

MAT1342/MAT464: Assignment II

Due in class Wednesday, February 4th 2009

1. Let G be a manifold which is at the same a group with the property that the group law

$$\begin{aligned} m : G \times G &\rightarrow G \\ (g, h) &\mapsto gh \end{aligned}$$

is a smooth map.

(a) Show that for each $g \in G$, the maps $L_g : G \rightarrow G$ and $R_g : G \rightarrow G$ given by left and right multiplication by g are diffeomorphisms.

(b) Show that the map

$$\begin{aligned} F : G \times G &\rightarrow G \times G \\ (g, h) &\mapsto (g, gh) \end{aligned}$$

is bijective. Show furthermore that its differential at any point is surjective and injective. Conclude by the inverse function theorem that it is a diffeomorphism.

(c) Using the result of (b), show that the map $\gamma : G \rightarrow G$ which to an element $g \in G$ associates its inverse $\gamma(g) := g^{-1}$ is a smooth map. Thus, this shows that G is a Lie group.

2. Let G be a discrete group of diffeomorphisms of a manifold M . We suppose that G acts freely and properly discontinuously on M . Recall that we say that G acts properly discontinuously on M if for $x \in M$ there exists a neighborhood U of x such that for any $g \in G$, $gU \cap U = \emptyset$ unless $g = e$ is the identity. In particular, you can use the fact this implies that given $x, y \in M$ with y not belonging to the orbit of x , there exists two neighborhoods V and W of x and y such that for all $g \in G$, $gV \cap W = \emptyset$.

(a) Show that the quotient M/G is a Hausdorff space. Recall that the topology of M/G is defined by declaring a set \mathcal{U} open if and only if $\pi^{-1}(\mathcal{U}) \subset M$ is open, where $\pi : M \rightarrow M/G$ is the canonical projection.

(b) Show that the Hausdorff space M/G admits a unique structure of smooth manifold such that the canonical projection $\pi : M \rightarrow M/G$ is smooth.

(c) Let $\mathbb{Z}_3 = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ be the group of cube roots of unity. It acts smoothly on the five dimensional sphere

$$\mathbb{S}^5 := \{(u, v, w) \in \mathbb{C}^3 \mid |u|^2 + |v|^2 + |w|^2 = 1\} \subset \mathbb{C}^3 \cong \mathbb{R}^6$$

by multiplication, $\lambda \cdot (u, v, w) := (\lambda u, \lambda v, \lambda w)$ for $\lambda \in \mathbb{Z}_3$. Show that in this way, \mathbb{Z}_3 in fact acts freely and properly. Conclude that S^5/\mathbb{Z}_3 is a smooth manifold. It is an example of lens space.

3. Do exercise 2.11a,b, and c in the book [1].
4. Let $\mathfrak{o}(n)$ denote the Lie algebra of $O(n)$. Recall that $\mathfrak{o}(n)$ can be identified with the space of skew-symmetric n by n matrices.
 - (a) Show that the bilinear form

$$\langle X, Y \rangle := \text{Tr}(XY^T), \quad X, Y \in \mathfrak{o}(n)$$

is positive definite. Show furthermore that it is invariant under the adjoint action of $O(n)$ onto its Lie algebra, that is,

$$\langle \text{Ad}_g(X), \text{Ad}_g(Y) \rangle = \langle X, Y \rangle \quad \forall g \in O(n).$$

(b) Using this bilinear form, we can define a Riemannian metric on $O(n)$ by:

$$g_A(V, W) := \langle A^{-1}V, A^{-1}W \rangle, \quad A \in O(n), \quad V, W \in T_A O(n).$$

Show that the metric g is left invariant, that is, $(L_A)_*g = g$ for all $A \in O(n)$, where

$$((L_A)_*g)_B(V, W) = g_{A^{-1}B}(TL_{A^{-1}}V, TL_{A^{-1}}W), \quad V, W \in T_B O(n).$$

(c) Using part (a), show the metric g is also right invariant, that is, $(R_A)_*g = g$ for all $A \in O(n)$. We say that the metric g is biinvariant.

(d) On the other hand, under the canonical embedding $O(n) \subset \text{GL}_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, $O(n)$ naturally inherits a Riemannian metric from the Euclidean metric of \mathbb{R}^{n^2} . Is it the same as the biinvariant metric g defined in (b)?

5. Let M be a compact manifold of dimension n and (\mathcal{U}_i, ϕ_i) , $i = 1, \dots, k$ an atlas of class \mathcal{C}^∞ for M . Using a partition of unity, show that M can be embedded as a submanifold of \mathbb{R}^{nk} .

REFERENCES

- [1] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, Berlin, 1993.