Reminder of Notation

Language is always $\mathcal{L}_{NT} = (0, S, +, \cdot, E, <)$.

$\mathfrak{N}$ is the natural numbers as $\mathcal{L}_{NT}$-structure $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot, E, <)$.

$N = \{N_1, \ldots, N_{11}\}$ is the set of axioms of Robinson Arithmetic.

For $a \in \mathbb{N}$, we let $\overline{a}$ stand for the variable-free term $\underbrace{SS\ldots S}_a 0$.

For a variable-free term $t$, we let $t^\mathfrak{N} \in \mathbb{N}$ stand for the interpretation of $t$ in $\mathfrak{N}$. (For example, $(SSS0 \cdot SS0)^\mathfrak{N}$ equals 6.)
The Power of Robinson Arithmetic

Robinson Arithmetic. The eleven axioms of $N$ are:

(N1) $(\forall x) \neg[Sx = 0]$
(N2) $(\forall x)(\forall y)[Sx = Sy \rightarrow x = y]$
(N3) $(\forall x)[x + 0 = x]$
(N4) $(\forall x)(\forall y)[x + Sy = S(x + y)]$
(N5) $(\forall x)[x \cdot 0 = 0]$
(N6) $(\forall x)(\forall y)[(x \cdot Sy) = (x \cdot y) + x]$
(N7) $(\forall x)[xE0 = S0]$
(N8) $(\forall x)(\forall y)[xE(Sy) = (xEy) \cdot x]$
(N9) $(\forall x)\neg[x < 0]$
(N10) $(\forall x)(\forall y)[x < Sy \leftrightarrow (x < y \lor x = y)]$
(N11) $(\forall x)(\forall y)[x < y \lor x = y \lor y < x]$. 
The Power of Robinson Arithmetic

Lemma 2.8.4. For all natural numbers $a$ and $b$:

1. If $a = b$, then $N \vdash \overline{a} = \overline{b}$.
2. If $a \neq b$, then $N \vdash \overline{a} \neq \overline{b}$.
3. If $a < b$, then $N \vdash \overline{a} < \overline{b}$.
4. If $a \not< b$, then $N \vdash \neg(\overline{a} < \overline{b})$.
5. $N \vdash \overline{a + b} = \overline{a} + \overline{b}$.
6. $N \vdash \overline{a \cdot b} = \overline{a} \cdot \overline{b}$.
7. $N \vdash \overline{a} E \overline{b} = \overline{a^b}$.

Lemma 5.3.10. $N \vdash (t = \overline{t^{\overline{t}}})$ for every variable-free term $t$. (Proof by induction on $t$, on blackboard.)

For example, if $t \equiv (SS0 + S0) \cdot SS0$, then this lemma tells us $N \vdash ((SS0 + S0) \cdot SS0 = SSSSSSSS0)$. 
Lemma 5.3.11 (Rosser’s Lemma). For every $a \in \mathbb{N}$,

$$\mathbb{N} \vdash (\forall x < a) [x = \overline{0} \lor x = \overline{1} \lor \cdots \lor x = \overline{a - 1}].$$

Proof by induction on $a$ (on blackboard)
**The Power of Robinson Arithmetic**

**Lemma 5.3.11 (Rosser’s Lemma).** For every $a \in \mathbb{N}$,

$$N \vdash \left( \forall x < a \right) \left[ x = 0 \lor x = 1 \lor \cdots \lor x = a - 1 \right].$$

Proof by induction on $a$ (on blackboard)

**Corollary 5.3.12.** For every $a \in \mathbb{N}$ and formula $\varphi(x)$,

$$N \vdash \left[ \forall x < a \varphi(x) \right] \leftrightarrow \left[ \varphi(0) \land \varphi(1) \land \cdots \land \varphi(a-1) \right]$$

that is,

$$\left[ \forall x < a \varphi \right] \leftrightarrow \left[ \varphi^x_0 \land \varphi^x_1 \land \cdots \land \varphi^x_{a-1} \right]$$

(Proof given as Exercise 11 in Section 5.3; solution on page 319. *Good exercise to try on your own!* )
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathfrak{M} \models \varphi$, then $\mathfrak{N} \vdash \varphi$.

In other words, $\mathfrak{N}$ proves every $\Sigma$-sentence which is true in $\mathfrak{M}$. 
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathfrak{M} \models \varphi$, then $N \vdash \varphi$.

In other words, $N$ proves every $\Sigma$-sentence which is true in $\mathfrak{M}$.

RECALL: As we have discussed before, $N$ does not prove every sentence which is true in $\mathfrak{M}$. In particular, $N \nvdash (\forall x)\neg[x < x]$ and $N \nvdash (\forall x)(\forall y)[x + y = y + x]$. 
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.

PROOF. Let $\varphi$ be a $\Sigma$-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of $\varphi$. 
Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $N \models \varphi$, then $N \vdash \varphi$.

**PROOF.** Let $\varphi$ be a $\Sigma$-sentence such that $N \models \varphi$. We argue by induction on the complexity of $\varphi$.

**Base case** $\varphi$ is atomic or $\neg$(atomic).

Suppose (for example) $\varphi$ is $t < u$.

Then $N \models \varphi$ means that $t^N < u^N$.

So by Lemma 2.8.4, $N \vdash t^N < u^N$.

By Lemma 5.3.10 (which we just proved), $N \models t = t^N$ and $N \models u = u^N$.

Therefore, $N \vdash t < u$ (using the (E3) axiom).
Proposition 5.3.13. If \( \varphi \) is a \( \Sigma \)-sentence such that \( \mathfrak{M} \models \varphi \), then \( \mathfrak{N} \vdash \varphi \).

PROOF. Let \( \varphi \) be a \( \Sigma \)-sentence such that \( \mathfrak{M} \models \varphi \). We argue by induction on the complexity of \( \varphi \).

Suppose \( \varphi \equiv (\alpha \lor \beta) \).

Without loss of generality, assume \( \mathfrak{M} \models \alpha \).

By induction hypothesis, \( \mathfrak{N} \vdash \alpha \).

Therefore, \( \mathfrak{N} \vdash \varphi \) by (PC) rule.
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathfrak{M} \models \varphi$, then $N \vdash \varphi$.

PROOF. Let $\varphi$ be a $\Sigma$-sentence such that $\mathfrak{M} \models \varphi$. We argue by induction on the complexity of $\varphi$.

NOTE: We do not need to consider the case $\varphi :\equiv \neg \alpha$, since $\Sigma$-sentences are not closed under negation.
Proposition 5.3.13. If \( \varphi \) is a \( \Sigma \)-sentence such that \( \mathcal{M} \models \varphi \), then \( N \vdash \varphi \).

**Proof.** Let \( \varphi \) be a \( \Sigma \)-sentence such that \( \mathcal{M} \models \varphi \). We argue by induction on the complexity of \( \varphi \).

Suppose \( \varphi \equiv (\exists y)\alpha \).

Since \( \mathcal{M} \models \varphi \), there exists \( a \in \mathbb{N} \) such that \( \mathcal{M} \models \alpha^y_a \).

Note that \( \alpha^y_a \) is a \( \Sigma \)-sentence with lower complexity than \( \varphi \) (that is, fewer \( \lor \) and \( \forall \) symbols). (NOTE: \( \alpha^y_a \) possibly has greater length as a string.)

By induction hypothesis, \( N \vdash \alpha^y_a \).

By (Q2) axiom: \( \vdash \alpha^y_a \rightarrow (\exists y)\alpha \). (Since \( a \) is variable-free, it is substitutable for \( y \) in \( \alpha \).)

Therefore, \( N \vdash \varphi \) by (PC) rule.
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathfrak{M} \models \varphi$, then $N \vdash \varphi$.

PROOF. Let $\varphi$ be a $\Sigma$-sentence such that $\mathfrak{M} \models \varphi$. We argue by induction on the complexity of $\varphi$.

Suppose $\varphi \equiv (\forall y < u)\alpha$ where $u$ is a variable-free term.

Since $\mathfrak{M} \models \varphi$, it follows that $\mathfrak{M} \models \alpha_y^a$ for every $a < u^\mathfrak{M}$.

By the induction hypothesis, $N \vdash \alpha_y^a$ for every $a < u^\mathfrak{M}$.

By Corollary 4.3.8 (the corollary of Rosser’s Lemma), we have

$$N \vdash [(\forall y < u^\mathfrak{M})\alpha] \leftrightarrow [\alpha_y^0 \land \alpha_y^1 \land \cdots \land \alpha_y^{u^\mathfrak{M} - 1}].$$

By (PC) rule, $N \vdash (\forall y < u^\mathfrak{M})\alpha$.

By Lemma 4.3.6, $N \vdash u = u^\mathfrak{M}$. This lets us derive $N \vdash (\forall y < u)\alpha$ as required.

Q.E.D.
Definable and Representable Sets

A set $A \subseteq \mathbb{N}^k$ is $\Sigma/\Pi/\Delta$-definable if there exists a $\Sigma/\Pi/\Delta$-formula $\varphi(x_1, \ldots, x_k)$ such that

- $\mathcal{M} \models \varphi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k) \in A$,
- $\mathcal{M} \models \neg \varphi(b_1, \ldots, b_k)$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$. 
Definable and Representable Sets

A set $A \subseteq \mathbb{N}^k$ is $\Sigma/\Pi/\Delta$-definable if there exists a $\Sigma/\Pi/\Delta$-formula $\varphi(x_1, \ldots, x_k)$ such that

- $\mathcal{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $\mathcal{N} \models \neg \varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

A set $A \subseteq \mathbb{N}^k$ is representable if there exists a formula $\varphi(x_1, \ldots, x_k)$ such that

- $N \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $N \models \neg \varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$. 
Definable and Representable Sets

A set $A \subseteq \mathbb{N}^k$ is \textbf{$\Sigma/\Pi/\Delta$-definable} if there exists a $\Sigma/\Pi/\Delta$-formula $\varphi(x_1, \ldots, x_k)$ such that

- $\mathfrak{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $\mathfrak{N} \models \neg \varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

A set $A \subseteq \mathbb{N}^k$ is \textbf{representable} if there exists a formula $\varphi(x_1, \ldots, x_k)$ such that

- $N \vdash \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $N \not\vdash \varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

A set $A \subseteq \mathbb{N}^k$ is \textbf{weakly representable} if there exists a formula $\varphi(x_1, \ldots, x_k)$ such that

- $N \vdash \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $N \not\models \varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$. 
Definable and Representable Sets

A function $f : A \rightarrow \mathbb{N}$ where $A \subseteq \mathbb{N}^k$ is definable or representable according to the corresponding set $\{(a_1, \ldots, a_k, b) : f(a_1, \ldots, a_k) = b\} \subseteq \mathbb{N}^{k+1}$.

Example. The function $a \mapsto a^2$ is $\Delta$-definable, since it is defined by the $\Delta$-formula $\varphi(x, y) : \equiv (y = x \cdot x)$ (or $(y = x E SS0)$).
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathcal{N} \models \varphi$, then $\mathcal{N} \vdash \varphi$.

Corollary 5.3.15. Every $\Delta$-definable set is representable.

This fact is extremely useful: it lets us show that various sets and functions are representable!
The Power of Robinson Arithmetic

Proposition 5.3.13. If $\varphi$ is a $\Sigma$-sentence such that $\mathcal{N} \models \varphi$, then $N \vdash \varphi$.

Corollary 5.3.15. Every $\Delta$-definable set is representable.

This fact is extremely useful: it lets us show that various sets and functions are representable!

Proof of Proposition $\Rightarrow$ Corollary:

Suppose $A \subseteq \mathbb{N}^k$ is defined by the $\Delta$-formula $\varphi(x_1, \ldots, x_n)$. Both $\varphi(x_1, \ldots, x_n)$ and $\neg \varphi(x_1, \ldots, x_n)$ are logically equivalent to $\Sigma$-formulas. Therefore, Proposition 5.3.13 implies

- $N \vdash \varphi(a_1, \ldots, a_k)$ (since $\mathcal{N} \models \varphi(a_1, \ldots, a_k)$) for every $(a_1, \ldots, a_k) \in A$
- $N \vdash \neg \varphi(b_1, \ldots, b_k)$ (since $\mathcal{N} \models \neg \varphi(b_1, \ldots, b_k)$) for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$. 
Various mathematical model of “computable” sets and functions were proposed in the 1930s:

- Turing machines (most intuitive model)
- Church’s λ-calculus
- Gödel’s recursive functions
- *representable functions*

Remarkably, all these models capture the same notion: a set $A \subseteq \mathbb{N}^k$ (or function $f : \mathbb{N}^k \to \mathbb{N}$) is *representable* iff it is λ-computable iff it is Turing computable iff it is recursive.

The equivalence of these various notions of “computable” is a mathematical theorem.
**Representable Functions and Computer Programs** (Section 5.4)

**Theorem.**

- If $A \subseteq \mathbb{N}^k$ is representable, then there is an algorithm (computer program) which, given $(a_1, \ldots, a_k) \in \mathbb{N}^k$ as input, determines in finite time whether or not $(a_1, \ldots, a_k) \in A$.

- Conversely, if there exists a computer program which determines membership in a set $A \subseteq \mathbb{N}^k$, then $A$ is representable.
Theorem.

- If $A \subseteq \mathbb{N}^k$ is representable, then there is an algorithm (computer program) which, given $(a_1, \ldots, a_k) \in \mathbb{N}^k$ as input, determines in finite time whether or not $(a_1, \ldots, a_k) \in A$.

- Conversely, if there exists a computer program which determines membership in a set $A \subseteq \mathbb{N}^k$, then $A$ is representable.

The Church-Turing Thesis. A function on the natural numbers is computable by a human being following an algorithm, ignoring resource limitations, if and only if it is computable by a Turing machine or any other equivalent notion (e.g., representability).