PDE II - Problem Set 1 (due: Jan 29)

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1. Weak vs strong convergence
Consider the sequence of functions $f_k(x) = \frac{1}{k}\sin(k\pi x)$, where $x \in (0,1)$.

- Show that f_k is uniformly bounded in $W^{1,2}((0,1))$.
- Show that $f_k \rightharpoonup 0$ in $W^{1,2}((0,1))$.
- Show that f_k does not converge strongly in $W^{1,2}((0,1))$.

2. Issues with a naive approach to the Plateau problem
Let $\Gamma \subset \mathbb{R}^3$ be a simple closed curve. Consider the class of functions

$$\mathcal{F}_\Gamma = \{u \in W^{1,2}(D^2,\mathbb{R}^3) | \; u|_{\partial D} \in C^0(\partial D, \mathbb{R}^3) \text{ is a weakly monotone parametrization of } \Gamma\}, \quad (0.1)$$

and the energy functional $E : \mathcal{F}_\Gamma \to \mathbb{R}$,

$$E[u] = \frac{1}{2}\int_{D^2} |\nabla u|^2 dx \quad (0.2)$$

Let $\mathcal{G}$ be the Mobius group of the disc,

$$\mathcal{G} = \left\{ \varphi(z) = e^{i\alpha}\frac{a+z}{1+\bar{a}z} | \alpha \in \mathbb{R}/2\pi\mathbb{Z}, |a| < 1 \right\}. \quad (0.3)$$

- Prove that $E[u] = E[u \circ \varphi]$ for all $u \in \mathcal{F}_\Gamma$, $\varphi \in \mathcal{G}$.
- Let $u \in \mathcal{F}_\Gamma$. Prove that there exists a sequence $\varphi_k \in \mathcal{G}$ such that $u \circ \varphi_k$ converges weakly in $W^{1,2}(D^2,\mathbb{R}^3)$ to a constant map.

3. The q-Laplacian
Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let $f : \bar{\Omega} \to \mathbb{R}$ be a smooth function. For $1 < q < \infty$, the q-Laplacian is the quasilinear 2nd order elliptic operator defined by

$$\triangle_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u). \quad (0.4)$$

- Prove the existence of a weak solution of the Dirichlet problem

$$-\triangle_q u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{0.5}$$

- Prove the uniqueness of weak solution of (0.5). You may assume $q \geq 2$.
- Give an example of a nonsmooth solution of (0.5) for q and Ω and f (smooth) of your choice.

4. A forth order variational elliptic equation

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let $f : \bar{\Omega} \to \mathbb{R}$ be a smooth function. Consider the energy functional

$$E[u] = \int_\Omega \left(\tfrac{1}{2}|\nabla^2 u|^2 + fu\right) dx, \tag{0.6}$$

where ∇^2 denotes the Hessian, subject to the boundary conditions

$$u = 0 \text{ on } \partial\Omega \quad \text{and} \quad \nabla_n u = 0 \text{ on } \partial\Omega. \tag{0.7}$$

- Suppose u is a smooth minimizer of E. Prove that u satisfies the Euler-Lagrange equation
$$-\Delta^2 u = f. \tag{0.8}$$
- Let $\mathcal{F} = \{u \in W^{2,2}(\Omega) | u \text{ satisfies (0.7) in the trace sense}\}$ and consider the energy (0.6) as a functional $E : \mathcal{F} \to \mathbb{R}$. Prove the existence of a minimizer.
- Prove that any $u \in \mathcal{F}$ with $E[u] = \inf_{v \in \mathcal{F}} E[v]$ is a weak solution of (0.8).
- Prove that any $u \in \mathcal{F}$ with $E[u] = \inf_{v \in \mathcal{F}} E[v]$ is smooth.

We will randomly select 2 questions, for which you will receive points $p_1, p_2, p_3 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.