PDE II - Problem Set 3 (due: March 12)

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1. Monotonicity of Boltzmann entropy
Let u be a positive solution of the heat equation on $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions. Show that the Boltzmann entropy $\mathcal{E}(t) := -\int_\Omega u(x,t)\log u(x,t)\,dx$ is nondecreasing in time.

2. Sobolev spaces involving time
Assume $u_k \rightharpoonup u$ in $L^2([0,T];H^1_0(\Omega))$ and $u_k' \rightharpoonup v$ in $L^2([0,T];H^{-1}(\Omega))$. Prove that $v = u'$.

3. Elliptic regularization for the heat equation
Consider the equation $\partial_t u_\varepsilon = \triangle u_\varepsilon + \varepsilon \partial_t^2 u_\varepsilon$ in $\Omega \times (0,T]$ with the boundary condition $u_\varepsilon|_{\partial\Omega\times[0,T]} = 0$ and the initial condition $u_\varepsilon|_{t=0} = g \in C^\infty_c(\Omega)$.

- Produce solutions of this equation using the calculus of variations (Hint: Look for a energy functional involving the measure $e^{-t/\varepsilon}dxdt$)
- Prove energy estimates that are uniform in ε.
- Explain how in the limit $\varepsilon \to 0$ we obtain a solution of the heat equation.

4. Maximum principle
Let $u(x,t)$ be a smooth solution of $\partial_t u = \triangle u + u^2$, where $x \in \mathbb{R}^n/\mathbb{Z}^n$ and $t \in [0,T]$. Assume that $|u|(x,t) \leq K$ for all $(x,t) \in \mathbb{R}^n/\mathbb{Z}^n \times [0,T]$. Prove that there exists a constant $C = C(K,T) < \infty$ such that

$$\max_{x\in\mathbb{R}^n/\mathbb{Z}^n} |\nabla u|^2(x,t) \leq \frac{C}{t}. \tag{0.1}$$

Hint: Consider the evolution of $f := t|\nabla u|^2 + \beta u^2$ for a suitable constant β and apply the maximum principle.