MAT1342/MAT464: Assignment III

Due in class Wednesday, February 25th 2009

1. (cf. §2.66 in [1]) On a Riemannian manifold (M, g) the Riemannian metric defines a canonical isomorphism

$$\begin{array}{rccc} \flat: & TM & \to & T^*M \\ & X & \mapsto & \xi := g(X, \cdot) \end{array}$$

between the tangent and the cotangent bundle with inverse map denoted $\sharp : T^*M \to TM$.

(a) If $(x^1, \ldots, x^n)$ are local coordinates on M and $X = X^i \frac{\partial}{\partial x^i}$, show that

$$\flat(X) = \xi_i dx^i, \quad \text{with } \xi_i = g_{ij} X^j$$

where $g_{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. This formula suggests the notation $X_i := g_{ij} X^j$, a subscript indicating that we are dealing with a covector, and a superscript indicating we are dealing with a vector. In this notation $\flat$ 'lowers' indices.

(b) Denote by g^{ij} the entries of the matrix which is the inverse of the matrix $[g_{ij}]$ with entries g_{ij}, thus $g^{ij} g_{jk} = \delta^i_k$ where $\delta^i_k = 1$ if $i = k$ and is zero if $i \neq k$. Show that for the covector $\xi = \xi_i dx^i$,

$$\sharp(\xi) = X^i \frac{\partial}{\partial x^i} \quad \text{with } X^i := g^{ij} \xi_j.$$

This formula suggests the notation $\xi^i := X^i = g^{ij} \xi_j$, so that $\sharp$ raises the index of ξ. Show that this notation is consistent with the previous one, namely, that $g_{ij} \xi^j = \xi_i$.

(c) Given two covectors $\xi, \eta \in T^*_m M$, $m \in M$, we can define their inner product by $g_m(\xi, \eta) := g_m(\sharp(\xi), \sharp(\eta))$. Show that in local coordinates with $\xi = \xi_i dx^i$ and $\eta = \eta_i dx^i$, we have

$$g_m(\xi, \eta) = g^{ij} \xi_i \eta_j = g_{ij} \xi^i \eta^j = \xi_i \eta^i.$$

2. Consider $\mathbb{R}^4$ with standard coordinates (x^0, x^1, x^2, x^3) where x^0 is a time variable and $\vec{x} := (x^1, x^2, x^3)$ are space variables. Consider on this space the trivial **complex** line bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & L := \mathbb{R}^4 \times \mathbb{C} \\ & & \big\downarrow \mathrm{pr}_1 \\ & & \mathbb{R}^4 \end{array}$$

Thus, a smooth section of L is just a smooth complex valued function on $\mathbb{R}^4$. A covariant derivative for L is a first order differential operator $\nabla : \Gamma(L) \to \Gamma(T^*\mathbb{R}^4 \otimes L)$ satisfying the Leibniz' rule for **complex** valued functions:

$$\nabla(fs) = df \otimes s + f\nabla s, \quad \forall f \in \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{C}), \quad \forall s \in \Gamma(L).$$

We say that a covariant derivative is **unitary** if it also preserves the natural Hermitian inner product in the fibres of L,

$$X(h(s_1, s_2)) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2), \; s_1, s_2 \in \Gamma(L),$$

for $X \in \Gamma(T\mathbb{R}^4)$, where $h_p(s_1(p), s_2(p)) := \overline{s_1(p)} s_2(p) \in \mathbb{C}$ for $p \in \mathbb{R}^4$.

(a) Thinking of $s \in \Gamma(L)$ as a smooth complex valued function, we can define a (trivial) covariant derivative by

$$D_X s := Xs, \quad X \in \Gamma(T\mathbb{R}^4).$$

Check that this covariant derivative is unitary. Check more generally that a unitary covariant derivative ∇ is of the form

(1) $$\nabla_X s = D_X s + \sqrt{-1}(A(X))s$$

where $A = A_\alpha dx^\alpha$ is a smooth section of $T^*\mathbb{R}^4$. This means the space of unitary covariant derivative is an affine space modelled on $\Gamma(T^*\mathbb{R}^4)$.

(b) Consider the group $\mathcal{G} := \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{S}^1)$, $\mathbb{S}^1 \subset \mathbb{C}$ with group law given by multiplication as complex valued functions. It is called the gauge group. It naturally acts on the space of sections of L by multiplication. Clearly, for $g \in \mathcal{G}$ and $s_1, s_2 \in \Gamma(L)$, we have that $h(gs_1, gs_2) = h(s_1, s_2)$. More generally, $\mathcal{G}$ acts on the space of unitary covariant derivatives: for $g \in \mathcal{G}$ and ∇ a unitary covariant derivative, define a new covariant derivative by $\nabla^g := g \cdot \nabla \cdot g^{-1}$, that is,

$$\nabla^g_X s := g(\nabla_X(g^{-1}s), \quad s \in \Gamma(L).$$

Show that indeed ∇^g is a unitary covariant derivative. How does the vector potential A in (1) transform under the action of an element $g \in \mathcal{G}$?

(c) We define the curvature of a unitary covariant derivative ∇ by

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

Check that for $g \in \mathcal{G}$, ∇^g has the same curvature as ∇. This means that gauge transformations preserve the curvature.

(d) Let ∇ be a unitary covariant derivative as in (1). Write the curvature as

$$R = \sum_{\alpha,\beta} \sqrt{-1} F_{\alpha\beta} dx^\alpha \otimes dx^\beta,$$

where $F_{\alpha\beta} := -\sqrt{-1} R(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta})$. **Show** that the 4 by 4 matrix defined by $F_{\alpha\beta}$ is skew-symmetric. Furthermore, **computes** its entries in terms of the components of the electric field

$$\vec{E} = (E_1, E_2, E_3), \; \vec{E} = \vec{D} A_0 - \frac{\partial \vec{A}}{\partial x^0}$$

and the components of the magnetic field

$$\vec{B} = (B_1, B_2, B_3), \quad \vec{B} = \vec{D} \times \vec{A} = \mathbf{curl}(\vec{A}),$$

with $\vec{A} = (A_1, A_2, A_3)$ and $\vec{D} = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$. The matrix $F_{\alpha\beta}$ is called the electromagnetic tensor.

3. Let $E \to M$ be a real vector bundle over M.
(a) Suppose that ∇^1 and ∇^2 are two covariant derivatives for the vector bundle E and that α_1 and α_2 are smooth functions such that $\alpha_1 + \alpha_2 = 1$. Then show that

$$\nabla := \alpha_1 \nabla^1 + \alpha_2 \nabla^2$$

is also a covariant derivative.
(b) Suppose now that the manifold M is countable at infinity. Show then that the vector bundle E can be equipped with a covariant derivative. *Hint*: Use a partition of unity...

4. A flat vector bundle is a vector bundle together with a covariant derivative whose curvature is zero.
(a) Show that a trivial vector bundle always admits a covariant derivative such that it becomes flat.
(b) Show that on a manifold of dimension 1, the curvature of any vector bundle with any covariant derivative is always zero.
(c) Give an example of a flat vector bundle which is not a trivial vector bundle.

References

[1] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, Berlin, 1993.