MAT1342/MAT464: Assignment V

Due in class Friday, April 3rd 2009

1. Let (M, g) be a Riemannian manifold.
(a) Show that the Ricci tensor R_{ij} satisfies the following identity:

$$\nabla^i R_{ij} = \frac{1}{2}\nabla_j s$$

where s is the scalar curvature. (*Hint*: it follows from the second Bianchi identity)
(b) The Riemannian manifold (M, g) is said to be Einstein if its Ricci tensor is proportional to the metric, that is, if there exists $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} = \lambda g.$$

If the dimension of (M, g) is greater than 2, show that (M, g) is Einstein if and only if

$$\mathrm{Ric} = \frac{s}{n} g$$

(c) Show that $(\mathbb{P}^n(\mathbb{C}), g_{FS})$ is Einstein, where g_{FS} is the Fubini-Study metric. (*Hint*: Have a look at the proof of proposition 3.21 in [1]).
(d) Using theorem 3.82 of [1], conclude that $(\mathbb{P}^1(\mathbb{C}), g_{FS})$ is isometric to $(\mathbb{S}^2, \lambda g_{\mathrm{can}})$ for some $\lambda > 0$ where g_{can} is the canonical metric on $\mathbb{S}^2$.
(e) Compute $\lambda > 0$ explicitly (*Hint*: Recall example 2.110 in [1]).
(f) Show that for $n > 0$, $(\mathbb{P}^n(\mathbb{C}), g_{FS})$ also has a positive sectional curvature. (*Hint*: Use O'Neil formula in theorem 3.61 of [1]).
(g) Show that for $n > 1$, $(\mathbb{P}^n(\mathbb{C}), g_{FS})$ does not have a constant sectional curvature. (*Hint*: You can use the fact $\mathbb{P}^n(\mathbb{C})$ is simply connected and is not homeomorphic to the sphere).

2. Let (M, g) be a Riemannian manifold of dimension n.
(a) If (M, g) has constant sectional curvature K, show that

$$\mathrm{Ric} = (n-1)Kg, \quad s = n(n-1)K.$$

(b) If the Ricci tensor is parallel, that is, if $\nabla \,\mathrm{Ric} \equiv 0$, show that the scalar curvature is constant.

(c) If $h = \lambda g$ is a rescaled version of the metric g where λ is a positive constant, describe the curvature tensor, the Ricci tensor, the curvature tensor and the sectional curvature of h in terms of the curvature tensor of g.

3. Read example 2.90 (the exponential map on Lie groups) in [1].

4. Show that the cone $\{(x, y, z) \in \mathbb{R}^3 \mid z > x^2 + y^2\}$ and the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0, x^2 + y^2 = 1\}$ with their naturally induced metrics are diffeomorphic, locally isometric but not globally isometric.

References

[1] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, Berlin, 1993.