Cardinality Part I

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S and T have the same cardinality if there exists \( f : S \rightarrow T \) one-to-one onto (i.e. a “pairing”) or one-to-one correspondence. We showed that \( |N| = |E| = |Q^+| \)

|S| = |N| iff S is an infinite set whose elements can be listed. We call such sets “countably infinite”, or say they have cardinality \( \aleph_0 \).

|0, 1| ⋄ \( \aleph_0 \)

**Proof** We’ll show no list can contain all numbers in [0,1].

Let \( a_{ij} \in \{0, 1 \} \) any digit other than 0, 9 or \( a_{jj} \)

Then \( x \) isn’t among numbers listed for it differs from the \( k \)th number listed in its \( k \)th place.

Therefore \( |(0, 1)| \neq \aleph_0 \)

We say \([0, 1]\) has the cardinality of the continuum, or \( |(0, 1)| = c \)

**Definition.** \(|S| \leq |T|\) (“The cardinality of S is less than or equal to the cardinality of T”) if there exists \( T_0 \subset T \) such that \(|S| = |T_0|\).

We say \(|S| < |T|\) if \(|S| \leq |T|\) and \(|S| \neq |T|\)

**Claim:** \(|N| < |(0, 1)|\). We just proved \(|N| \neq |(0, 1)|\).

\(\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, .010, \cdots\}\)

This is an easier way:

Let \( T_0 = \{1, 1/2, 1/3, 1/4, \cdots\} \subset [0, 1] \)

Let \( f : N \rightarrow T_0 \) by \( f(n) = 1/n \).

Since \( f \) is one-to-one onto and onto, \(|N| = |T_0|\).

Therefore \(|N| \leq |T| = |(0, 1)|\), or \(|N| \leq \aleph_0 < c\).

We defined \(|S| \leq |T|\) to mean \(|S| = |T_0|\) for some \( T_0 \subset T\).

Suppose that also \(|T| \leq |S|\). Must \(|S| = |T|\) ?

\(|S| \leq |T|\) means there exists \( f : S \rightarrow T\), \( f \) one-to-one (not necessarily onto)

\(|T| \leq |S|\) means there exists \( g : T \rightarrow S\), \( g \) one-to-one

\(|S| = |T|\) means there exists \( h : S \rightarrow T\), \( h \) one-to-one and onto

**Theorem. (Schroeder-Bernstein or Cantor-Bernstein Theorem)**

If \(|S| \leq T\) and \(|T| \leq |S|\), then \(|S| = |T|\).
Theorem. If $a < b$ and $c < d$, then $|[a, b]| = |[b, d]|$ and $|(a, b)| = |(c, d)|$

Proof: Let $f(x) = c \left( \frac{x - b}{a - b} \right) + d \left( \frac{x - a}{b - a} \right)$. Then $f : [a, b] \Rightarrow [c, d]$ one-to-one and onto
$f : (a, b) \Rightarrow (c, d)$ one-to-one and onto

\[ |(\pi, 3\pi/2), [0,1]| \leq |(\pi, 3\pi)| = |[\pi + 0.1, 3\pi - 0.1]| \leq |(\pi, 3\pi)| \]

S-B $\Rightarrow |(\pi, 3\pi/2)| = |[\pi, 3\pi]|$

Corollary. If $a < b$ and $c < d$, then $|[a, b]| = |(c, d)| = |(c, d)| = |(d, c)| = |(d, c)|$. The cardinalities of any intervals (closed or not) are equal.

\[ f(x) = \tan x \]
\[ f : (\pi - \frac{\pi}{2}, \pi + \frac{\pi}{2}) \rightarrow \mathbb{R}, \text{ one-to-one and onto} \]
Therefore $|(-\pi, \pi/2)| = |\mathbb{R}|$
Therefore $|\mathbb{R}| = |[0,1]| = c.$

$[0,1] \times [0,1] = \{ (x, y) : x \in [0,1], y \in [0,1] \}$
Let $S = [0,1] \times [0,1]$ be the unit square.
To see $|\mathbb{R}| \leq |\mathbb{R}|$
Let $S_0 = \{ (x, y) \in S : y = 0 \}$.
Let $f : [0,1] \rightarrow S_0$ by $f(x) = (x, 0)$.
Therefore $|[0,1]| = |S_0| = |[0,1]| \leq |\mathbb{R}|$

Is $|S| = |[0,1]|$?
Represent points in $S$ as infinite decimals:
$(x, y) = (a_1a_2a_3\cdots ,.a_1b_2b_3\cdots )$
Choose all 9’s in ambiguous cases.
Let $f : S \Rightarrow [0,1]$ by $f(a_1a_2a_3\cdots ,.b_1b_2b_3\cdots ) = a_1b_1a_2a_3b_2a_3b_3a_4b_4\cdots$
$f$ is one-to-one (but not onto).
For example, $1.70707070707\cdots$ is not in the range of $f$; it would have to come from
$(.1000\cdots ,777\cdots )$, but this is written as $(.0999\cdots ,777\cdots )$.
Since $f$ is one-to-one, $|S| \leq |[0,1]|$.
Schroeder-Bernstein $\Rightarrow |S| = |[0,1]| = c.$

Theorem. If $|S_i| = c$ for $i = 1, 2, 3, \cdots$, then $\bigcup_{i=1}^{\infty} S_i = c$
$(\bigcup_{i=1}^{\infty} S_i = S_1 \cup S_2 \cup S_3 \cdots )$

Proof: Clearly $|\bigcup_{i=1}^{\infty} S_i| \geq c$, since $S_i \subset \bigcup_{i=1}^{\infty} S_i$.
Write $\bigcup_{i=1}^{\infty} S_i = S_1 \cup (S_2 \setminus S_1) \cup (S_3 \setminus (S_1 \cup S_2)) \cup (S_4 \setminus (S_1 \cup S_2 \cup S_3)) \cdots$ as a disjoint union.

Can construct $f : \bigcup_{i=1}^{\infty} S_i \rightarrow \mathbb{R}$ as follows: let $f$ on $S_1$ be any one-to-one function from $S_1$ to $(0,1)$; $f$ on $S_2 \setminus S_1$ is any one-to-one function from $S_2 \setminus S_1$ onto $(1,2)$, etc.
Then $f : \bigcup_{i=1}^{\infty} S_i \rightarrow \mathbb{R}$ is one-to-one.
Therefore $|\bigcup_{i=1}^{\infty} S_i| \leq |\mathbb{R}|$, S-B $\Rightarrow |\bigcup_{i=1}^{\infty} S_i| = |\mathbb{R}| = c$

In words: a countable union of sets of cardinality $c$ has cardinality $c$.
Countable numbers of squares of unit sides covers $\mathbb{R}^2$, so $|\mathbb{R}^2| = c$.
**Theorem.** Let $S = $ set of all sets of real numbers (ie. the collection of subsets of $\mathbb{R}$). Then $|S| > c$ (ie $|S| > |\mathbb{R}|$).

**Proof:** First, $|\mathbb{R}| \leq |S|$.

For each $x \in \mathbb{R}$, let $f(x) = \{x\}$ (singleton subset of $\mathbb{R}$)

If $S_0 = \{\text{all singleton subsets of } \mathbb{R}\}$, $f : \mathbb{R} \to S_0$ one-to-one and onto.

$|\mathbb{R}| \leq |S|$ by definition.

Must show: $|S| \neq |\mathbb{R}|$

Suppose there exists $g : \mathbb{R} \to S$, and show $g$ can’t be onto.

For $x \in \mathbb{R}$, $g(x)$ is a subset of $\mathbb{R}$.

Let $T = \{x \in \mathbb{R} : x \notin g(x)\}$.

**Claim:** there is no $y \in \mathbb{R}$ such that $g(y) = T$.

For if $g(y) = T$, is $y \in g(y)$ or not?

If $y \in g(y)$, then $y \notin T$ (=g(y)). Contradiction.

Therefore $y \notin g(y)$.

But if $y \notin g(y)$, $y \in T$, so $y \in g(y)$. Another contradiction.

Therefore there is no such $y$, and so $g$ is not onto.

The cardinality of $S$, we call $2^c$. Therefore $2^c > c$. 