Name: ________________________ Tutorial: ______ ID#: _______________

Midterm of MAT234

University of Toronto

Tuesday, March 4^{th} 2008, 2:10-4pm

Tutorial	Time	location	TA
TUT 01	M 16-17:30	GB 304	Stephan Stoyan
TUT 02	M 16-17:30	SF 2202	Qingan (Andy) Zhang
TUT 03	Tue 9-10:30	GB 404	Jing Wang
TUT 04	Tue 9-10:30	GB 412	Jonathan Lesage

Last name	Exam location
A-D	MR 101
E-K	MR103
L-N	MR215
O-Z	MR 202

Problems:	1	2	3	4	5	Total
Points:						

There are five problems. Do all work on these pages. No Calculators, cell phones or notes may be used. However, a sheet of formulas (Two sided $8\frac{1}{2} \times 11$) is allowed. The point value (out of 100) of each problem is marked in the margin.

1.

(10pts) (a) Find the most general function $M(x, y)$ for which the equation

$$M(x, y)dx + (3x^2y^2 + x + 1)dy = 0$$

is exact.

(10pts) (b) Solve this equation.

2.

(10pts) (a) Find a solution of the form $y = x^n$ for the differential equation

$$x^2(1-x)\frac{d^2y}{dx^2} + 2x(2-x)\frac{dy}{dx} + 2(1+x)y = 0, \quad \text{with } x > 2.$$

(10pts) (b) Using the method of the reduction of order, find the general solution of this differential equation.

3. If the impressed voltage of a RLC-circuit is given by $E(t) = E_0 \cos \omega t$, the differential equation satisfied by the current passing through the circuit is

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = -E_0\omega \sin \omega t.$$

(15pts) (a) The amplitude of the steady-state solution $I_p(t) = -A\sin(\omega t - \delta)$ is given by

$$A(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\omega\right)^2 + R^2}}.$$

What is the frequency of resonance, that is, the frequency for which the amplitude is maximal?

(5pts) Suppose the impressed voltage $E(t) = E_0 \cos(\omega t)$ comes from a weak electromagnetic signal of angular frequency $\omega = 100$ Hz. If the inductance is $L = 1H$ and the resistance is $R = 100\ \Omega$, how should one chooses the capacitance C to maximize the reception of this signal?

4. Consider the mechanical vibration system satisfying the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0.$$

(5pts) (a) If $\vec{\mathbf{y}}(t) = \begin{pmatrix} x(t) \\ \frac{dx}{dt}(t) \end{pmatrix}$, find a 2×2 matrix $\mathbf{A}$ such that $\frac{d\vec{\mathbf{y}}}{dt} = \mathbf{A}\vec{\mathbf{y}}$.

(5pts) (b) What are the eigenvalues of $\mathbf{A}$?

(5pts) (c) For each eigenvalue of $\mathbf{A}$, find an associated (nonzero) eigenvector.

(5pts) (d) Give an explicit formula for $e^{t\mathbf{A}}$.

5. Consider the first order linear system

$$\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}, \quad \text{with } \mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(15pts) (a) Find three linearly independent solutions of the form $\vec{x}(t) = e^{\lambda t}\vec{v}$ where $\vec{v}$ is a constant vector.

(5pts) (b) Find the unique solution $\vec{x}(t)$ satisfying the initial condition

$$\vec{\mathbf{x}}(0) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$