Problem 1. Come up with a sentence in the language $\mathcal{L} = \{+, \cdot\}$ that is true in the real numbers $\mathbb{R}$ but false in the rational numbers $\mathbb{Q}$ (viewing both $\mathbb{R}$ and $\mathbb{Q}$ as $\mathcal{L}$-structures where $+$ and $\cdot$ have the usual interpretation).

Note: The language $\mathcal{L}$ does not include any constant symbols, so neither may your sentence!

Problem 2. A *linear order* is a structure $\mathfrak{A}$ in the language $\mathcal{L}_{LO} = \{<\}$ which satisfies the usual axioms for linear orders:

- $\forall x\ \neg(x < x)$,
- $\forall x \forall y\ (x = y \vee x < y \vee y < x)$,
- $\forall x \forall y \forall z\ (x < y \wedge y < z) \to (x < z)$.

We say that a linear order $\mathfrak{A}$ is *well-founded* if it contains no infinite descending chain, that is, there is no infinite sequence of elements $a_1, a_2, a_3, \ldots$ such that $\ldots <^{\mathfrak{A}} a_3 <^{\mathfrak{A}} a_2 <^{\mathfrak{A}} a_1$.

Prove that no set of $\mathcal{L}_{LO}$-sentences Σ has the property that, for all linear orders $\mathfrak{A}$,

$$\mathfrak{A} \models \Sigma \iff \mathfrak{A} \text{ is well-founded.}$$

(This shows that the class of well-founded linear orders is not axiomatizable in first-order logic.)

Problem 3. A set S of vertices in graph G is said to be:

- a *clique* if every two elements of S are adjacent in G,
- an *independent set* if every two elements of S are non-adjacent in G.

The Infinite Ramsey Theorem states that every infinite graph G contains an infinite clique or an infinite independent set.

Using the Compactness Theorem, show that the Infinite Ramsey Theorem implies the following *Finite Ramsey Theorem*: for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ (depending on k) such that every finite graph with at least n vertices contains a clique of size k or an independent set of size k.

Problem 4. Prove the Infinite Ramsey Theorem.

(Hint: Let G be an infinite graph with vertex set V_0. Consider an arbitrary vertex $v_0 \in V_0$ and observe that there is an infinite subset $V_1 \subset V_0$ such that either v_0 is adjacent all elements of V_1, or v_0 is non-adjacent to all elements of V_1. Based on this observation, inductively construct a sequence $v_0, v_1, v_2, \ldots$.)