Existence & Uniqueness

Thm Let $\gamma_0 : S^1 \to \mathbb{R}^2$ be an embedded curve. Then there exists a unique smooth solution $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$ of the CSF

$$\partial_t \gamma = \partial_s^2 \gamma , \quad \gamma|_{t=0} = \gamma_0$$

defined on a maximal time interval $[0,T)$. Moreover, T is characterized by

$$\sup_{S^1 \times [0,T)} |\kappa|(x,t) = \infty .$$

Note: CSF is not strictly parabolic:

$$\partial_t \gamma = \partial_s^2 \gamma = \frac{1}{|\partial_x \gamma|} \partial_x \left(\frac{1}{|\partial_x \gamma|} \partial_x \gamma \right)$$

$$= \frac{1}{|\partial_x \gamma|^2} \left(\partial_x^2 \gamma - \left\langle \frac{\partial_x \gamma}{|\partial_x \gamma|}, \partial_x^2 \gamma \right\rangle \frac{\partial_x \gamma}{|\partial_x \gamma|} \right)$$

In components $\gamma = (\gamma^1, \gamma^2)$:

$$\begin{pmatrix} \partial_t \gamma^1 \\ \partial_t \gamma^2 \end{pmatrix} = \frac{1}{|\partial_x \gamma|^4} \underbrace{\begin{pmatrix} (\partial_x \gamma^2)^2 & \partial_x \gamma^1 \partial_x \gamma^2 \\ \partial_x \gamma^1 \partial_x \gamma^2 & (\partial_x \gamma^1)^2 \end{pmatrix}}_{\text{positive semidefinite, but not positive definite } (\operatorname{tr} > 0,\ \det = 0)} \begin{pmatrix} \partial_x^2 \gamma^1 \\ \partial_x^2 \gamma^2 \end{pmatrix}$$

Thus, the standard theory for strictly parabolic systems cannot be applied.

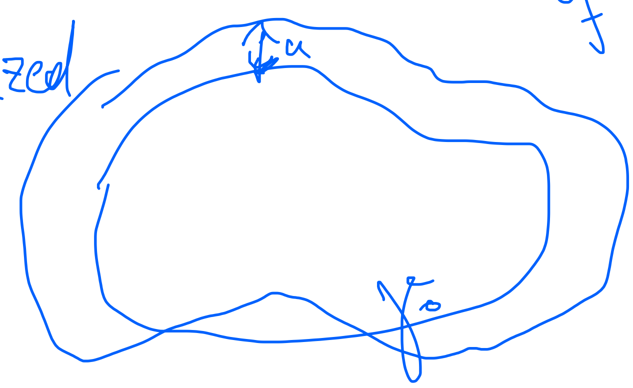
Such degeneracies are quite typical for geometric PDEs. For CSF, the underlying reason is that only the normal component of the velocity is geometrically meaningful.

To overcome this degeneracy we have to fix the parametrization/gauge:

$$\tilde{\gamma}(x,t) = \gamma_0(x) + u(x,t)\, N(x)$$

normal of γ_0

Assume γ_0 parametrized by arclength.



<u>Goal:</u> Show u satisfies nondegenerate parabolic PDE.

Compute:

unit normal/tangent of γ_0

$$\tilde{\gamma}' = u' N + (1 - k u) T$$

curvature of γ_0

$$\tilde{\gamma}'' = \left(u'' + k(1 - k u)\right) N - (k' u + 2 k u') T.$$

$$\tilde{\kappa} := \text{curvature of } \tilde{\gamma}$$

$$= |\tilde{\gamma}'|^{-3} \det(\tilde{\gamma}', \tilde{\gamma}'')$$

$$= \frac{(1-ku)u'' + 2ku'^2 + k'uu' - 2k^2u + k^3u^2 + k}{\left((1-ku)^2 + u'^2\right)^{3/2}}$$

$$\tilde{N} := \text{unit normal of } \tilde{\gamma}$$

$$= \tilde{\gamma}'/|\tilde{\gamma}'| \text{ rotated by } \pi/2$$

$$= \frac{(1-ku)N - u'T}{\left((1-ku)^2 + u'^2\right)^{1/2}}.$$

Finally, $\Gamma_t = \tilde{\gamma}(S^1, t)$ moves by CSF

$$\iff \langle \tilde{N}, u_t N \rangle = \tilde{\kappa}.$$

Hence,

$$u_t = \frac{u'' + (1-ku)^{-1}(2ku'^2 + k'uu' - 2k^2u + k^3u^2 + k)}{\left((1-ku)^2 + u'^2\right)}$$

This is a quasilinear strictly parabolic PDE (as long as say $|k u| \leq 1/2$), thus has a unique solution on some interval $[0, \varepsilon)$.

In general, $\tilde{\gamma}$ only solves CSF up to tangential motion, i.e.

$$\partial_t \tilde{\gamma} = \kappa \tilde{N} + f \partial_x \tilde{\gamma},$$

where $f = f(x, t)$.

Finally, let $\gamma(x,t) = \tilde{\gamma}(\varphi_t(x), t)$, where $\varphi_t : S^1 \to S^1$ solves

$$\begin{cases} \dfrac{d}{dt} \varphi_t(x) = - f(x,t) \dfrac{\partial_x \tilde{\gamma}(x,t)}{\partial_x \tilde{\gamma}(\varphi_t(x), t)} \\ \varphi_0(x) = x \end{cases}$$

Then $\partial_t \gamma = \partial_s^2 \gamma$.

This shows short time existence & uniqueness

Now let $\{\Gamma_t\}_{t \in [0,T)}$ be a solution on a maximal interval $[0,T)$.

If $\limsup_{t \to T} \sup_{\Gamma_t} |\kappa| < \infty$,

then by gradient est from last lecture $|\nabla^j \kappa| \leq C_j$.

$\Rightarrow$ Can pass to smooth limit Γ_T.

Applying short time existence with initial condition Γ_T, can continue evolution until $T+\varepsilon$ $\lightning$ T maximal. $\boxed{/\!/}$