

MAT1061: Assignment VI

Due in class Friday, April 3rd 2009

1. Consider the reaction-diffusion equation

$$\begin{cases} \dot{\mathbf{u}} - \Delta \mathbf{u} = \mathbf{f}(\mathbf{u}) & \text{in } \mathcal{U}_T \\ \mathbf{u} = 0 & \text{on } \partial\mathcal{U} \times [0, T] \\ \mathbf{u} = \mathbf{g} & \text{on } \mathcal{U} \times \{0\}, \end{cases} \quad (1)$$

where $\mathbf{u} := (u_1, u_2, u_3)$ represents the concentrations of three types of molecules, $\mathbf{g} = (g_1, g_2, g_3) \in C_c^\infty(\mathcal{U}; \mathbb{R}^3)$ is a non-negative function giving the initial concentrations and $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by

$$\mathbf{f}(u_1, u_2, u_3) = (-2u_1u_2, -u_1u_2, 2u_1u_2).$$

Here, \mathcal{U} is a connected bounded open set in \mathbb{R}^3 with smooth boundary.

(a) Choose $\delta > 0$ such that $\sup |\mathbf{g}| < \delta$. Fix $R > 3\delta$ and consider a new smooth map $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that \mathbf{F} has compact support contained in the ball of radius $2R$ and $\mathbf{F}(\mathbf{u}) = \mathbf{f}(\mathbf{u})$ for $|\mathbf{u}| \leq R$. Show that (1) with \mathbf{f} replaced by \mathbf{F} admits a unique weak solution $\mathbf{v} = (v_1, v_2, v_3)$ for positive time.

(b) Show that the solution \mathbf{v} found in (a) is in fact smooth.

(c) Consider the compact set $K \subset \mathbb{R}^3$ defined by

$$K := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq R\}.$$

By applying the maximum principle to the functions v_1, v_2, v_3 , and $w := v_1 + v_2 + v_3$, show that for all $x \in \mathcal{U}$ and $t > 0$, $\mathbf{v}(t, x) \in K$. Conclude that \mathbf{v} is also a solution to the reaction-diffusion equation (1), but with \mathbf{f} this time instead of the function \mathbf{F} .

(d) Is this solution unique?

(e) The equation could be seen as a simple (maybe naive...) model for the chemical reaction



with u_1 giving the concentration of bihydrogen H_2 , u_2 giving the concentration of bioxygen O_2 and u_3 giving the concentration of molecules of water H_2O . Is this model compatible with the famous principle of Lavoisier (inspired by Anaxagoras of Clazomenae) “Nothing is lost, nothing is created, everything is transformed”? What boundary condition would enforce this principle inside \mathcal{U} ?

2. Let (M, g) be a compact connected Riemannian manifold of dimension $n \geq 3$. Consider the Yamabe equation

$$-c\Delta_g\varphi + R_g(x)\varphi = \tilde{R}(x)\varphi^{N-1}, \quad c := \frac{4(n-1)}{n-2}, \quad N = \frac{2n}{n-2}, \quad (2)$$

where \tilde{R} is a given smooth function on M . We have that \tilde{R} is the scalar curvature of a Riemannian metric $\tilde{g} = \varphi^{\frac{4}{n-2}}g$ in the conformal class of g if and only if (2) has a strictly positive smooth solution φ . In what follows, we will suppose that the Yamabe constant of (M, g) is strictly negative.

(a) Consider the operator

$$L(\varphi) := -c\Delta_g\varphi + R_g(x)\varphi. \quad (3)$$

By considering the functional

$$I[\varphi] = \int_M (c|\nabla\varphi|^2 + R_g\varphi^2)dg \quad (4)$$

with space of admissible functions

$$\mathcal{A} := \{\varphi \in H^1(M) \mid \|\varphi\|_{L^2(M)} = 1\},$$

show that $\nu = \inf_{\varphi \in \mathcal{A}} I(\varphi)$ is the smallest eigenvalue of L with eigenfunction given by a strictly positive smooth function $\Psi \in \mathcal{C}^\infty(M)$. Show furthermore that ν is negative. (*Hint*: It is the same variational approach as the Yamabe equation, but it is easier since L is linear).

(b) Suppose now that the function \tilde{R} is strictly negative. With Ψ the eigenfunction of part (a), show that for $c_+ > 0$ chosen large enough and $c_- > 0$ chosen small enough, $u_\pm := c_\pm\Psi$ are respectively a supersolution and a subsolution of (2),

$$Lu_+ \geq \tilde{R}u_+^{N-1}, \quad Lu_- \leq \tilde{R}u_-^{N-1}.$$

(c) Show that there exists a constant $\lambda > 0$ such that the function $z \mapsto \tilde{R}(x)z^{N-1} + (\lambda - R_g(x))z$ is nondecreasing for all $x \in M$ and $z \in [0, \sup u_+]$.

(d) Write $u_0 := u_-$. Given u_k , define inductively u_{k+1} by

$$-c\Delta_g u_{k+1} + \lambda u_{k+1} = \tilde{R}u_k^{N-1} - R_g(x)u_k + \lambda u_k \quad (5)$$

Show that for all $k \in \mathbb{N}$, $u_- \leq u_k \leq u_{k+1} \leq u_+$.

(e) Describe what should be a weak solution of (2). Show that the functions u_k converge to a weak solution u of (2) as $k \rightarrow +\infty$.

(f) Show that the weak solution u is smooth and strictly positive.

3. Do problem 2 of Chapter 9 in [1].

4. Do problem 3 of Chapter 9 in [1].

References

- [1] L.C. Evans, *Partial differential equations*, American Mathematical Society, Providence, Rhode Island, 1998.