$$(*)\begin{cases} (\partial_t + L)u = f & \text{in } \Omega \times (0,T] \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{on } \Omega \times \{t=0\} \end{cases}$$

$\{w_k\}_{k=1}^{\infty}$ ONB for $-\Delta$ on $H_0^1(\Omega)$.

Approximate solutions $u_N(t) = \sum_{k=1}^{N} d_N^k(t)\, w_k$

$$(*)_N \begin{cases} \langle u_N', w_k \rangle + B[u_N, w_k; t] = \langle f, w_k \rangle \\ d_N^k(0) = \langle g, w_k \rangle \end{cases} \quad (k = 1, \dots, N)$$

<u>Thm</u> (uniform energy estimates for approximators)

$\exists\, C < \infty$ depending on Ω, T, bounds for L coeffs, but <u>not</u> on N, st:

$$\|u_N\|_{L^\infty L^2} + \|u_N\|_{L^2 H_0^1} + \|u_N'\|_{L^2 H^{-1}}$$

$$\leq C \left(\|f\|_{L^2 L^2} + \|g\|_{L^2} \right)$$

Proof $(*)_N \Rightarrow \langle u_N', u_N \rangle + B[u_N, u_N; t] = \langle f, u_N \rangle$

$\cdot)$ $B[u_N, u_N; t] \geq \beta \| u_N \|^2_{H^1_0} - \gamma \| u_N \|^2_{L^2}$ $\begin{pmatrix} \beta > 0 \\ \gamma \geq 0 \end{pmatrix}$

$\cdot)$ $|\langle f, u_N \rangle| \leq \frac{1}{2} \| f \|^2_{L^2} + \frac{1}{2} \| u_N \|^2_{L^2}$.

$$\Rightarrow \underbrace{\frac{d}{dt} \| u_N \|^2_{L^2} + 2\beta \| u_N \|^2_{H^1_0} \leq \| f \|^2_{L^2} + (1 + 2\gamma) \| u_N \|^2_{L^2}}_{(*)}$$

Gronwall's lemma $\Rightarrow$

$$\max_{t \in [0,T]} \| u_N(t) \|^2_{L^2} \leq C \left(\| g \|^2_{L^2} + \| f \|^2_{L^2 L^2} \right)$$

$$\left[x, y \geq 0, \quad \dot{x} \leq C x + y \Rightarrow x(t) \leq e^{Ct} \left(x(0) + \int_0^t y \right) \right]$$

this proves the $L^\infty L^2$ - estimate.

Integrate $(*)$ from 0 to T:

$$\Rightarrow \| u_N \|^2_{L^2 H^1_0} \leq C \left(\| g \|^2_{L^2} + \| f \|^2_{L^2 L^2} \right)$$

Finally, to estimate $\|u_N'\|_{L^2H^{-1}}$ let $v \in H_0^1$ with $\|v\|_{H_0^1} \le 1$.

Decompose $v = v_N + v^\perp$,

where $v_N \in \operatorname{span}\{w_k\}_{k=1}^N$ & $(v^\perp, w_k) = 0 \ (k=1,\dots,N)$

$$\underbrace{\langle u_N', v_N \rangle}_{= \langle u_N', v \rangle} + B[u_N, v_N; t] = \langle f, v_N \rangle$$

can estimate using $\|v\|_{H_0^1} \le 1$

$$\Rightarrow |\langle u_N', v \rangle| \le C\left(\|f\|_{L^2} + \|u_N\|_{H_0^1}\right)$$

$$\Rightarrow \|u_N'\|_{H^{-1}} \le C\left(\|f\|_{L^2} + \|u_N\|_{H_0^1}\right)$$

$$\Rightarrow \|u_N'\|^2_{L^2H^{-1}} \le C\left(\|f\|_{L^2L^2} + \|g\|_{L^2}\right)$$

$\boxed{//}$

<u>Thm</u> (existence) There exists a (unique) weak solution of $(*)$. Moreover,

$$\|u\|_{L^\infty L^2} + \|u\|_{L^2 H^1_0} + \|u'\|_{L^2 H^{-1}} \le C\left(\|f\|_{L^2 L^2} + \|g\|_{L^2}\right).$$

<u>Proof</u> •) uniqueness already last time

•) u_N bdd in $L^\infty L^2 \cap L^2 H^1_0$

u_N' bdd in $L^2 H^{-1}$

$$\Rightarrow \begin{cases} u_{N_\ell} \rightharpoonup u \text{ in } L^2 H^1_0 \\ u_{N_\ell}' \rightharpoonup u' \text{ in } L^2 H^{-1} \\ u_{N_\ell} \overset{*}{\rightharpoonup} u \text{ in } L^\infty L^2 \end{cases}$$

cf homework

i.e. $\iint u_{N_\ell} v \, dx dt \to \iint u v \, dx dt \quad \forall v \in L^1 L^2$.

·) lower semicontinuity

$\Rightarrow$ energy estimate holds for u.

<u>show eqn holds weakly:</u>

Let $v \in C^1 H_0^1$, $v(t) = \sum_{k=1}^{N} d^k(t) w_k$

$$\Rightarrow \int_0^T \left(\langle u_N', v \rangle + B[u_N, v; t] \right) dt = \int_0^T \langle f, v \rangle dt$$

$$\Rightarrow \int_0^T \left(\langle u', v \rangle + B[u, v; t] \right) dt = \int_0^T \langle f, v \rangle dt$$

density $\Rightarrow$ holds for all $v \in L^2 H_0^1$.

$$\Rightarrow \langle u', v \rangle + B[u, v; t] = \langle f, v \rangle$$

$\forall v \in H_0^1$ a.e. t.

<u>Show initial cond. attained:</u>

$u \in L^2 H_0^1$, $u' \in L^2 H^{-1} \Rightarrow u \in C L^2$.

(by Banach valued L^p calculus)

•) $$\int_0^T \left(-\langle v', u \rangle + B[u, v; t] \right) dt = \int_0^T \langle f, v \rangle dt + \langle u(0), v(0) \rangle$$

$\forall v \in C^1 H_0^1$ with $v(T) = 0$.

•) $$\int_0^T \left(-\langle v', u_N \rangle + B[u_N, v; t] \right) dt = \int_0^T \langle f, v \rangle dt + \langle u_N(0), v(0) \rangle$$

$$\underset{N \to \infty}{\Rightarrow} \int_0^T \left(\langle -v', u \rangle + B[u, v; t] \right) dt = \int_0^T \langle f, v \rangle dt + \langle g, v(0) \rangle$$

$$\Rightarrow \langle u(0) - g, v(0) \rangle = 0$$

v arbitrary $\Rightarrow u(0) = g$ $\boxtimes$

Higher regularity

Thm (i) If $g \in H^1_0$, then

$$u \in L^\infty H^1_0 \cap L^2 H^2, \quad u' \in L^2 L^2$$

with the estimate

$$\|u\|_{L^\infty H^1_0} + \|u\|_{L^2 H^2} + \|u'\|_{L^2 L^2}$$

$$\leq C \left(\|f\|_{L^2 L^2} + \|g\|_{H^1_0} \right)$$

(ii) If in addition $g \in H^2$, $f' \in L^2 L^2$, then

$$\|u\|_{L^\infty H^2} + \|u'\|_{L^\infty L^2} + \|u'\|_{L^2 H^1_0} + \|u''\|_{L^2 H^{-1}}$$

$$\leq C \left(\|f\|_{H^1 L^2} + \|g\|_{H^2} \right).$$

Proof (sketch for $(\partial_t - \Delta) u = f$)

make precise using Galerkin approx; see Evans.

(i) $$\frac{d}{dt}\frac{1}{2}\int|\nabla u|^2 = \int\langle\nabla u, \nabla u'\rangle$$

$$= \int\langle\nabla u, \nabla\Delta u\rangle + \int\langle\nabla u, \nabla f\rangle$$

$$= -\int|\nabla^2 u|^2 - \int\underbrace{\langle\Delta u, f\rangle}_{\text{absorb}}$$

$$\Rightarrow \sup_{t\in[0,T]}\int_\Omega|\nabla u|^2dx + \int_0^T\int_\Omega\left(|\nabla^2 u|^2 + u'^2\right)dx\,dt$$

$$\leq C\left(\int_0^T\int_\Omega f^2\,dx\,dt + \int_\Omega|\nabla g|^2dx\right)$$

$$\boxed{\begin{aligned}\left|\int\langle\Delta u, f\rangle\right| &\leq \frac{1}{2n}\int(\Delta u)^2 + \frac{n}{2}\int f^2\\ &\leq \frac{1}{2}\int|\nabla^2 u|^2 + \frac{n}{2}\int f^2.\end{aligned}}$$

(ii) differentiate equ wrt to t:

$$(\partial_t - \Delta)\hat{u} = \hat{f} \qquad \hat{u} = \partial_t u, \ \hat{f} = \partial_t f$$

$$\hat{u}(0) = \hat{g} \qquad \hat{g} = \Delta g + f(0)$$

·) $$\frac{d}{dt}\frac{1}{2}\int \hat{u}^2 = \int \hat{u}\,\partial_t \hat{u} = \int \hat{u}\Delta\hat{u} + \int \hat{f}\hat{u}$$

$$\leq -\int |\nabla \hat{u}|^2 + \frac{1}{2}\int \hat{u}^2 + \frac{1}{2}\int \hat{f}^2$$

Gronwall / integrate $\Rightarrow$

$$\sup_{t\in[0,T]} \int \hat{u}^2 dx + \int_0^T \int |\nabla \hat{u}|^2 dx\,dt \leq C\left(\int_0^T \int \hat{f}^2 dx\,dt + \int |D^2 g|^2 + f^2(0)\right)$$

·) $$\int |D^2 u|^2 dx = \int (\Delta u)^2 dx$$

$$= \int (\hat{u} - f)^2 dx \leq \int (f^2 + \hat{u}^2)\,dx$$

$\cdot)\ \max_{t\in[0,T]} \|f(t)\|_{L^2} \underset{\substack{\text{calc}\\\text{thm.}}}{\leq} C\left(\|f\|_{L^2L^2} + \|\tilde{f}\|_{L^2L^2}\right)$

$$\Rightarrow \sup_{t\in[0,T]} \int \left(u_t^2 + |D^2u|^2\right) dx + \int_0^T \int |Du_t|^2 dx\,dt$$

$$\leq C\left(\int_0^T \int (f^2 + f_t^2)\, dx\,dt + \int |D^2 g|^2\right)$$

Finally, $\|u''\|_{L^2H^{-1}} \leq \|\Delta u'\|_{L^2H^{-1}} + \|f'\|_{L^2H^{-1}}$

$$\leq C\left(\|u'\|_{L^2H^1_0} + \|f'\|_{L^2L^2}\right)$$

$\boxed{\text{q.}}$

By induction (& localization):

$\cdot)$ If $g\in C^\infty$, $f\in C^\infty$, then $u\in C^\infty(\Omega_T)$

$\cdot)$ If $g\in C^\infty(\overline{\Omega})$, $f\in C^\infty(\overline{\Omega}_T)$ and if the compatibility conditions

$$g_0 = g \in H_0^1, \quad g_1 = f(0) - Lg_0 \in H_0^1, \quad \cdots \quad g_k = \frac{d^{k-1}}{dt^{k-1}} f(0) - Lg_{k-1} \in H_0^1$$

hold, then $u \in C^\infty(\bar{\Omega}_T)$.

(see Evans for details)

Here we of course also assume that L has C^∞ coeffs.