PDE II - Problem Set 2 (due: Feb 12)

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1. Boundary Schauder Estimates
Consider the upper half space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$, and let $\alpha \in (0,1)$. Prove that there exists a constant $C = C(n, \alpha) < \infty$ such that

$$|D^2 u|_\alpha \leq C |\triangle u|_\alpha \tag{0.1}$$

for every $u \in C^{2,\alpha}(\overline{\mathbb{R}^n_+})$ with $u|_{\partial \mathbb{R}^n_+} = 0$.

2. An ε-regularity theorem
Let $n = 4$, and suppose that $u \in H^1(B_1)$ satisfies

$$-D_i(a_{ij} D_j u) \leq u^2 \tag{0.2}$$

weakly. Assume that

$$a_{ij}\xi_i\xi_j \geq \lambda |\xi|^2 \quad \text{and} \quad \|a_{ij}\|_{L^\infty(B_1)} \leq \Lambda \tag{0.3}$$

for some constants $\lambda > 0$, $\Lambda < \infty$. Prove that there exist constants $\varepsilon = \varepsilon(\lambda, \Lambda) > 0$ and $C = C(\lambda, \Lambda) < \infty$ such that

$$\|u\|_{L^2(B_1)} \leq \varepsilon \quad \Rightarrow \quad \|u\|_{L^\infty(B_{1/2})} \leq C \|u\|_{L^2(B_1)}. \tag{0.4}$$

3. Constraint minimizers
Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a smooth domain. Use the method of constraint minimizers (Evans 8.4.1) to prove the existence of a nontrivial solution $u \in H^1_0(\Omega)$ of

$$-\triangle u = |u|^{q-1} u \tag{0.5}$$

for $1 < q < \frac{n+2}{n-2}$.

4. Three distinct solutions
Let $\Omega \subset \mathbb{R}^3$ be a smooth domain and let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ be the eigenvalues of $-\triangle : H^1_0(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$. Consider for $f \in L^2(\Omega)$ the problem

$$-\triangle u = \lambda u - u^3 + f, \qquad u|_{\partial\Omega} = 0. \tag{0.6}$$

Prove that for each $\lambda \in (\lambda_1, \lambda_2)$ there is a constant $c(\lambda) > 0$ such that for $\|f\|_{L^2(\Omega)} \leq c(\lambda)$, there are at least three distinct solutions of this problem.

We will randomly select 2 questions, for which you will receive points $p_1, p_2, p_3 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.