MAT1061: Assignment III

Due in class Friday, Frebruary 13th 2009

1. Do probem 6 of Chapter 7 in [1].

2. Do problem 7 of Chapter 7 in [1].

3. Let $g(t)$, $t \in [0,T]$ be a smooth family of Riemannian metrics on a compact manifold M.

(a) Let $C \in \mathbb{R}$ be given. Establish the following dichotomy: If u and v are $\mathcal{C}^2$ functions on $[0,T] \times M$ such that $u(0,m) \geq v(0,m)$ for all $m \in M$, then exactly one the following two possiblities happens:

(i) $u(t,m) \geq v(t,m)$ for all $(t,m) \in [0,T] \times M$;

(ii) there exists $(t,m) \in (0,T] \times M$ such that

$$\begin{aligned} u(t,m) &< v(t,m) \\ \nabla u(t,m) &= \nabla v(t,m) \\ \Delta_{g(t)} u(t,m) &\geq \Delta_{g(t)} v(t,m) \\ \tfrac{d}{dt} u(t,m) &\leq \tfrac{d}{dt} v(t,m) - C(v(t,m) - u(t,m)) \end{aligned}$$

where $\Delta_{g(t)}$ is the Laplacian associated to the metric $g(t)$.

Hint: By replacing u, v with $(u-v), 0$, you can assume that $v = 0$. By replacing u with $e^{Ct}u$, you may also assume that $C = 0$.

(b) Suppose now that the functions u and v above are such that

$$\begin{aligned} \tfrac{d}{dt} u(t,m) &\geq \Delta_{g(t)} u(t,m) + F(t,m,u(t,m)), \\ \tfrac{d}{dt} v(t,m) &\leq \Delta_{g(t)} v(t,m) + F(t,m,v(t,m)), \end{aligned}$$

for all $(t,m) \in [0,T] \times M$, where $F : [0,T] \times M \times \mathbb{R} \to \mathbb{R}$ is a function uniformly Lipschitz in the last variable, that is, there exists a constant $K > 0$ such that for all $(t,m) \in [0,T] \times M$,

$$|F(t,m,x_1) - F(t,m,x_2)| \leq K|x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

Show that if $u(0,m) \geq v(0,m)$ for all $m \in M$, then in fact $u(t,m) \geq v(t,m)$ for all $(t,m) \in [0,T] \times M$.

Hint: Choose the constant C in **(a)** to be greater than the Lipschitz constant of F.

(c) Prove that on a compact Riemannan surface Σ with Riemannian metric g, the normalized Ricci flow

$$\begin{cases} \frac{d\varphi}{dt} = e^{-\varphi}(\Delta_g \varphi - R_g) + \overline{r}, \\ \varphi(0) \equiv 0, \end{cases} \tag{1}$$

with $g(t) : e^{\varphi(t)} g$ has at most one smooth solution. *Hint*: If φ_1 and φ_2 are two such solutions, try to show that $\varphi_1 \geq \varphi_2$ and $\varphi_2 \geq \varphi_1$.

4. Let $(\Sigma, g(t))$ be a solution of the normalized Ricci flow on a compact surface Σ with $g(0) = g$.

(a)(optional) Show that the the norm square of the scalar curvature satisfies the evolution equation

$$\frac{d}{dt}|\nabla R|^2 = \Delta |\nabla R|^2 - 2|\nabla\nabla R|^2 + (4R - 3r)|\nabla R|^2,$$

where the norms, the covariant derivatives and the Laplacian are in terms of the metric $g(t)$. For instance,

$$|\nabla\nabla R|^2 = g^{ij}(t) g^{kl}(t) (\nabla_i \nabla_k R)(\nabla_j \nabla_l R).$$

Hint: Use the identities $\nabla\Delta = \Delta\nabla - \frac{1}{2} R \nabla$ and $\Delta|\nabla R|^2 = 2\langle \Delta\nabla R, \nabla R\rangle + 2|\nabla\nabla R|^2$ where $\langle X, Y\rangle := g^{ij}(t) X_i Y_j$.

(b) If Σ has negative Euler characteristic, conclude that there exists a positive constant C_1 such that

$$|\nabla R|^2 \leq C_1 e^{\frac{rt}{2}}$$

for all $(t, \sigma) \in [0, \infty) \times \Sigma$.

References

[1] L.C. Evans, *Partial differential equations*, American Mathematical Society, Providence, Rhode Island, 1998.