Huisken's monotonicity formula and applications

Recall that under CSF the total length satisfies

$$\frac{d}{dt}\int_{\Gamma_t} ds = -\int_{\Gamma_t} \kappa^2\, ds \le 0.$$

Huisken discovered a scale invariant version (much more useful for singularity analysis, where one wants to rescale by $\lambda_i \to \infty$):

Let $X_0 = (x_0, t_0)$ be a point in space-time. Consider the Gaussian weight:

$$\rho_{X_0}(x,t) = \frac{1}{(4\pi(t_0-t))^{1/2}}\, e^{-\frac{|x-x_0|^2}{4(t_0-t)}} \qquad (t<t_0)$$

Note that this localizes at scales $|x-x_0|^2 \lesssim (t_0-t)$.

[If $|x-x_0|^2 \gg t_0-t$, then $e^{-\cdots}$ is really small]

We can think of $\int_\Gamma \rho_{x_0}\, ds$ as "weighted length".

Thm (Huisken's monotonicity formula)

If $\Gamma_t \subset \mathbb{R}^2$ evolves by CSF, then

$$\boxed{\frac{d}{dt}\int_{\Gamma_t} \rho_{x_0}\, ds = -\int_{\Gamma_t} \left| \kappa + \frac{\langle \gamma, N \rangle}{2(t_0-t)} \right|^2 \rho_{x_0}\, ds}$$

$(t < t_0)$

Key properties: wlog $(x_0, t_0) = (0,0)$.

- Invariant under parabolic rescaling $x' = \lambda x$, $t' = \lambda^2 t$

  (check this!)

- $\frac{d}{dt} \int_{\Gamma_t} \rho_{x_0} \, ds \leq 0$ with

  "$= 0$" $\iff$ $\kappa - \frac{\langle \gamma, N \rangle}{2t} = 0$

  $\iff$ $\underbrace{\Gamma_t = \sqrt{|t|} \, \Gamma_{-1}}_{\text{self similarly shrinking solution.}}$ $(t \leq 0)$

  ↑ Homework

[for many PDEs if you find a scale-invariant monotone quantity that is constant exactly on self-similar solutions, that's great!]

Proof wlog $X_0 = (0,0)$.

Key identity:

$$\left(\frac{d}{dt} + \partial_s^2 - \varkappa^2\right)\rho = -\left|\varkappa - \frac{\langle \gamma, N\rangle}{2t}\right|^2 \rho$$

Using the key identity & $\frac{d}{dt} ds = -\varkappa^2 ds$ one gets

$$\frac{d}{dt}\int_{\Gamma_t} \rho \, ds = \int_{\Gamma_t}\left(\left(-\partial_s^2 + \cancel{\varkappa^2}\right)\rho - |\ldots|^2 \rho - \cancel{\varkappa^2}\right) ds$$

goes away after integration

$$= -\int_{\Gamma_t} |\ldots|^2 \rho \, ds .$$

So it remains to prove the key identity:

-) $\frac{d}{dt}\rho = \partial_t \rho + \langle \nabla\rho, \kappa N\rangle$

-) $\underbrace{\partial_s \rho}_{\text{tangential gradient}} = \nabla\rho - \langle \nabla\rho, N\rangle N$

-) $\partial_s^2 \rho = \langle T, \nabla_T \partial_s \rho\rangle$

$$= \langle T, \nabla_T \nabla\rho\rangle + \kappa\langle N, \nabla\rho\rangle$$

Thus:

$$\left(\frac{d}{dt} + \partial_s^2\right)\rho = \partial_t \rho + \langle T, \nabla_T \nabla\rho\rangle + 2\kappa\langle N, \nabla\rho\rangle$$

$$= \underbrace{\partial_t \rho + \langle T, \nabla_T \nabla\rho\rangle + \frac{\langle N, \nabla\rho\rangle^2}{\rho}}_{\equiv 0. \text{ (check this!)}} - \left|\kappa - \frac{\langle N, \nabla\rho\rangle}{\rho}\right|^2 \rho + \kappa^2 \rho$$

$$= - \left|\kappa - \frac{\langle N, \nabla\rho\rangle}{\rho}\right|^2 \rho + \kappa^2 \rho \quad \boxed{\square}$$

Application: study singularities via blowup analysis

$\Gamma_t \subset \mathbb{R}^2$, $t \in [0, T)$ CSF

of closed embedded curves defined on maximal time interval.

Recall $\limsup_{t \to T} \max_{\Gamma_t} |K| = \infty$. $(*)$

Assume for now the singularity forms with the so-called type I blowup rate

$$\max_{\Gamma_t} |\kappa| \leq \frac{C}{\sqrt{T-t}} \qquad (\mathrm{I})$$

(will justify this next week).

We say $x_0 \in \mathbb{R}^2$ is a blowup point if $\exists\, t_i \to T$ and $p_i \in \Gamma_{t_i}$ s.t.

$$|\kappa|(p_i) \to \infty, \quad p_i \to x_0 .$$

(by $(*)$ there exists at least one blowup point)

Now rescale by $\lambda > 0$ with center (x_0, T), i.e. consider the parabolically rescaled flow

$$\Gamma^{\lambda}_t := \lambda \cdot \left(\Gamma_{T + \lambda^{-2} t} + x_0 \right),$$

where $t \in [-\lambda^2 T, 0)$.

<u>Claim</u> For $\lambda \to \infty$ the flows $\{\Gamma^{\lambda}_t\}_{t \in [-\lambda^2 T, 0)}$ converge smoothly to the family of round shrinking circles $\{\partial B_{\sqrt{-2t}}(0)\}_{t \in (-\infty, 0)}$.

Proof After rescaling (I) gives

$$\max_{\Gamma_t^\lambda} |\nabla \kappa| \leq \frac{C}{\sqrt{-t}}, \quad t \in [-\lambda^2 T, 0)$$

In particular, given any compact interval $I \subset (-\infty, 0)$, for λ large enough Γ_t^λ is defined for $t \in I$ and has

$$|K| \leq C(I) \text{ for } t \in I.$$

derivative estimates

$$\Rightarrow \quad |\nabla^\ell \kappa| \leq C_\ell(I).$$

Thus, for any sequence $\lambda_i \to \infty$, we can find a subsequence λ_{i_k}

such that $\{\Gamma_t^{\lambda_{i_k}}\}$ converges smoothly to a limit $\{\Gamma_t^\infty\}_{t\in(-\infty,0)}$.

Now analyze limit Γ_t^∞:

- by construction ancient CSF.
- embedded with multiplicity-one (see next week)
- nonempty (Exer: check this using definition of blowup point).
- By Huisken's monotonicity formula for every $t_1 < t_2 < 0$ we have

$$\int_{t_1}^{t_2}\int_{\Gamma_t^\lambda} \left|\kappa - \frac{\langle \gamma, N\rangle}{2t}\right|^2 \rho \, ds\, dt$$

$$= \left[-\int_{\Gamma_t} \rho \, ds\right]_{t=T-t_1/\lambda^2}^{T-t_2/\lambda^2} \xrightarrow{\lambda\to\infty} 0.$$

Thus $\{\Gamma_t^\infty\}_{t\in(-\infty,0)}$ is self-similarly shrinking: $\kappa - \frac{\langle \gamma, N\rangle}{2t} \equiv 0$,

and hence $\Gamma_t^\infty = \sqrt{|t|}\,\Gamma_{-1}^\infty$,

where Γ_{-1}^∞ satisfies

$$\kappa + \frac{\langle \gamma, N\rangle}{2} = 0$$

HW $\Rightarrow$ Γ_{-1}^∞ must be a circle of radius $\sqrt{2}$,

or straight line (cannot happen by local regularity theorem).

Limit unique $\Rightarrow$ x_0 unique & subsequential convergence entails full convergence $\boxed{\,//\,}$

Application : Local regularity theorem

Notation:

$$\Theta(\{\Gamma_t\}, (x_0,t_0), r) := \int_{\Gamma_{t_0-r^2}} \rho_{(x_0,t_0)}\, ds$$

$$P_r(X) = B_r(x) \times (t-r^2, t], \quad X = (x,t).$$

Thm $\exists\, \varepsilon > 0,\ C < \infty$ universal st:

If $\{\Gamma_t\}_{t\in(t_0-2r^2, t_0)}$ is a CSF

with

$$\sup_{\bar{X}_0 \in P_r(X_0)} \Theta(\{\Gamma_t\}, \bar{X}_0, r) < 1+\varepsilon,$$

then

$$\sup_{P_{r/2}(X_0)} |\kappa| \le C/r.$$

Interpretation:

$\underbrace{\text{weakly close to line}}_{\text{integral sense}} + \text{CSF} \Rightarrow \underbrace{\text{strongly close to line}}_{C^2}$

Proof: If not, $\exists \{\Gamma_t^j\}_{t \in (-2,0]}$ with

$$\sup_{X_0 \in P_1(0)} \Theta(\{\Gamma_t^j\}, X_0, 1) < 1 + \frac{1}{j},$$

but $|K|(X_j) > j$

for some $X_j \in P_{1/2}(0)$.

By point selection, we can find

$Y_j \in P_{3/4}(0)$ with $K_j = |K|(Y_j) > j$

and $\sup_{P_{j/10K_j}(Y_j)} |K| \le 2K_j$ (*)

Indeed, fix j. If $Y_j^0 = X_j$ already satisfies $(*)$ with $K_j^0 = |K|(Y_j^0)$ we are done. If not, $\exists\, Y_j^1 \in P_{j/10K_j^0}(Y_j^0)$ with $K_j^1 = |K|(Y_j^1) > 2K_j^0$.

If Y_j^1 satisfies $(*)$, done.

If not $\exists\, Y_j^2 \in P_{j/10K_j^1}(Y_j^1)$ with $K_j^2 = |K|(Y_j^2) > 2K_j^1$, etc.

Note that $\frac{1}{2} + \frac{j}{10K_j^0}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) < 3/4$

Hence, iteration terminates after finitely many steps & last point lies in $P_{3/4}(0)$ and satisfies $(*)$.

Now, let $\{\hat{\Gamma}_t^j\}$ be the flows obtained by shifting p_j to origin and per. rescaling by $K_j = |\kappa|(p_j)$.

$\hat{\Gamma}_t^j$ satisfies:

$$|\kappa|(0,0) = 1$$

$$\sup_{P_{j/10}(0)} |\kappa| \leq 2$$

Hence, we can smoothly pass to a limit $\{\Gamma_t^\infty\}_{t\in(-\infty,0]}$.

On the one hand $|\kappa|(0,0) = 1$, hence Γ_t^∞ is not flat.

On the other hand, since $(\ast\ast) \leq 1 + 1/j$ by Huisken's monotonicity formula the limit must be a flat line ↯ $\boxed{}$