Mathematical Induction

Original Notes adopted from September 11, 2001(Week 1)

Natural Numbers
1,2,3,4,5,6… n = { 1,2,3,4,… }

Principle of Mathematical Induction
If $S \subset$ n such that
a) $1 \in S$
b) $(k+1) \in S$ whenever $k \in S$, then s = n.

Eg. Prove: $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Let $S = \{m: 1^2 + 2^2 + 3^2 + \ldots + m^2 = \frac{m(m+1)(2m+1)}{6}$

To show: S = n.

By Mathematical Induction, suffices to show
a) $1 \in S$
b) *b) $(k+1) \in S$ whenever $k \in S$*

Is $1 \in S$? $1^2 = \frac{1(1+1)(2+1)}{6}$

$1 = 1$

Suppose $k \in S$
$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Must show:
$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+2+1)}{6}$

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$$
$$= \frac{(k+1)}{6}[2k^2 + 7k + 6]$$
$$= \frac{(k+1)}{6}[(2k+3)(k+2)]$$
$$= \frac{(k+1)(k+2)(2k+2+1)}{6}$$

which is the formula for n = k +1, so $(k+1) \in S$. $\therefore$ s has properties a) & b), so S = n.

	n!	3^n		n!	3n
n = 1	1	3	n = 5	120	243
n = 2	2	9	n = 6	720	729
n = 3	6	27	n = 7	5040	2187
n = 4	24	81			

Thm: n! > 3^n for n $\geq$ 7

Can start induction anywhere Ie.
For any $n_0 \in$ N, if S $\subset$N such that
a) $n_0 \in$ S
b) (k+1) $\in$ S whenever k $\in$ S & k $\geq$n0
then S $\supset$ {n$\in$N : n$\geq n_0$}

To prove n! > 3^n for n $\geq$ 7 use above.
Let S = {m $\in$ N: M! > 3^m}, 7 $\in$ S (we checked).
Suppose k! > 3^k, Must show: (k+1)! > 3^{k+1}
We have k! > 3^k, Multiply both sides by k + 1.
We get (k + 1)! > 3^{k+1} * (k+1)
Since k $\geq$ 7, k + 1 > 3, so
3^k(k+1) > 3^k * 3 = 3^{k+1}
$\therefore$ (k +1) ! > 3^{k+1}, so S $\supset$ { m: m $\geq$ 7 }

<u>Well Ordering Principle</u>: Every subset of N other than $\emptyset$ has a smallest element.

Assume Well-ordering Principle.
Thm: If S $\subset$ N such that
a) 1 $\in$ S
b) (k + 1) $\in$ S whenever k $\in$ S, then S = N.

Proof:
Let T = { n $\in$ M : n $\notin$ S } ("The complement of S")
Show T $\neq\emptyset$, T would have a smallest element, say n_1,
n $\neq$ 1, since 1 $\in$ S.
$\therefore$ (n_1 – 1) $\in$ N

n_1 – 1 < n_1 & n_1 least element of T $\Rightarrow$ n1 – 1 $\notin$ T
$\therefore$ n_1 - 1 $\in$ S By b), (n_1 - 1) + 1 $\in$ S, So n_1 $\in$ S.

In N, a divides b (written a|b)
If b = ac for some c $\in$ N.

Defn : p $\in$ N is prime if only divisors of p are p & 1, & p $\neq$ 1. Prime: 2,3,4,7,11,13

<u>Lemma</u>: If n is a natural number & n $\neq$ 1 & n is not a prime number, then n is a product of prime numbers.
180 = 9 x 10 x 2 = 3 x 3 x 5 x 2 x 2.

Principle of Complete Mathematical Induction.

> If $S \subset N$ such that:
> a) $1 \in S$
> b) $(k+1) \in S$ whenever $\{1,2,3, ..., k\} \subset S$, then $S = N$.

Proof of Lemma:
Let S = { n: Lemma holds for n }. Show S = N. Use Complete Induction.
Assume $\{1,2,...k\} \subset S$. Show $(k + 1) \in S$.

If k + 1 is prime, $k + 1 \in S$.
If k + 1 is not prime, then k + 1 = m * n with m, n not 1 or k+1.
m <= k, n <=k $m \in S, n \in S$.

Each of m & n is either prime or product of primes,
so k + 1 is the product of the primes that multiply to m and
the primes to n. $\therefore$ $k + 1 \in S$, so S = N.

Corollary: If $n \in N$, & $n \neq 1$, then n is divisible by a prime.
Theorem: There is no largest prime number.

Proof: Let p be a prime number.
Multiply all the primes from 2 up to p together and then add 1.
Let M = (2*3*5*7*11...p) + 1

M > P, so if M is prime, we're done.
M >1. Suppose M is not prime, By corollary q| M (q divides M) for some prime q.

We want to show: q > P (that finishes the proof).

$q \neq 2$ since M leaves remainder 1 upon division by 2
For any prime r <= p, M leaves remainder 1 upon division by r.
$\therefore$ $q \neq r$ for any r <= p. $\therefore$ q > p, finishing the proof.

Twin primes; p, p+2 both primes: Is there a biggest pair of twin primes? – Unknown.
2,3 5,7 11,13 17,19 23,29