Curve shortening flow

I Basics

draw a curve in $\mathbb{R}^2$

$\gamma(s)$ parametrization by arclength, i.e. $\left| \frac{d}{ds} \gamma(s) \right| = 1$

$\vec{t}(s) := \frac{d}{ds} \gamma(s)$ unit tangent vector

$\vec{\kappa}(s) := \frac{d^2}{ds^2} \gamma(s)$ curvature vector

$\langle \vec{t}, \vec{t} \rangle = 1$ (unit normal)

$d = \frac{d}{ds}$

$\langle \vec{t}, \vec{\kappa} \rangle = 0$

hence can write $\vec{\kappa} = \kappa \vec{N}$
A one-parameter family of curves $\Gamma_t \subset \mathbb{R}^2$ evolves by curve shortening flow (CSF) if

$$\frac{\partial P}{\partial t} = \overrightarrow{\kappa}(p) \quad \forall p \in \Gamma_t, \forall t \in I$$

(normal velocity at $p$)
Ex: Round shrinking circle

\[ \Gamma_t = \partial B_{r(t)} \subset \mathbb{R}^2 \]

\[ \dot{r} = -\frac{1}{r}, \quad r(0) = R \]

\[ \Rightarrow r(t) = \sqrt{R^2 - 2t}, \quad t \in (-\infty, R^2/2). \]

Ex: Grim reaper

\[ \Gamma_t = \text{graph} \left( -\log \cos p \right) + t \]

\[ p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \]

\[ t \in (-\infty, \infty) \]
Ex: paperclip

\[ \cosh y(t) = e^{-t} \cos x(t) \]

Ex: hairclip

\[ \sinh y(t) = e^{-t} \cos x(t) \]

\[ \Gamma_0 \]
\[ y = y(\cdot, t) : S' \times I \to \mathbb{R}^2 \]

time-dependent embedding of circle into plane; \( \Gamma_t = y(S', t) \)

Setting \( p = y(x, t) \) our PDE becomes

\[ \partial_t y(x, t) = \kappa(x, t) N(x, t) \tag{CSF} \]

Since \( \kappa = \frac{d}{d s^2} |y| \), this is a

(weakly) parabolic PDE.

Nonlinear, since \( d s = |\frac{dy}{dx}| d x \).

\[ A(t) : \text{area enclosed by } \Gamma_t \]

\[ \frac{d}{d t} A(t) = - \int_{\Gamma_t} \kappa d s = -2 \pi \]
\[ A(t) = A(0) - 2\pi t, \]

in particular \( T \leq A(0)/2\pi \)

\[ \uparrow \]

Maximal existence time

Thm (Grayson 87) The CSF starting at any closed embedded curve in the plane exists until \( T = \frac{A(0)}{2\pi} \) and converges to a round point.

We will prove this at the end of the course.
Comparison principle ("maximum principle")

\[ \{ \Gamma_t \}_{t \in [t_0, t_1]} \quad \text{CSFs} \]
\[ \{ \Gamma'_t \}_{t \in [t_0, t_1]} \]

\[ \Gamma_{t_0} \cap \Gamma'_{t_0} = \emptyset \implies \Gamma_t \cap \Gamma'_t = \emptyset \quad \forall t \geq t_0 \]

In fact, \( \frac{\partial}{\partial t} \text{dist} (\Gamma_t, \Gamma'_t) \geq 0 \)

Evolution of length

\[ L(t) = \int_{\Gamma_t} ds \]
\[ = \int_{S^1} \left( \langle \partial_x y, \partial_x y \rangle \right)^{1/2} dx \]

\[ \implies \frac{\partial}{\partial t} L(t) = \int_{S^1} \langle \partial_x \partial_t y, \frac{1}{2} \rangle \, dx \]
where $\vec{T} = \frac{\partial x}{\partial x} \xi$ is the unit tangent

$$\Rightarrow \frac{d}{dt} L(t) = \int_{S'} \left< \partial_s (\kappa \vec{N}), \vec{T} \right> ds$$

$$= \int_{S'} \left( -\kappa \left< \partial_s \vec{N}, \vec{T} \right> + \partial_s \kappa \left< \vec{N}, \vec{T} \right> \right) ds$$

$$= -\kappa \frac{d}{dx} \int_{S'} ds$$

$$\Rightarrow \frac{d}{dt} L(t) = -\int_{\Gamma_t} \kappa^2 ds$$

Hence, CSF is the downwards gradient flow of the length functional.
Evolution of curvature

If $\Gamma_t$ evolves by CSF, then

$$\kappa_t = \kappa_{ss} + \kappa^3.$$ 

Proof: Work with parametrization

$s + 12s^2) = 1$, $\langle 2x_s, T \rangle = 0$ at $(x_0, t_0)$. Then at $(x, t)$ we can compute

$$\kappa_t = \frac{\partial}{\partial t} \left( \frac{1}{2s^2} \langle 2x_s, N \rangle \right)$$

$$= 2_t \langle 2x_s, N \rangle - 2 \langle T, 2x_s \rangle \langle 2x_s, N \rangle$$

at $(x_0, t_0)$

$$= \langle 2x_s, 2x_s, N \rangle - 2 \kappa \langle T, 2x_s \rangle$$

$$= 2_{\kappa}^2 + 2 \kappa \langle 2x_s, N \rangle + \kappa \langle 2x_s, N \rangle - 2 \kappa^2 \langle T, 2x_s \rangle$$

$$= \kappa_{ss}$$

$$= -\langle 2x_s, 2x_s, N \rangle = -\kappa^2.$$
Cor: Convexity is preserved under CSF, i.e. $\lambda > 0$ at $t = 0 \implies \lambda > 0$ for $t \geq 0$.

More precisely, if $\lambda_{\min}(t) := \min_{t} \lambda$, is positive at $t = 0$, then

$$
\lambda_{\min}(t) \geq \lambda_{\min}(0),
$$

hence

$$
\lambda_{\min}(t) \geq \frac{\lambda_{\min}(0)}{1 - 2t \lambda_{\min}(0)}.
$$

In part, $T \leq \sqrt{2 \lambda_{\min}(0)}$.

Proof: Use max. principle. $\blacksquare$

Thm (Derivative estimates)

There exists $C_{e} = C_{e}(K, T) < \infty$ such that if $\{F_{t}\}_{t \in [0, T]}$ is a CSF with $\sup_{t \in [0, T]} \sup_{|k| \leq K} |\partial_{s}^{2} k| \leq K$,

then

$$
\sup_{t \in [0, T]} |\partial_{s}^{2} k| \leq C_{e}/t^{e/2}.
$$
Proof.

1) \( \kappa_t - \kappa_{ss} = \kappa^3 \)

\[ \begin{align*}
(\partial_t - \partial_s^2) \kappa^2 &= -2\kappa_s^2 + 2\kappa (\partial_t - \partial_s^2 \kappa) \\
&= -2\kappa_s^2 + 2\kappa^4 \\
\end{align*} \]

2) \( (\kappa_t)_s = \kappa_{sss} + 3\kappa^2 \kappa_s \)

\[ \begin{align*}
&= (\kappa_s)_t - \kappa^2 \kappa_s \text{ (see HW)} \\
&\Rightarrow (\partial_t - \partial_s^2) \kappa_s = 4\kappa^2 \kappa_s \\
&\Rightarrow (\partial_t - \partial_s^2) \kappa_s^2 = -2\kappa_{ss}^2 + 8\kappa^2 \kappa_s^2 \\
\end{align*} \]

Combine eqns to get

\[ (\partial_t - \partial_s^2) (t\kappa_s^2 + \beta \kappa^2) \]

\[ \leq (8t\kappa^2 + 1 - 2\beta) \kappa_s^2 + 2\beta \kappa^4 \leq 2\beta \kappa^4, \]

provided we choose \( \beta \geq (8t \kappa^2 + 1)/2 \).
max princ ⇒ \( tK_s^2 \leq 13K^2 + 23K^4 T \),
which proves the estimate for \( l = 1 \).
Higher der: ind. on \( l \) (exew) \( \Box \)

Thus (Hamilton's Harnack inequality)
If \( \Gamma_t \) is a convex solution of the CSF,
then \( \frac{K_t}{K} - \frac{K_s^2}{K^2} + \frac{1}{2t} \geq 0 \).

Proof: We argue similarly as in the proof of the Li-Yau Harnack for the heat equation:
Let \( f := \log K, F := t(f_s^2 - f_t) \)
Want to show: \( F \leq \frac{1}{2} \) \( \forall t \in [0, \tau) \).
\[ F \leq \frac{1}{2} \text{ holds for } t \text{ small.} \]

Compute

\[ F_{ss} = t \left( 2t f_{ssss} + 2f^2_{ss} - (ft)_{ss} \right) \]

\[ = t \left( 2f_{ss} f_{ssss} + 2f^2_{ss} - f_{ss} \right)_{tt} + 2\kappa^2 f_{ss} + 2\kappa \right) \]

\[ \left[ \partial_t, \partial_s \right] = \kappa^2 \partial_s \]

\[ \kappa_t = \kappa_{ss} + \kappa^3 \Rightarrow f_{ss} = -F/t - \kappa^2 \]

Thus

\[ F_{ss} = -2f_{ss} + \frac{2\kappa^2}{t} - \frac{F}{t} + F \]

\[ -4t\kappa^2 f^2_{ss} + 4F\kappa^2 + 2t\kappa^4 + 2t\kappa^2 f_t - 2\kappa^2 F - 2t\kappa^4 + 2\kappa^2 f^2_{ss} \]

\[ = 0 \]

\[ \Rightarrow F_{ss} - F_t = -2f_{ss} F + \frac{1}{t} F(2F - 1) \]

If there is a maximum point \((x_0, t_0)\) with \( F(x_0, t_0) > \frac{1}{2} \), then

\[ 0 \geq (F_{ss} - F_t) \bigg|_{(x_0, t_0)} > 0 \quad \checkmark \]
Cor. If \( \{ \Gamma_t \}_{t \in (-\infty, T)} \) is an ancient convex CSF, then
\[
\frac{\kappa_t}{\kappa} - \frac{\kappa_s^2}{\kappa^2} \geq 0.
\]
In fact, \( \kappa_t \geq 0 \).

**Thm.** Any eternal convex CSF \( \{ \Gamma_t \}_{t \in (-\infty, \infty)} \)
with \( \kappa \) a critical point somewhere in spacetime, must be a translating soliton, i.e. \( \exists V \in \mathbb{R}^2 \) s.t. \( \Gamma_t = \Gamma_0 + tV \).

**Homework:** \( \Gamma_t = \) grim reaper.

**Proof of Thm.** \( \kappa_t = 0 = \kappa_s \) at \( (x_0, t_0) \).

Harmonic quantity \( Z := \frac{\kappa_t}{\kappa} - \frac{\kappa_s^2}{\kappa^2} \)
satisfies \( Z \geq 0 \) & \( Z(x_0, t_0) = 0 \).
Strict max principle $\Rightarrow Z \equiv 0 \quad \forall t \leq t_0$, i.e. $\mathcal{Z}_t = \frac{\mathcal{Z}_0^2}{\kappa}$.

Consider $V := -\frac{\kappa_0}{\kappa} \mathbf{T} + \kappa \mathbf{N}$.

Then $V_s = \left(-\frac{\kappa_0}{\kappa} + \frac{\kappa_0^2}{\kappa^2} - \kappa^2 \right) \mathbf{T} + (\kappa_0 - \kappa_0) \mathbf{N}$

$\equiv 0$.

Similarly, $V_t \equiv 0$, i.e. $V \in \mathbb{R}^2$ is a constant vector.

$\langle V, N \rangle = \kappa$

$\Rightarrow \mathcal{P}_t = \mathcal{P}_{t_0} + (t-t_0) V$ for $t \leq t_0$.

Uniqueness $\Rightarrow \mathcal{P}_t = \mathcal{P}_{t_0} + tV \quad \forall t \in \mathbb{R}$.