MAT1342/MAT464: Assignment I

Due in class Wednesday, January 21st 2009

1. We denote by $\mathbb{P}^n\mathbb{C}$ the space of complex lines of $\mathbb{C}^{n+1}$ passing by the origin. In other words, $\mathbb{P}^n\mathbb{C}$ is the set of equivalence classes of points in $\mathbb{C}^{n+1} \setminus \{0\}$ with equivalence given by

$$x \sim y \iff \exists\, \lambda \in \mathbb{C}^* \text{ such that } x = \lambda y.$$

(a) Show that $\mathbb{P}^n\mathbb{C}$ with its induced topology is a Hausdorff space and show that it is naturally a smooth manifold of (real) dimension $2n$ by exhibiting an atlas of class $\mathcal{C}^\infty$.
(b) Let $[z_0 : z_1 : z_2] \in \mathbb{P}^2\mathbb{C}$ denote the equivalence class of $(z_0, z_1, z_2) \in \mathbb{C}^3 \setminus \{0\}$, where $z_k = x_k + iy_k \in \mathbb{C}$. Show that the set

$$\mathcal{Q} = \{[z_0 : z_1 : z_2] \in \mathbb{P}^2\mathbb{C} \quad | \quad z_0^2 + z_1^2 + z_2^2 = 0\}$$

is a submanifold of $\mathbb{P}^2\mathbb{C}$ of dimension 2.

2. Consider the set H of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$.
(a) Show that H is a Lie group (it is called the Heisenberg group). Determine it Lie algebra and show that its exponential map is a diffeomorphism from its Lie algebra to H.
(b) Show with a counter example that the exponential map is not always a (global) homeomorphism from the Lie algebra of a Lie group onto this Lie group.

3. The algebra of Quaternions $\mathbb{H}$ is a non-commutative algebra of dimension 4 over $\mathbb{R}$. It admits a basis of 4 elements $1, i, j, k$ where 1 is the unit satisfying the relations

$$ij = k = -ji,\ ki = j = -ik,\ jk = i = -kj,\ i^2 = j^2 = k^2 = -1.$$

A general element is of the form $q = q_0 + q_1 i + q_2 j + q_3 k$ where q_0, q_1, q_2, q_3 are real numbers. We denote by

$$\overline{q} := q_0 - q_1 i - q_2 j - q_3 k$$

the conjugate of q, and by $|q|$ the norm of q defined by

$$|q|^2 := \sum_{m=0}^{3} q_m^2.$$

Using the basis $1, i, j, k$, we can identify $\mathbb{H}$ with $\mathbb{R}^4$ as real vector spaces.

(a) Show that the set of unit quaternions defined by[1]

$$\mathbb{S}^3 = \{q \in \mathbb{H} \quad | \quad |q| = 1 \}.$$

is a Lie group with group law induced by the product of $\mathbb{H}$. (*Hint*: Show first that $|q|^2 = q\overline{q}$ and $p \cdot q = \overline{\overline{q} \cdot \overline{p}}$ for $p, q \in \mathbb{H}$.)

(b) Describe the Lie algebra and the left invariant vector fields of the Lie group $\mathbb{S}^3$. What can you say about the tangent bundle of $\mathbb{S}^3$?

(c) Show that the map

$$q = q_0 + q_1 i + q_2 j + q_3 k \mapsto \begin{pmatrix} q_0 + iq_1 & q_2 + iq_3 \\ -q_2 + iq_3 & q_0 - q_1 i \end{pmatrix}$$

induces a diffeomorphism $\Psi : \mathbb{S}^3 \to \mathrm{SU}(2)$ which is at the same time a group homomorphism. Thus, as Lie groups, S^3 and $\mathrm{SU}(2)$ are equivalent.

(d) Denote by $\mathrm{Im}(\mathbb{H}) = \{q \in \mathbb{H} \mid q_0 = 0\} \cong \mathbb{R}^3$ the space of imaginary quaternions. Show that $\mathbb{S}^3$ acts by conjugation on $\mathrm{Im}(\mathbb{H})$,

$$\Phi(q)\xi := q\xi\overline{q}, \quad q \in \mathbb{S}^3, \xi \in \mathrm{Im}(\mathbb{H}).$$

Under the identification $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$, show that this induces a a group homomorphism $\Phi : \mathbb{S}^3 \to \mathrm{SO}(3)$.

(e) Show that the map $\Phi : \mathbb{S}^3 \to \mathrm{SO}(3)$ is a submersion and an immersion. Show also it is surjective (you may assume $\mathrm{SO}(3)$ is compact and connected if needed) and that $\Phi(q_1) = \Phi(q_2)$ if and only if $q_1 = \pm q_2$. This shows that $\mathbb{S}^3 \cong \mathrm{SU}(2)$ is the double cover of the group $\mathrm{SO}(3)$. The group $\mathbb{S}^3 \cong \mathrm{SU}(2)$ is also called the spin group of dimension 3 and is denoted $\mathrm{Spin}(3)$.

4. Show that the definition of the differential of a smooth map given in 1.36 of [1] does not depend on the choice of charts.

References

[1] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer-Verlag, Berlin, 1993.

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[1] The notation is chosen to indicate that under the identification $\mathbb{H} \cong \mathbb{R}^4$, $\mathbb{S}^3$ corresponds to the sphere of dimension 3.