PDE II – Schauder estimates

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In this lecture, we consider linear second order differential operators in non-divergence form

$$Lu(x) = a^{ij}(x)D^2_{ij}u(x) + b^i(x)D_iu(x) + c(x)u(x). \tag{0.1}$$

for functions u on a smooth domain $\Omega \subset \mathbb{R}^n$. We assume that the coefficients a^{ij}, b^i and c are Hölder continuous for some $\alpha \in (0,1)$, i.e.

$$\|a^{ij}\|_{C^\alpha(\Omega)}, \|b^i\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)} \leq \Lambda \tag{0.2}$$

for some $\Lambda < \infty$. We assume that the operator L is uniformly elliptic, i.e. that there exists a constant $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n. \tag{0.3}$$

Let us also recall the definition of the Hölder norms,

$$\|u\|_{C^{k,\alpha}(\Omega)} = \sum_{|\gamma|\leq k} \|D^\gamma u\|_{L^\infty(\Omega)} + \sum_{|\gamma|=k} |D^\gamma u|_\alpha, \tag{0.4}$$

where $|\cdot|_\alpha$ denotes the α-Hölder constant on Ω,

$$|f|_\alpha = \sup_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha}. \tag{0.5}$$

The goal of this lecture is to prove the following interior estimate.

Theorem 0.1 (Interior Schauder estimate)
There exists a constant $C = C(n,\alpha,\lambda,\Lambda) < \infty$ with the following significance. If L is a linear second order differential operators of the form (0.1) on $\Omega = B_2(0)$ satisfying the Hölder continuity assumption (0.2) and the ellipticity assumption (0.3), then

$$\|u\|_{C^{2,\alpha}(B_1(0))} \leq C\left(\|Lu\|_{C^\alpha(B_2(0))} + \|u\|_{L^\infty(B_2(0))}\right). \tag{0.6}$$

Remark. For convenience we consider the case $B_1(0) \subset B_2(0) = \Omega$, but of course this implies that a similar estimate (with another constant C) holds for any $\Omega' \Subset \Omega$.

Remark. Historically, Schauder estimates have been proved first by carefully estimating the Newtonian potential $u = \Gamma * f$ associated with the Newtonian kernel $\Gamma(x) = c_n|x|^{2-n}$ $(n \neq 2)$ respectively $\Gamma(x) = (2\pi)^{-1}\log|x|$ $(n = 2)$, see e.g. [2, Chap. 4, 6]. We present instead a more modern blowup argument due to Leon Simon [3].

The core of the proof is to establish the estimate for the Laplacian on $\mathbb{R}^n$:

Theorem 0.2 (Fundamental Schauder estimate)
There exists a constant $C = C(\alpha, n) < \infty$ such that

$$|D^2 u|_\alpha \le C|\triangle u|_\alpha. \tag{0.7}$$

for every $u \in C^{2,\alpha}(\mathbb{R}^n)$.

For the proof of Theorem 0.2 we need the following lemma:

Lemma 0.3 (Liouville type lemma)
Let $C < \infty, \varepsilon > 0$. If $u : \mathbb{R}^n \to \mathbb{R}$ is a harmonic function with $\sup_{B_r(0)}|u| \le Cr^{3-\varepsilon}$ for all $r < \infty$, then u is a quadratic polynomial.

Proof of Lemma 0.3. Since u is harmonic, we have the derivative estimates

$$|D^\gamma u(x_0)| \le \frac{C_k}{r^{|\gamma|+n}}\|u\|_{L^1(B(x_0,r))}, \tag{0.8}$$

see e.g. [1, p. 29]. Using the growth assumption and sending $r \to \infty$ this implies that $D^\gamma u(x_0) = 0$ whenever $|\gamma| > 2$, and thus proves the claim. □

Proof of Theorem 0.2. If the assertion doesn't hold, then there exists a sequence $u_\ell \in C^{2,\alpha}(\mathbb{R}^n)$ such that

$$|D^2 u_\ell|_\alpha > \ell|\triangle u_\ell|_\alpha. \tag{0.9}$$

After replacing u_ℓ by $\lambda_\ell u_\ell$, where $\lambda_\ell = |D^2 u_\ell|_\alpha^{-1}$, we can assume that

$$|D^2 u_\ell|_\alpha = 1, \quad |\triangle u_\ell|_\alpha < \ell^{-1}. \tag{0.10}$$

By the pigeon-hole principle, there exist $i, j, k \in \{1, \dots, n\}$ such that for infinitely many ℓ there are $x_\ell \in \mathbb{R}^n$ and $h_\ell > 0$ such that

$$\frac{|D^2_{ij} u_\ell(x_\ell + h_\ell e_k) - D^2_{ij} u_\ell(x_\ell)|}{h_\ell^\alpha} \ge \frac{1}{2n^3}. \tag{0.11}$$

We now shift x_ℓ to the origin and rescale suitably by h_ℓ, i.e. we consider

$$\tilde{u}_\ell(x) = h_\ell^{-2-\alpha} u_\ell(x_\ell + h_\ell x). \tag{0.12}$$

For $\tilde{u}_\ell$ the formulas (0.10) and (0.11) take the form

$$|D^2 \tilde{u}_\ell|_\alpha = 1, \quad |\triangle \tilde{u}_\ell|_\alpha < \ell^{-1}, \quad |D^2_{ij}\tilde{u}_\ell(e_k) - D^2_{ij}\tilde{u}_\ell(0)| \ge \frac{1}{2n^3}. \tag{0.13}$$

After adding a suitable second order polynomial, we can assume that

$$\tilde{u}_\ell(0) = 0, \quad D\tilde{u}_\ell(0) = 0, \quad D^2\tilde{u}_\ell(0) = 0, \tag{0.14}$$

and still retain the estimates (0.13). By compactness, after passing to a subsequence we can assume that $u_\ell \to u$ in C^2_{loc}. The limit u has the properties

$$u(0)=0,\ Du(0)=0,\ D^2u(0)=0,\ |D^2u|_\alpha \le 1,\ \triangle u=0,\ D^2_{ij}u(e_k)\neq 0. \tag{0.15}$$

By Lemma 0.3 we conclude that u is a second order polynomial, and thus that D^2u is constant; this contradicts (0.15). □

Proof of Theorem 0.1. Applying Theorem 0.2 after a linear change of coordinates we see that there exists a constant $C_1 = C_1(\alpha,n,\lambda) < \infty$ such that if $A = (a^{ij})$ is a positive definite symmetric matrix with eigenvalues bounded between λ and λ^{-1}, then

$$|D^2v|_\alpha \le C_1|a^{ij}D^2_{ij}v|_\alpha. \tag{0.16}$$

for every $v \in C^{2,\alpha}(\mathbb{R}^n)$.
Now let L be as in the statement of the theorem. Given a point $x_0 \in B_1 = B_1(0)$ and a function $v \in C^{2,\alpha}(B_\rho(x_0))$ ($\rho < 1$), we can freeze the coefficients a^{ij}, namely we can write

$$a^{ij}(x_0)D^2_{ij}v = Lv - (a^{ij} - a^{ij}(x_0))D^2_{ij}v - b^iD_iv - cv. \tag{0.17}$$

Using the rule $|fg|_\alpha \le \|f\|_{L^\infty}|g|_\alpha + |f|_\alpha\|g\|_{L^\infty}$ and assumption (0.2) we see that

$$|(a^{ij} - a^{ij}(x_0))D^2_{ij}v|_\alpha \le \Lambda\rho^\alpha|D^2v|_\alpha + \Lambda\|D^2v\|_{L^\infty(B_\rho(x_0))}. \tag{0.18}$$

Choosing ρ small enough such that $C_1\Lambda\rho^\alpha \le 1/2$ from (0.16) – (0.18) we obtain

$$|D^2v|_\alpha \le C_2\left(|Lv|_\alpha + \|v\|_{C^2(B_\rho(x_0))}\right) \tag{0.19}$$

for some $C_2 = C_2(n,\alpha,\lambda,\Lambda) < \infty$. Applying (0.19) for $v = \xi u$ where ξ is a suitable cutoff function, and with various center points $x_0 \in B_{3/2}(0)$, we infer that

$$\|u\|_{C^{2,\alpha}(B_1)} \le C_3\left(\|Lu\|_{C^\alpha(B_2)} + \|u\|_{C^2(B_2)}\right) \tag{0.20}$$

for some $C_3 = C_3(n,\alpha,\lambda,\Lambda) < \infty$.
The final step is to replace the C^2-norm on the right hand side of (0.20) by the L^∞-norm. To this end, we recall the interpolation inequality (c.f. Assignment 2)

$$\|v\|_{C^2(B_1)} \le \varepsilon\|v\|_{C^{2,\alpha}(B_1)} + C_\varepsilon\|v\|_{L^\infty(B_1)}, \tag{0.21}$$

where $\varepsilon > 0$ is as small as we want and $C_\varepsilon = C_\varepsilon(n,\alpha) < \infty$. Consider

$$Q := \sup_{x\in B_2} d(x,\partial B_2)^2|D^2u(x)|. \tag{0.22}$$

Given $x_0 \in B_2$, let $\rho = \frac{1}{3}d(x_0,\partial B_2)$ and consider the rescaled function

$$\tilde{u}(x) = u(x_0+\rho x), \qquad x \in B_2. \tag{0.23}$$

It solves the equation

$$\tilde{L}\tilde{u}(x) = \rho^2 Lu(x_0 + \rho x), \tag{0.24}$$

where

$$\tilde{L} = a^{ij}(x_0 + \rho x)D^2_{ij} + \rho b^i(x_0 + \rho x) + \rho^2 c(x_0 + \rho x). \tag{0.25}$$

Thus, taking also into account that $\rho \leq 1$, estimate (0.20) gives

$$\|\tilde{u}\|_{C^{2,\alpha}(B_1)} \leq C_3 \left(\|\rho^2 Lu(x_0 + \rho\,\cdot)\|_{C^\alpha(B_2)} + \|u(x_0 + \rho\,\cdot)\|_{C^2(B_2)} \right) \tag{0.26}$$

$$\leq C_3 \left(\|Lu\|_{C^\alpha(B_2)} + \|u\|_{L^\infty(B_2)} + \rho^2 \|D^2 u\|_{L^\infty(B_{2\rho}(x_0))} \right). \tag{0.27}$$

Putting things together, and choosing $\varepsilon = \min\{1, \frac{1}{18}C_3^{-1}\}$, this implies

$$\tfrac{1}{9} d(x_0, \partial B_2)^2 |D^2 u|(x_0) \tag{0.28}$$

$$\leq \rho^2 \|D^2 u\|_{L^\infty(B_\rho(x_0))} \tag{0.29}$$

$$= \|D^2 \tilde{u}\|_{L^\infty(B_1)} \tag{0.30}$$

$$\leq \varepsilon \|\tilde{u}\|_{C^{2,\alpha}(B_1)} + C_\varepsilon \|\tilde{u}\|_{L^\infty(B_1)} \tag{0.31}$$

$$\leq (C_3 + C_\varepsilon) \left(\|Lu\|_{C^\alpha(B_2)} + \|u\|_{L^\infty(B_2)} \right) + \varepsilon C_3 \rho^2 \|D^2 u\|_{L^\infty(B_{2\rho}(x_0))} \tag{0.32}$$

$$\leq (C_3 + C_\varepsilon) \left(\|Lu\|_{C^\alpha(B_2)} + \|u\|_{L^\infty(B_2)} \right) + \tfrac{1}{18} Q \tag{0.33}$$

Since $x_0 \in B_2$ was arbitrary, we infer that

$$\tfrac{1}{4} \sup_{x \in B_{3/2}} |D^2 u| \leq Q \leq 18(C_3 + C_\varepsilon) \left(\|Lu\|_{C^\alpha(B_2)} + \|u\|_{L^\infty(B_2)} \right). \tag{0.34}$$

Combined with (0.20) (with B_2 replaced by $B_{3/2}$), this proves the theorem. □

References

[1] L.C. Evans, *Partial Differential Equations*, AMS, 2010.

[2] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.

[3] L. Simon, *Schauder estimates by scaling*, Calc. Var. PDE 5(5):391–407, 1997.