

PDE II - Problem Set 4 (due: March 31)

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1. Graphical solutions of the curve shortening flow

- Let $u(x, t)$ be a solution of the PDE $u_t = (\arctan u_x)_x$. Prove that $\Gamma_t = \text{graph}(x \mapsto u(x, t))$ evolves by curve shortening flow $\partial_t p = \vec{\kappa}(p)$, $p \in \Gamma_t$. Hint: Recall that $\partial_t p$ denotes the velocity in *normal* direction.
- Using the previous part, check that the grim reaper and the hairclip indeed satisfy the curve shortening flow equation.
- Prove that the grim reaper is up to scaling and rigid motion the only complete eternal selfsimilarly translating convex solution of the curve shortening flow.

2. Parabolic rescaling and selfsimilarly shrinking solutions

- Let $\gamma : S^1 \times [0, T) \rightarrow \mathbb{R}^2$ be a solution of the curve shortening flow. Given $\lambda > 0$ we rescale space by the factor λ and time by the factor λ^2 , i.e. we consider $\gamma^\lambda : S^1 \times [0, \lambda^2 T) \rightarrow \mathbb{R}^2$, defined as

$$\gamma^\lambda(\theta, t) := \lambda\gamma(\theta, \lambda^{-2}t). \quad (0.1)$$

Show that γ^λ is again a solution of the curve shortening flow.

- Let $\gamma : S^1 \times (-\infty, 0) \rightarrow \mathbb{R}^2$ be the family of round shrinking circles given by $\gamma(\theta, t) = \sqrt{-2t}(\cos \theta, \sin \theta)$. Show that γ is invariant under parabolic rescaling, i.e. that $\gamma^\lambda = \gamma$ for every $\lambda > 0$.
- Let $\gamma : S^1 \times (-\infty, 0) \rightarrow \mathbb{R}^2$ be an ancient solution of the curve shortening flow whose image $\Gamma_t := \gamma(S^1, t)$ is invariant under parabolic rescaling, i.e. such that $\lambda \cdot \Gamma_t = \Gamma_{\lambda^2 t}$ for every $\lambda > 0$. Show that $\Gamma_t = \sqrt{-t}\Gamma_{-1}$ and that Γ_{-1} satisfies the equation

$$\kappa + \frac{1}{2}\langle \gamma, N \rangle = 0. \quad (0.2)$$

- Prove that the only closed embedded curve which satisfies (0.2) is the round circle of radius $\sqrt{2}$.

3. Evolution equations under curve shortening flow

Suppose $\gamma : S^1 \times [0, T) \rightarrow \mathbb{R}^2$ evolves by curve shortening flow, i.e. $\partial_t \gamma = \kappa N = \partial_s^2 \gamma$, where $\partial_s = |\frac{\partial \gamma}{\partial x}|^{-1} \partial_x$ denotes differentiation with respect to arc length.

- Prove the commutator identity $[\partial_t, \partial_s] = \kappa^2 \partial_s$.
- Use the commutator identity to compute $\partial_t \partial_s^2 \gamma$. Use your result to derive the evolution equations $\partial_t \kappa = \partial_s^2 \kappa + \kappa^3$ and $\partial_t N = -(\partial_s \kappa) T$.
- Show that $(\partial_t - \partial_s^2) \partial_s \kappa = 4\kappa^2 \partial_s \kappa$ and $(\partial_t - \partial_s^2) \partial_s^2 \kappa = 5\kappa^2 \partial_s^2 \kappa + 8\kappa(\partial_s \kappa)^2$.
- Prove by induction that

$$(\partial_t - \partial_s^2) \partial_s^\ell \kappa = (\ell + 3) \kappa^2 \partial_s^\ell \kappa + \sum c_{ijk}^{(\ell)} \partial_s^i \kappa \partial_s^j \kappa \partial_s^k \kappa, \quad (0.3)$$

where $c_{ijk}^{(\ell)}$ are nonnegative integers and the sum is over $0 \leq i \leq j \leq k \leq \ell - 1$ with $i + j + k = \ell$.

4. Derivative estimates for the curve shortening flow

Prove that there are constants $C_\ell = C_\ell(K, T) < \infty$ such that if $\gamma : S^1 \times [0, T) \rightarrow \mathbb{R}$ is a curve shortening flow with $\sup_{S^1 \times [0, T)} |\kappa|(x, t) \leq K$, then

$$\sup_{x \in S^1} |\partial_s^\ell \kappa|(x, t) \leq C_\ell t^{-\ell/2} \quad \forall t \in (0, T). \quad (0.4)$$

Hint: Consider the evolution of $F_\ell = t^\ell |\partial_s^\ell \kappa|^2 + \sum_{i=0}^{\ell-1} \beta_i^{(\ell)} t^i |\partial_s^i \kappa|^2$ for suitable constants $\beta_i^{(\ell)}$, and proceed by induction.