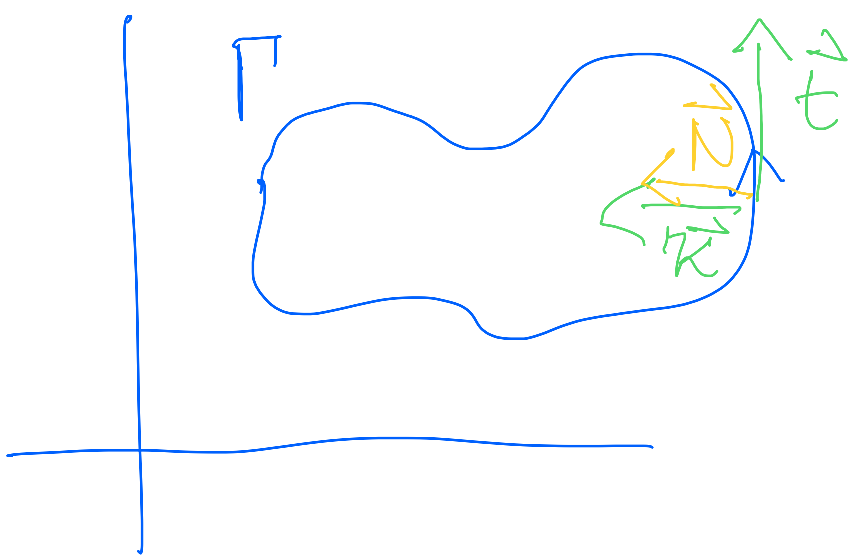
Curve shortening flow

① Basics

curve in $\mathbb{R}^2$



$\gamma(s)$ parametrization by arclength, i.e. $\left|\frac{d}{ds}\gamma(s)\right| = 1$

$\vec{t}(s) := \frac{d}{ds}\gamma(s)$ unit tangent vector

$\vec{\kappa}(s) := \frac{d^2}{ds^2}\gamma(s)$ curvature vector

$\left(\vec{N} = \text{unit normal} = \frac{\pi}{2}\text{-rotation of } \vec{t}\right)$

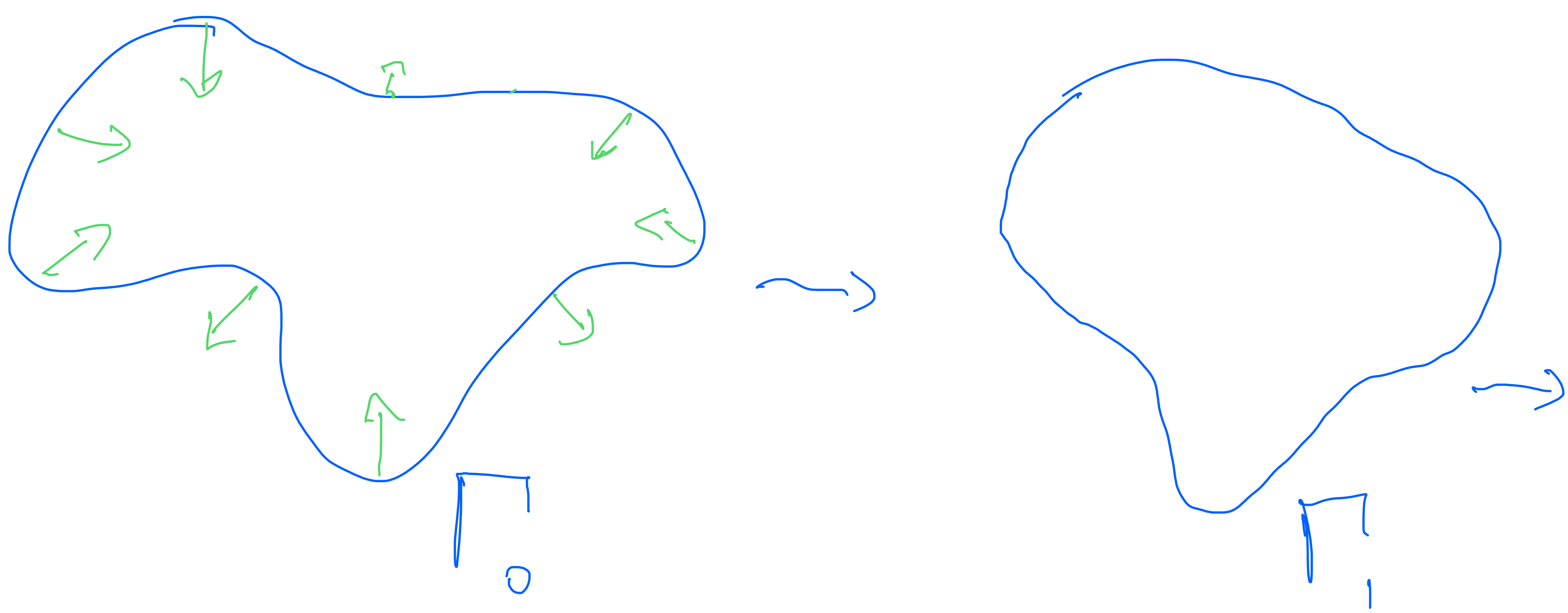
$$\langle \vec{t}, \vec{t} \rangle \equiv 1$$

$$\frac{d}{ds} \Rightarrow \langle \vec{t}, \vec{\kappa} \rangle = 0,$$

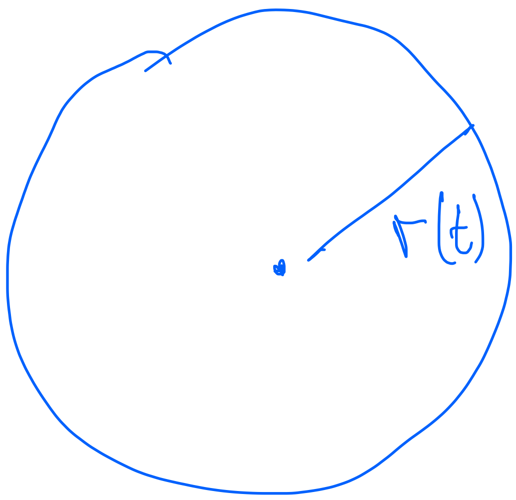
hence can write $\vec{\kappa} = \kappa \vec{N}$

A one-parameter family of curves $\Gamma_t \subset \mathbb{R}^2$ evolves by curve shortening flow (CSF) if

$$\underbrace{\partial_t p}_{\text{(normal velocity at } p\text{)}} = \vec{\kappa}(p) \qquad \forall p \in \Gamma_t,\ \forall t \in I$$



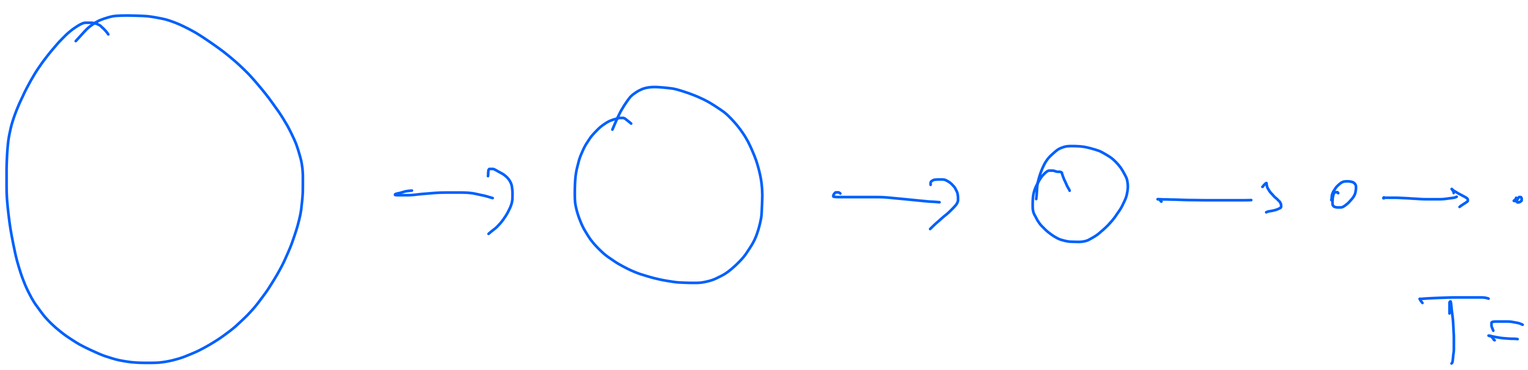
Ex: Round shrinking circle



$$\Gamma_t = \partial B^2_{r(t)} \subset \mathbb{R}^2$$

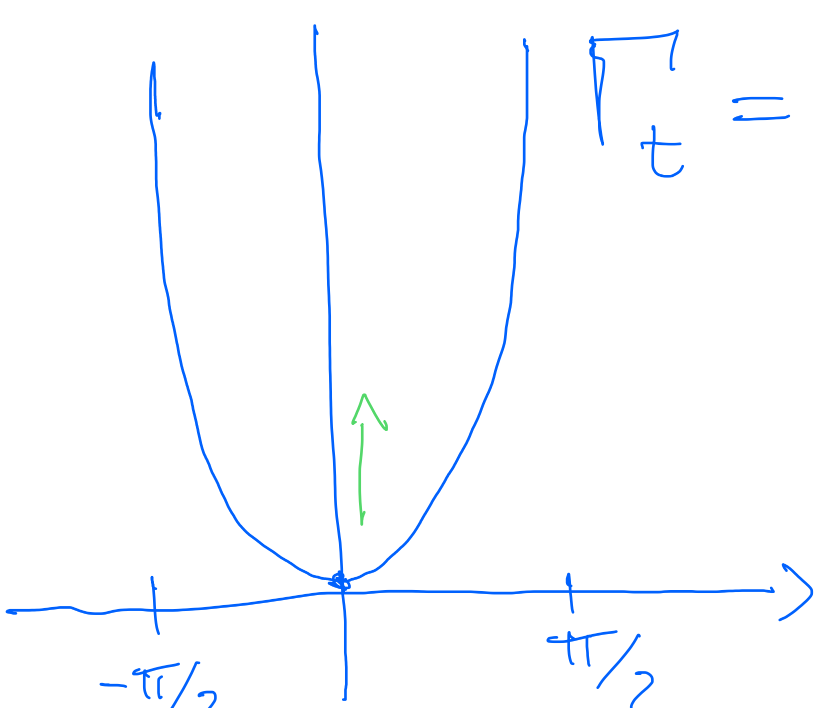
$$\dot r = -\frac{1}{r}, \quad r(0) = R$$

$$\Rightarrow r(t) = \sqrt{R^2 - 2t}, \quad t \in (-\infty, R^2/2).$$



$$T = R^2/2$$

Ex: Grim reaper

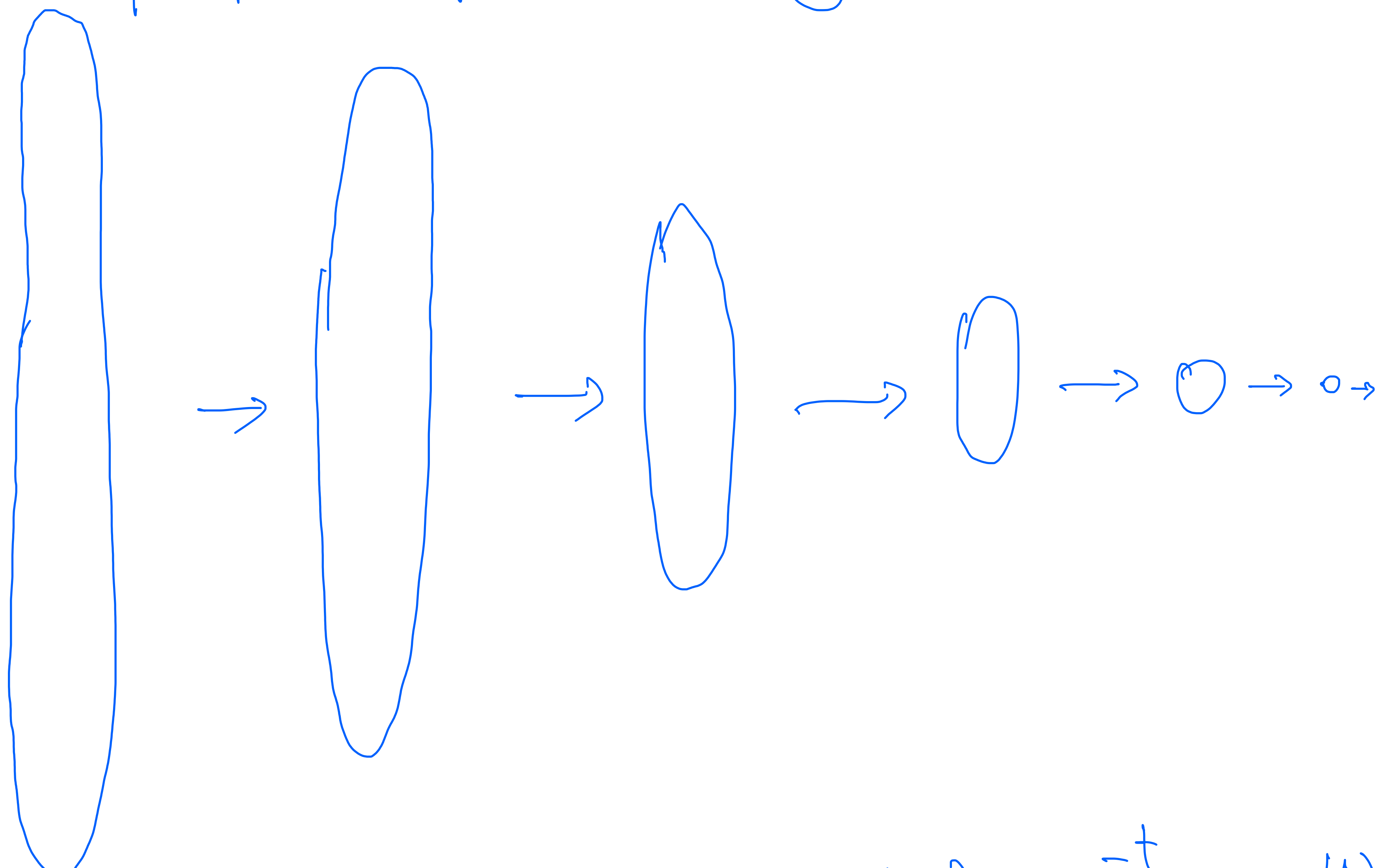


$$\Gamma_t = \operatorname{graph}(-\log\cos p) + t$$

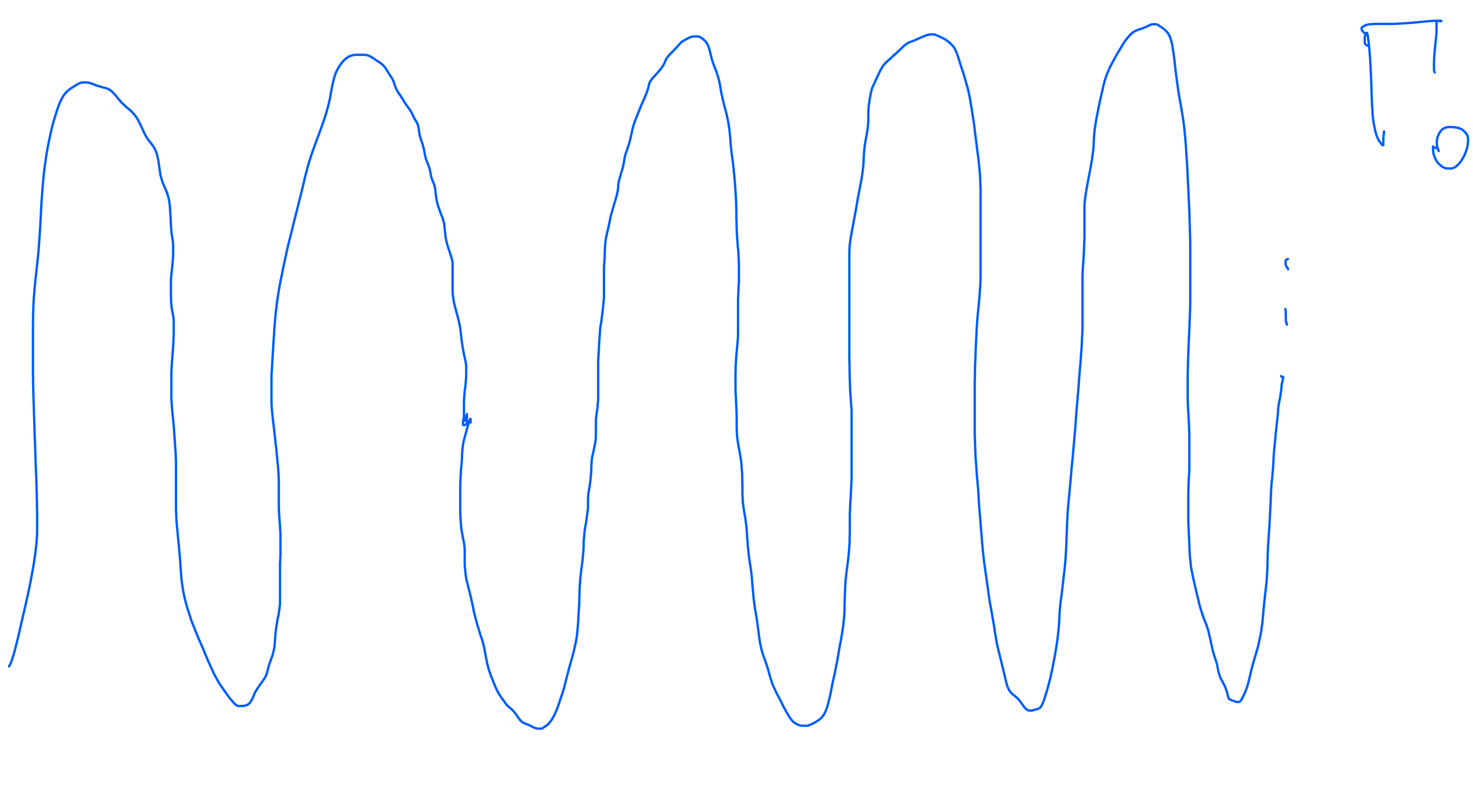
$$p \in (-\pi/2, \pi/2)$$

$$t \in (-\infty, \infty)$$

Ex: paperclip $\quad \cosh \, y(t) = e^{-t} \cos x(t)$



Ex: hairclip $\quad \sinh \, y(t) = e^{-t} \cos x(t)$



$\gamma = \gamma(\cdot, t) : S^1 \times I \to \mathbb{R}^2$

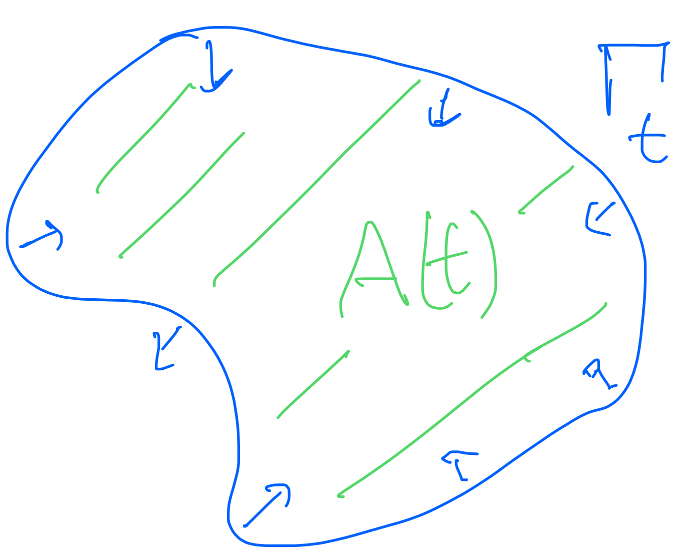
time-dependent embedding of circle into plane; $\Gamma_t = \gamma(S^1, t)$

Setting $p = \gamma(x,t)$ our PDE becomes

$$\boxed{\partial_t \gamma(x,t) = \kappa(x,t)\, N(x,t)} \qquad \text{(CSF)}$$

Since $\kappa = \frac{d^2}{ds^2}\gamma$, this is a (weakly) parabolic PDE.

Nonlinear, since $ds = \left|\frac{d\gamma}{dx}\right| dx$.



$A(t) :=$ area enclosed by Γ_t

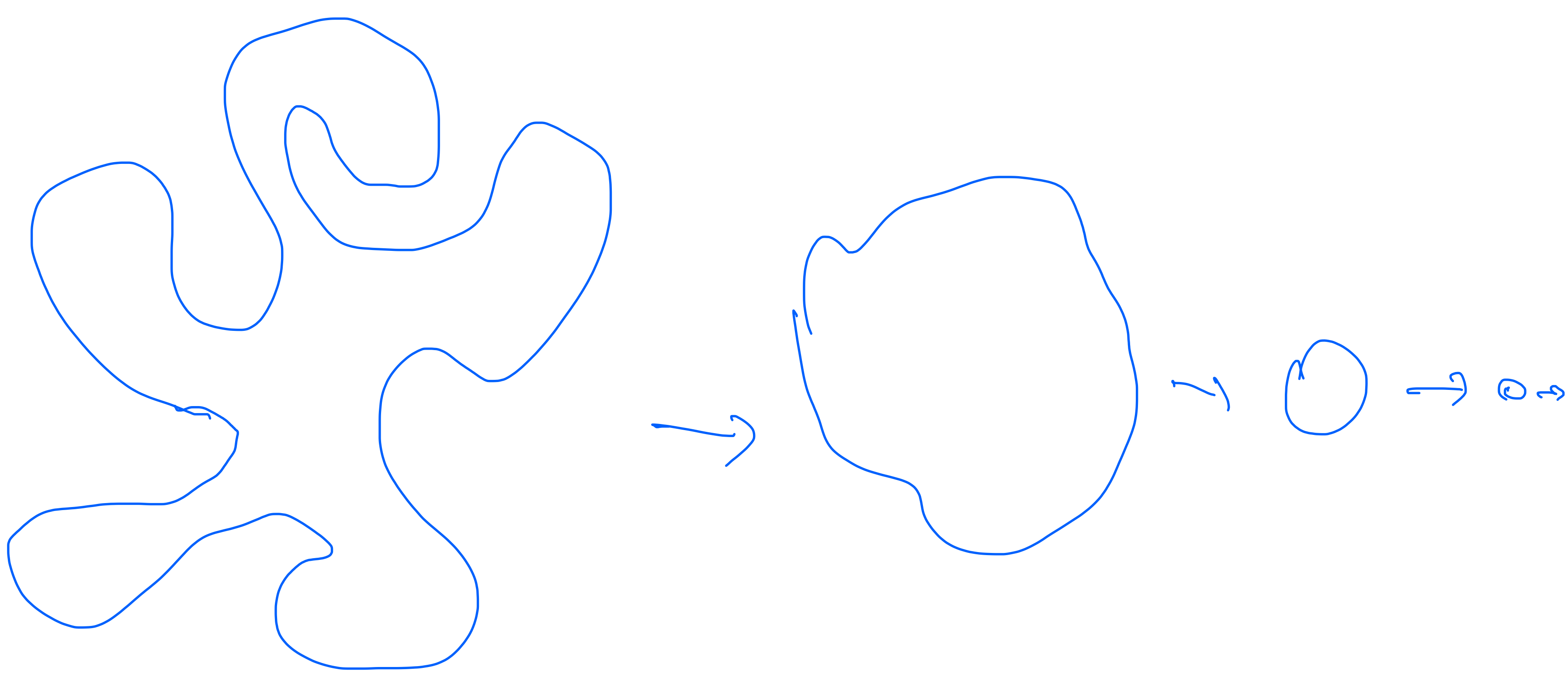
$$\frac{d}{dt} A(t) = -\int_{\Gamma_t} \kappa \, ds = -2\pi$$

$$\Rightarrow A(t) = A(0) - 2\pi t,$$

in particular $T \leq A(0)/2\pi$

maximal existence time

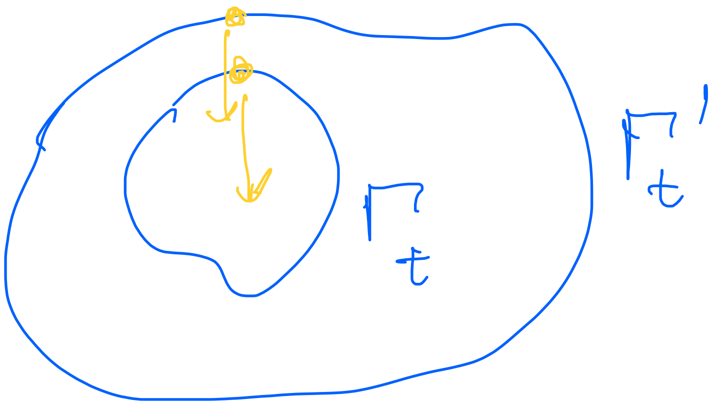
Thm (Grayson 87) The CSF starting at any closed embedded curve in the plane exists until $T = \frac{A(0)}{2\pi}$ and converges to a round point.



Γ_0

We will prove this at the end of the course.

Comparison principle ("maximum principle")

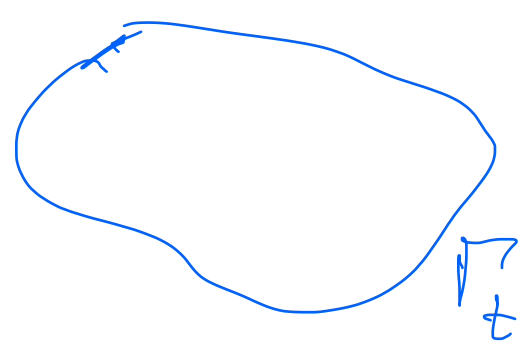


$\{\Gamma_t\}_{t\in[t_0,t_1]}$
$\{\Gamma_t'\}_{t\in[t_0,t_1]}$ CSFs

$$\Gamma_{t_0} \cap \Gamma_{t_0}' = \phi \implies \Gamma_t \cap \Gamma_t' = \phi \quad \forall t \geq t_0$$

In fact, $\frac{d}{dt}\operatorname{dist}(\Gamma_t, \Gamma_t') \geq 0$

Evolution of length



$$L(t) = \int_{\Gamma_t} ds$$

$$= \int_{S^1} \langle \partial_x \gamma, \partial_x \gamma \rangle^{1/2} dx$$

$$\implies \frac{d}{dt} L(t) = \int_{S^1} \langle \partial_x \partial_t \gamma, \vec{T} \rangle \, dx,$$

where $\vec{T} = \dfrac{\partial_x \gamma}{|\partial_x \gamma|}$ is the unit tangent

$$\Rightarrow \frac{d}{dt} L(t) = \int_{S^1} \langle \partial_s (\kappa \vec{N}), \vec{T} \rangle \, dx$$

$$= \int_{S^1} \Big(-\kappa \underbrace{\langle \partial_s \vec{N}, \vec{T} \rangle}_{= -\kappa \frac{ds}{dx}} + \partial_s \kappa \underbrace{\langle \vec{N}, \vec{T} \rangle}_{=0} \Big) dx$$

$$\Rightarrow \frac{d}{dt} L(t) = -\int_{\Gamma_t} \kappa^2 \, ds$$

Hence, CSF is the downwards gradient flow of the length functional.

Evolution of curvature

If Γ_t evolves by CSF, then

$$\kappa_t = \kappa_{ss} + \kappa^3 .$$

Proof Work with parametrization st $|\partial_x \gamma| = 1$, $\langle \partial_x^2 \gamma, T \rangle = 0$ at (x_0, t_0).

Then at (x,t) we can compute

$$\kappa_t = \frac{\partial}{\partial t}\left(|\partial_x \gamma|^{-2} \langle \partial_x^2 \gamma, N \rangle \right)$$

$$\underset{\text{at } (x_0,t_0)}{=} \partial_t \langle \partial_x^2 \gamma, N \rangle - 2\langle T, \partial_x \partial_t \gamma \rangle \langle \partial_x^2 \gamma, N \rangle$$

$$= \langle \partial_x^2 \underbrace{\partial_t \gamma}_{\kappa N}, N \rangle - 2\kappa \langle T, \partial_x \underbrace{\partial_t \gamma}_{\kappa N} \rangle$$

$$= \underbrace{\partial_x^2 \kappa}_{= \kappa_{ss}} + 2 \partial_x \kappa \underbrace{\langle \partial_x N, N \rangle}_{=0} + \kappa \underbrace{\langle \partial_x^2 N, N \rangle}_{= -\langle \partial_x N, \partial_x N \rangle = -\kappa^2} - 2\kappa^2 \underbrace{\langle T, \partial_x N \rangle}_{= -\kappa}$$

$\square$

Cor Convexity is preserved under CSF,

i.e. $\kappa > 0$ at $t = 0 \implies \kappa > 0$ $\forall t \geq 0$.

More precisely, if $\kappa_{min}(t) := \min_{\Gamma_t} \kappa$ is positive at $t = 0$, then

$$\dot{\kappa}_{min}(t) \geq \kappa_{min}^3(t),$$

hence $\kappa_{min}(t) \geq \dfrac{\kappa_{min}(0)}{1 - 2t\kappa_{min}^2(0)}$.

In part, $T \leq 1/2\kappa_{min}^2(0)$.

Proof Use max. principle. $\square$

Thm (Derivative estimates)

$\exists C_\ell = C_\ell(K,T) < \infty$ st if $\{\Gamma_t\}_{t\in[0,T)}$ is a CSF with $\sup_{t\in[0,T)} \sup_{\Gamma_t} |\kappa| \leq K$,

then $\sup_{\Gamma_t} |\partial_s^\ell \kappa| \leq C_\ell / t^{\ell/2}$.

Proof
•) $\kappa_t - \kappa_{ss} = \kappa^3$

•) $$\underbrace{(\partial_t - \partial_s^2)\kappa^2} = -2\kappa_s^2 + 2\kappa\underbrace{(\partial_t - \partial_s^2\kappa)}_{=\kappa^3}$$

$$= \underbrace{-2\kappa_s^2 + 2\kappa^4}$$

•) $$\underbrace{(\kappa_t)_s} = \kappa_{sss} + 3\kappa^2\kappa_s$$

$$= (\kappa_s)_t - \kappa^2\kappa_s \quad \text{(see HW)}$$

$$\Rightarrow (\partial_t - \partial_s^2)\kappa_s = 4\kappa^2\kappa_s$$

$$\Rightarrow \underbrace{(\partial_t - \partial_s^2)\kappa_s^2 = -2\kappa_{ss}^2 + 8\kappa^2\kappa_s^2}$$

Combine equs to get

$$(\partial_t - \partial_s^2)(t\kappa_s^2 + \beta\kappa^2)$$

$$\leq (8t\kappa^2 + 1 - 2\beta)\kappa_s^2 + 2\beta\kappa^4 \leq 2\beta K^4,$$

provided we choose $\beta \geq (8TK^2+1)/2$.

max princ $\Rightarrow$ $t\kappa_s^2 \leq \beta K^2 + 2\beta K^4 T$,

which proves the estimate for $\ell = 1$.

Higher der: ind. on ℓ (exer) $\square$

Thm (Hamilton's Harnack inequality)

If Γ_t is a convex solution of the CSF, then

$$\frac{\kappa_t}{\kappa} - \frac{\kappa_s^2}{\kappa^2} + \frac{1}{2t} \geq 0.$$

Proof We argue similarly as in the proof of the Li-Yau Harnack for the heat equation:

Let $f := \log \kappa$, $F := t(f_s^2 - f_t)$

Want to show: $F \leq \frac{1}{2}$ $\forall t \in [0, T)$.

$F \leq 1/2$ holds for t small.

Compute

$\cdot)$ $F_{ss} = t\left(2 f_s f_{sss} + 2 f_{ss}^2 - (f_t)_{ss}\right)$

$$= t\left(2 f_s f_{sss} + 2 f_{ss}^2 - (f_{ss})_t + 2\kappa^2 f_{ss} + 2\kappa^2 f_s^2\right)$$

$$\underset{[\partial_t, \partial_s] = \kappa^2 \partial_s}{\uparrow}$$

$\cdot)$ $\kappa_t = \kappa_{ss} + \kappa^3 \Rightarrow f_{ss} = -F/t - \kappa^2$

Thus

$$F_{ss} = -2 f_s F_s + \frac{2F^2}{t} - \frac{F}{t} + F_t \underbrace{- 4 t \kappa^2 f_s^2 + 4 F \kappa^2 + 2 t \kappa^4 + 2 t \kappa^2 f_t - 2\kappa^2 F - 2 t \kappa^4 + 2 t \kappa^2 f_s^2}_{= 0}$$

$$\Rightarrow F_{ss} - F_t = -2 f_s F_s + \frac{1}{t} F (2F - 1)$$

If there is a maximum point (x_0, t_0) with $F(x_0, t_0) > 1/2$, then

$$0 \geq (F_{ss} - F_t)\big|_{(x_0, t_0)} > 0 \quad ↯$$

$\boxed{12}$

Cor If $\{\Gamma_t\}_{t\in(-\infty,T)}$ is an ancient convex CSF, then

$$\frac{\kappa_t}{\kappa} - \frac{\kappa_s^2}{\kappa^2} \geq 0.$$

In part, $\kappa_t \geq 0$.

Thm Any eternal convex CSF $\{\Gamma_t\}_{t\in(-\infty,\infty)}$ s.t. κ has a critical point somewhere in spacetime, must be a translating soliton, i.e. $\exists V\in\mathbb{R}^2$ st $\Gamma_t = \Gamma_0 + tV$.

Homework: Γ_t = grim reaper.

Proof of thm $\kappa_t = 0 = \kappa_s$ at (x_0,t_0).

Harnack quantity $Z := \frac{\kappa_t}{\kappa} - \frac{\kappa_s^2}{\kappa^2}$

satisfies $Z \geq 0$ & $Z(x_0,t_0) = 0$.

strict max principle $\Rightarrow Z \equiv 0 \quad \forall t \le t_0$,

i.e. $\kappa_t = \frac{\kappa_s^2}{\kappa}$.

Consider $V := -\frac{\kappa_s}{\kappa} T + \kappa N$.

Then $V_s = \left(-\frac{\kappa_{ss}}{\kappa} + \frac{\kappa_s^2}{\kappa^2} - \kappa^2\right) T + (\kappa_s - \kappa_s) N$

$\equiv 0$.

Similarly, $V_t \equiv 0$, i.e. $V \in \mathbb{R}^2$ is a constant vector.

$\langle V, N \rangle = \kappa$

$\Rightarrow \Gamma_t = \Gamma_{t_0} + (t - t_0) V$ for $t \le t_0$

uniqueness $\Rightarrow \Gamma_t = \Gamma_0 + tV \quad \forall t$ $\square$