

# MAT1061: Assignment I

Due Friday, January 16 2009 in class

1. (cf. problem 4, p.345 in Chapter 6 of [1]) Consider the equation

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n, \quad (1)$$

where  $f \in L^2(\mathbb{R}^n)$  and  $c : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function with  $c(0) = 0$  and  $c'$  bounded.

(a) Explain in what sense a function  $u \in H^1(\mathbb{R}^n)$  with compact support can be interpreted as a weak solution of (1) (*Hint: You can use the result of problem 16, p.291 in Chapter 5 of [1]*).

(b) If  $u \in H^1(\mathbb{R}^n)$  has compact support and is a weak solution of (1), show that in fact  $u \in H^2(\mathbb{R}^n)$ .

2. (multiplicativity for Sobolev spaces) Let  $\mathcal{U}$  be a bounded open set in  $\mathbb{R}^n$  with smooth boundary.

(a) For  $p \in [1, \infty)$  and  $k \in \mathbb{N}$  such that  $k > \frac{n}{p}$ , show that there exists a **continuous** bilinear map

$$m : W^{k,p}(\mathcal{U}) \times W^{k,p}(\mathcal{U}) \rightarrow W^{k,p}(\mathcal{U})$$

which is given by the usual multiplication  $m(u, v) = uv$  whenever  $u$  and  $v$  are smooth functions.

*Hint:* The main step in the proof consists in estimating the  $L^p$  norm of  $D^\alpha u D^\beta v$  for  $|\alpha + \beta| \leq k$  in terms of the Sobolev norms of  $u$  and  $v$ . Consider the cases

- (i)  $k - |\alpha| > \frac{n}{p}$ ,
- (ii)  $k - |\beta| > \frac{n}{p}$ ,
- (iii)  $k - |\alpha| \leq \frac{n}{p}$  and  $k - |\beta| \leq \frac{n}{p}$ ,

separately.

(b) If  $u \in H^k(\mathcal{U})$  is a weak solution of

$$\Delta u = u^4$$

with  $k > \frac{n}{2}$ , show that in fact  $u$  is a smooth function.

3. Consider the heat equation on  $\mathbb{R}^n$ ,

$$u_t - \Delta u = 0, \quad u(0, x) = g(x), \quad t \geq 0, x \in \mathbb{R}^n. \quad (2)$$

(a) Let  $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$  be the Fourier transform of  $f$ . Suppose  $g \in \mathcal{S}(\mathbb{R}^n)$  is a Schwartz function and  $u$  is a solution of (2) so that for each  $t \geq 0$ ,  $u(t, \cdot)$  is in  $\mathcal{S}(\mathbb{R}^n)$ . Taking the Fourier transform (in space) on both side of (2), show that

$$\hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{g}.$$

(b) Taking the inverse Fourier transform show then that  $u$  is given by

$$u(t, x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy, \quad (x \in \mathbb{R}^n, t > 0).$$

(c) Suppose the solution above is such that

$$\sup_{x \in \mathbb{R}^n, t \in [0, S]} (1 + |x|^2)^N |D^\alpha u(t, x)| < \infty$$

for all  $S > 0$ ,  $N \in \mathbb{N}$  and  $\alpha \in \mathbb{N}_0^{n+1}$ . Show that

$$v(t) := \int_{\mathbb{R}^n} u^2(t, x) dx$$

is a non-increasing function of  $t$ .

*Hint:* Multiply the heat equation by  $u$  and integrate over  $[t_1, t_2] \times \mathbb{R}^n$ .

4. Do problem 5, p.425 of Chapter 7 in [1].

## References

- [1] L.C. Evans, *Partial differential equations*, American Mathematical Society, Providence, Rhode Island, 1998.