Name: ____________________ Recitation: Tu Th ID#: ______________

Solutions to MAT303: Midterm I

Wednesday, October 12 2005

Problems:	1	2	3	4	5	Total
Points:						

There are five problems. Do all work on these pages. No Calculators, cell phones or notes may be used. The point value (out 100) of each problem is marked in the margin.

1. In a small town of 100 persons, there are $P(t)$ persons having the flu after t days. Assume that the rate of increase of $P(t)$ satisfies the differential equation:

$$\frac{dP}{dt} = \frac{kP(100-P)}{100},$$

where k is some constant.

(5pts) (a) Should k be positive or negative? Explain your answer.

The constant k should be positive, since the flu will propagate and one therefore expects the number P(t) of persons having the flu will increase.

(10pts) (b) Find the solution if at time $t = 0$ only one person has the flu.

This is a separable equation (also a Bernoulli equation):

$$\frac{dP}{P(100-P)} = \frac{kdt}{100}.$$

Hence, integrating both side, we get:

$$\begin{aligned} \tfrac{kt}{100} &= \int \tfrac{dP}{P(100-P)} = \int \left(\tfrac{1}{P} + \tfrac{1}{100-P}\right) \tfrac{dP}{100} \\ &= \tfrac{1}{100} \left(\log|P| - \log|100-P|\right) + C = \tfrac{1}{100} \log\left|\tfrac{P}{100-P}\right| + C. \end{aligned}$$

Since P and $100 - P$ are positive, this implies that

$$\frac{P}{100-P} = C_1 e^{kt} \quad \Rightarrow \quad P = \frac{100C_1}{e^{-kt} + C_1}.$$

In order to have $P(0) = 1$, we need $C_1 = \frac{1}{99}$, so we finally get:

$$P(t) = \frac{100}{1 + 99e^{-kt}}.$$

(5pts) (c) When t gets large, does $P(t)$ approach some fixed value? If yes, what is this value? Yes,

$$\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty} \frac{100}{1+99e^{-kt}} = \frac{100}{1+0} = 100,$$

which is the total population of the village.

2.

(10pts) (a) Find the solution of the initial value problem:

$$\frac{dv}{dx} - \frac{6}{x}v = 2x, \quad v(1) = 0.$$

This is a linear equation with $P(x) = -\frac{6}{x}$ and $Q(x) = 2x$. Since

$$\int P(x)dx = \int -\frac{6}{x}dx = -6\log|x|,$$

the integrating factor is given by $\rho(x) = |x|^{-6} = x^{-6}$. Hence,

$$\rho(x)v(x) = \int \rho(x)Q(x)dx = \int \frac{2dx}{x^5} = \frac{-1}{2x^4} + C$$

and $v(x) = -\frac{x^2}{2} + Cx^6$. To get $v(1) = 0$ we need $C = \frac{1}{2}$, so the final answer is:

$$v(x) = \frac{x^6 - x^2}{2}.$$

(10pts) (b) Find the solution of the initial value problem:

$$(1+x)\frac{dy}{dx} = 4y, \quad y(0) = 1.$$

This is a separable equation:

$$\frac{dy}{4y} = \frac{dx}{1+x}.$$

Integrating on both sides we get:

$$\frac{1}{4}\log|y| = \log|1+x| + C \quad \Rightarrow \quad |y| = K|1+x|^4 = K(1+x)^4, \quad K = e^C > 0.$$

Thus, $y = K(1+x)^4$ with K possibly negative. To get $y(0) = 1$, we need $K = 1$, so the final answer is:

$$y(x) = (1+x)^4.$$

3.

(10pts) (a) Check that the following differential equation is exact and find its general solution:

$$\underbrace{(4x-y)}_{M}\,dx+\underbrace{(6y-x)}_{N}\,dy=0.$$

$$\frac{\partial M}{\partial y}=-1=\frac{\partial N}{\partial x}\quad\Rightarrow\quad\text{the equation is exact.}$$

Hence, for $F(x,y)$ such that $\frac{\partial F}{\partial x}=M$, $\frac{\partial F}{\partial y}=N$, we will have,

$$F(x,y)=\int M dx=\int(4x-y)dx=2x^2-yx=g(y).$$

Hence,

$$6y-x=N=\frac{\partial F}{\partial y}=-x+g'(y)\quad\Rightarrow\quad g'(y)=6y\quad\Rightarrow\quad g(y)=\int 6ydy=3y^2+C.$$

Taking $C=0$, we get $F(x,y)=2x^2-yx+3y^2$, so the general solution is

$$2x^2-yx+3y^2=C,\quad\text{where C is an arbitrary constant.}$$

(5pts) (b) Show that the following equation is not exact:

$$\underbrace{-(xy+y^2)}_{M}\,dx+\underbrace{x^2}_{N}\,dy=0.$$

$$\frac{\partial M}{\partial y}=-x-2y\neq 2x=\frac{\partial N}{\partial x}\quad\Rightarrow\quad\text{the equation is not exact.}$$

(10pts) (c) Find the general solution of 3.(b).

It is a homogeneous equation (also a Bernoulli equation). Putting $v=\frac{y}{x}$, the equation becomes (after some computation):

$$\frac{dv}{dx}=\frac{v^2}{x}\quad\Rightarrow\quad\frac{dv}{v^2}=\frac{dx}{x}.$$

Integrating on both sides, we get

$$-\frac{1}{v}=\log|x|+C\quad\Rightarrow\quad y(x)=xv(x)=\frac{-x}{C+\log|x|}.$$

4.

(5pts) (a) Show that the functions $\cos x$ and $\sin x$ are linearly independent functions on the real line by looking at their Wronskian at $x = 0$.

$$W(\cos 0, \sin 0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

so $\cos x$ and $\sin x$ are linearly independent on the real line.

(10pts) (b) Find the solution of the initial value problem

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 0, \quad x(0) = 0, \ \frac{dx}{dt}(0) = 1.$$

The characteristic polynomial is $r^2 + 6r + 10$ and its roots are $r = -3 \pm i$. Therefore, the general solution is of the form

$$x(t) = e^{-3t}(A\cos x + B\sin x).$$

Thus,

$$x'(t) = e^{-3t}[(-3A + B)\cos t + (-3B - A)\sin t].$$

Hence,

$$\begin{array}{rll} 0 = x(0) & = A & \\ 1 = x'(0) & = -3A + B \end{array} \quad \Rightarrow \quad \begin{array}{r} A = 0 \\ B = 1 \end{array}$$

so the solution is

$$x(t) = e^{-3t}\sin t.$$

(10pts) (c) Find the general solution of $Ly = 0$ if

$$L = (D-1)(D^2 + 4D + 4), \quad D = \frac{d}{dx}.$$

The characteristic polynomial is

$$(r-1)(r^2 + 4r + 4) = (r-1)(r+2)^2,$$

so $r = 1$ and $r = -2$ are the two roots, $r = -2$ having multiplicity 2. The general solution is therefore of the form

$$y(x) = Ae^x + Be^{-2x} + Cxe^{-2x},$$

where A, B and C are arbitrary constants.

5. Suppose that the air resistance of a drop of water (of mass m) in free fall is proportional to the cube of its velocity, so that its equation of motion is

$$m\frac{d^2h}{dt^2} = -k\left(\frac{dh}{dt}\right)^3 - mg,$$

where h is the height of the drop of water, k is some positive constant and $g = 9.8m/s^2$ is the gravitational acceleration.

(10pts) (a) What is the terminal velocity of the drop of water (the velocity it ultimately reaches)?

The velocity will stop to change when the force of friction will exactly compensate for the force of gravity. So the terminal velocity v_T is such that

$$\begin{aligned} 0 = -kv_T^3 - mg \;&\Rightarrow\; v_T^3 = -\tfrac{mg}{k}, \\ &\Rightarrow\; v_T = -\left(\tfrac{mg}{k}\right)^{\frac{1}{3}}. \end{aligned}$$

Thus, the terminal velocity is $v_T = -\left(\frac{mg}{k}\right)^{\frac{1}{3}}$. It is negative since the drop of water is going down.