MAT1342/MAT464: Assignment IV

Due Friday, March 13th 2009

1. Let $\pi : E \to M$ be a real vector bundle on a manifold M and let ∇^E be a covariant derivative for this vector bundle.
(a) If N is another manifold and $f : N \to M$ is smooth map, define the pull-back of E, denoted f^*E, to be the vector bundle over N with fibre above $x \in N$ given by:

$$f^*E_x := E_{f(x)}$$

Check that indeed f^*E defined in this way inherits a structure of vector bundle from the one of E. Check furthermore that there is a unique covariant derivative ∇^{f^*E} on f^*E such that for any section $e \in \Gamma(E)$, $\nabla^{f^*E}(f^*e) = f^*(\nabla^E e)$, that is,

$$\nabla^{f^*E}_X(e \circ f) = (\nabla_{f_*X} e) \circ f \quad \forall X \in \Gamma(TN).$$

The covariant derivative ∇^{f^*E} can be interpreted as the pull-back of the covariant derivative ∇^E.
(b) If $E = E_1 \oplus E_2$ is the direct sum of two vector bundles, denote by $P_i : E \to E_i$, $i \in \{1, 2\}$ the corresponding projections on each factor, and by $\iota_i : E_i \hookrightarrow E$ the corresponding inclusions. Show that the operators

$$\nabla^{E_i}_X := P_i \nabla^E_X \iota_i : \Gamma(E_i) \to \Gamma(E_i), \quad X \in \Gamma(TM),$$

define covariant derivatives for E_1 and E_2. Is it necessarily true that $\nabla^E = \nabla^{E_1} \oplus \nabla^{E_2}$?
2. Do exercise 2.57 of [1].
3. Do exercise 2.83b,d,e of [1].

References

[1] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, Berlin, 1993.