Name: ______________________ Recitation: Tu Th ID#: ______________

# MAT303: Solutions of Midterm II

Wednesday, November 16 2005

| Problems: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | | | | | |

There are four problems. Do all work on these pages. No Calculators, cell phones or notes may be used. However, a sheet of formulas ($8\frac{1}{2} \times 11$ recto-verso) is allowed. The point value (out of 100) of each problem is marked in the margin.

**1.**

(10pts) (a) Find a particular solution to the differential equation

$$\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = 3x - 1$$

The complementary solution is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x + c_3.$$

So we should look for a particular solution of the form $y_p(x) = x(Ax+B)$. Plugging this into the equation, we get

$$3x - 1 = 8Ax + 4B \quad \Longrightarrow \quad A = \frac{3}{8},\ B = -\frac{1}{4}.$$

Hence, $y_p(x) = \frac{3}{8}x^2 - \frac{1}{4}x$ is a particular solution.

(5pts) (b) Show that $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ using the identity $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

$$\begin{aligned}\cos^2\theta &= \left(\tfrac{e^{i\theta}+e^{-i\theta}}{2}\right)^2 = \tfrac{e^{2i\theta}+2+e^{-2i\theta}}{4} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{e^{i2\theta}+e^{-i2\theta}}{2}\right) = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta.\end{aligned}$$

(10pts) (c) Find a particular solution to the differential equation

$$\frac{d^2y}{dx^2} + y = \cos^2 x = \frac{1+\cos 2\theta}{2}.$$

The complementary solution is of the form $y_c(x) = c_1 \cos x + c_2 \sin x$. So, using the method of undetermined coefficients, we should look for a particular solution of the form

$$y_p(x) = A + B\cos 2x + C\sin 2x \quad \Longrightarrow \quad \frac{d^2y_p}{dx^2} = -4B\cos 2x - 4C\sin 2x.$$

Plugging this into the equation, we get

$$\frac{1+\cos 2x}{2} = -4B\cos 2x - 4C\sin 2x + A + B\cos 2x + C\sin 2x$$

which implies that $A = \frac{1}{2},\ B = -\frac{1}{6},\ C = 0$.

Hence, $y_p(x) = \frac{1}{2} - \frac{1}{6}\cos 2x$ is a particular solution. It is also possible to find a particular solution using the method of variation of parameters without too much effort.

**2.** In a damped forced oscillations system $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \cos \omega t$, the amplitude of the steady periodic solution $x_p(t) = C\cos(\omega t - \alpha)$ is given by

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

(5pts) (a) By analogy, in the RLC system $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \cos \omega t$, what is the amplitude $A(\omega)$ of the steady periodic solution $Q_p(t) = A\cos(\omega t - \alpha)$?

$$A(\omega) = \frac{E_0}{\sqrt{(\frac{1}{C} - L\omega^2)^2 + (R\omega)^2}}$$

(10pts) (b) For which values of $R$ (in terms of $L$ and $C$) does the RLC system above exhibits a phenomenon of practical resonance for the amplitude of the steady periodic solution $Q_p(t)$?

There is a frequency of practical resonance provided $A(\omega)$, seen as a function of $\omega$, has a maximum. This will happen if $f(u) = \left(\frac{1}{C} - Lu\right)^2 + R^2 u$ has a maximum for $u = \omega^2 > 0$. But

$$\left.\begin{array}{l} \frac{df}{du} = -2\left(\frac{1}{C} - Lu\right)L + R^2 \\ \frac{d^2f}{du^2} = 2L^2 \geq 0 \end{array}\right\} \Longrightarrow u_{min} = \frac{1}{2L}\left(\frac{2}{C} - \frac{R^2}{L}\right).$$

This minimum will be attained for $u_{min} > 0$ provided $\frac{R^2}{L} < \frac{2}{C}$. Hence, there is a practical resonance provided $R < \sqrt{\frac{2L}{C}}$.

(5pts) (c) In that case, what is the frequency of practical resonance in terms of $R$, $L$ and $C$?

$$\omega_r = \sqrt{u_{min}} = \sqrt{\frac{1}{2L}\left(\frac{2}{C} - \frac{R^2}{L}\right)}.$$

(5pts) (d) Is it the same as the frequency of practical resonance for the amplitude of the steady periodic current $I_p(t) = \frac{dQ_p}{dt}(t)$? Explain.

No, the frequency of practical resonance for $I_p$ is $\omega = \frac{1}{\sqrt{LC}}$ which is different. In fact, whatever are $R$, $L$ and $C$, $I_p$ always has a frequency of practical resonance, which is not the case for $Q_p$.

**3.** Consider the mass-spring-dashpot system satisfying the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0.$$

(5pts) (a) If $\vec{\mathbf{y}}(t) = \begin{pmatrix} x(t) \\ \frac{dx}{dt}(t) \end{pmatrix}$, find a $2 \times 2$ matrix $\mathbf{A}$ such that $\frac{d\vec{\mathbf{y}}}{dt} = \mathbf{A}\vec{\mathbf{y}}$.

$$\left.\begin{array}{rcl} \frac{dx}{dt} & = 0x & + \quad \frac{dx}{dt} \\ \frac{d}{dt}\left(\frac{dx}{dt}\right) & = -5x & - \quad 2\frac{dx}{dt} \end{array}\right\} \Longrightarrow \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}.$$

(5pts) (b) What are the eigenvalues of $\mathbf{A}$?

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{pmatrix} = -\lambda(-2-\lambda) + 5 \\ &= \lambda^2 + 2\lambda + 5 \implies \lambda = \tfrac{-2 \pm \sqrt{4-4(5)}}{2} = -1 \pm 2i. \end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$.

(5pts) (c) For each eigenvalue of $\mathbf{A}$, find an associated (nonzero) eigenvector.

$$\lambda_1 : \begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\mathbf{A} - \lambda_1\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-2i & 1 \\ -5 & -1-2i \end{pmatrix} = \begin{pmatrix} x - 2ix + y \\ -5x - y - 2iy \end{pmatrix},$$

so $y = (-1+2i)x$ and $\vec{\mathbf{v}}_\mathbf{1} = \begin{pmatrix} 1 \\ -1+2i \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -1+2i$. To get an eigenvector for $\lambda_2$, it suffices to take the complex conjugate of $\vec{\mathbf{v}}_\mathbf{1}$:

$$\vec{\mathbf{v}}_\mathbf{2} = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix} \text{ is an eigenvector of } \lambda_2 = -1 - 2i.$$

(5pts) (d) What is the general solution of the linear system $\frac{d\vec{\mathbf{y}}}{dt} = \mathbf{A}\vec{\mathbf{y}}$ ?

$$\vec{\mathbf{v}}_\mathbf{1} = \vec{\mathbf{u}}_\mathbf{1} + i\vec{\mathbf{w}}_\mathbf{1} \text{ with } \vec{\mathbf{u}}_\mathbf{1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\mathbf{w}}_\mathbf{1} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Thus, the general solution is

$$\begin{aligned} \vec{\mathbf{y}}(t) = e^{-t}\Bigg[c_1 &\left(\begin{pmatrix} \cos 2t \\ -\cos 2t \end{pmatrix} - \begin{pmatrix} 0 \\ 2\sin 2t \end{pmatrix}\right) \\ &+ c_2\left(\begin{pmatrix} \sin 2t \\ -\sin 2t \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos 2t \end{pmatrix}\right)\Bigg]. \end{aligned}$$

(5pts) (e) Is it critically damped, overdamped or underdamped? Justify your answer.

It is underdamped since there are oscillations.

**4.** Consider the first order linear system

$$\frac{d\vec{\mathbf{x}}}{dt} = \mathbf{A}\vec{\mathbf{x}}, \quad \text{with } \mathbf{A} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}.$$

(5pts) (a) What are the eigenvalues of $\mathbf{A}$?

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 4 \\ 0 & 4 & -\lambda \end{pmatrix} \\ &= -\lambda[(-\lambda)(-\lambda) - 16] - (3)[3(-\lambda)] \\ &= -\lambda[\lambda^2 - 16 - 9] = -\lambda[\lambda^2 - 25] = -\lambda(\lambda+5)(\lambda-5), \end{aligned}$$

so the eigenvalues are $\lambda_1 = -5$, $\lambda_2 = 0$ and $\lambda_2 = 5$.

(10pts) (b) For each eigenvalue, find an associated (nonzero) eigenvector.

$$\lambda_1 : \mathbf{0} = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x+3y \\ 3x+5y+4z \\ 4y+5z \end{pmatrix} \Longrightarrow \begin{aligned} x &= \tfrac{-3y}{5} \\ z &= \tfrac{-4y}{5} \end{aligned}$$

so $\vec{\mathbf{v}}_1 = \begin{pmatrix} -3 & 5 & -4 \end{pmatrix}^T$ is an eigenvector with eigenvalue $\lambda_1 = -5$ .

$$\lambda_2 : \mathbf{0} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y \\ 3x+4z \\ 4y \end{pmatrix} \Longrightarrow \begin{aligned} y &= 0 \\ z &= \tfrac{-3x}{4} \end{aligned}$$

so $\vec{\mathbf{v}}_2 = \begin{pmatrix} 4 & 0 & -3 \end{pmatrix}^T$ is an eigenvector with eigenvalue $\lambda_2 = 0$ .

$$\lambda_3 : \mathbf{0} = \begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5x+3y \\ 3x-5y+4z \\ 4y-5z \end{pmatrix} \Longrightarrow \begin{aligned} x &= \tfrac{3y}{5} \\ z &= \tfrac{4y}{5} \end{aligned}$$

so $\vec{\mathbf{v}}_3 = \begin{pmatrix} 3 & 5 & 4 \end{pmatrix}^T$ is an eigenvector with eigenvalue $\lambda_1 = -5$ .

(10pts) (c) What is the solution if initially $\vec{\mathbf{x}}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$?

The general solution is $\vec{\mathbf{x}}(t) = c_1 e^{-5t}\vec{\mathbf{v}}_\mathbf{1} + c_2\vec{\mathbf{v}}_\mathbf{2} + c_3 e^{5t}\vec{\mathbf{v}}_\mathbf{3}$ where $c_1$, $c_2$ and $c_3$ are arbitrary constants. At $t = 0$, this gives

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \vec{\mathbf{x}}(0) = c_1 \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\begin{matrix} 0 = -3c_1 + 4c_2 + 3c_3 \\ 1 = 5c_1 + 5c_3 \\ 0 = -4c_1 - 3c_2 + 4c_3 \end{matrix} \quad &\Longrightarrow \left\{ \begin{array}{l} c_1 = \frac{1}{5} - c_3 \\ c_2 = 7c_1 - 7c_3 \\ 0 = -4c_1 - 3(7c_1 - 7c_3) + 4c_3 \end{array} \right. \\
&\Longrightarrow \left\{ \begin{array}{l} c_1 = \frac{1}{5} - c_3 \\ 0 = -25c_1 + 25c_3 \\ c_2 = 7c_1 - 7c_3 \end{array} \right. \\
&\Longrightarrow c_1 = c_3 = \tfrac{1}{10},\ c_2 = 0.
\end{aligned}$$

Hence, the solution of this initial value problem is $\vec{\mathbf{x}}(t) = \frac{e^{-5t}\vec{\mathbf{v}}_\mathbf{1} + e^{5t}\vec{\mathbf{v}}_\mathbf{3}}{10}$.