Gödel Numbers of Terms and Formulas

We assign a unique number to each symbol in $\mathcal{L}_{NT}$ as follows:

\[
\begin{align*}
\neg & \quad 1 \\
\lor & \quad 3 \\
\forall & \quad 5 \\
= & \quad 7 \\
0 & \quad 9 \\
S & \quad 11 \\
+ & \quad 13 \\
\cdot & \quad 15 \\
E & \quad 17 \\
< & \quad 19 \\
( & \quad 21 \\
) & \quad 23 \\
v_i & \quad 2i
\end{align*}
\]

Suppose $s \equiv s_1 \ldots s_n$ is a string of symbols, which constituting a well-formed term or formula of $\mathcal{L}_{NT}$.

Naively, we could encode $s$ by the number $\langle \#(s_1), \ldots, \#(s_n) \rangle$ where $\#(s_i)$ is the number corresponding to the symbol $s_i$.

However, it much better to encode $s$ according to the inductive type of terms and formulas.
Def 5.7.1. For each term $t$ and formula $\varphi$, the Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are defined as follows:

\[-\alpha = \langle 1, \ulcorner \alpha \urcorner \rangle \]
\[(\alpha \lor \beta) = \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle \]
\[(\forall v_i)(\alpha) = \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle \]
\[= t_1 t_2 = \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \]
\[0 = \langle 9 \rangle \]
\[S t = \langle 11, \ulcorner t \urcorner \rangle \]
\[+ t_1 t_2 = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \]
\[\cdot t_1 t_2 = \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \]
\[E t_1 t_2 = \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \]
\[< t_1 t_2 = \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \]
\[v_i = \langle 2i \rangle. \]
**Def 5.7.1.** For each term \( t \) and formula \( \varphi \), the Gödel numbers \( \ulcorner t \urcorner \) and \( \ulcorner \varphi \urcorner \) are defined as follows:

\[
\begin{align*}
\ulcorner \neg \alpha \urcorner &= \langle 1, \ulcorner \alpha \urcorner \rangle \\
\ulcorner (\alpha \lor \beta) \urcorner &= \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle \\
\ulcorner (\forall v_i)(\alpha) \urcorner &= \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle \\
\ulcorner t_1 t_2 \urcorner &= \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner 0 \urcorner &= \langle 9 \rangle \\
\ulcorner St \urcorner &= \langle 11, \ulcorner t \urcorner \rangle \\
\ulcorner + t_1 t_2 \urcorner &= \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner \cdot t_1 t_2 \urcorner &= \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner Et_1 t_2 \urcorner &= \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner < t_1 t_2 \urcorner &= \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner v_i \urcorner &= \langle 2i \rangle.
\end{align*}
\]

**Obs.** \( \ulcorner t \urcorner \) and \( \ulcorner \varphi \urcorner \) are never divisible by 7. (Why?)
Def 5.7.1. For each term $t$ and formula $\varphi$, the Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are defined as follows:

- $\ulcorner \neg \alpha \urcorner = \langle 1, \ulcorner \alpha \urcorner \rangle$
- $\ulcorner (\alpha \lor \beta) \urcorner = \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle$
- $\ulcorner (\forall v_i)(\alpha) \urcorner = \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle$
- $\ulcorner = t_1 t_2 \urcorner = \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
- $\ulcorner 0 \urcorner = \langle 9 \rangle$
- $\ulcorner S t \urcorner = \langle 11, \ulcorner t \urcorner \rangle$

- $\ulcorner + t_1 t_2 \urcorner = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
- $\ulcorner \cdot t_1 t_2 \urcorner = \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
- $\ulcorner E t_1 t_2 \urcorner = \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
- $\ulcorner < t_1 t_2 \urcorner = \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
- $\ulcorner v_i \urcorner = \langle 2i \rangle$.

Example. $\ulcorner = 0 S 0 \urcorner = \langle 7, \ulcorner 0 \urcorner, \ulcorner S 0 \urcorner \rangle$

- $= \langle 7, \langle 9 \rangle, \langle 11, \langle 9 \rangle \rangle \rangle$
- $= \langle 7, 2^{10}, \langle 11, 2^{10} \rangle \rangle = 2^8 3^{1025} 5^{(2^{123^{1025}} + 1)}$.

Notice how fast $\ulcorner S S S S S 0 \urcorner$ grows:

$\ulcorner S S S S S 0 \urcorner = \langle 11, \langle 11, \langle 11, \langle 11, \langle 9 \rangle \rangle \rangle \rangle \rangle = 2^{123^{2^{123^{2^{123^{2^{123^{10}}}}}}}}$.
Next Steps (Section 5.8)

$\Delta$-definability of sets

\[\text{TERTMS} := \{\lbrack t \rbrack: \text{terms } t\} = \{a \in \mathbb{N} : a = \lbrack t \rbrack \text{ for some term } t\},\]

\[\text{FORMULAS} := \{\lbrack \varphi \rbrack: \text{formulas } \varphi\} = \{a \in \mathbb{N} : a = \lbrack \varphi \rbrack \text{ for some formula } \varphi\}.\]
}\text{Definition of \textbf{Terms}} = \{ \neg t \downarrow : t \text{ is a term} \}

\begin{center}
\begin{tabular}{|l|}
\hline
\(\neg t\downarrow = \langle 1, \neg t \rangle\) & \(= t_1 t_2 \downarrow = \langle 7, t_1 \downarrow, t_2 \downarrow \rangle\) & \(+ t_1 t_2 \downarrow = \langle 13, t_1 \downarrow, t_2 \downarrow \rangle\) & \(< t_1 t_2 \downarrow = \langle 19, t_1 \downarrow, t_2 \downarrow \rangle\) \\
\(\alpha \lor \beta \downarrow = \langle 3, \alpha \downarrow, \beta \downarrow \rangle\) & \(0 \downarrow = \langle 9 \rangle\) & \(\cdot t_1 t_2 \downarrow = \langle 15, t_1 \downarrow, t_2 \downarrow \rangle\) & \(v_i \downarrow = \langle 2i \rangle\) \\
\(\forall v_i (\alpha) \downarrow = \langle 5, \forall v_i \downarrow, \alpha \downarrow \rangle\) & \(St \downarrow = \langle 11, t \downarrow \rangle\) & \(Et_1 t_2 \downarrow = \langle 17, t_1 \downarrow, t_2 \downarrow \rangle\) & \hline
\end{tabular}
\end{center}
**Δ-Definition of Terms** = \{\text{\textasciitilde}t \mid t \text{ is a term}\}

| \text{\textasciitilde}\alpha \rangle = \langle 1, \alpha \rangle | \text{\textasciitilde}t_{1}t_{2} \rangle = \langle 7, \text{\textasciitilde}t_{1}, \text{\textasciitilde}t_{2} \rangle | \text{\textasciitilde}+t_{1}t_{2} \rangle = \langle 13, \text{\textasciitilde}t_{1}, \text{\textasciitilde}t_{2} \rangle | \text{\textasciitilde}<t_{1}t_{2} \rangle = \langle 19, \text{\textasciitilde}t_{1}, \text{\textasciitilde}t_{2} \rangle |
| \text{\textasciitilde}(\alpha \lor \beta) \rangle = \langle 3, \text{\textasciitilde}\alpha, \text{\textasciitilde}\beta \rangle | \text{\textasciitilde}0 \rangle = \langle 9 \rangle | \text{\textasciitilde}t_{1}t_{2} \rangle = \langle 15, \text{\textasciitilde}t_{1}, \text{\textasciitilde}t_{2} \rangle | \text{\textasciitilde}v_{i} \rangle = \langle 2i \rangle |
| \text{\textasciitilde}(\forall v_{i})(\alpha) \rangle = \langle 5, \text{\textasciitilde}v_{i}, \text{\textasciitilde}\alpha \rangle | \text{\textasciitilde}St \rangle = \langle 11, \text{\textasciitilde}t \rangle | \text{\textasciitilde}Et_{1}t_{2} \rangle = \langle 17, \text{\textasciitilde}t_{1}, \text{\textasciitilde}t_{2} \rangle |

Recall the inductive definition of an \( \mathcal{L}_{NT} \)-term \( t \): it is either

- a variable symbol \( v_{i} \),
- \( St_{1} \) where \( t_{1} \) is term,
- the constant symbol \( 0 \),
- \( +t_{1}t_{2} \) or \( \cdot t_{1}t_{2} \) or \( Et_{1}t_{2} \) where \( t_{1}, t_{2} \) are terms.
\[ \text{\textbf{\Delta-Definition of Terms}} = \{ \lceil t \rceil : t \text{ is a term} \} \]

\[
\begin{array}{l}
\lceil \lnot \alpha \rceil = \langle 1, \lceil \alpha \rceil \rangle \\
\lceil \theta t_2 \rceil = \langle 7, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\lceil + t_1 t_2 \rceil = \langle 13, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\lceil < t_1 t_2 \rceil = \langle 19, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\lceil (\alpha \lor \beta) \rceil = \langle 3, \lceil \alpha \rceil, \lceil \beta \rceil \rangle \\
\lceil 0 \rceil = \langle 9 \rangle \\
\lceil \cdot t_1 t_2 \rceil = \langle 15, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\lceil v_i \rceil = \langle 2i \rangle \\
\lceil (\forall v_i)(\alpha) \rceil = \langle 5, \lceil v_i \rceil, \lceil \alpha \rceil \rangle \\
\lceil St \rceil = \langle 11, \lceil t \rceil \rangle \\
\lceil Et_1 t_2 \rceil = \langle 17, \lceil t_1 \rceil, \lceil t_2 \rceil \rangle \\
\end{array}
\]

Recall the inductive definition of an \( \mathcal{L}_{NT} \)-term \( t \): it is either

- a variable symbol \( v_i \),
- \( St_1 \) where \( t_1 \) is term,
- the constant symbol \( 0 \),
- \( + t_1 t_2 \) or \( \cdot t_1 t_2 \) or \( Et_1 t_2 \) where \( t_1, t_2 \) are terms.

Let’s start with \( \Delta \)-definition of

\[
\text{\textbf{Variables}} := \{ \lceil v_i \rceil : i = 1, 2, \ldots \} \quad (= \{ 2^{2i+1} : i = 1, 2, \ldots \}).
\]

by the formula

\[
\text{Variable}(x) \equiv (\exists y < x)[\text{Even}(y) \land (0 < y) \land (x = 2^Sy)].
\]
**Δ-Definition of Terms** = \{ \lnot t \triangledown : t \text{ is a term} \}

<table>
<thead>
<tr>
<th>Term ( \triangledown )</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lnot \alpha \triangledown ) = (1, ( \lnot \alpha ))</td>
<td>( \lnot t_1 t_2 \triangledown = \langle 7, t_1 \triangledown, t_2 \triangledown )</td>
</tr>
<tr>
<td>( \alpha \lor \beta \triangledown ) = (3, ( \alpha \triangledown, \beta \triangledown )</td>
<td>( +t_1 t_2 \triangledown = \langle 13, t_1 \triangledown, t_2 \triangledown )</td>
</tr>
<tr>
<td>0 \triangledown = \langle 9 \rangle</td>
<td>( &lt;t_1 t_2 \triangledown = \langle 19, t_1 \triangledown, t_2 \triangledown )</td>
</tr>
<tr>
<td>( \forall v_i (\alpha) \triangledown ) = (5, ( \forall v_i \triangledown, \alpha \triangledown )</td>
<td>( St \triangledown = \langle 11, t \triangledown )</td>
</tr>
<tr>
<td>( Et_1 t_2 \triangledown ) = (17, ( t_1 \triangledown, t_2 \triangledown )</td>
<td></td>
</tr>
</tbody>
</table>

Recall the inductive definition of an \( \mathcal{L}_{NT} \)-term \( t \): it is either

- a variable symbol \( v_i \),
- \( St_1 \) where \( t_1 \) is term,
- the constant symbol 0,
- \( +t_1 t_2 \) or \( \cdot t_1 t_2 \) or \( Et_1 t_2 \) where \( t_1, t_2 \) are terms.

We would like to write:

\[
\text{Term}(x) \equiv \text{Variable}(x) \lor x = 2^{10} \lor (\exists y < x)[\text{Term}(y) \land x = \overline{2^{12}, 3^{Sy}}^{\langle 11, y \rangle}]
\]

However, there is a problem with this “\( \Delta \)-formula”: It is a not legitimate formula of first-order logic! Note the circular use of the subformula \( \text{Term}(y) \).
**Definition of Terms** = \{ \overline{t} : t \text{ is a term} \}

**Definition.** A *term construction sequence* for a term \( t \) is a finite sequence of terms \( (t_1, \ldots, t_\ell) \) such that \( t_\ell \equiv t \) and, for each \( k \in \{1, \ldots, \ell\} \), the term \( t_k \) is either

- a variable symbol,
- the constant symbol \( 0 \),
- \( St_j \) for some \( j < k \), or
- \( +t_it_j \) or \( \cdot t_it_j \) or \( Et_it_j \) for some \( i, j < k \).
**Δ-Definition of Terms** = \{「t」: t is a term\}

**Definition.** A *term construction sequence* for a term $t$ is a finite sequence of terms $(t_1, \ldots, t_\ell)$ such that $t_\ell \equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term $t_k$ is either

- a variable symbol,
- the constant symbol $0$,
- $St_j$ for some $j < k$, or
- $+t_it_j$ or $\cdot t_it_j$ or $Et_it_j$ for some $i, j < k$.

**Example.** $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$. 
**Definition.** A *term construction sequence* for a term $t$ is a finite sequence of terms $(t_1, \ldots, t_\ell)$ such that $t_\ell \equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term $t_k$ is either

- a variable symbol,
- the constant symbol 0,
- $St_j$ for some $j < k$, or
- $+t_it_j$ or $\cdot t_it_j$ or $Et_it_j$ for some $i, j < k$.

**Example.** $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$.

**Lemma.** Every term $t$ has a term construction sequence of length at most the number of symbols in $t$.

(Easy proof by induction.)
Δ-Definition of Terms = \{\langle t \rangle : t \text{ is a term}\}

Definition. A term construction sequence for a term \( t \) is a finite sequence of terms \((t_1, \ldots, t_\ell)\) such that \( t_\ell \equiv t \) and, for each \( k \in \{1, \ldots, \ell\} \), the term \( t_k \) is either

- a variable symbol,
- the constant symbol 0,
- \( St_j \) for some \( j < k \), or
- \(+t_it_j\) or \( \cdot t_it_j\) or \( Et_it_j \) for some \( i, j < k \).

Key to defining Terms: We will write a Δ-formula defining the set

\[ \text{TermConSeq} = \{(c, a) : c = \langle \langle t_1 \rangle, \ldots, \langle t_\ell \rangle \rangle \text{ and } a = \langle t_\ell \rangle \text{ where } (t_1, \ldots, t_\ell) \text{ is a term construction sequence}\} \]
\textbf{Δ-Definition of Terms} = \{ \varphi t^\downarrow : t \text{ is a term} \}

\textit{TermConSeq}(c, a) \equiv \\
\text{Codenumber}(c) \land (\exists \ell < c) \left[ \text{Length}(c, \ell) \land \text{IthElement}(a, \ell, c) \land \\
(\forall k \leq \ell)(\exists e_k < c) \left[ \text{IthElement}(e_k, k, c) \land \\
\left\{ \begin{array}{l}
\text{Variable}(e_k) \\
\lor e_k = 2^{10} \\
\lor (\exists j < k)(\exists e_j < c)[\text{IthElement}(e_j, j, c) \land e_k = 2^{12} \cdot 3^{S e_j}] \\
\lor \ldots \end{array} \right. \right] \right] \\
\text{``} e_k \text{ is } \varphi Se_j \text{``} \right]}

\textbf{Key to defining Terms:} We will write a Δ-formula defining the set

\textbf{TermConSeq} = \{(c, a) : c = \langle \varphi t_1^\downarrow, \ldots, \varphi t_\ell^\downarrow \rangle \text{ and } a = \varphi t_\ell^\downarrow \text{ where } \\
(t_1, \ldots, t_\ell) \text{ is a term construction sequence} \}. 
Δ-Definition of Terms = \{\overline{t} : t \text{ is a term}\}

Now there is an obvious way to define Term(a):

\[ \text{Term}(a) \equiv (\exists c) \text{TermConSeq}(c, a). \]

To make this a Δ-formula, we need an upper bound on c as a function of a.
**Δ-Definition of Terms** $= \{ \lceil t \rceil : t \text{ is a term} \}$

Now there is an obvious way to define $\text{Term}(a)$:

$$
\text{Term}(a) :\equiv (\exists c) \text{TermConSeq}(c, a).
$$

To make this a $\Delta$-formula, we need an upper bound on $c$ as a function of $a$.

Suppose $a = \lceil t \rceil$. Another easy lemma by induction: The number of symbols in $t$ is at most $a$. Therefore, there exists a term construction sequence $(t_1, \ldots, t_\ell)$ for $t$ with length $\leq a$. We may assume that each $t_k$ is a subterm of $t$, so that $\lceil t_k \rceil \leq \lceil t \rceil = a$ for all $k \in \{1, \ldots, \ell\}$.

Let $c := \langle \lceil t_1 \rceil, \ldots, \lceil t_\ell \rceil \rangle$. We have

$$
c = 2^{\lceil t_1 \rceil + 1}3^{\lceil t_2 \rceil + 1} \cdots (p_\ell)^{\lceil t_\ell \rceil + 1} \leq (p_\ell)^{\lceil t_1 \rceil + \cdots + \lceil t_\ell \rceil + \ell} \leq (p_\ell)^{\ell a + \ell} \leq (p_a)^{a^2 + a} \leq (a + 1)^{a^3}
$$

using the (easy) fact that the $a^{th}$ prime number $p_a$ is at most $(a + 1)^a$. 

**Δ-Definition of Terms** = \{\( \lceil t \rceil : t \text{ is a term} \}\}

Now there is an obvious way to define \( \text{Term}(a) \):

\[
\text{Term}(a) :\equiv (\exists c) \text{TermConSeq}(c, a).
\]

To make this a \( \Delta \)-formula, we need an upper bound on \( c \) as a function of \( a \).

We may therefore take

\[
\text{Term}(a) :\equiv (\exists c \leq (a + 1)^3) \text{TermConSeq}(c, a).
\]
Construction Sequences for Other Recursive Definitions

In a similar way, using the notion of a formula construction sequence, we get a \( \Delta \)-definition of the set

\[
\text{FORMULAS} = \{ \Gamma \varphi \upharpoonright : \varphi \text{ is a formula} \}.
\]

Definition. A formula construction sequence for a formula \( \varphi \) is a finite sequence of terms \( (\varphi_1, \ldots, \varphi_\ell) \) such that \( \varphi_\ell \equiv \varphi \) and, for each \( k \in \{1, \ldots, \ell\} \), the term \( \varphi_k \) is either

- \( \varphi = t_1 t_2 \) for some terms \( t_1 \) and \( t_2 \)
- \( \varphi < t_1 t_2 \) for some terms \( t_1 \) and \( t_2 \)
- \( \neg \varphi_j \) for some \( j < k \)
- \( \varphi_i \lor \varphi_j \) for some \( i, j < k \)
- \( (\forall x)(\varphi_i) \) for some \( i < k \) and \( x \in Vars \)
Construction Sequences for General Recursive Definitions

This idea is very general: using an appropriate notion of construction sequence, we get a $\Delta$-definition of any recursively defined set or function.

Suppose want a $\Delta$-formula $\text{Factorial}(x, y)$ defining the function $\text{FACTORIAL} : \mathbb{N} \rightarrow \mathbb{N}$ (i.e., the $\{(a, b) \in \mathbb{N}^2 : b = a!\}$).

Key idea: Write a $\Delta$-formula defining the set

$\text{FACTORIAL}\text{ConstSeq} := \{(a, b, c) \in \mathbb{N}^3 : b = a! \text{ and } c = \langle 0!, 1!, 2!, \ldots, a! \rangle\}$

using $\text{Codenumber}$, $\text{Length}$, $\text{IthElement}$ as subformulas. (Details in tutorial.)

Homework Problem (PSET 4). Write down a $\Delta$-formula defining the function $\text{FIBONACCI} : \mathbb{N} \rightarrow \mathbb{N}$.
Next Steps (Sections 5.11–5.12)

The following are $\Delta$-definable:

**LogicalAxiom** := \{\[\Gamma \varphi \vdash : \varphi \text{ is a logical axiom}\}

**RuleOfInference** := \{\{(\[\Gamma \gamma_1 \vdash, \ldots, \Gamma \gamma_n \vdash\], \[\Gamma \varphi \vdash\]) : (\{\gamma_1, \ldots, \gamma_n\}, \varphi) \text{ is a rule of inference}\}\}

**Axiom**$_N$ := \{\[\Gamma N_1 \vdash, \ldots, \Gamma N_{11} \vdash\}\}

**Deduction**$_N$ := \{\{(\[\Gamma \delta_1 \vdash, \ldots, \Gamma \delta_1 \vdash\], \[\Gamma \varphi \vdash\]) : (\delta_1, \ldots, \delta_n) \text{ is a deduction from } N \text{ of } \varphi\}\}.

Important $\Delta$-definable functions:

\[
\begin{align*}
\text{Num}(a) & := \[\Gamma a \vdash, \\
\text{TermSub}(\[\Gamma u \vdash, \Gamma x \vdash, \Gamma t \vdash\]) & := \[\Gamma u^x \vdash, \\
\text{Sub}(\[\Gamma \varphi \vdash, \Gamma x \vdash, \Gamma t \vdash\]) & := \[\varphi^x \vdash.
\end{align*}
\]
\textbf{\(\Delta\)-DEFINABLE SETS AND FUNCTIONS}

Using an appropriate notion of “construction sequence”, we get a \(\Delta\)-formula \(\text{Num}(x, y)\) which defines the function

\[
\text{Num}(a) := \lnot \exists b.
\]

\textbf{Δ-DEFINABLE SETS AND FUNCTIONS}

Using an appropriate notion of “construction sequence”, we get a \( \Delta \)-formula \( \text{Num}(x, y) \) which defines the function

\[
\text{Num}(a) := \neg \bar{a}^\gamma.
\]

This means:

\begin{itemize}
  \item \( \mathcal{N} \models \text{Num}(\bar{a}, \bar{b}) \) for all \( (a, b) \in \mathbb{N}^2 \) such that \( b = \neg \bar{a}^\gamma \)
  \item \( \mathcal{N} \models \neg \text{Num}(\bar{a}, \bar{b}) \) for all \( (a, b) \in \mathbb{N}^2 \) such that \( b \neq \neg \bar{a}^\gamma \)
\end{itemize}
**Δ-DEFINABLE SETS AND FUNCTIONS**

Similarly (by a more complicated “construction sequence”), there is a Δ-formula $\text{Sub}(x_1, x_2, x_3, y)$ which defines the function

$$\text{Sub}(\exists \varphi \neg, \exists x \neg, \exists t \neg) := \exists \varphi^x \neg.$$
**Δ-DEFINABLE SETS AND FUNCTIONS**

Similarly (by a more complicated “construction sequence”), there is a $\Delta$-formula $Sub(x_1, x_2, x_3, y)$ which define the function

$$Sub(\varphi^\frown, x^\frown, t^\frown) := \varphi_x^\frown.$$ 

This means: for all $(a, b, c, d) \in \mathbb{N}^4$,

- $\mathcal{M} \models Sub(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ if $a = \varphi^\frown$ and $b = x^\frown$ and $c = t^\frown$ and $d = \varphi_x^\frown$ for some formula $\varphi$ and variable symbol $x$ and term $t$
- $\mathcal{M} \models \neg Sub(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ otherwise.
\textbf{Δ-DEFINABLE SETS AND FUNCTIONS}

Using \(\Delta\)-formulas \(\text{Num}(x, y)\) and \(\text{Sub}(x_1, x_2, x_3, y)\) (among other useful \(\Delta\)-formulas such as \textit{Free} and \textit{Substitutable}), we get \(\Delta\)-formulas defining sets:

\begin{align*}
\text{LogicalAxiom} & := \{ \varphi^\neg : \varphi \text{ is a logical axiom} \} \\
\text{Axiom}_N & := \{ \textit{\neg}N_1 \neg, \ldots, \textit{\neg}N_{11} \neg \}, \\
\text{RuleOfInf} & := \{ (c, a) : c = \langle \textit{\neg}\gamma_1 \neg, \ldots, \textit{\neg}\gamma_n \neg \rangle \text{ and } a = \textit{\neg}\varphi \neg \}
\text{where } (\{\gamma_1, \ldots, \gamma_n\}, \varphi) \text{ is a rule of inference} \}.
\end{align*}

Finally, we get a \(\Delta\)-formula \(\text{Deduction}_N(y, z)\) which defines the set

\begin{align*}
\text{Deduction}_N & := \{ (c, a) : c = \langle \textit{\neg}\delta_1 \neg, \ldots, \textit{\neg}\delta_1 \neg \rangle \text{ and } a = \textit{\neg}\varphi \neg \}
\text{where } (\delta_1, \ldots, \delta_n) \text{ is a deduction from } N \text{ of } \varphi \}.
\end{align*}
The $\Sigma$-formula $Thm_N(x)$

The set

$$THM_N := \{ \uparrow \varphi \uparrow : N \vdash \varphi \}$$

is defined by $\Sigma$-formula

$$Thm_N(x) :\equiv (\exists y)\, Deduction_N(y, x).$$
The $\Sigma$-formula $Thm_N(x)$

The set

$$\text{THM}_N := \{ \exists \varphi \upharpoonright : N \vdash \varphi \}$$

is defined by $\Sigma$-formula

$$Thm_N(x) \equiv (\exists y)\text{Deduction}_N(y, x).$$

This means: for every $a \in \mathbb{N}$,

- $\mathfrak{n} \models Thm(\bar{a})$ if $a = \exists \varphi \upharpoonright$ for some formula $\varphi$ such that $N \vdash \varphi$,

  (In this case, $N \vdash Thm(\bar{a})$ since $N$ proves every $\Sigma$-sentence which is true in $\mathfrak{m}$ by Proposition 5.3.13.)

- $\mathfrak{n} \models \neg Thm(\bar{a})$ otherwise.

  (We cannot conclude that $N \vdash \neg Thm(\bar{a})$ since $\neg Thm(\bar{a})$ is (equivalent to) a $\Pi$-sentence.)
The $\Sigma$-formula $Thm_N(x)$

The set

$$THM_N := \{ \varphi \vdash N \vdash \varphi \}$$

is defined by $\Sigma$-formula

$$Thm_N(x) \equiv (\exists y) Deduction_N(y, x).$$

There is no obvious way to rewrite $Thm_N(x)$ as a $\Delta$-sentence (in fact, this is impossible). For instance, we cannot replace $(\exists y)$ with $(\exists y < x^{x^x})$, since this would imply that every formula of length $\ell$ provable by $\varphi$ has a deduction of length $< \ell^{\ell^\ell}$ (which is false).
Representable $\Rightarrow$ $\Sigma$-Definable

Previously, we showed that every $\Delta$-definable set is representable.

(This is a straightforward corollary of Proposition 5.3.13: $N$ proves every $\Sigma$-sentence which is true in $\mathcal{M}$.)

Next, we show that every representable set is $\Sigma$-definable.