ $\qquad$ $\boxed{\varphi \in \Sigma' \Longrightarrow \mathfrak{A} \models \varphi}$

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Assume $\varphi \in \Sigma'$. To show $\mathfrak{A} \models \varphi$, we must show

$$\begin{array}{rll} & \mathfrak{A} \models \varphi[s] & \text{for every } s : \mathit{Vars} \to A, \\ \text{that is,} & \mathfrak{A} \models \alpha[s[x|a]] & \text{for every } s : \mathit{Vars} \to A \text{ and } a \in A. \end{array}$$

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Fix any $s : \mathit{Vars} \to A$ and $a = [t]_\sim \in A$. We have

$$\begin{aligned} \mathfrak{A} \models \alpha[s[x|a]] &\iff \mathfrak{A} \models \alpha[s[x|\overline{s}(t)]] && (\text{since } \overline{s}(t) = [t]_\sim = a) \\ &\iff \mathfrak{A} \models \alpha_t^x[s] && (\text{by Theorem 2.6.2}) \\ &\iff \mathfrak{A} \models \alpha_t^x && (\text{since } \alpha_t^x \text{ is a sentence}). \end{aligned}$$

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Since $\varphi \in \Sigma'$, we have $\Sigma' \vdash \forall x \alpha$, hence $\Sigma \vdash \alpha_t^x$ by (Q1) axiom and (PC) rule. (OBS: Since t is variable-free, it is substitutable for x.)

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Therefore, $\alpha_t^x \in \Sigma'$. By induction hypothesis applied to α_t^x, it follows that $\mathfrak{A} \models \alpha_t^x$. Hence, $\mathfrak{A} \models \varphi$.

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Assume $\varphi \notin \Sigma'$. Then $\neg\varphi \in \Sigma'$, hence $\Sigma' \vdash \neg\varphi :\equiv \neg\forall x \alpha$, hence $\Sigma' \vdash \exists x \neg\alpha$.

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The set $\widehat{\Sigma}$ (and hence also Σ') contains a Henkin axiom

$$(\exists x \neg\alpha) \to (\neg\alpha)^x_c.$$

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Hence $\Sigma' \vdash (\exists x \neg\alpha) \to (\neg\alpha)^x_c$. Therefore, by (PC) rule (modus ponens),

$$\Sigma' \vdash (\neg\alpha)^x_c.$$

By the induction hypothesis applied to the sentence $(\neg\alpha)^x_c$, we have $\mathfrak{A} \models (\neg\alpha)^x_c$.

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By the induction hypothesis applied to the sentence $(\neg\alpha)^x_c$, we have $\mathfrak{A} \models (\neg\alpha)^x_c$.

It follows that $\mathfrak{A} \models \exists x \neg\alpha$, hence $\mathfrak{A} \not\models \neg\exists x \neg\alpha$, hence $\mathfrak{A} \not\models \varphi$ as required.

Q.E.D.

✓ **Model Existence Theorem.** Every consistent set of sentences has a model.

✓ **Completeness Theorem.** If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

□ **Compactness Theorem.** $\Sigma \models \varphi$ iff $\Sigma_0 \models \varphi$ for some finite $\Sigma_0 \subseteq \Sigma$.

The Compactness Theorem

COMPACTNESS THEOREM (Theorem 3.3.1) Let Σ be any set of formulas. Then Σ has a model (a.k.a. it is "satisfiable") if, and only if, every finite subset of Σ has a model.

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($\Leftarrow$): Assume that Σ has no model. We will show that some finite $\Sigma_0 \subseteq \Sigma$ has no model.

Since Σ has no model, this means $\Sigma \models \bot$. By the Completeness Theorem (or the contrapositive of the Model Existence Theorem), it follows that $\Sigma \vdash \bot$.

That is, there exists a deduction $(\delta_1, \ldots, \delta_n)$ of $\bot$ from Σ.

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That is, there exists a deduction $(\delta_1, \ldots, \delta_n)$ of $\bot$ from Σ.

Let $\Sigma_0 := \Sigma \cap \{\delta_1, \ldots, \delta_n\}$. Observe that Σ_0 is a finite set and $\Sigma_0 \vdash \bot$ (since the same sequence $(\delta_1, \ldots, \delta_n)$ is a deduction of $\bot$ from Σ_0).

Therefore, $\Sigma_0 \models \bot$ by the Soundness Theorem. This means that Σ_0 has no model, as required.

COMPACTNESS THEOREM (Theorem 3.3.1) Let Σ be any set of formulas. Then Σ has a model (a.k.a. it is "satisfiable") if, and only if, every finite subset of Σ has a model.

Corollary 3.2.2. $\Sigma \models \varphi$ iff $\Sigma_0 \models \varphi$ for some finite subset Σ_0 of Σ.

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Corollary 3.2.2. $\Sigma \models \varphi$ iff $\Sigma_0 \models \varphi$ for some finite subset Σ_0 of Σ.

Why is this called "Compactness Theorem"?

Topological definition of a *compact space*: every open cover has a finite subcover. The Compactness Theorem implies compactness of a certain topological space (whose points are the complete theories of a language $\mathcal{L}$).

Applications of Compactness

Definition. The *theory* of a structure $\mathfrak{A}$ is the set $Th(\mathfrak{A})$ of all formulas true in $\mathfrak{A}$:

$$Th(\mathfrak{A}) := \{\varphi : \mathfrak{A} \models \varphi\}.$$

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Definition. If $\mathfrak{A}$ is an $\mathcal{L}_{NT}$-structure such that $\mathfrak{A} \equiv \mathbb{N}$, then we call $\mathfrak{A}$ a "model of arithmetic". (This means that $\mathfrak{A}$ is indistinguishable from $\mathbb{N}$ from the perspective of first-order logic.)

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Theorem (Example 3.3.3). There exists a non-standard model of arithmetic (that is, a model of arithmetic which is non-isomorphic to the standard model $\mathbb{N}$).

Theorem. There exists a non-standard model of arithmetic.

Proof. Let $\mathcal{L} := \mathcal{L}_{NT} \cup \{c\}$ and let

$$\Theta := Th(\mathbb{N}) \cup \{0 < c, S0 < c, SS0 < c, \dots\}.$$

OBS: Every finite subset $\Theta_0 \subset \Theta$ has a model.

Therefore, by the Compactness Theorem, Θ has a model $\mathfrak{A}$.

The reduct $\mathfrak{A}|_{\mathcal{L}}$ (forgetting the constant c) is elementarily equivalent to $\mathbb{N}$.

Note that $\mathfrak{A}$ is non-isomorphic to $\mathbb{N}$, since it contains "infinite" elements. $\square$

This theorem shows that no set of axioms completely characterizes $\mathbb{N}$ up to isomorphism.